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**Life span and the problem of optimal
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Abstract

We reconsider the optimal population size problem in a continuous time economy populated by homogenous cohorts with a fixed life span. This assumption is combined with a linear production function in the labor input and standard rearing costs. A general social welfare function is specified, admitting the Millian and Benthamite cases as polar parameterizations. It is shown that if the lifetime is low enough, population is asymptotically driven to extinction whatever the utility function and the level of inter-generational altruism. Moreover, population is driven to extinction at finite time whatever the values of lifetime and altruism provided the utility function is negative. When the utility function is positive, it is shown that the Millian welfare function leads to optimal extinction at finite time whatever the lifetime. In contrast, the Benthamite case is much more involved: for isoelastic positive utility functions, it gives rise to two threshold lifetime values, say $T_0 < T_1$: below T_0 , finite time extinction is optimal; above T_1 , balanced growth paths are optimal. In between, asymptotic extinction is optimal. Last, intermediate welfare functions are studied, resulting in more complex optimal consumption and fertility dynamics compared to the two polar cases but delivering similar optimal extinction features.

Key words: Optimal population size, Benthamite Vs Millian criterion, finite lives, optimal extinction, optimal control of infinite dimensioned problems

JEL numbers: D63, D64, C61, 0 40

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1 Introduction

As recently outlined by Dasgupta (2005), the question of optimal population size traces back to antiquity. For example, Plato concluded that the number of citizens in the ideal city-state is 5,040, arguing that it is divisible by every number up to ten and have as many as 59 divisors, which would allow for the population to "... suffice for purposes of war and every peacetime activity, all contracts for dealings, and for taxes and grants" (cited in Dasgupta, 2005). A considerable progress has been made since then! A fundamental contribution to this normative debate is due to Edgeworth (1925) who considered the implications of total utilitarianism, originating from the classical Benthamite welfare function, for population and standard of living, in comparison with the alternative average utilitarianism associated with the Millian welfare function (see also Dasgupta, 1969). Edgeworth was the first to claim that total utilitarianism leads to a bigger population size and lower standard of living. Subsequent literature has aimed to study the latter claim in different frameworks. A first important inspection is due to Nerlove, Razin and Sadka (1985), who examined the robustness of Edgeworth's claim to parental altruism. They show that the claim still holds when the utility function of adults is increasing in the number of children and/or the utility of children.¹

The analysis uses simple arguments within a static model. Dynamic extensions were considered later. A question arises as to the robustness of Edgeworth's claim when societies experience long periods (say infinite time periods) of economic growth. Two endogenous growth papers with apparently opposite conclusions are worth mentioning here.² Razin and Yuen (1995) confirm Edgeworth's claim in an endogenous growth model driven by human capital accumulation with an explicit trade-off between economic growth and demographic growth deriving from an underlying time allocation between education and children rearing. In contrast, Palivos and Yip (1993) showed that Edgeworth's claim cannot hold for the realistic parameterizations of their model. Palivos and Yip used the framework of endogenous growth driven by an AK production function. The determination of optimal population size relies on the following trade-off: on one hand, the utility function depends explicitly on population growth rate; on the other, population growth has the standard linear dilution effect on physical capital accumu-

¹A connected philosophical literature is population ethics, as illustrated by the writings of Parfit for example (see Parfit, 1984).

²A more recent contribution to the optimal population size literature within the Ramsey framework can be found in Arrow et al. (2010).

lation. Palivos and Yip proved that in such a framework the Benthamite criterion leads to a smaller population size and a higher growth rate of the economy provided the intertemporal elasticity of substitution is lower than one (consistently with empirical evidence), that is when the utility function is negative. Indeed, a similar result could be generated in the setting of Razin and Yuen (1995) when allowing for negative utility functions.³

This paper is a contribution to the literature of optimal population size under endogenous growth in line with Palivos and Yip, and Razin and Yuen. It has the following three distinctive features:

1. First of all, it departs from the current literature by bringing into the analysis the life span of individuals. We shall assume that all individuals of all cohorts live a fixed amount of time, say T . The value of T will be shown to be crucial in the optimal dynamics and asymptotics of the model. Early exogenous increases in life expectancy have been invoked to be at the dawn of modern growth in several economic theory and historical demography papers (see for example Boucekkine et al., 2002), explaining a substantial part of the move from demographic and economic stagnation to the contemporaneous growth regime. We shall examine the normative side of the story. Our study can be also understood as a normative appraisal of natural selection. Admittedly, a large part of the life spans of all species is the result of a complex evolutionary process (see the provocative paper of Galor and Moav, 2007). Also it has been clearly established that for many species life span correlates with mass, genome size, and growth rate, and that these correlations occur at differing taxonomic levels (see for example Goldwasser, 2001).⁴ Our objective here is to show that the lifetime value is a dramatic determinant of optimal population size, which could be naturally connected to more appealing issues like for example the determinants of species' extinction. This point is made clear hereafter.
2. Second, in comparison with the AK models surveyed above which do not display transitional dynamics, our AK-like model does display transitional dynamics because of the finite lifetime assumption (just like in the AK vintage capital model studied in Boucekkine et al., 2005, and Fabbri and Gozzi, 2008). Optimal population extinction at finite time

³See also Boucekkine and Fabbri (2010).

⁴Of course, part of the contemporaneous increase of humans' life span is, in contrast, driven by health spending and medical progress. We shall abstract from the latter aspect, and take a fully exogenous view of life spans. See Arrow et al. (2010) for a model with health expenditures allowing to endogenize life spans.

or asymptotically can be therefore studied. Extinction is an appealing topic that has been much more explored in natural sciences than in economics. A few authors have already tried to investigate it both positively or normatively. On the positive side, one can mention the literature of the Easter Island collapse, and in particular the work of de la Croix and Dottori, (2008).⁵ On the normative side, one can cite the early work of Baranzini and Bourguignon (1995) or more recently Boucekkine and de la Croix (2009). Interestingly enough, the former considers a stochastic environment inducing an uncertain lifetime but the modelling leads to the standard deterministic framework once the time discounting is augmented with the (constant) survival probability.

3. Third, in order to address analytically the dynamic issues mentioned above, we shall consider a minimal model in the sense that we do not consider neither capital accumulation (as in Palivos and Yip) nor natural resources (as in Makdissi, 2001, and more recently Boucekkine and de la Croix, 2009). We consider one productive input, population (that's labor), and, in contrast to Palivos and Yip, the instantaneous utility function does not depend on population growth rate, that is we remove intratemporal (or instantaneous since time is continuous) altruism. Nonetheless, we share with the latter constant returns to scale: we therefore have an AN model with N the population size.⁶ By taking this avenue, population growth and economic growth will coincide in contrast to the previous related AK literature (and in particular to Razin and Yuen, 1995). However, we shall show clearly that the difference between the outcomes of the Millian and Benthamite cases is much sharper regarding optimal dynamics than long-term growth (which is the focus of the related existing AK literature).

Resorting to AN production functions and removing intratemporal altruism and capital accumulation has the invaluable advantage to allow for (non-trivial) analytical solutions to the optimal dynamics in certain parametric conditions. In particular, we shall provide optimal dynamics in closed-form in the two polar cases where the welfare function is Millian Vs Benthamite. It is important to notice here that considering finite lifetimes changes substantially the mathematical nature of the optimization problem under study. Because the induced state equations are no longer ordinary differential equations but delay differential equations, the problem is infinitely dimensioned.

⁵Again the literature on the Easter Island collapse is much more abundant in natural sciences and applied mathematics, see for example Basener and Ross (2004).

⁶Boucekkine and de la Croix (2009) have decreasing returns to labor, infinite lifetime and natural resources which depletion depends on population size.

Problems with these characteristics are tackled in Boucekkine et al. (2005), Fabbri and Gozzi (2008) and recently by Boucekkine, Fabbri and Gozzi (2010). We shall follow the dynamic programming approach used in the two latter papers. Because some of the optimization problems studied in this paper present additional peculiarities, a nontrivial methodological progress has been made along the way. The main technical details on the dynamic programming approach followed are however reported in the appendix given the complexity of the material.

Main findings

Several findings will be enhanced along the way.

1. A major outcome is that population and therefore economic growth (given the AN production structure) are optimal only if the individuals' lifetime is large enough. Otherwise, extinction is optimal at least asymptotically. Moreover, and in accordance with unified growth theory (Galor and Moav, 2002, and Boucekkine, de la Croix and Licandro, 2002), a larger lifetime does not only allow to escape from finite time or asymptotic extinction, it increases the optimal rate of demographic and economic growth.
2. While a large enough life span is necessary for growth to be optimal, it is not sufficient. We show two relatively robust cases of optimal finite time extinction. Indeed, the latter occurs when instantaneous utility is negative whatever the value of the life span. It also occurs under the Millian social welfare function independently of the sign of the utility function and the life span value.
3. Indeed, the optimal outcomes crucially depend on the welfare function. Once restricted to positive utility function, we highlight dramatic differences between the Millian and Benthamite cases in terms of optimal dynamics, which is to our knowledge a first contribution to this topic (the vast majority of the papers in the topic only are working on balanced growth paths). As mentioned just above, while the Millian welfare function leads to optimal population extinction at finite time, the Benthamite case does deliver a much more complex picture. We identify the existence of two threshold values for individuals' lifetime, say $T_0 < T_1$: below T_0 , finite time extinction is optimal; above T_1 , balanced growth paths (at positive rates) are optimal. In between, asymptotic extinction is optimal.
4. We also study the optimal dynamics for some intermediate welfare functions. We show that most of the conclusions drawn for the Benthamite

case hold for the intermediate welfare functions. In particular, we highlight the crucial role of individuals' lifetime in the shape of optimal outcomes. This said, we also clearly show that the optimal dynamics in the intermediate cases are much more complex: for example while the optimal fertility is constant in the Benthamite case, it shows up transitional dynamics in the intermediate cases. Moreover, while per-capita consumption is independent of the initial procreation profile in the Benthamite case, it does depend on initial conditions in the intermediate case: when intertemporal altruism is maximal, the social planner abstracts from the initial conditions when fixing optimal consumption level.

5. Because of the simple structure of the model, the predictions for optimal fertility rates and long-run level population sizes are not always compatible with the stylized facts of the demographic transition. For example, since the unique production input is labor, and utility only depends on consumption, a larger life span leads to a bigger fertility rate in our model. Also our setting implies that when finite-time extinction is ruled out, the societies which historically have the larger population levels end up *ceteris paribus* with the same status in the long-run. But this divergence property is accompanied by convergence in standards of living as captured by per capita consumption, an extreme form of it being generated with the Benthamite social welfare function. When long-term growth is optimal, which disqualifies *de facto* the Millian case, the social planner will bring consumption per capita to an optimal long-run level, independent of the initial conditions, which is not the case in standard AK theory.

Of course, our analysis is purely normative, and should not be evaluated roughly in terms of stylized facts reproduction. This said, we believe that the lessons extracted from our AN model are a rather good benchmark even from the positive point of view. The paper is organized as follows. Section 2 describes the optimal population model, gives some technical details on the maximal admissible growth and the boundedness of the associated value function, and displays some preliminary major results relating both finite-time and asymptotic extinction to either the value of individuals' lifetime or some characteristics of the utility functions assumed. Section 3 derives the optimal dynamics corresponding to the Millian Vs Benthamite cases. Section 4 studies the case of an intermediate welfare function. Section 5 concludes. The Appendices A and B are devoted to collect most of the proofs.

2 The optimal population size problem

2.1 The model

Let us consider a population in which every cohort has a fixed finite life span equal to T . Assume for simplicity that all the individuals remain perfectly active (i.e. they have the same productivity and the same procreation ability) along their life time. Moreover assume that, at every moment t , if $N(t)$ denotes the size of population at t , the size $n(t)$ of the cohort born at time t is bounded by $M \cdot N(t)$, where $M > 0$ measures the maximal (time-independent) biological reproduction capacity of an individual.

The dynamic of the population size $N(t)$ is then driven by the following delay differential equation (in integral form):

$$N(t) = \int_{t-T}^t n(s) ds, \quad (1)$$

and

$$n(t) \in [0, MN(t)], \quad t \geq 0. \quad (2)$$

The past history of $n(r) = n_0(r) \geq 0$ for $r \in [-T, 0)$ is known at time 0: it is in fact the initial datum of the problem. This features the main mathematical implication of assuming finite lives. Pointwise initial conditions, say $n(0)$, are no longer sufficient to determine a path for the state variable, $N(t)$. Instead, an initial function is needed. The problem becomes infinitely dimensioned, and the standard techniques do not immediately apply. Summarizing, (1) becomes:

$$N(t) = \int_{t-T}^t n(s) ds, \quad n(r) = n_0(r) \geq 0 \text{ for } r \in [-T, 0), \quad N(0) = \int_{-T}^0 n(r) dr. \quad (3)$$

Note that the constraint (2) together with the positivity of n_0 ensure the positivity of $N(t)$ for all $t \geq 0$.

We consider a closed economy, with a unique consumption good, characterized by a labor-intensive aggregate production function exhibiting constant returns to scale, that is

$$Y(t) = aN(t). \quad (4)$$

Note that by equation (1) we are assuming that individuals born at any date t start working immediately after birth. Delaying participation into the labor market is not an issue but adding another time delay into the model will only complicate unnecessarily the computations without altering

substantially our findings. Note also that there is no capital accumulation in our model. The linearity of the production technology is necessary to generate long-term growth, it is also adopted in the related bulk of papers surveyed in the introduction. If decreasing returns were introduced, that is $Y(t) = aN^\alpha$ with $\alpha < 1$, growth will vanish, and we cannot in such a case connect life span with economic and demographic growth. This said, we shall comment along the way on how the results of the paper could be altered if one switches from constant to decreasing returns, namely from endogenous to exogenous growth.

Output is partly consumed, and partly devoted to raising the newly born cohort, say rearing costs. In this benchmark we assume that the latter costs are linear in the size of the cohort, which leads to the following resource constraint:

$$Y(t) = N(t)c(t) + bn(t) \quad (5)$$

where $b > 0$. Again we could have assumed that rearing costs are distributed over time but consistently with our assumption of immediate participation in the labor market, we assume that these costs are paid once for all at time of birth. On the other hand, the linearity of the costs is needed for the optimal control problem considered above to admit closed-form solutions. As it will clear along the way, this assumption is much more innocuous than the AN production function considered. This seems rather natural: if extinction is optimal for linear costs, this property is likely to hold *a fortiori* for stronger strictly convex costs.

Let us describe now accurately the optimal control problem handled. The controls of the problem are $n(\cdot)$ and $c(\cdot)$ but, using (4) and (5), one obtains

$$n(t) = \frac{(a - c(t))N(t)}{b}. \quad (6)$$

so we have only to choose $c(t)$. Since we want both per-capita consumption and the size of new cohorts to remain positive, we need to ensure:

$$0 \leq c(t) \leq a. \quad (7)$$

In other words we consider the controls $c(\cdot)$ in the set⁷

$$\mathcal{U}_{n_0} := \{c(\cdot) \in L^1_{\text{loc}}(0, +\infty; \mathbb{R}_+) : \text{eq. (7) holds for all } t \geq 0\}.$$

⁷The space $L^1_{\text{loc}}(0, +\infty; \mathbb{R}_+)$ appearing in the definition of \mathcal{U}_{n_0} is defined as $L^1_{\text{loc}}(0, +\infty; \mathbb{R}_+) := \{f: [0, +\infty) \rightarrow \mathbb{R}_+ : f \text{ measurable and } \int_0^T |f(x)| dx < +\infty, \forall T > 0\}$.

We shall consider the following social welfare function to be maximized within the latter set of controls:

$$\int_0^{+\infty} e^{-\rho t} u(c(t)) N^\gamma(t) dt, \quad (8)$$

where $\rho > 0$ is the time discount factor, $u: (0, +\infty) \rightarrow \mathbb{R}$ is a continuous, strictly increasing and concave function, and $\gamma \in [0, 1]$ is a parameter allowing to capture the altruism of the social planner. More precisely, γ measures the degree of *intertemporal altruism* of the planner in that the term $N^\gamma(t)$ is a determinant of the discount rate at which the welfare of future generations is discounted. While intratemporal welfare is not considered here (as mentioned in the introduction section) in order to extract closed-form solution to optimal dynamics, intertemporal altruism is kept to study the two polar cases outlined above: indeed, $\gamma = 0$ covers the case of average utilitarianism, that's the Millian social welfare function, and $\gamma = 1$ is the Benthamite social welfare function featuring total utilitarianism. We shall also solve an intermediate case, $0 < \gamma < 1$, in Section 4. Our modeling of intertemporal altruism is nowadays quite common. Recently, Strulik (2005) and Bucci (2008) have studied the impact of population growth on economic growth within endogenous growth settings, keeping population growth exogenous and introducing intertemporal altruism as above. In particular, Strulik (2005) shows that population growth rate has a positive impact on economic growth through the latter discounting (or time preference) channel. In our framework, population growth is endogenous in line with Palivos and Yip (1993).

2.2 Maximal admissible growth

We begin our analysis by giving a sufficient condition ensuring the boundedness of the value function of the problem. The arguments used are quite intuitive so we mostly sketch the proofs.⁸

Consider the state equation (3) with the constraint (7). Given an initial datum $n_0(\cdot) \geq 0$ (and then $N_0 = \int_{-T}^0 n_0(r) dr$), we consider the admissible control defined as $c_{MAX} \equiv 0$. This control obviously gives the maximal population size allowed, associated with $n_{MAX}(t) = \frac{a}{b} N(t)$ by equation (6): it is the control/trajectory in which all the resources are assigned to raising the newly born cohorts with nothing left to consumption. Call the trajectory related to such a control $N_{MAX}(\cdot)$. By definition $N_{MAX}(\cdot)$ is a solution to

⁸The reader interested in technical details in the proofs of Lemma 2.1 and Proposition 2.1 is reported to Propositions 2.1.6, 2.1.10 and 2.1.11 in Fabbri and Gozzi (2008).

the following delay differential equation (written in integral form):

$$N_{MAX}(t) = \int_{(t-T) \wedge 0}^0 n_0(s) ds + \frac{a}{b} \int_{(t-T) \vee 0}^t N_{MAX}(s) ds. \quad (9)$$

The characteristic equation of such a delay differential equation is⁹

$$z = \frac{a}{b} (1 - e^{-zT}). \quad (10)$$

It can be readily shown (see e.g. Fabbri and Gozzi, 2008, Proposition 2.1.8) that if $\frac{a}{b}T > 1$, the characteristic equation has a unique strictly positive root ξ . This root belongs to $(0, \frac{a}{b})$ and it is also the root with maximal real part. If $\frac{a}{b}T \leq 1$, then all the roots of the characteristic equation have non-positive real part and the root with maximal real part is 0. In that case, we define $\xi = 0$. We have that (see for example Diekmann et al., 1995, page 34), for all $\epsilon > 0$,

$$\lim_{t \rightarrow \infty} \frac{N_{MAX}(t)}{e^{(\xi+\epsilon)t}} = 0, \quad (11)$$

and that the dynamics of $N_{MAX}(t)$ are asymptotically driven by the exponential term corresponding to the root of the characteristic equation with the largest real part. This natural property is therefore also shown to hold for our infinitely dimensioned law of motion. As it will be shown later, this result drives the optimal economy to extinction when individuals' lifetime is low enough. At the minute, notice that since $N_{MAX}(\cdot)$ is the trajectory obtained when all the resources are diverted from consumption, it is the trajectory with the largest population size. More formally, one can write:

Lemma 2.1 *Consider a control $\hat{c}(\cdot) \in \mathcal{U}_{n_0}$ and the related trajectory $\hat{N}(\cdot)$ given by (1). We have that*

$$\hat{N}(t) \leq N_{MAX}(t), \quad \text{for all } t \geq 0.$$

The previous lemma, coupled with property (11), straightforwardly implies the following sufficient condition for the value function of the problem to be bounded:

Proposition 2.1 *The following hypothesis*

$$\rho > \gamma\xi \quad (12)$$

is sufficient to ensure that the value function

$$V(n_0) := \sup_{\hat{c}(\cdot) \in \mathcal{U}_{n_0}} \int_0^{+\infty} e^{-\rho t} u(\hat{c}(t)) \hat{N}^\gamma(t) dt$$

⁹As for any linear dynamic equation (in integral or differential form), the characteristic equation is obtained by looking at exponential solutions, say e^{zt} , of the equation.

is finite (again we denoted with $\hat{N}(\cdot)$ the trajectory related to the control $\hat{c}(\cdot)$).

The proofs of the two results above follow the line of Propositions 2.1.10 and 2.1.11 in Fabbri and Gozzi (2008), proving that $V(n_0) < +\infty$ using a bound for every admissible strategy and $V(n_0) > -\infty$ using that for at least for an admissible strategy the welfare function is finite.

We are now ready to provide the first important result of the paper highlighting the case of asymptotic extinction.

2.3 A preliminary extinction result

We provide now a general extinction property inherent to our model. Recall that when $\frac{a}{b}T \leq 1$, all the roots of the characteristic equation of the dynamic equation describing maximal population, that is equation (9), have non-positive real part, which may imply that maximal population goes to zero asymptotically (asymptotic extinction). The next proposition shows that this is actually the case for any admissible control in the case where $\frac{a}{b}T < 1$.

Proposition 2.2 *If $\frac{a}{b}T < 1$ then every admissible control drives the system to extinction.*

Proof. Thanks to Lemma 2.1 it is enough to prove the statement for $N_{MAX}(t)$. Let us take $\bar{t} \in \arg \max_{s \in [T, 2T]} N_{MAX}(s)$ (the argmax is non-void because N_{MAX} is continuous on $[0, +\infty)$). We have that $N_{MAX}(\bar{t}) = \frac{a}{b} \int_{\bar{t}-T}^{\bar{t}} N_{MAX}(s) ds \leq a/b(2T - \bar{t}) \max_{s \in [0, T]} N_{MAX}(s) + a/b(\bar{t} - T)N_{MAX}(\bar{t})$ so $N_{MAX}(\bar{t}) \leq \frac{a/b(2T - \bar{t})}{1 - a/b(\bar{t} - T)} \max_{s \in [0, T]} N_{MAX}(s)$. Observe that, for all $\bar{t} \in [T, 2T]$ we have that $\frac{a/b(2T - \bar{t})}{1 - a/b(\bar{t} - T)} \in [0, \frac{a}{b}T]$, so $\max_{s \in [T, 2T]} N_{MAX}(s) \leq \frac{a}{b}T \max_{s \in [0, T]} N_{MAX}(s)$. In the same way we can prove that, for all positive integer n , $\max_{s \in [nT, (n+1)T]} N_{MAX}(s) \leq (\frac{a}{b}T)^n \max_{s \in [0, T]} N_{MAX}(s)$. Since, by hypothesis, $(\frac{a}{b}T) < 1$ we have that $\lim_{t \rightarrow +\infty} N_{MAX}(t) = 0$ and then the claim. \square

The value of individuals lifetime is therefore crucial for the optimal (and non-optimal) population dynamics. If this value is not large enough, then, independently of the welfare function, that's independently of the strength of intertemporal altruism (or parameter γ), population will vanish asymptotically. This property is consistent with the theories emphasizing the role of decreasing mortality, or equivalently increasing life expectancy, in the development process (see Galor and Weil, 1999, Galor and Moav, 2002, and Boucekkine, de la Croix and Licandro, 2002). The optimal population size is asymptotically zero and the economy is not sustainable in the long run

if individuals' lifetime does not exceed a threshold value, equal to $\frac{b}{a}$: the larger the productivity of these individuals, the lower this threshold is, and the larger the rearing costs, the larger the threshold is.¹⁰ An originally non-sustainable economy can be made sustainable by two types of exogenous impulses: technological shocks (via a or b) or demographic shocks (via T).

Notice that the extinction property outlined above is asymptotic. Could we have finite time optimal extinction in our model? The next section shows that this possibility exists and depends crucially on the welfare function chosen.

Proposition 2.3 *If $u(a) \leq 0$ then the optimal strategy is $n^*(\cdot) \equiv 0$ so the system is driven to extinction at the finite time T .*

Proof. Consider the admissible strategy $n^*(\cdot) \equiv 0$. Then the associated welfare value is

$$\int_0^T e^{-\rho t} u(a) [N^*(t)]^\gamma dt \leq 0$$

Take any other admissible strategy $\hat{n}(\cdot)$. Since $\hat{c}(t) \leq a$ and u is increasing we have $u(c(t)) \leq u(a) \leq 0$ for every $t \in [0, T]$ and $u(c(t)) < u(a) \leq 0$ when $c(t) \neq a$. Moreover it must be, by (1),

$$\hat{N}(t) \geq N^*(t).$$

Then the claim follows. □

Corollary 2.1 *Consider the case $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$ with $\sigma > 1$. Then by Proposition 2.3, the system is driven to extinction at the finite time T .*

The corollary indicates that the most standard utility functions can fall in the somewhat trivial finite time extinction case of Proposition 2.3. Moreover, it shows that it is so for rather realistic values of σ (involving intertemporal elasticities of substitution below unity, as stipulated by Palivos and Yip). Clearly, the sign of the utility function is crucial in our set-up. This is by no means a specific property. A very clear illustration of this point is in Baranzini and Bourguignon (1995) for example.¹¹ In this paper, we shall require

$$u(a) > 0. \tag{13}$$

¹⁰If $T = \frac{b}{a}$, not all the admissible trajectories drive the system to extinction: indeed if we have for example the constant initial datum $N(t) = 1$ for all $t < 0$ or $n(t) = a/b$ for all $t < 0$, the (admissible) maximal control $N_{MAX}(t)$ allows to maintain the population constant equal to 1 for every t .

¹¹The more recent literature on optimal growth with endogenous discounting has also integrated this point, see for example Schumacher (2009).

A very standard isoelastic utility function meeting this condition for any $\sigma > 0$ (thus including $\sigma > 1$) is: $u(c) = \frac{c^{1-\sigma}-1}{1-\sigma}$, which degenerates into the logarithmic utility function when σ goes to 1. For this function, whatever $\sigma > 0$, it is always possible to find an interval of a 's values ensuring condition (13). A more general class of function sharing this property is $u(c) = \frac{c^{1-\sigma}-R}{1-\sigma}$, where $R \geq 0$ plays the role of minimal consumption.¹² We shall consider the cases $R = 0$ and $R = 1$ in the main text of this paper. The Appendix studies the problem for any $R \geq 0$.

Remark 2.1 Before getting to the analysis of the Millian Vs Benthamite social welfare function, let us discuss briefly the robustness of our results in this section to departures from the linearity assumptions made on the cost and production functions. Introducing a strictly convex rearing function, say replacing bn by bn^β with $\beta > 1$, will obviously not alter the message of the extinction Proposition 2.2 and 2.3. Things are apparently more involved if we move from the linear production function $Y = aN$ to $Y = aN^\alpha$, with $\alpha < 1$. First note that in such a case the resource constraint (5) becomes

$$aN(t)^\alpha = Y(t) = N(t)c(t) + bn(t)$$

that is

$$c(t) = aN(t)^{\alpha-1} - b\frac{n(t)}{N(t)}. \quad (14)$$

The trajectory of maximum population growth (found taking $c(t) \equiv 0$) is now the solution of

$$\dot{N}_{MAX}(t) = \frac{a}{b} (N_{MAX}^\alpha(t) - N_{MAX}^\alpha(t-T))$$

This equation has two equilibrium points: $N_0 = 0$ which is unstable, and $N_1 > 0$ which is asymptotically stable and attracts all positive data. This implies that the existence result of Proposition 2.1 holds for all $\rho > 0$ and the result of Proposition 2.2 does not hold.

While the asymptotic extinction optimal outcome vanishes in the absence of endogenous growth, the finite time extinction result in Proposition 2.3 remains unaffected if we assume the stronger condition (that is satisfied for example when $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$) that $u(c) \leq 0$ for all $c > 0$. In that case the monotonicity arguments used in proof remain untouched and the system is again driven to finite time extinction. So in particular Corollary 2.1 holds without any changes.

¹²Baranzini and Bourguignon (1995) consider a similar class of utility functions.

3 The Benthamite Vs Millian case

In this section, we perform the traditional comparison between the outcomes of the polar Benthamite Vs Millian cases. Nonetheless, our comparison sharply departs from the existing work (like in Nerlove et al., 1985, or Palivos and Yip, 1993) in that we are able to extract a closed-form solution to optimal dynamics, and therefore we compare the latter. Traditional comparison work only considers steady states.¹³ This focus together with the finite lifetime specification allows to derive several new results. In first place, we shall show consistently with the asymptotic extinction property highlighted above, that the value of individuals' life time is crucial for the shape of optimal dynamics. Second we are able to identify a much stronger asymmetry between the Benthamite and Millian cases: in our AN setting with finite lives, the Millian case systematically leads to optimal finite-time extinction while the Benthamite welfare function is compatible with several asymptotic configurations (including asymptotic but not finite-time extinction) depending on the value of the lifetime T .

3.1 The Millian case: $\gamma = 0$

This case can be treated straightforwardly. Indeed, in the absence of intertemporal altruism, the functional (8) reduces to

$$\int_0^{+\infty} e^{-\rho t} u(c(t)) dt. \quad (15)$$

and, since we can freely choose $c(t) \in [0, a]$ for all $t \geq 0$, the following claim is straightforward:

Proposition 3.1 *Consider the problem of maximizing (8) with $\gamma = 0$ subject to the state equation (3) and the constraint (7). Then the optimal control is given by $c^*(t) \equiv a$, so that, from (6), $n^*(t) \equiv 0$.*

Since the objective function depends only on consumption, and since it is increasing in the latter, the optimal control $c^*(t) \equiv a$, or equivalently $n^*(t) \equiv 0$, is obvious: in the Millian case, it is always optimal to not procreate. A direct implication of this property is finite-time extinction:

Corollary 3.1 *For the solution of the optimal control problem described in Proposition 3.1, population extinction occurs at a certain time $\bar{t} \leq T$.*

¹³As mentioned above, Palivos and Yip have an AK model, so their model does not display transition dynamics.

Some comments are in order here. In our set-up, the absence of intertemporal altruism makes procreation sub-optimal at any date. And this property is independent of the deep parameters of the problem: it is independent of the value of individuals' lifetime, T , of the value of intertemporal elasticity of substitution (determined by σ , and of the technological parameters, a and b). One would think that a higher enough labor productivity, a , and/or a lower enough marginal cost, b , would make procreation optimal at least along a transition period. This does not occur at all. Much more than in the AK model built up by Palivos and Yip, our benchmark enhances the implications of intertemporal altruism, which will imply a much sharper distinction between the outcomes of the Millian Vs Benthamite cases. This will be clarified in the next section. Before, it is worth pointing out that Proposition 3.1 is robust to departures from linearity. Indeed, the finite-time extinction result does not at all depend on the linear cost function, $bn(t)$, adopted. Even if we consider a more general cost $C(n(t))$, the behavior of the system does not change in the Millian case: in this case the production would be again equal to $Y(t) = aN(t)$, resulting in $c(t) = a - C(n(t))/N(t)$, so, any admissible function $C(\cdot)$ would work (for example $C(0) = 0$ and $C(\cdot)$ increasing and strictly convex): again optimal $c(t)$ should be picked in the interval $[0, a]$, and as before, one would have to choose $c(t) = a$ or $n(t) = 0$, leading to finite-time extinction.

Last but not least, it is worth pointing out that the optimal finite time extinction property identified here holds also under decreasing returns: the result described in Proposition 3.1 can be replicated without changes. Again only per-capita consumption enters the utility function and again the highest per-capita consumption is obtained taking $n \equiv 0$. Note that, differently from the linear case, here the per-capita consumption is not bounded by a but, when the population approaches to extinction, thanks to (14), tends to infinity, so in a sense the incentive to choose $n = 0$ is even greater.

3.2 The Benthamite case: $\gamma = 1$

We now come to the Benthamite case. This case is much more complicated than the first one. In particular, the mathematics needed to characterize the optimal dynamics is complex, relying on advanced dynamic programming techniques in infinite-dimensional Hilbert spaces. Technical details are given in Appendix A. To get a quick idea of the method, we summarize here the steps taken.

1. First of all, we have to define a convenient functional Hilbert space. We denote by $L^2(-T, 0)$ the space of all functions f from $[-T, 0]$ to \mathbb{R}

that are Lebesgue measurable and such that $\int_{-T}^0 |f(x)|^2 dx < +\infty$. It is an Hilbert space when endowed with the scalar product $\langle f, g \rangle_{L^2} = \int_{-T}^0 f(x)g(x) dx$. We consider the Hilbert space $M^2 := \mathbb{R} \times L^2(-T, 0)$ (with the scalar product $\langle (x_0, x_1), (z_0, z_1) \rangle_{M^2} := x_0 z_0 + \langle x_1, z_1 \rangle_{L^2}$).

2. Then we translate our initial optimal control problem of a delay differential equation as an optimal control problem of an ordinary differential equation in this infinite dimensional Hilbert space.
3. Finally we write the corresponding Hamilton-Jacobi-Bellman equation in the Hilbert space, and we seek for explicit expressions for the value function, which in turn gives the optimal feedback in closed form.

We shall also illustrate along the way to which extent our methodology and results are sensitive to the specification of the utility function. By condition (13), we already know that the sign of the utility function is crucial in the problem. As announced before, we study in this section the optimal outcomes of the Benthamite case when the utility function takes the form

$$u(c) = \frac{c^{1-\sigma} - R}{1 - \sigma},$$

when $R = 0, 1$.

Some preliminary manipulations are needed. First we need to rewrite the optimal control problem using $n(\cdot)$ as a control instead of $c(\cdot)$: using (4) and (5) we obtain

$$c(t) = \frac{aN(t) - bn(t)}{N(t)}. \quad (16)$$

Since we want per-capita consumption to remain positive, we need $n(t) \leq \frac{a}{b}N(t)$, so that:

$$0 \leq n(t) \leq \frac{a}{b}N(t) \quad t \geq 0. \quad (17)$$

The previous constraint can be rewritten by requiring $n(t)$ to be in the set

$$\mathcal{V}_{n_0} := \{n(\cdot) \in L^1_{\text{loc}}(0, +\infty; \mathbb{R}_+) : \text{conditions (17) hold for all } t \geq 0\}. \quad (18)$$

3.2.1 The case $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$

If we choose $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$ for $\sigma \in (0, 1)$ (for $\sigma > 1$ see Corollary 2.1, note that for $\sigma \in (0, 1)$ the condition (13) is always satisfied) the functional (8) can be

rewritten as

$$\int_0^{+\infty} e^{-\rho t} \frac{(aN(t) - bn(t))^{1-\sigma}}{1-\sigma} N^{\gamma-(1-\sigma)}(t) dt. \quad (19)$$

If $\gamma = 1$, the functional simplifies into

$$\int_0^{+\infty} e^{-\rho t} \frac{(aN(t) - bn(t))^{1-\sigma}}{1-\sigma} N^\sigma(t) dt. \quad (20)$$

For the value-function to be bounded, we can use the general sufficient condition (12): when $\gamma = 1$, it amounts to

$$\rho > \xi. \quad (21)$$

Recall that we have $\xi = 0$ when (10) does not have any strictly positive roots, i.e. when $\frac{a}{b}T \leq 1$. Moreover if we define

$$\beta := \frac{a}{b}(1 - e^{-\rho T}). \quad (22)$$

then equation (21) implies (see e.g. Fabbri and Gozzi 2008, equation (15))

$$\rho > \beta \iff \frac{\rho}{1 - e^{-\rho T}} > \frac{a}{b}. \quad (23)$$

The following theorem states a sufficient parametric condition ensuring the existence of an optimal control and characterize it.

Theorem 3.1 *Consider the functional (20) with $\sigma \in (0, 1)$. Assume that (21) holds and let β given by (22). Then there exists a unique optimal control $n^*(\cdot)$.*

- If

$$\beta \leq \rho(1 - \sigma) \iff \frac{\rho}{1 - e^{-\rho T}} \leq \frac{a}{b} \cdot \frac{1}{1 - \sigma}, \quad (24)$$

then the optimal control is $n^*(\cdot) \equiv 0$ and we have extinction at time T .

- If

$$\beta > \rho(1 - \sigma) \iff \frac{\rho}{1 - e^{-\rho T}} > \frac{a}{b} \cdot \frac{1}{1 - \sigma}, \quad (25)$$

then, setting

$$\theta := \frac{a}{b} \cdot \frac{\beta - \rho(1 - \sigma)}{\beta\sigma} = \frac{a}{b} \left[\frac{1}{\sigma} - \frac{\rho(1 - \sigma)}{\beta\sigma} \right] = \frac{1}{\sigma} \frac{a}{b} + \frac{\rho}{1 - e^{-\rho T}} \left(1 - \frac{1}{\sigma} \right) \quad (26)$$

we have $\theta \in (0, \frac{a}{b})$. The optimal control $n^*(\cdot)$ and the related trajectory $N^*(\cdot)$ satisfy

$$n^*(t) = \theta N^*(t). \quad (27)$$

Along the optimal trajectory the per-capita consumption is constant. Its value is

$$c^*(t) = \frac{aN^*(t) - bn^*(t)}{N^*(t)} = a - b\theta \in (0, a). \quad (28)$$

Moreover the optimal control $n^*(\cdot)$ is the unique solution of the following delay differential equation

$$\begin{cases} \dot{n}(t) = \theta (n(t) - n(t - T)), & \text{for } t \geq 0 \\ n(0) = \theta N_0 \\ n(s) = n_0(s), & \text{for all } s \in [-T, 0). \end{cases} \quad (29)$$

The proof is in Appendix A. In contrast to the Millian case, there is now room for optimal procreation, and therefore for both demographic and economic growth. When $\gamma = 1$, intertemporal altruism is maximal, and such an ingredient may be strong enough in certain circumstances (to be specified) to offset the anti-procreation forces isolated in the analysis of the Millian case. Some comments on the optimal control identified are in order here.

1. In the Benthamite case, finite time extinction is still possible. This occurs when parameter β is low enough. By definition (see definition (22)), this parameter measures a kind of adjusted productivity of the individual: productivity, a , is adjusted for the fact that individuals live a finite life (through the term $1 - e^{-\rho T}$), and also for the rearing costs they have to pay along their lifetime. If this adjusted productivity parameter is too small, the economy goes to extinction at finite time. And this possibility is favored by larger time discount rates and intertemporal elasticities of substitution (under $\sigma < 1$). Longer lives, better productivity and lower rearing costs can allow to escape from this scenario, although even in such cases, the economy is not sure to avoid extinction asymptotically (see Proposition 3.2 below). In particular, it is readily shown that condition (24), ruling out finite time extinction, is fulfilled if and only if $T > T_0$, where T_0 is the threshold value induced by (24), which depends straightforwardly on the parameters of the model.
2. Notice that θ , that's the optimal constant size of new cohorts as a proportion of total population, can be interpreted as a reproduction

or fertility rate. The theorem opens the door for the optimal reproduction rate to be strictly positive in the Benthamite case in contrast to the Millian case where this rate is optimally zero. It is also worth mentioning that the optimal fertility rate is constant over time, just like consumption per capita. As it will be clear in Section 4, this is a distinctive property of the Benthamite case in our framework. In the intermediate case considered later, we show that optimal fertility and consumption show up transitory dynamics.

3. By (26), one can see that the reproduction rate is increasing in labor marginal productivity, a , and decreasing in the marginal rearing cost, b . One can also observe that as T rises, the reproduction rate goes up since $\sigma < 1$. Though our analysis is normative, it is worth mentioning here that the latter result is inconsistent with the demographic transition picture according to which increments in life expectancy are followed by lower reproduction or fertility rates. In our setting, it is optimal to increase population size if adjusted productivity β rises, which is the case when for example individuals' lifetime goes up (or their labor productivity, a , is stimulated). For optimal dynamics to coincide with the demographic transition, our setting should be enriched. An obvious avenue is to allow for another input than labor to partially disconnect production from labor availability. This goes beyond the objectives of this paper.
4. Equation (29) gives the optimal dynamics of cohort's size $n(t)$. This linear delay differential equation is similar to the one analyzed by Boucekkine et al. (2005) and Fabbri and Gozzi (2008). The dynamics depends on the initial function, $n_0(t)$ and on the parameters θ and T in a way that will be described below. Notice here that optimal trajectories of the demographic variables show transition dynamics while optimal consumption per capita is constant. This property is also identified in the AK vintage capital model studied in Boucekkine et al. (2005).

We now dig deeper in the dynamics properties and asymptotics of optimal trajectories. The following proposition summarizes the key points.

Proposition 3.2 *Consider the functional (20) with $\sigma \in (0, 1)$. Assume that (21) and (25) hold, so $\theta \in (0, \frac{a}{b})$. Then*

- If $\theta T < 1$ then $n^*(t)$ (and then $N^*(\cdot)$) goes to 0 exponentially.
- If $\theta T > 1$ then the characteristic equation of (29)

$$z = \theta (1 - e^{-zT}), \quad (30)$$

has a unique strictly positive solution h belonging to $(0, \theta)$ while all the other roots have negative real part. Moreover¹⁴

$$\lim_{t \rightarrow \infty} \frac{n^*(t)}{e^{ht}} = \frac{\theta}{1 - T(\theta - h)} \int_{-T}^0 (1 - e^{(-s-T)h}) n_0(s) ds > 0$$

and

$$\lim_{t \rightarrow \infty} \frac{N^*(t)}{e^{ht}} = \frac{1 - e^{-hT}}{h} \frac{\theta}{1 - T(\theta - h)} \int_{-T}^0 (1 - e^{(-s-T)h}) n_0(s) ds > 0$$

The proof is in Appendix B. The proposition above highlights the dynamic and asymptotic properties of the optimal control when finite time extinction is ruled out, that it is when $T > T_0$. Abstracting away at the minute from the fact that θ depends on T , one can see that the proposition adds another threshold value on individuals' lifetime: we have an asymptotic extinction property if and only if individuals' lifetime is below a threshold consistently with Proposition 2.2.¹⁵ Notice that extinction here is asymptotic in contrast to the Millian case where optimal extinction takes place at finite time whatever the individuals' lifetime,¹⁶ or to the Benthamite case where optimal extinction occurs at finite time when $T < T_0$. The proposition uncovers the existence of a second threshold, $T_1 > T_0$, defined by $T = \frac{1}{\theta}$, such that asymptotic extinction is optimal when $T_0 < T < T_1$, while population and economic growth are optimal when $T > T_1$.

The argument can be made more accurate once accounting for the dependence of θ on T . Actually, function $T \mapsto T\theta(T)$ is strictly increasing in T , at least as long as $\theta(T)$ remains in $[0, \frac{a}{b})$, which is the interval in which our main theorem works. In particular if $\sigma \in (0, 1)$, which is the parametric case considered in this sub-section, this property is obvious because both $\theta(T)$ and T are increasing in T .¹⁷ This validates the analysis before on the existence of a threshold value T_1 . One can readily show that indeed $T_0 < T_1$. This allows to formulate the following important result.

Corollary 3.2 *Under the conditions of Theorem 3.1, there exist two threshold values for individuals' lifetime, T_0 and T_1 , $0 < T_0 < T_1$ such that:*

1. for $T < T_0$, finite-time extinction is optimal,

¹⁴Observe that $(1 - e^{(-s-T)h})$ is always positive for $s \in [-T, 0]$ and the constant $\frac{1}{1 - T(\theta - h)}$ can be easily proved to be positive too.

¹⁵Here the threshold is $\frac{1}{\theta}$, which is even larger than the threshold identified in Proposition 2.2.

¹⁶Of course, in this case, the longer the lifetime, the later extinction will take place.

¹⁷If $\sigma > 1$ then by computing the derivative $\frac{d}{dT} T\theta(T)$ one can easily see that it is always greater than $\frac{1}{\sigma} \frac{a}{b}$.

2. for $T_0 < T < T_1$, asymptotic extinction is optimal,
3. for $T > T_1$, economic and demographic growth (at positive rate) is optimal.

Proposition 3.2 brings indeed further important results. If individuals' lifetime is large enough (i.e. above the threshold T_1), then both the cohort size and population size will grow asymptotically at a strictly positive rate. In other words, these two variables will go to traditional balanced growth paths (BGPs). As in standard endogenous growth theory, the levels of the BGPs depend notably on the initial conditions, here the initial function $n_0(t)$. Proposition 3.2 derives explicitly these long-run levels such that their dependence on the initial datum is explicitly given. Three more aspects should be mentioned here.

1. A first one has to do with the shape of the optimal paths. One would like to know how they look like once growth is taken out, that is after detrending. It can be readily shown that detrended trajectories are oscillatory as demonstrated by Boucekkine et al. (2005) for delay differential equations similar to (29). The mechanism behind is the so-called replacement echoes, which is induced by the finite life characteristic.
2. A second observation concerns the precise role of initial conditions. Proposition 3.2 shows that once finite-time extinction is ruled out and since the growth rate h is such that $1 - T(\theta - h) > 0$, the long-run level of cohort and population sizes are positively correlated with the "historical" values of the cohort size (that is $n_0(t)$'s values for $t < 0$). Every thing equal elsewhere, the countries with the largest historical values will end up with the largest long-run population levels. So once finite-time extinction is ruled out, the process of optimal economic and demographic development designed here will not alter the historical ranking in terms of population levels. Since output only depends on labor input, the same conclusion can be made for output levels. This said, and given that both optimal per capita consumption and fertility rates are constant and independant of the initial data, the Benthamite case does also generate convergence outcomes: even if two countries differ in their historical demography, they will be assigned the same amount of consumption and children per capita by the social planner. This is a quite peculiar property for an AK-type model, it is driven by its endogenous fertility component.
3. The last and crucial aspect has to do with the correlation between the BGP growth rate, h , and individuals' lifetime, T . The unified growth

literature has been inspecting this relationship from different theoretical and empirical perspectives (see Galor and Moav, 2002, Boucekkine, de la Croix and Licandro, 2002, and Hazan and Zoaby, 2006). The following proposition gives the prediction of our model regarding this relationship. Not surprisingly, and in line with Boucekkine et al. (2002), our model predicts that the optimal growth rate is an increasing function of individuals' lifetime.

Proposition 3.3 *Suppose that the hypotheses of Proposition 3.2 are satisfied, then h is strictly increasing in T : a higher T implies a higher growth rate h .*

Before getting to the next section, it is worth commenting a bit on what would deliver the Benthamite case in the absence of growth, that is when the production function has the form $Y(t) = aN^\alpha(t)$ with $\alpha < 1$. Needless to say, in such a case, long-term growth being ruled out, the picture cannot be replicated by construction. Recall that, in the decreasing returns case, the trajectory of maximum population growth (found taking $c(t) \equiv 0$) is given by

$$\dot{N}_{MAX}(t) = \frac{a}{b} (N_{MAX}^\alpha(t) - N_{MAX}^\alpha(t - T)),$$

which has two equilibrium points, $N_0 = 0$ which is unstable, and $N_1 > 0$ which is asymptotically stable. In such a case, one expects finite time extinction to never occur, and in the absence of growth, convergence to an equilibrium point N_2 smaller than the maximal one N_1 to set in.

3.2.2 The case $u(c) = \frac{c^{1-\sigma}-1}{1-\sigma}$

We analyze now what happens if we choose $u(c) = \frac{c^{1-\sigma}-1}{1-\sigma}$, $\sigma > 0$ (and $\sigma \neq 1$). To satisfy the condition (13) we need

$$\frac{a^{1-\sigma} - 1}{1 - \sigma} > 0. \tag{31}$$

For any positive value of σ , there exists an interval of a 's values such that the latter condition is checked. One could therefore accommodate any value of σ in contrast to the previous sub-section where this parameter was restricted to be below unity. We now show how the properties outlined in the previous sub-section can be reproduced with this utility function. Since the economic mechanisms and properties are the same as in the previous sub-section, we will mainly (and briefly) highlight the technical steps taken.

The functional (8) can be rewritten, when $\gamma = 1$ as

$$\begin{aligned} \int_0^{+\infty} e^{-\rho t} \frac{\left(\frac{aN(t)-bn(t)}{N(t)}\right)^{1-\sigma} - 1}{1-\sigma} N(t) dt \\ = \int_0^{+\infty} e^{-\rho t} \left(\frac{(aN(t) - bn(t))^{1-\sigma} N^\sigma(t)}{1-\sigma} - \frac{N(t)}{1-\sigma} \right) dt \end{aligned} \quad (32)$$

The next proposition is the counterpart of Theorem 3.1 when the utility function is $u(c) = \frac{c^{1-\sigma}-1}{1-\sigma}$.

Theorem 3.2 *Suppose that (21) (and then (23)) holds, with $\sigma > 0$ ($\sigma \neq 1$). Assume (31) to be satisfied and call α_1 the unique positive solution (recall that $\rho - \beta > 0$ from (23)) of*

$$0 = g(\alpha) := \alpha(\rho - \beta) + \frac{1}{1-\sigma} - \frac{\sigma}{1-\sigma} \left(\frac{\beta}{a}\right)^{1-1/\sigma} \alpha^{1-1/\sigma}.$$

If

$$\frac{\rho}{\beta} a^{1-\sigma} < \frac{a^{1-\sigma} - 1}{1-\sigma} \quad (33)$$

holds then there exist a unique optimal control/trajectory. Setting

$$\theta_1 := \frac{a}{b} \left(1 - (\alpha_1 \beta)^{-\frac{1}{\sigma}}\right) \in \left(0, \frac{a}{b}\right), \quad (34)$$

and the optimal control $n^*(\cdot)$ and the related trajectory $N^*(\cdot)$ satisfy the following equation:

$$n^*(t) = \theta_1 N^*(t). \quad (35)$$

If

$$\frac{\rho}{\beta} a^{1-\sigma} \geq \frac{a^{1-\sigma} - 1}{1-\sigma} \quad (36)$$

is satisfied then there exist a unique optimal control/trajectory. The optimal control $n^*(\cdot)$ is identically zero and we have finite time extinction up to time T .

As one can see, the condition (33) of the Theorem ruling out finite time extinction is the counterpart of condition (25) in Theorem 3.1.¹⁸ The two theorems deliver indeed the same kind of optimal outcomes. As before, one

¹⁸Needless to say, condition (33) gives condition (25) if the right hand side term $a^{1-\sigma} - 1$ is replaced by $a^{1-\sigma}$.

can also prove that, when (33) is satisfied, along the optimal trajectory the per-capita consumption is constant and its value is

$$c^*(t) = \frac{aN^*(t) - bn^*(t)}{N^*(t)} = a - b\theta_1 \in (0, a). \quad (37)$$

Moreover the optimal control $n^*(\cdot)$ is the unique solution of the following delay differential equation

$$\begin{cases} \dot{n}(t) = \theta_1 (n(t) - n(t - T)), & \text{for } t \geq 0 \\ n(0) = \theta_1 N_0 \\ n(s) = n_0(s), & \text{for all } s \in [-T, 0). \end{cases} \quad (38)$$

Finally, and similarly to the case of Subsection 3.2.1 we find

Proposition 3.4 *Suppose that (21) (and then (23)) holds, with $\sigma > 0$ ($\sigma \neq 1$). Assume that (31) and (33) are satisfied, so θ_1 , defined in (34), is in $(0, \frac{a}{b})$. Then*

- If $\theta_1 T < 1$ then $n^*(t)$ (and then $N^*(\cdot)$) goes to 0 exponentially.
- If $\theta_1 T > 1$ then the characteristic equation of (38)

$$z = \theta_1 (1 - e^{-zT}), \quad (39)$$

has a unique strictly positive solution h_1 belonging to $(0, \theta_1)$ while all the other roots have negative real part. Moreover

$$\lim_{t \rightarrow \infty} \frac{n^*(t)}{e^{h_1 t}} = \frac{\theta_1}{1 - T(\theta_1 - h_1)} \int_{-T}^0 (1 - e^{(-s-T)h_1}) n_0(s) ds > 0$$

and

$$\lim_{t \rightarrow \infty} \frac{N^*(t)}{e^{h_1 t}} = \frac{1 - e^{-h_1 T}}{h_1} \frac{\theta_1}{1 - T(\theta_1 - h_1)} \int_{-T}^0 (1 - e^{(-s-T)h_1}) n_0(s) ds > 0.$$

Not surprisingly, the economic properties of the optimal solutions remain the same as in the previous section:

Proposition 3.5 *Under the hypotheses of Proposition 3.4, both θ_1 and h_1 are increasing in T .*

4 An intermediate case

In this section we study the intermediate case $\gamma = 1 - \sigma$, with $0 < \sigma < 1$. A crucial question arising from the findings of the previous section is how the huge gap between the outcomes of the Millian and the Benthamite cases is altered when the intertemporal altruism parameter γ varies in $(0, 1)$. The answer is that, for $\gamma \in (0, 1)$, the optimal dynamics show substantially the same qualitative properties as the Benthamite case studied in Section 3.2. This fact can be seen (with some hard mathematical work) also in the case $\gamma \neq 1 - \sigma$ studying the qualitative properties of the optimal dynamics through the dynamic programming approach.

We consider here the intermediate case $\gamma = 1 - \sigma$ since it is a good and “cheap” way to address such crucial question. Indeed from the mathematical point of view, and in contrast to the case $\gamma = 1$ handled above (and to the case $\gamma \neq 1 - \sigma$), the case $\gamma = (1 - \sigma)$ leads to the same infinitely dimensioned optimal control problem solved out explicitly by Fabbri and Gozzi (2008). Moreover, by varying σ in $(0, 1)$, one can extract some insightful lessons on the outcomes of our optimal control problem for any γ in $(0, 1)$.

As in the previous section, we reformulate the optimal control problem using $n(\cdot)$ as a control in the set

$$\mathcal{V}_{n_0} := \{n(\cdot) \in L^1_{\text{loc}}(0, +\infty; \mathbb{R}_+) : \text{conditions (17) hold for all } t \geq 0\},$$

while the objective function becomes

$$\int_0^{+\infty} e^{-\rho t} \frac{(aN(t) - bn(t))^{1-\sigma}}{1-\sigma} dt. \quad (40)$$

Also, as discussed in Subsection 2.2, we call ξ the unique strictly positive root of equation

$$z = \frac{a}{b} (1 - e^{-zT}),$$

if it exists, otherwise we pose $\xi = 0$. From Subsection 2.2, we know that $\xi > 0$ if individuals’ lifetime is large enough: $T > \frac{b}{a}$. The condition (12) needed for the boundedness of the value function becomes:

$$\rho > \xi(1 - \sigma). \quad (41)$$

It is then possible to characterize the optimal control of our problem as follows:

Theorem 4.1 *Consider the optimal control problem driven by (3), with constraint (17) and functional (40). If (41) and the following condition (needed to rule out corner solutions)*

$$\frac{\rho - \xi(1 - \sigma)}{\sigma} \leq \frac{a}{b} \quad (42)$$

are satisfied, then, along the unique optimal trajectory $n^(\cdot)$ and the related optimal trajectory $N^*(\cdot)$, we have*

$$\frac{a}{b}N^*(t) - n^*(t) = \Lambda e^{gt}$$

where

$$g := \frac{\xi - \rho}{\sigma} \quad (43)$$

and

$$\Lambda := \left(\frac{\rho - \xi(1 - \sigma)}{\sigma} \cdot \frac{a}{b\xi} \right) \left(\int_{-T}^0 (1 - e^{\xi r}) n_0(r) dr \right).$$

Moreover $n^*(\cdot)$ is characterized as the unique solution of the following delay differential equation:

$$\begin{cases} \dot{n}(t) = \frac{a}{b} (n(t) - n(t - T)) - g\Lambda e^{gt}, & t \geq 0 \\ n(0) = \frac{a}{b} (N_0 - \Lambda) \\ n(r) = n_0(r), & r \in [-T, 0). \end{cases}$$

The proof is in Appendix B, it is a simple adaptation of previous work of Fabbri and Gozzi (2008). Two important comments should be already made. First of all, one can see that the properties extracted in the theorem above are not applicable to the limit case $\gamma = 1$ because this amounts to study the limit case $\sigma = 0$: in the latter case, magnitudes, like the growth rate g given in equation (43), are not defined. In contrast, the theorem can be used to study possible dynamics of optimal controls when γ is close to zero, or when σ is close to one (but not equal to 1 of course). When $\gamma = 0$, we know from Section 3.1 that we have optimal extinction at finite-time whatever the value of $\sigma > 0$. Theorem 4.1 shows that when γ is close to zero (but not equal to zero), finite-time extinction is not the unique optimal outcome: population may even grow at a rate close to $g = \xi - \rho$ which might well be positive if the lifetime T is large enough (see a finer characterization below). Indeed the intermediate case considered here delivers the same qualitative properties as the Benthamite case studied in Section 3.2. Condition (42) rules out finite time extinction as an optimal outcome: if it is not verified, we get as in Section 3.2 a case for

optimal finite time extinction. Since the root ξ is an increasing function of life span T (see Proposition 4.2 below), one can also interpret condition (42) as putting a first threshold value for the latter below which finite extinction is optimal. Above this first threshold, either sustainable positively growing or asymptotically vanishing populations (and economies) are optimal. In particular, note that when $T < \frac{b}{a}$, $\xi = 0$ and therefore $g < 0$: in this case we necessarily have asymptotic extinction. Sustainable growth is not guaranteed even if $T > \frac{b}{a}$ because even if in this case the root $\xi > 0$, it is not necessarily bigger than ρ for g to be necessarily positive. Just like in the Benthamite case, there should exist a second threshold value of life span above which positive growth is optimal.

Other than this, the theorem is an accurate description of the optimal population and cohort sizes dynamics. As in the case $\gamma = 1$, optimal $n(t)$ follows a delay differential equation. The delayed nature of the dynamic motions involved induce oscillatory optimal paths following the same principle as in the Benthamite case explored above. The main difference between the two cases is the term $g\Lambda e^{gt}$. The implications of this term for the asymptotic properties of the model are not all immediate, we give them in the next proposition. Note already that the dynamics are clearly more complex than in the Benthamite case: while optimal consumption per capital and the fertility rates are constant and independent of the initial procreation profile in the Benthamite case, their optimal paths do depend on initial conditions in the intermediate case. This may reflect the specificity of the Benthamite case: when intertemporal altruism is maximal, the social planner abstracts from the initial conditions when fixing optimal consumption level and the fertility rate. Under intermediate altruism, the planner takes into account the initial data, and the optimal dynamics of the latter variables do adjust to this data. The next proposition shows the asymptotic properties of this adjustment.

Proposition 4.1 *Under the hypotheses of Theorem 4.1 the following limits exist*

$$\lim_{t \rightarrow \infty} \frac{n^*(t)}{e^{gt}} =: n_L$$

and

$$\lim_{t \rightarrow \infty} \frac{N^*(t)}{e^{gt}} =: N_L.$$

Moreover, if $g \neq 0$ we have:

$$n_L = \frac{\Lambda}{\frac{a}{bg}(1 - e^{-gT}) - 1}$$

and

$$N_L = \frac{b}{a}(n_L + \Lambda) = \frac{\Lambda(1 - e^{-gT})}{\frac{a}{b}(1 - e^{-gT}) - g} = n_L \cdot \frac{1 - e^{-gT}}{g}.$$

In particular, if $\rho > \xi$ in the long run $N(t)$ and $n(t)$ go to zero exponentially, if $\rho < \xi$ they grow exponentially with rate g defined in (43), if $\rho = \xi$ they stabilize respectively to n_L and N_L . Moreover

$$\lim_{t \rightarrow \infty} c^*(t) = \lim_{t \rightarrow \infty} \frac{aN^*(t) - bn^*(t)}{N^*(t)} = a - \frac{bg}{1 - e^{-gT}}.$$

The proposition shows that, as in the Benthamite case and despite the extra non-autonomous term, the economy will converge to a balanced growth path at rate g given in equation (43). As before, the long-run levels corresponding to total population and cohort sizes depend on the initial procreation profile via the parameter Λ . It should be noted here that despite the latter feature, both per capita consumption and the fertility rate are independent of the parameter Λ in the long-run. Therefore, and though the two latter variables do show up transition dynamics, they converge to magnitudes which are independent of the initial conditions, contrary to the traditional AK model. So the intermediate cases studied here give rise to a weaker form of convergence in the standards of living compared to the Benthamite case. This said, in all cases where growth is optimal in the long run, we have the same picture: differences in historical demography yield different long term optimal population sizes but identical optimal per capita consumption and fertility rates in the long run. Other than this, the proposition highlights the possibilities to have either sustainable, constant (stagnant) or asymptotically declining economies as already pointed out before. It is possible to reformulate these properties in terms of the lifetime, T , as explained in the proposition below.

Proposition 4.2 *Under the hypotheses of Theorem 4.1, fixed a and b , the constant ξ and then the growth rate g are strictly increasing in $T \in (\frac{b}{a}, +\infty)$. Moreover, once the constants a and b are chosen, writing g as function of T : “ $g(T)$ ” we have that*

$$\lim_{T \rightarrow \frac{b}{a}^+} g(T) = \frac{-\rho}{\sigma}$$

and

$$\lim_{T \rightarrow +\infty} g(T) = \frac{\frac{a}{b} - \rho}{\sigma}.$$

So, if $\frac{a}{b} > \rho$, there exists $\bar{T} \in (\frac{b}{a}, +\infty)$ such that, for every $T > \bar{T}$ the growth rate of the population g is positive while, for every $T < \bar{T}$, g is negative.¹⁹

As in the Benthamite case, the growth rate of the economic is an increasing function of individuals' lifetime. Moreover, there exists a threshold value for this variable such that above this value, the economy will converge to a balanced growth path (at a positive growth rate), while it goes to asymptotic extinction below it. The results are thus qualitatively identical to the Benthamite case. We can also show that the two cases deliver the same prediction as to the impact of life expectancy on the fertility rate, although the proof in the intermediate cases is much more sophisticated. The following local argument makes the point.

Proposition 4.3 *Under the hypotheses of Theorem 4.1 fix a, b, ρ, σ . Assume $a/b > \rho$. Let \bar{T} be, as in Proposition 4.2, the unique value such that $g(\bar{T}) = 0$. Then n_L/N_L is locally increasing in \bar{T} i.e. if T is close to \bar{T} (i.e. if g is close to 0) an increase of T causes an increase of the ratio n_L/N_L .*

5 Conclusion

In this paper, we have introduced the realistic assumption of finite lives into an otherwise standard optimal population size problem. By taking advantage of some recent developments in the optimization of infinite-dimensional problems, we have been able to fully characterize the optimal dynamics of the resulting problems. Within a very simple AN setting, we have highlighted the role of the value of individuals' lifetime in optimal dynamics, and the highly differentiated optimal outcomes of the Millian Vs Benthamite cases. We have also characterized finely the implications of some intermediate welfare functions.

Of course, our analytical approach cannot be trivially adapted to handle natural extensions of our model (through the introduction of capital accumulation or natural resources for example, or the incorporation of nonlinear production functions). We believe however that this first step into the analysis of optimal dynamics in optimal population size problems is an important enrichment of the ongoing debate. It is especially interesting because it follows from a very natural assumption: individuals have finite lives, and this feature can only be crucial for the outcomes of the optimal population size problem.

¹⁹Needless to say, the threshold \bar{T} plays exactly the role as the threshold T_1 in Corollary 3.2.

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A The case $\gamma = 1$: the proofs of Theorem 3.1, Theorem 3.2

We denote by $L^2(-T, 0)$ the space of all functions f from $[-T, 0]$ to \mathbb{R} that are Lebesgue measurable and such that $\int_{-T}^0 |f(x)|^2 dx < +\infty$. It is an Hilbert space when endowed with the scalar product $\langle f, g \rangle_{L^2} = \int_{-T}^0 f(x)g(x) dx$. We consider the Hilbert space $M^2 := \mathbb{R} \times L^2(-T, 0)$ (with the scalar product $\langle (x_0, x_1), (z_0, z_1) \rangle_{M^2} := x_0 z_0 + \langle x_1, z_1 \rangle_{L^2}$). Following Bensoussan et al. (2007) Chapter II-4 and in particular Theorem 5.1²⁰, given an admissible control $n(\cdot)$ and the related trajectory $N(\cdot)$, if we define $x(t) = (x_0(t), x_1(t)) \in M^2$ for all $t \geq 0$ as

$$\begin{cases} x_0(t) := N(t) \\ x_1(t)[r] := -n(t - T - r), \quad \text{for all } r \in [-T, 0), \end{cases} \quad (44)$$

we have that $x(t)$ satisfy the following evolution equation in M^2 :

$$\dot{x}(t) = A^*x(t) + B^*n(t). \quad (45)$$

where A^* is the adjoint of the generator of a C_0 -semigroup²¹ A defined as²²

$$\begin{cases} D(A) \stackrel{\text{def}}{=} \{(\psi_0, \psi_1) \in M^2 : \psi_1 \in W^{1,2}(-T, 0), \psi_0 = \psi_1(0)\} \\ A: D(A) \rightarrow M^2, \quad A(\psi_0, \psi_1) \stackrel{\text{def}}{=} (0, \frac{d}{ds}\psi_1) \end{cases} \quad (46)$$

and B^* is the adjoint of $B: D(A) \rightarrow \mathbb{R}$ defined as $B(\psi_0, \psi_1) := (\psi_1[0] - \psi_1[-T])$. Moreover, using the new variable $x \in M^2$ defined in (44) we can rewrite the welfare functionals (20) (when $R = 0$) and (32) (when $R = 0$) as

$$\begin{aligned} & \int_0^{+\infty} e^{-\rho t} \frac{\left(\frac{ax_0(t) - bn(t)}{x_0(t)}\right)^{1-\sigma} - R}{1 - \sigma} x_0(t) dt \\ &= \int_0^{+\infty} e^{-\rho t} \left(\frac{(ax_0(t) - bn(t))^{1-\sigma} x_0^\sigma(t)}{1 - \sigma} - \frac{Rx_0(t)}{1 - \sigma} \right) dt. \end{aligned} \quad (47)$$

Here we consider the optimal control problem for all the family of functionals varying $R \geq 0$ and not only for $R = 0$ and $R = 1$. Consequently we consider a more general formulation for the condition (31) given by:

$$\frac{a^{1-\sigma} - R}{1 - \sigma} > 0. \quad (48)$$

²⁰The result is originally due to Vinter and Kwong (1981)

²¹See e.g. Pazy (1983) for a standard reference to the argument.

²² $W^{1,2}(-T, 0)$ is the set $\{f \in L^2(-T, 0) : \partial_\omega f \in L^2(-T, 0)\}$ where $\partial_\omega f$ is the distributional derivative of f .

Our optimal control problem of maximizing the welfare functional (32) and (20) over the set \mathcal{V}_{n_0} in (18) with the state equation (1) can be equivalently rewritten as the problem of maximizing the functional above over the same set \mathcal{V}_{n_0} in (18) and with the state equation (45). The value function V depends now on the new variable x that can be expressed in term of the datum n_0 using (44) for $t = 0$. The associated Hamilton-Jacobi-Bellman equation for the unknown v is²³:

$$\rho v(x) = \langle x, ADv \rangle_{M^2} + \sup_{n \in [0, \frac{a}{b}x_0]} \left(nBDv(x) + \frac{(ax_0 - bn)^{1-\sigma}}{1-\sigma} x_0^\sigma \right) - \frac{R}{1-\sigma} x_0. \quad (49)$$

As far as

$$BDv > a^{-\sigma} b \quad (50)$$

the supremum appearing in (49) is a maximum and the unique maximum point is strictly positive (since $x_0 > 0$) and is

$$n_{max} := \frac{a}{b} \left(1 - \left(\frac{BDv(x)}{a^{-\sigma} b} \right)^{-1/\sigma} \right) x_0 \quad (51)$$

so (49) can be rewritten as

$$\rho v(x) = \langle x, ADv \rangle_{M^2} + \frac{a}{b} x_0 BDv(x) + \frac{\sigma}{1-\sigma} x_0 \left(\frac{1}{b} BDv(x) \right)^{1-\frac{1}{\sigma}} - \frac{R}{1-\sigma} x_0. \quad (52)$$

When

$$BDv \leq a^{-\sigma} b \quad (53)$$

then the supremum appearing in (49) is a maximum and the unique maximum point is $n_{max} := 0$. In this case (49) can be rewritten as

$$\rho v(x) = \langle x, ADv \rangle_{M^2} + \frac{(a^{1-\sigma} - R)x_0}{1-\sigma} \quad (54)$$

We expect that the value function of the problem is a (the) solution of the HJB equation. Since it is not hard to see that the value function is 1-homogeneous, we look for a linear solution of the HJB equation. In the simpler case $R = 0$ (see Subsection 3.2.1) we find the following:

Proposition A.1 *Suppose that (21) (and then (23)) holds, $R = 0$ and $\sigma \in (0, 1)$. If*

$$\beta > \rho(1 - \sigma) \quad (55)$$

then the function

$$v(x) := \alpha_1 \left(x_0 + \int_{-T}^0 x_1(r) e^{\rho r} dr \right) \quad (56)$$

²³ Dv is the Gateaux derivative.

where

$$\alpha_1 = a^{1-\sigma} \frac{1}{\beta} \left(\frac{1-\sigma}{\sigma} \cdot \frac{\rho-\beta}{\beta} \right)^{-\sigma}$$

is a solution of (52) in all the points s.t. $x_0 > 0$.

On the other side, if

$$\beta \leq \rho(1-\sigma) \quad (57)$$

then the function

$$v(x) := \alpha_2 \left(x_0 + \int_{-T}^0 x_1(r) e^{\rho r} dr \right) \quad (58)$$

where

$$\alpha_2 = \frac{a^{1-\sigma}}{\rho(1-\sigma)}$$

is a solution of (54) in all the points s.t. $x_0 > 0$.

Proof. Let $i = 1, 2$. We first observe that the function v is C^1 (since it is linear). Setting $\phi(r) = e^{\rho r}$, $r \in [-T, 0]$ we see that its first derivative is constant and is

$$Dv(x) = \alpha_i(1, \phi) \quad \text{for all } x \in M^2$$

Looking at (46) we also see that such derivative belongs to $D(A)$ so that all the terms in (49) make sense. We have $ADv(x) = (0, \alpha_i \rho \phi)$ and $BDv(x) = \alpha_i(1 - e^{-\rho T})$. Then, thanks to (55) (or (57)) we have that (50) (or (53)) is satisfied and (49) can be written in the form (52) (or in the form (54)) with $R = 0$. To verify the statement we have only to check directly: the left hand side of (52) (or of (54)) is equal to $\rho \alpha_i (x_0 + \langle x_1, \phi \rangle_{L^2})$ while the right hand side is, for $i = 1$

$$\begin{aligned} & \langle x_1, \alpha_1 \rho \phi \rangle_{L^2} + \frac{a}{b} x_0 \alpha_1 (1 - e^{-\rho T}) + \frac{\sigma}{1-\sigma} x_0 \left(\frac{1}{b} \alpha_1 (1 - e^{-\rho T}) \right)^{1-\frac{1}{\sigma}} \\ &= \langle x_1, \alpha_1 \rho \phi \rangle_{L^2} + x_0 \alpha_1 \beta + \frac{\sigma}{1-\sigma} x_0 \left(\frac{\alpha_1 \beta}{a} \right)^{1-\frac{1}{\sigma}} \\ &= \langle x_1, \alpha_1 \rho \phi \rangle_{L^2} + x_0 \frac{\alpha_1 \beta}{a} \left[1 + \frac{\sigma}{1-\sigma} \left(\frac{\alpha_1 \beta}{a} \right)^{-\frac{1}{\sigma}} \right] \end{aligned}$$

Since the expression in square brackets is equal to $a\rho/\beta$ thanks to the definition of α_1 , we have the claim for $i = 1$. For $i = 2$ the right hand side is (using the expression of α_2 above)

$$\langle x_1, \alpha_2 \rho \phi \rangle_{L^2} + \frac{a^{1-\sigma}}{1-\sigma} x_0 = \langle x_1, \alpha_2 \rho \phi \rangle_{L^2} + \alpha_2 \rho x_0$$

and this proves the claim for $i = 2$. □

For the case $R > 0$ (in Subsection 3.2.2, there we considered $R = 1$, here we take a generic $R \geq 0$) we have the following:

Proposition A.2 *Suppose that (21) (and then (23)) holds, $R > 0$ and $\sigma > 0$ (with $\sigma \neq 1$). Assume (48) (that gives (31) when $R = 1$) to be satisfied and call α_1^R the unique positive solution (recall that $\rho - \beta > 0$ from (23)) of*

$$0 = g(\alpha) := \alpha(\rho - \beta) + \frac{R}{1 - \sigma} - \frac{\sigma}{1 - \sigma} \left(\frac{\beta}{\alpha}\right)^{1-1/\sigma} \alpha^{1-1/\sigma}.$$

If

$$\frac{\rho}{\beta} a^{1-\sigma} < \left(\frac{a^{1-\sigma} - R}{1 - \sigma}\right) \quad (59)$$

then the function

$$v(x) := \alpha_1^R \left(x_0 + \int_{-T}^0 x_1(r) e^{\rho r} dr\right) \quad (60)$$

is a solution of (52) in all the points s.t. $x_0 > 0$.

On the other side, if

$$\frac{\rho}{\beta} a^{1-\sigma} \geq \left(\frac{a^{1-\sigma} - R}{1 - \sigma}\right) \quad (61)$$

then the function

$$v(x) := \alpha_2^R \left(x_0 + \int_{-T}^0 x_1(r) e^{\rho r} dr\right) \quad (62)$$

where

$$\alpha_2^R = \frac{a^{1-\sigma} - R}{\rho(1 - \sigma)}$$

is a solution of (54) in all the points s.t. $x_0 > 0$.

Proof. We sketch the proof because it is similar to that of Proposition A.1. Setting $\phi(r) = e^{\rho r}$, $r \in [-T, 0]$ we see that $Dv(x) = \alpha_i^R(1, \phi)$ for all $x \in M^2$, $ADv(x) = (0, \alpha_i^R \rho \phi)$ and $BDv(x) = \alpha_i^R(1 - e^{-\rho T})$. Let us first look at the case $i = 1$. We observe that (59) implies that

$$\frac{a^{1-\sigma}}{1 - \sigma} \left[\frac{\rho}{\beta}(1 - \sigma) - \left(1 - \frac{R}{a^{1-\sigma}}\right) \right] = g\left(\frac{a^{1-\sigma}}{\beta}\right) < 0,$$

now, since g is strictly increasing and $g(\alpha_1^R) = 0$, we have $\alpha_1^R > \frac{a^{1-\sigma}}{\beta}$ that is equivalent to (50). Analogously (61) ensures that (53) is satisfied. So the HJB can be written in the form (52) [resp. (54)]. To verify the statement we have only to check directly: the left hand side of (52) [resp. (54)] is equal to $\rho \alpha_i^R (x_0 + \langle x_1, \phi \rangle_{L^2})$ while the right hand side is, for $i = 1$

$$\begin{aligned} \langle x_1, \alpha_1^R \rho \phi \rangle_{L^2} + \frac{a}{b} x_0 \alpha_1^R (1 - e^{-\rho T}) + \frac{\sigma}{1 - \sigma} x_0 \left(\frac{1}{b} \alpha_1^R (1 - e^{-\rho T})\right)^{1-\frac{1}{\sigma}} - \frac{R}{1 - \sigma} x_0 \\ = \rho \alpha_1^R (x_0 + \langle x_1, \phi \rangle_{L^2}) \quad (63) \end{aligned}$$

thanks to the fact that $g(\alpha_1^R) = 0$. So we have the claim for $i = 1$.

For $i = 2$ the right hand side is (using the expression of α_2^R above)

$$\langle x_1, \alpha_2^R \rho \phi \rangle_{L^2} + \frac{a^{1-\sigma}}{1-\sigma} x_0 - \frac{R}{1-\sigma} x_0 = \langle x_1, \alpha_2^R \rho \phi \rangle_{L^2} + \alpha_2^R \rho x_0$$

and this proves the claim for $i = 2$. \square

Once we have a solution of the Hamilton-Jacobi-Bellman equation we can prove that it is the value function and so use it to find a solution of our optimal control problem in feedback form.

Theorem A.1 *Suppose that (21) (and then (23)) holds, $R = 0$ and $\sigma \in (0, 1)$. If (55) holds then the function v defined in (56) is the value function V and there exist a unique optimal control/trajectory. The optimal control $n^*(\cdot)$ and the related trajectory $x^*(\cdot)$ satisfy the following equation:*

$$n^*(t) = \frac{a}{b} \left(1 - (\alpha_1 \beta)^{-\frac{1}{\sigma}} \right) x_0^*(t) = \theta x_0^*(t) \quad (64)$$

where θ is given by (26). If (57) is satisfied then the function v defined in (58) is the value function V and there exist a unique optimal control/trajectory. The optimal control $n^*(\cdot)$ is identically zero.

Proof. The proof follows the arguments of the one of Proposition 2.3.2. in Fabbri and Gozzi (2008) with various modifications due to peculiarity of our problem. We do not write the details for brevity. \square

Proof of Theorem 3.1. Theorem 3.1 is nothing but Theorem A.1 once we write again $N^*(\cdot)$ instead of $x_0^*(\cdot)$. In particular (64) becomes (27). Finally, if we write $N^*(t)$ as $\int_{t-T}^t n(s) ds$ and we take the derivative we obtain (29). \square

And in the same way we have:

Theorem A.2 *Suppose that (21) (and then (23)) holds, $R > 0$ and $\sigma > 0$ (with $\sigma \neq 1$). Assume (48) to be satisfied. If (59) holds then the function v defined in (60) is the value function V and there exist a unique optimal control/trajectory. The optimal control $n^*(\cdot)$ and the related trajectory $x^*(\cdot)$ satisfy the following equation:*

$$n^*(t) = \frac{a}{b} \left(1 - (\alpha_1^R \beta)^{-\frac{1}{\sigma}} \right) x_0^*(t) = \theta_1 x_0^*(t). \quad (65)$$

If (61) is satisfied then the function v defined in (62) is the value function V and there exist a unique optimal control/trajectory. The optimal control $n^*(\cdot)$ is identically zero.

Proof of Theorem 3.2. Follows from Theorem A.2 once we write again $N^*(\cdot)$ instead of $x_0^*(\cdot)$. \square

B Other proofs

Proof of Proposition 3.2. Since $n^*(\cdot)$ solves (29) it can be written (see Diekmann et al., 1995, page 34) as a series

$$n^*(t) = \sum_{j=1}^{\infty} p_j(t) e^{\lambda_j t}$$

where $\{\lambda_j\}_{j=1}^{+\infty}$ are the roots of the characteristic equation (30) (studied in Fabbri and Gozzi, 2008, Proposition 2.1.8) and $\{p_j\}_{j=1}^N$ are \mathbb{C} -valued polynomial. If $\theta T > 1$, as already observed in Subsection 2.2 there exists a unique strictly positive root $\lambda_1 = h$. Moreover $h \in (0, \theta)$ and it is also the root with biggest real part (and it is simple). The polynomial p_1 associated to h is a constant (since h is simple) and can be computed explicitly (see for example Hale and Lunel (1993) Chapter 1, in particular equations (5.10) that gives the expansion of the fundamental solution and Theorem 6.1) obtaining

$$p_1(t) \equiv \frac{\theta}{1 - T(\theta - h)} \int_{-T}^0 \left(1 - e^{(-s-T)h}\right) n_0(s) ds$$

this gives the limit for $n(t)^*/e^{ht}$. The limit for $N(t)^*/e^{ht}$ follows from the relation $N^*(t) = \int_{t-T}^t n^*(s) ds$.

If $\theta T < 1$ each λ_j , for $j \geq 2$, has negative real while $\lambda_1 = 0$ is the only real root. But again if we compute explicitly the polynomial p_1 (again a constant value) related to the root 0 we have

$$p_1(t) \equiv \frac{\theta N_0 + (-\theta) \int_{-T}^0 n_0(r) dr}{1 + \theta T} = \frac{\theta N_0 - \theta N_0}{1 + \theta T} = 0.$$

so only the contributions of the roots with negative real parts remain. This concludes the proof. \square

Proof of Proposition 3.3. We use the implicit function theorem. Define

$$F(\lambda, T) = \theta(T)(1 - e^{-T\lambda}) - \lambda.$$

Given T such that $\Theta(T)T > 1$ one has that $F(\lambda, T)$ is concave in λ , $F(0, T) = 0$ and $F(h, T) = 0$ for $h \in (0, \theta(T))$. So it must be

$$\left. \frac{\partial}{\partial \lambda} F(\lambda, T) \right|_{\lambda=h} = \theta(T)T e^{-Th} - 1 < 0.$$

Moreover, since by the definition of θ in (26) we easily get $\theta'(T) > 0$, we have:

$$\frac{\partial F(h, T)}{\partial T} = \theta'(T)(1 - e^{-Th}) + \theta(T)h e^{-Th} > 0$$

Now, by the implicit function theorem we have

$$\frac{dh}{dT} = -\frac{\partial F}{\partial T} \left(\frac{\partial F}{\partial \lambda} \Big|_{\lambda=h} \right)^{-1} > 0$$

and this concludes the proof. \square

Proof of Proposition 3.4. It can be proved as Proposition 3.2. \square

Proof of Proposition 3.5. We use again the implicit function theorem. Again we consider here a generic $R > 0$, the claim of the Proposition follows taking $R = 1$. From the definition of α_1^R we have

$$0 = g(\alpha_1^R) := \alpha_1^R(\rho - \beta) + \frac{R}{1 - \sigma} - \frac{\sigma}{1 - \sigma} \left(\frac{\beta}{a} \right)^{1-1/\sigma} (\alpha_1^R)^{1-1/\sigma}. \quad (66)$$

If we call $K := \alpha_1^R \beta$ we can rewrite (66) as

$$0 = \tilde{g}(\beta, K) = K\rho/\beta - K + \frac{R}{1 - \sigma} - \frac{\sigma}{1 - \sigma} \left(\frac{K}{a} \right)^{1-1/\sigma}.$$

We compute now

$$\frac{dK}{d\beta} = -\frac{\partial \tilde{g}}{\partial \beta} \left(\frac{\partial \tilde{g}}{\partial K} \right)^{-1} = +\frac{\rho}{\beta^2} K \left(\frac{\rho}{\beta} - 1 + \frac{K^{-1/\sigma}}{a^{1-1/\sigma}} \right) > 0$$

(the last inequality follows from the fact that $\rho > \beta$). Finally, thanks to the form of θ_1 given in (34), $\frac{d\theta_1}{d\beta} > 0$ and then, since β is strictly increasing in T , $\frac{d\theta_1}{dT} > 0$. This concludes the first part of the claim. The claim that h_1 is increasing in T can be proved using the same arguments used in the proof of Proposition 3.3 and the fact, already proved, that $\frac{d\theta_1}{dT} > 0$. This concludes the proof. \square

Proof of Theorem 4.1. The statements follows from Lemma 2.3.3 and Theorem 2.3.4 of Fabbri and Gozzi (2008): here we have the control variable n instead of i and the state variable N instead of k . The state equation is the same. To rewrite the objective functional exactly in the form of the problem treated in Fabbri and Gozzi (2008) we only need to write

$$aN(t) - bn(t) = b \left(\frac{a}{b} N(t) - n(t) \right)$$

so the functional becomes

$$b^{1-\sigma} \int_0^{+\infty} e^{-\rho t} \frac{\left(\frac{a}{b} N(t) - n(t) \right)^{1-\sigma}}{1 - \sigma} dt.$$

The constant $b^{1-\sigma}$ as it does not changes the optimal trajectories. Dropping it the functional is the same as the one of Fabbri and Gozzi (2008) where the constant a is substituted here by $\frac{a}{b}$. \square

Proof of Proposition 4.1. Arguing as in the proof of Theorem 4.1 the statement is equivalent to that of Proposition 2.3.5 in Fabbri and Gozzi (2008). \square

Proof of Proposition 4.2. Define $F(z) = \left(\frac{a}{b} - \frac{a}{b}e^{-zT}\right) - z$. For a fixed T , the function $F: \mathbb{R} \rightarrow \mathbb{R}$, $\xi \mapsto F(\xi)$, is concave and it can be seen with elementary arguments that, when $\frac{a}{b}T > 1$, it has exactly two zeros, the first in 0 and the second in ξ and so $F'(\xi) < 0$ i.e. $(T\frac{a}{b}e^{-\xi T} - 1) < 0$. To show that ξ (and then g) is increasing in T in the interval $T \in (b/a, +\infty)$ we can apply the implicit function theorem:

$$\frac{d\xi}{dT} = - \left(\frac{\partial F}{\partial z} \Big|_{z=\xi} \right)^{-1} \left(\frac{\partial F}{\partial T} \right) = - \left(T\frac{a}{b}e^{-\xi T} - 1 \right)^{-1} \left(\xi\frac{a}{b}e^{-\xi T} \right) > 0. \quad (67)$$

From what we have said above we easily have $\frac{d\xi}{dT} > 0$ when $T > \frac{b}{a}$. So, since $\xi(T)$ (and then $g(T)$) is continuous in T and strictly increasing, there exist the two limits $\underline{\xi} := \lim_{T \rightarrow \frac{b}{a}^+} \xi(T)$ and $\bar{\xi} := \lim_{T \rightarrow +\infty} \xi(T)$. Since $F(\xi) = 0$, F is continuous in ξ and ξ is continuous in T and bounded, we have

$$0 = \lim_{T \rightarrow +\infty} F(\xi(T)) = F(\bar{\xi}) = \frac{a}{b} - \frac{a}{b} \lim_{T \rightarrow +\infty} e^{-\bar{\xi}T} - \bar{\xi}.$$

This implies $\bar{\xi} = \frac{b}{a}$ and so $\lim_{T \rightarrow +\infty} g(T) = \frac{\frac{b}{a} - \rho}{\sigma}$. The same argument allows to get the statement when $T \rightarrow \frac{b}{a}^+$. \square

Proof of Proposition 4.3. We have $n_L/N_L = \frac{g(T)}{1 - e^{g(T)T}}$ if $T \neq \bar{T}$ and it is equal to $1/\bar{T}$ if $T = \bar{T}$. Since ξ is strictly increasing in T we have that $T(\xi)$ is strictly increasing in ξ . Indeed $T(\xi) = \frac{1}{\xi} \ln \left(\frac{a/b}{a/b - \xi} \right)$. We denote by $b(\xi) := \frac{g(\xi)}{1 - e^{-T(\xi)g(\xi)}}$. To get the proof it is enough to prove that $b(\xi)$ is locally increasing in $\xi = \rho$. We have

$$b'(\xi) = \frac{g'(\xi)(1 - e^{T(\xi)g(\xi)}) - g(\xi)e^{T(\xi)g(\xi)}(T'(\xi)g(\xi) + T(\xi)g'(\xi))}{1 - e^{T(\xi)g(\xi)}}.$$

Since $g(\xi) = \frac{\xi - \rho}{\sigma}$ then we easily see that $b'(\rho) = 0$. To get the result we have to study the sign of the numerator of $b'(\xi)$ for ξ closed to ρ . We call it $B(\xi)$. By straightforward (yet annoying) computations we find that $B'(\rho) = 0$ while $B''(\rho) = T^2/\sigma^2 > 0$. This gives the claim. \square