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# Quasi-rational solutions of the NLS equation and rogue waves

<sup>+</sup>Pierre Gaillard, <sup>+</sup> Université de Bourgogne, Dijon, France :  
e-mail: Pierre.Gaillard@u-bourgogne.fr,

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## Abstract

We degenerate solutions of the NLS equation from the general formulation in terms of theta functions given in [17], to get quasi-rational solutions of NLS equations. For this we establish a link between Fredholm determinants and Wronskians. We give solutions of the NLS equation as a quotient of two wronskian determinants. In the limit when some parameter goes to 0, we recover Akhmediev's solutions given recently [1, 2, 3, 4, 5, 6, 7]. It gives a new approach to get the well known rogue waves.

## 1 Introduction

The solution of the NLS equation

$$iv_t + v_{xx} + 2|v|^2v = 0, \quad (1)$$

is given in terms of theta function by (see [17])

$$v(x, t) = \frac{\theta_3(x, t)}{\theta_1(x, t)} \exp(2it - i\varphi), \quad (2)$$

where

$$\theta_r(x, t) = \sum_{k \in \{0;1\}^{2N}} \exp\left\{ \sum_{\mu > \nu, \mu, \nu=1}^{2N} \ln \left( \frac{\gamma_\nu - \gamma_\nu}{\gamma_\nu + \gamma_\mu} \right)^2 k_\mu k_\nu \right\} \quad (3)$$

$$+(\sum_{\nu=1}^{2N} i\kappa_{\nu}(x-x_{0\nu})-2\delta_{\nu}(t-t_{0\nu})+(r-1)\ln\frac{\gamma_{\nu}-i}{\gamma_{\nu}+i}+\sum_{\mu=1, \mu\neq\nu}^{2N}\ln\left|\frac{\gamma_{\nu}+\gamma_{\mu}}{\gamma_{\nu}-\gamma_{\mu}}\right|+\pi i\epsilon_{\nu})k_{\nu}\}, \quad r=1, 3.$$

In this formula, the quantities  $\kappa_{\nu}$ ,  $\delta_{\nu}$ ,  $\gamma_{\nu}$  are functions of the parameters  $\lambda_{\nu}$ ,  $\nu=1, \dots, 2N$ ,  $N$  being a positive integer,  $N \geq 1$  and they are given by the following equations,

$$\kappa_{\nu} = 2\sqrt{1-\lambda_{\nu}^2}, \quad \delta_{\nu} = \kappa_{\nu}\lambda_{\nu}, \quad \gamma_{\nu} = \sqrt{\frac{1-\lambda_{\nu}}{1+\lambda_{\nu}}}, \quad \epsilon_{\nu} \in \{0; 1\}. \quad (4)$$

$\varphi$ ,  $x_{0\nu}$ ,  $t_{0\nu}$ ,  $\lambda_{\nu}$ , and  $\kappa_{\nu}$ ,  $\delta_{\nu}$ ,  $\gamma_{\nu}$ ,  $\nu=1 \dots 2N$  are real numbers satisfying the conditions

$$\begin{aligned} x_{0N+j} &= x_{0j}, & t_{0N+j} &= t_{0j}, & 0 < \lambda_j < 1, \\ \lambda_{N+j} &= -\lambda_j, & \kappa_{N+j} &= \kappa_j, & \delta_{N+j} &= -\delta_j, & \gamma_{N+j} &= 1/\gamma_j, & j &= 1 \dots N. \end{aligned}$$

## 2 Expression of solutions of NLS equation in terms of Fredholm determinant

### 2.1 Relation between $\theta$ and Fredholm determinant

The function  $\theta_r$  defined in (3) can be rewritten with a summation in terms of subsets of  $[1, \dots, 2N]$

$$\begin{aligned} \theta_r(x, t) &= \sum_{J \subset \{1, \dots, 2N\}} \prod_{\nu \in J} (-1)^{\epsilon_{\nu}} \prod_{\nu \in J, \mu \notin J} \left| \frac{\gamma_{\nu} + \gamma_{\mu}}{\gamma_{\nu} - \gamma_{\mu}} \right| \\ &\times \exp\left\{ \sum_{\nu \in J} i\kappa_{\nu}(x - x_{0\nu}) - 2\delta_{\nu}(t - t_{0\nu}) + x_{r, \nu} \right\}, \end{aligned}$$

with

$$x_{r, \nu} = (r-1) \ln \frac{\gamma_{\nu} - i}{\gamma_{\nu} + i}, \quad 1 \leq j \leq 2N, \quad (5)$$

in particular

$$\begin{aligned} x_{r, j} &= (r-1) \ln \frac{\gamma_j - i}{\gamma_j + i}, \quad 1 \leq j \leq N, \\ x_{r, N+j} &= -(r-1) \ln \frac{\gamma_j - i}{\gamma_j + i} - (r-1)i\pi, \quad 1 \leq j \leq N. \end{aligned} \quad (6)$$

In the preceding formula,  $\epsilon_j$  can be chosen arbitrary, for example it can be fixed in the following way :

$\epsilon_j = j$  for  $1 \leq j \leq N$ , and  $\epsilon_j = j + 1$  for  $N + 1 \leq j \leq 2N$ .

In a recent paper, Kirillov and Van Diejen have given determinantal formulas for zonal spherical functions on hyperboloids. In particular, they compute  $\det(I + A)$ , where  $I$  is the unit matrix and  $A = (a_{jk})_{1 \leq j, k \leq 2N}$  the matrix defined as :

$$a_{jk} = \frac{2\epsilon_j K_j}{K_j + K_k} \prod_{l \neq j} \left| \frac{K_j + K_l}{K_j - K_l} \right| \exp(-2K_j x), \quad (7)$$

where  $\epsilon_j \in \{-1; +1\}$ .

Then it is shown that  $\det(I + A)$  has the following form

$$\det(I + A) = \sum_{J \subset \{1, \dots, N\}} \prod_{j \in J} \epsilon_j \prod_{j \in J, k \notin J} \left| \frac{K_j + K_k}{K_j - K_k} \right| \exp(-2x \sum_{j \in J} K_j). \quad (8)$$

Using the same strategy, we can compute  $\det(I + A_r)$  where  $A_r = (a_{\nu\mu})_{1 \leq \nu, \mu \leq 2N}$  is the matrix defined as :

$$a_{\nu\mu} = \frac{2(-1)^{\epsilon_\nu} \gamma_\nu}{\gamma_\nu + \gamma_\mu} \prod_{\eta \neq \nu} \left| \frac{\gamma_\nu + \gamma_\eta}{\gamma_\nu - \gamma_\eta} \right| \exp(iK_\nu(x - x_{0\nu}) - 2\delta_\nu(t - t_{0\nu}) + x_{r,\nu}). \quad (9)$$

Then  $\det(I + A_r)$  has the following form

$$\det(I + A_r) = \sum_{J \subset \{1, \dots, 2N\}} \prod_{\nu \in J} (-1)^{\epsilon_\nu} \prod_{\nu \in J, \mu \notin J} \left| \frac{\gamma_\nu + \gamma_\mu}{\gamma_\nu - \gamma_\mu} \right| \exp(iK_\nu(x - x_{0\nu}) - 2\delta_\nu(t - t_{0\nu}) + x_{r,\nu}). \quad (10)$$

From the beginning of this section,  $\tilde{\theta}$  has the same expression as in (10) so, we have clearly the equality

$$\theta_r = \det(I + A_r). \quad (11)$$

Then the solution of NLS equation takes the form

$$v(x, t) = \frac{\det(I + A_3(x, t))}{\det(I + A_1(x, t))} \exp(2it - i\varphi). \quad (12)$$

## 2.2 Example

If we consider the case where  $N = 1$  we get :

$$\begin{aligned}\theta_3(x, t) &= 1 - \exp\left(-2 \ln \frac{\gamma_1 - i}{\gamma_1 + i} + \ln \frac{1 + \gamma_1^2}{1 - \gamma_1^2} + iK_1(x - x_{01}) + 2\delta_1(t - t_{01})\right) \\ &\quad - \exp\left(-2 \ln \frac{\gamma_1 - i}{\gamma_1 + i} + \ln \frac{1 + \gamma_1^2}{1 - \gamma_1^2} + iK_1(x - x_{01}) - 2\delta_1(t - t_{01})\right) \\ &\quad + \exp(iK_1(x - x_{01}) + 2\delta_1(t - t_{01})).\end{aligned}$$

It is clearly the expression of  $\det(I + A_3)$ .

$$\begin{aligned}\theta_1(x, t) &= 1 - \exp\left(\ln \frac{1 + \gamma_1^2}{1 - \gamma_1^2} + iK_1(x - x_{01}) + 2\delta_1(t - t_{01})\right) \\ &\quad - \exp\left(\ln \frac{1 + \gamma_1^2}{1 - \gamma_1^2} + iK_1(x - x_{01}) - 2\delta_1(t - t_{01})\right) \\ &\quad + \exp(iK_1(x - x_{01}) + 2\delta_1(t - t_{01})).\end{aligned}$$

We refine the expression of  $\det(I + A_1)$ .

Then we can write clearly the solution of NLS equation in the form

$$v(x, t) = \frac{\det(I + A_3(x, t))}{\det(I + A_1(x, t))} \exp(2it - i\varphi). \quad (13)$$

## 3 Expression of solutions of NLS equation in terms of wronkian determinant

### 3.1 Link between Fredholm determinants and wronskians

We consider the following functions

$$\begin{aligned}\phi_\nu(y) &= \sin(K_\nu(x - x_{0\nu})/2 + i\delta_\nu(t - t_{0\nu}) - ix_{r,\nu}/2 + \gamma_\nu y), \quad 1 \leq \nu \leq N, \\ \phi_\nu(y) &= \cos(K_\nu(x - x_{0\nu})/2 + i\delta_\nu(t - t_{0\nu}) - ix_{r,\nu}/2 + \gamma_\nu y), \quad N + 1 \leq \nu \leq 2N.\end{aligned} \quad (14)$$

These functions depend on  $r$ . We must denote them  $\phi_\nu^r(y)$ . For simplicity, in this section we denote them  $\phi_\nu(y)$ .

We use the following notations :

$\Theta_\nu = K_\nu(x - x_{0\nu})/2 + i\delta_\nu(t - t_{0\nu}) - ix_{r,\nu}/2 + \gamma_\nu y$ ,  $1 \leq j \leq 2N$ .  
 $W_r(y) = W(\phi_1, \dots, \phi_{2N})$  is the wronskian  $W_r(y) = \det[(\partial_y^{\mu-1} \phi_\nu)_{\nu, \mu \in [1, \dots, 2N]}]$ .  
We consider the matrix  $D_r = (d_{\nu\mu})_{\nu, \mu \in [1, \dots, 2N]}$  defined by

$$d_{\nu\mu} = \frac{2(-1)^{\epsilon_\nu \gamma_\nu}}{\gamma_\nu + \gamma_\mu} \prod_{\eta \neq \nu} \left| \frac{\gamma_\nu + \gamma_\mu}{\gamma_\nu - \gamma_\mu} \right| \exp(i\kappa_\nu(x - x_{0\nu}) - 2\delta_\nu(t - t_{0\nu}) + x_{r,\nu}),$$

$$1 \leq \nu \leq 2N, \quad 1 \leq \mu \leq 2N,$$

with

$$x_{r,\nu} = (r - 1) \ln \frac{\gamma_\nu - i}{\gamma_\nu + i}.$$

Then we have the following statement

**Theorem 3.1**

$$\det(I + D_r) = k_r(0) \times W_r(\phi_1, \dots, \phi_{2N})(0), \quad (15)$$

where

$$k_r(y) = \frac{2^{2N} \exp(i \sum_{\nu=1}^{2N} \Theta_\nu)}{\prod_{\nu=2}^{2N} \prod_{\mu=1}^{\nu-1} (\gamma_\nu - \gamma_\mu)}.$$

**Proof :** We start to remove the factor  $(2i)^{-1} e^{i\Theta_\nu}$  in each row  $\nu$  in the wronskian  $W_r(y)$  for  $1 \leq \nu \leq 2N$ .

Then

$$W_r = \prod_{\nu=1}^{2N} e^{i\Theta_\nu} (2i)^{-N} (2)^{-N} \times W_{r1}, \quad (16)$$

with

$$W_{r1} = \begin{vmatrix} (1 - e^{-2i\Theta_1}) & i\gamma_1(1 + e^{-2i\Theta_1}) & \dots & (i\gamma_1)^{2N-1}(1 + (-1)^{2N} e^{-2i\Theta_1}) \\ (1 - e^{-2i\Theta_2}) & i\gamma_2(1 + e^{-2i\Theta_2}) & \dots & (i\gamma_2)^{2N-1}(1 + (-1)^{2N} e^{-2i\Theta_2}) \\ \vdots & \vdots & \vdots & \vdots \\ (1 - e^{-2i\Theta_{2N}}) & i\gamma_{2N}(1 + e^{-2i\Theta_{2N}}) & \dots & (i\gamma_{2N})^{2N-1}(1 + (-1)^{2N} e^{-2i\Theta_{2N}}) \end{vmatrix}$$

The determinant  $W_{r1}$  can be written as

$$W_{r1} = \det(\alpha_{jk} e_j + \beta_{jk}),$$

where  $\alpha_{jk} = (-1)^k (i\gamma_j)^{k-1}$ ,  $e_j = e^{-2i\theta_j}$ , and  $\beta_{jk} = (i\gamma_j)^{k-1}$ ,  $1 \leq j \leq N$ ,  $1 \leq k \leq 2N$ ,

$\alpha_{jk} = (-1)^{k-1} (i\gamma_j)^{k-1}$ ,  $e_j = e^{-2i\theta_j}$ , and  $\beta_{jk} = (i\gamma_j)^{k-1}$ ,  $N + 1 \leq j \leq 2N$ ,

$1 \leq k \leq 2N$ .

Denoting  $U = (\alpha_{ij})_{i,j \in [1, \dots, 2N]}$ ,  $V = (\beta_{ij})_{i,j \in [1, \dots, 2N]}$ , the determinant of  $U$  is clearly equal to

$$\det(U) = (i)^{\frac{2N(2N-1)}{2}} \prod_{2N \geq l > m \geq 1} (\gamma_l - \gamma_m). \quad (17)$$

Then we use the following Lemma

**Lemma 3.1** *Let  $A = (a_{ij})_{i,j \in [1, \dots, N]}$ ,  $B = (b_{ij})_{i,j \in [1, \dots, N]}$ ,  $(H_{ij})_{i,j \in [1, \dots, N]}$ , the matrix formed by replacing the  $j$ th row of  $A$  by the  $i$ th row of  $B$*

*Then*

$$\det(a_{ij}x_i + b_{ij}) = \det(a_{ij}) \times \det\left(\delta_{ij}x_i + \frac{\det(H_{ij})}{\det(a_{ij})}\right) \quad (18)$$

**Proof :** For  $\tilde{A} = (\tilde{a}_{ji})_{i,j \in [1, \dots, N]}$  the transposed matrix in cofactors of  $A$ , we have the well known formula  $A \times {}^t \tilde{A} = \det A \times I$ .

So it is clear that  $\det(\tilde{A}) = (\det(A))^{N-1}$ .

The general term of the product  $(c_{ij})_{i,j \in [1, \dots, N]} = (a_{ij}x_i + b_{ij})_{i,j \in [1, \dots, N]} \times (\tilde{a}_{ji})_{i,j \in [1, \dots, N]}$  can be written as

$$\begin{aligned} c_{ij} &= \sum_{s=1}^N (a_{is}x_i + b_{is}) \times \tilde{a}_{js} \\ &= x_i \sum_{s=1}^N a_{is} \tilde{a}_{js} + \sum_{s=1}^N b_{is} \tilde{a}_{js} \\ &= \delta_{ij} \det(A)x_i + \det(H_{ij}). \end{aligned}$$

We get

$$\det(c_{ij}) = \det(a_{ij}x_i + b_{ij}) \times (\det(A))^{N-1} = (\det(A))^N \times \det\left(\delta_{ij}x_i + \frac{\det(H_{ij})}{\det(A)}\right).$$

$$\text{Thus } \det(a_{ij}x_i + b_{ij}) = \det(A) \times \det\left(\delta_{ij}x_i + \frac{\det(H_{ij})}{\det(A)}\right).$$

□

Using the previous lemma (18), we get :

$$\det(\alpha_{ij}e_i + \beta_{ij}) = \det(\alpha_{ij}) \times \det\left(\delta_{ij}e_i + \frac{\det(H_{ij})}{\det(\alpha_{ij})}\right),$$

where  $(H_{ij})_{i,j \in [1, \dots, N]}$  is the matrix formed by replacing the  $j$ th row of  $U$  by the  $i$ th row of  $V$  defined previously.

We compute  $\det(H_{ij})$  and we get

$$\det(H_{ij}) = (-1)^{\frac{2N(2N+1)}{2}+1} (i)^{\frac{2N(2N-1)}{2}} \prod_{2N \geq l > m \geq 1, l \neq j, m \neq j} (\gamma_l - \gamma_m) \prod_{l < j} (\gamma_k - \gamma_l) \prod_{l > j} (\gamma_k - \gamma_l). \quad (19)$$

We can simplify the quotient  $q = \frac{\det(H_{ij})}{\det(\alpha_{ij})}$  :

$$\begin{aligned} q &= \frac{(-1)^{\epsilon(j)} \prod_{l \neq j} (\gamma_l + \gamma_k)}{(-1)^{k-1} \prod_{l \neq k} (\gamma_l - \gamma_k)} \\ &= \frac{(-1)^{\epsilon(j)-k+1} 2\gamma_k \prod_{l \neq k} (\gamma_l + \gamma_k)}{(\gamma_j + \gamma_k) \prod_{l \neq k} (\gamma_l - \gamma_k)} \\ &= \frac{(-1)^{\epsilon(j)} 2\gamma_k}{(\gamma_j + \gamma_k)} \prod_{l \neq k} \left| \frac{\gamma_l + \gamma_k}{\gamma_l - \gamma_k} \right|. \end{aligned} \quad (20)$$

So  $\det(\delta_{jk}e_j + \frac{\det(H_{jk})}{\det(\alpha_{jk})})$  can be expressed as

$$\det(\delta_{jk}e_j + \frac{\det(H_{jk})}{\det(\alpha_{jk})}) = \prod_{j=1}^{2N} e^{-2i\Theta_j} \det(\delta_{jk} + \frac{(-1)^{\epsilon(j)} 2\gamma_k}{(\gamma_j + \gamma_k)} \prod_{l \neq k} \left| \frac{\gamma_l + \gamma_k}{\gamma_l - \gamma_k} \right| e^{2i\Theta_j}).$$

Then dividing each column  $k$  by  $\gamma_k$ ,  $1 \leq k \leq N$  and multiplying each row  $j$  by  $\gamma_j$ ,  $1 \leq j \leq N$ , we get

$$\det(\delta_{jk}e_j + \frac{\det(H_{jk})}{\det(\alpha_{jk})}) = \prod_{j=1}^{2N} e^{-2i\Theta_j} \det(\delta_{jk} + \frac{(-1)^{\epsilon(j)} 2\gamma_j}{(\gamma_j + \gamma_k)} \prod_{l \neq k} \left| \frac{\gamma_l + \gamma_k}{\gamma_l - \gamma_k} \right| e^{2i\Theta_j}),$$

and therefore the wronskian

$$W_r(\phi_1, \dots, \phi_N)(0)$$

can be written as

$$\prod_{j=1}^{2N} e^{i\Theta_j|_{y=0}} (2)^{-2N} (i)^{\frac{2N(2N-2)}{2}} \prod_{j=2}^{2N} \prod_{i=1}^{j-1} (\gamma_j - \gamma_i) \prod_{j=1}^{2N} e^{-2i\Theta_j|_{y=0}} \det(I + D_r)$$

It follows that

$$\det(I + D_r) = \frac{e^{i \sum_{j=1}^{2N} \Theta_j|_{y=0}} 2^{2N}}{\prod_{j=2}^{2N} \prod_{i=1}^{j-1} (\gamma_j - \gamma_i)} W_r(\phi_1, \dots, \phi_{2N})(0) = k_r(0) W_r(\phi_1, \dots, \phi_{2N})(0). \quad (21)$$

□

So, the solution of NLS equation takes the form

$$v(x, t) = \frac{k_3(0)W_3(0)}{k_1(0)W_1(0)} \exp(2it - i\varphi) \quad (22)$$

and we get the following result



**Theorem 3.2** *The function  $v$  defined by*

$$v(x, t) = \frac{W_3(\phi_1, \dots, \phi_{2N})(0)}{W_1(\phi_1, \dots, \phi_{2N})(0)} \exp(2it - i\varphi).$$

*is solution of the NLS equation (1)*

$$iv_t + v_{xx} + 2|v|^2v = 0,$$

*where*

$W_r(y) = W(\phi_1, \dots, \phi_{2N})$  *is the wronskian*  $W_r(y) = \det[(\partial_y^{\mu-1} \phi_\nu)_{\nu, \mu \in [1, \dots, 2N]}]$ , *and*  $\phi_\nu$  *the functions given by*

$$\phi_\nu(y) = \sin(K_\nu(x - x_{0\nu})/2 + i\delta_\nu(t - t_{0\nu}) - ix_{r,\nu}/2 + \gamma_\nu y), \quad 1 \leq \nu \leq N,$$

$$\phi_\nu(y) = \cos(K_\nu(x - x_{0\nu})/2 + i\delta_\nu(t - t_{0\nu}) - ix_{r,\nu}/2 + \gamma_\nu y), \quad N+1 \leq \nu \leq 2N,$$

$\kappa_\nu, \delta_\nu, \gamma_\nu$  *parameters defined by (4) for*  $\nu \in [1, \dots, 2N]$

$$\kappa_\nu = 2\sqrt{1 - \lambda_\nu^2}, \quad \delta_\nu = \kappa_\nu \lambda_\nu, \quad \gamma_\nu = \sqrt{\frac{1 - \lambda_\nu}{1 + \lambda_\nu}}, \quad \epsilon_\nu \in \{0; 1\},$$

*where*  $\lambda_\nu$  *is an arbitrary parameter such that*  $0 < \lambda_j < 1$  *and*  $\lambda_{N+j} = -\lambda_j$  *for*  $j \in [1, \dots, N]$ .

### 3.2 Example

If we consider the simplest case where  $N = 1$ , we get the following results. The wronskian  $W_3(0)$  is equal to

$$W_3(0) = \frac{1}{\gamma_1(1 + \lambda_1)} \times [\lambda_1 \cos(K_1(x - x_{01})) - \cosh(2\delta_1(t - t_{01}) - x_1)].$$

The Fredholm determinant  $F_3(0) = \det(I + A_3)$  takes the form

$$F_3(0) = \frac{2e^{iK_1x}}{\lambda_1} \times [\lambda_1 \cos(K_1(x - x_{01})) - \cosh(2\delta_1(t - t_{01}) - x_1)].$$

So we have the equality

$$\det(I + A_3) = C \times W_3,$$

with

$$C = \frac{K_1 e^{iK_1 x}}{\lambda_1}.$$

If we compute

$$k_3(0) = \frac{2^{2N} \exp(i \sum_{\nu=1}^{2N} \Theta_\nu|_{y=0})}{\prod_{\nu=2}^{2N} \prod_{\mu=1}^{\nu-1} (\gamma_\nu - \gamma_\mu)}$$

we get :

$$\exp(i \sum_{\nu=1}^{2N} \Theta_\nu|_{y=0}) = e^{iK_1 x};$$

$$\prod_{\nu=2}^{2N} \prod_{\mu=1}^{\nu-1} (\gamma_\nu - \gamma_\mu) = \gamma_2 - \gamma_1 = \frac{1}{\gamma_1} - \gamma_1 = \sqrt{\frac{1+\lambda_1}{1-\lambda_1}} - \sqrt{\frac{1-\lambda_1}{1+\lambda_1}} = \frac{2\lambda_1}{\sqrt{1-\lambda_1^2}} = \frac{2\lambda_1}{K_1/2} = \frac{4\lambda_1}{K_1};$$

So,

$$k_3(0) = \frac{2^{2N} \exp(i \sum_{\nu=1}^{2N} \Theta_\nu)}{\prod_{\nu=2}^{2N} \prod_{\mu=1}^{\nu-1} (\gamma_\nu - \gamma_\mu)} = \frac{K_1 e^{iK_1 x}}{\lambda_1}$$

and we have the equality

$$\det(I + A_3) = k_3(0) \times W_3.$$

## 4 Construction of quasi-rational solutions of NLS equation

### 4.1 Taking the limit when the parameters $\lambda_j \rightarrow 1$ for $1 \leq j \leq N$ and $\lambda_j \rightarrow -1$ for $N+1 \leq j \leq 2N$

In the following, we show how we can obtain quasi-rational solutions of NLS equation by a rough estimation.

We consider the parameter  $\lambda_j$  written in the form  $1 - \epsilon^2 c_j^2$ . For simplicity, we denote  $d_j$  the term  $\frac{c_j}{\sqrt{2}}$ .

When  $\epsilon$  goes to 0, we realize limited expansions at order  $2N - 1$  of  $\kappa_j$ ,  $\gamma_j$ ,  $\delta_j$ ,  $x_{r,j}$ ,  $\kappa_{N+j}$ ,  $\gamma_{N+j}$ ,  $\delta_{N+j}$ ,  $x_{r,N+j}$ , for  $1 \leq j \leq N$ . For example, in the case where  $N = 1$ , it gives :

$$\begin{aligned}
\kappa_j &= 4d_j\epsilon + O(\epsilon^2), & \gamma_j &= d_j\epsilon + O(\epsilon^2), & \delta_j &= 4d_j\epsilon + O(\epsilon^2), \\
x_{r,j} &= (r-1)(2id_j\epsilon + O(\epsilon^2)), \\
\kappa_{N+j} &= 4d_j\epsilon + O(\epsilon^2), & \gamma_{N+j} &= 1/(d_j\epsilon) - (d_j\epsilon)/2 + O(\epsilon^2), & \delta_{N+j} &= -4d_j\epsilon + O(\epsilon^2), \\
x_{r,N+j} &= -(r-1)(2id_j\epsilon + O(\epsilon^2)), \\
1 &\leq j \leq N.
\end{aligned}$$

Then we realize limited expansions at order  $2N - 1$  of the functions  $\phi_j^r(0)$  and  $\phi_{N+j}^r(0)$ , for  $1 \leq j \leq N$  :

$$\begin{aligned}
\phi_j^1(0) &= \sin E_j, \\
\phi_j^3(0) &= \sin D_j, \\
\phi_{N+j}^1(0) &= \sin E'_j, \\
\phi_{N+j}^3(0) &= \sin D'_j,
\end{aligned}$$

with

$$\begin{aligned}
D_j &= K_j(x - x_{0j})/2 + i\delta_j(t - t_{0j}) - ix_j/2, \\
E_j &= K_j(x - x_{0j})/2 + i\delta_j(t - t_{0j}), \\
D'_j &= K_j(x - x_{0j})/2 - i\delta_j(t - t_{0j}) + ix_j/2, \\
E'_j &= K_j(x - x_{0j})/2 - i\delta_j(t - t_{0j}).
\end{aligned}$$

For example, in the case where  $N = 1$ , it gives :

$$\begin{aligned}
D_j &= 2\epsilon d_j(x - x_{0j} + 2i(t - t_{0j}) + 1) + O(\epsilon^2), \\
E_j &= 2\epsilon d_j(x - x_{0j} + 2i(t - t_{0j})) + O(\epsilon^2), \\
D'_j &= 2\epsilon d_j(x - x_{0j} - 2i(t - t_{0j}) - 1) + O(\epsilon^2), \\
E'_j &= 2\epsilon d_j((x - x_{0j} - 2i(t - t_{0j}))) + O(\epsilon^2),
\end{aligned}$$

and

$$\begin{aligned}
\phi_j^1(0) &= 2\epsilon d_j(x - x_{0j} + 2i(t - t_{0j})) + O(\epsilon^2), \\
\phi_j^3(0) &= 2\epsilon d_j(x - x_{0j} + 2i(t - t_{0j}) + 1) + O(\epsilon^2), \\
\phi_{N+j}^1(0) &= 1 + O(\epsilon^2), \\
\phi_{N+j}^3(0) &= 1 + O(\epsilon^2).
\end{aligned}$$

Then according to the relation

$$v(x, t) = \frac{W_3(\phi_1, \dots, \phi_N)(0)}{W_1(\phi_1, \dots, \phi_N)(0)} \exp(2it - i\varphi).$$

the solution of NLS equation takes the form :

$$v(x, t) = \exp(2it - i\varphi) \frac{N(x, t, \epsilon)}{D(x, t, \epsilon)}$$

$$= \exp(2it - i\varphi) \left| \begin{array}{cccccc} \sin D_1 & \dots & \sin D_N & \cos D'_1 & \dots & \cos D'_N \\ \gamma_1 \cos D_1 & \dots & \gamma_N \cos D_N & -\frac{1}{\gamma_1} \sin D'_1 & \dots & -\frac{1}{\gamma_N} \sin D'_N \\ \gamma_1^2 \sin D_1 & \dots & \gamma_N^2 \sin D_N & \frac{1}{\gamma_1} \cos D'_1 & \dots & \frac{1}{\gamma_N} \cos D'_N \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \gamma_1^{2N-1} \cos D_1 & \dots & \gamma_N^{2N-1} \cos D_N & -\frac{1}{\gamma_1^{2N-1}} \sin D'_1 & \dots & -\frac{1}{\gamma_N^{2N-1}} \sin D'_N \\ \hline \sin E_1 & \dots & \sin E_N & \cos E'_1 & \dots & \cos E'_N \\ \gamma_1 \cos E_1 & \dots & \gamma_N \cos E_N & -\frac{1}{\gamma_1} \sin E'_1 & \dots & -\frac{1}{\gamma_N} \sin E'_N \\ \gamma_1^2 \sin E_1 & \dots & \gamma_N^2 \sin E_N & \frac{1}{\gamma_1} \cos E'_1 & \dots & \frac{1}{\gamma_N} \cos E'_N \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \gamma_1^{2N-1} \cos E_1 & \dots & \gamma_N^{2N-1} \cos E_N & -\frac{1}{\gamma_1^{2N-1}} \sin E'_1 & \dots & -\frac{1}{\gamma_N^{2N-1}} \sin E'_N \end{array} \right|$$

We use the classical developments

$$\sin z = \sum_{j=1}^N \frac{(-1)^{j-1} z^{2j-1}}{(2j-1)!} + O(z^{2N}) \text{ and } \cos z = \sum_{j=0}^{2N} \frac{(-1)^j z^{2j}}{(2j)!} + O(z^{2N+1}),$$

for  $z$  in a neighborhood of 0.

Using the fact that  $D_j = E_j + \alpha_j(\epsilon)$ ,  $D'_j = E'_j - \alpha_j(\epsilon)$ , and addition formulas for trigonometric functions, we can expand the numerator  $N(x, t, \epsilon)$  along the columns; it takes the form of

$$\prod_{j=1}^N \cos^2(\alpha_j(\epsilon)) D(x, t, \epsilon) + A.$$

If we divide each term of this sum by  $D(x, t)$ , when  $\epsilon$  tends to 0, the first term tends to 1; in each other term of the sum of A, we can multiply each fraction by some power of  $\gamma_j$  to get a finite limit when  $\epsilon$  goes to 0.

So we get really a non trivial and non degenerate quotient of two polynomials in  $x$  and  $t$  by an exponential.

Practically, we keep only the terms of lower degree in  $\epsilon$ , in numerator and denominator which expressed as powers of  $x$  and  $t$

So we get in this way a quasi-rational solution of the NLS equation.

So we get the following result

**Theorem 4.1** *The function  $v$  defined by*

$$v(x, t) = \lim_{\epsilon \rightarrow 0} \frac{W_3(\phi_1, \dots, \phi_{2N})(0)}{W_1(\phi_1, \dots, \phi_{2N})(0)} \exp(2it - i\varphi).$$

*is a quasi-rational solution of the NLS equation (1)*

$$iv_t + v_{xx} + 2|v|^2v = 0,$$

where

$W_r(y) = W(\phi_1, \dots, \phi_{2N})$  is the wronskian  $W_r(y) = \det[(\partial_y^{\mu-1} \phi_\nu)_{\nu, \mu \in [1, \dots, 2N]}]$ , and  $\phi_\nu$  the functions given by

$$\phi_\nu(y) = \sin(K_\nu(x - x_{0\nu})/2 + i\delta_\nu(t - t_{0\nu}) - ix_{r,\nu}/2 + \gamma_\nu y), \quad 1 \leq \nu \leq N,$$

$$\phi_\nu(y) = \cos(K_\nu(x - x_{0\nu})/2 + i\delta_\nu(t - t_{0\nu}) - ix_{r,\nu}/2 + \gamma_\nu y), \quad N + 1 \leq \nu \leq 2N,$$

$\kappa_\nu, \delta_\nu, \gamma_\nu$  parameters defined by (4) for  $\nu \in [1, \dots, 2N]$

$$\kappa_\nu = 2\sqrt{1 - \lambda_\nu^2}, \quad \delta_\nu = \kappa_\nu \lambda_\nu, \quad \gamma_\nu = \sqrt{\frac{1 - \lambda_\nu}{1 + \lambda_\nu}}, \quad \epsilon_\nu \in \{0; 1\},$$

where  $\lambda_j = 1 - \epsilon^2 c_j^2$  is an arbitrary parameter such that  $0 < \lambda_j < 1$  and  $\lambda_{N+j} = -\lambda_j$ ,  $c_j$  an arbitrary real number, for  $j \in [1, \dots, N]$ ,

If we want to recover exact solution of Akhmediev type, we must more precise in the estimations.

## 4.2 Examples

For brevity, in the following we choose  $x_{0i} = 0$ ,  $t_{0i} = 0$  and  $\varphi = 0$  for  $1 \leq i \leq 2N$ .

If we consider the case  $N = 1$ , from (22), we realize a development at order 1 of  $W_3$  and  $W_1$  in  $\epsilon$ . The solution of NLS equation can be written as

$$v(x, t) = -\frac{(-4x^2 - 16t^2 + 16it + 3)}{4x^2 + 16t^2 + 1} \exp(2it).$$

Then  $v$  takes the form

$$v(x, t) = 1 - 4 \frac{1 + 4it}{1 + 16t^2 + 4x^2} \exp(2it).$$

If we make the following changes of variables  $t \rightarrow X/2$  and  $x \rightarrow T$ , we get exactly first order Akhmediev's solution (see [1]).

$$v(x, t) = \left(1 - \frac{4(1 + 2ix)}{1 + 4x^2 + 4t^2}\right) \exp(ix).$$

In the case  $N = 2$ , we must realize a development at order 3 in  $\epsilon$ . From (22), the solution of NLS equation can be written as

$$v(x, t) = \frac{N(x, t)}{D(x, t)} \exp(2it),$$

with

$$\begin{aligned} N(x, t) = & 5760x^2t^2 - 3072x^2t^4 + 1872t^2 - 64x^6 - 45 - 4096t^6 + 180x^2 + 144x^4 - 768x^4t^2 \\ & + 768ix^4t + 8448t^4 - 1152ix^2t + 1536it^3 - 720it + 12288it^5 + 6144it^3x^2, \end{aligned}$$

and

$$D(x, t) = -1152x^2t^2 + 3072x^2t^4 + 1584t^2 + 64x^6 + 9 + 4096t^6 + 108x^2 + 48x^4 + 768x^4t^2 + 6912t^4.$$

If we make the following changes of variables  $t \rightarrow X/2$  and  $x \rightarrow T$ , we get

$$v(x, t) = \frac{N_1(x, t)}{D_1(x, t)} \exp(ix),$$

with

$$\begin{aligned} N_1(x, t) = & 1440x^2t^2 - 192x^4t^2 + 468x^2 - 64t^6 - 45 - 64x^6 + 180t^2 + 144t^4 - 192x^2t^4 - 576it^2x \\ & + 528x^4 + 384ix^5 + 192ix^3 - 360ix + 384it^4x + 768ix^3t^2, \end{aligned}$$

and

$$D_1(x, t) = -288x^2t^2 + 192x^4t^2 + 396x^2 + 64t^6 + 9 + 64x^6 + 108t^2 + 48t^4 + 192x^2t^4 + 432x^4.$$

It can be reduced exactly at the second order Akhmediev's solution (see [1]).

$$v(x, t) = 1 + \frac{3/8 - 3t^2 - 2t^4 - 9x^2 - 10x^4 - 12t^2x^2 + ix(15/4 + 6t^2 - 4t^4 - 2x^2 - 4x^4 - 8t^2x^2)}{3/32 + 9/8t^2 + 1/2t^4 + 2/3t^6 + 33/8x^2 + 9/2x^4 + 2/3x^6 - 3t^2x^2 + 2t^4x^2 + 2t^2x^4} \exp(ix).$$

This method can be continued to get quasi-rational solutions of NLS equation at any order.

## 5 Conclusion

Here is given a new formulation of solutions of NLS equation that gives as particular case, Akhmediev's solutions. These solutions expressed as a quotient of wronskians are similar to a previous result of Eleonski, Krichever and Kulagin [12]. Their formulation can also be seen as a quotient of two wronkians but is more complicated because it needs recursion operations. However, it can be used to get solutions of NLS equation (see [11]).

This new formulation can give explicitly a set of non singular solutions of NLS equation which can be obtained by a simpler way than with the Krichever method.

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