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Quasi-rational solutions of the NLS equation and rogue waves

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Abstract

We degenerate solutions of the NLS equation from the general formulation in terms of theta functions given in [19], to get quasi-rational solutions of NLS equations. For this we establish a link between Fredholm determinants and wronskians. We give solutions of the NLS equation as a quotient of two wronskian determinants. In the limit when some parameter goes to 0, we recover Peregrine's solution [32], and also Akhmediev's solutions given recently [1, 2, 3, 4, 5, 6, 8]. It gives a new approach to get the well known rogue waves.

1 Introduction

The nonlinear Schrödinger equation (NLS) equation plays a fundamental role in hydrodynamics, in particular for waves in deep water in the context of the rogue waves or in non-linear optics.

In this paper, we will study solutions of NLS equation from the theta formulation given by A. Its [19] and give a new representation in terms of wronskians. The solutions take the form of a quotient of two wronskians of elementary functions depending on a certain number of parameters.

If we take wronskians of order $2N$, we will call these solutions, solutions of NLS of order N .

Then, we take the limit when some parameter goes to 0 to get quasi-rational solutions of NLS equation.

At order 1, we recover the well known Peregrine's solution to the focusing

NLS equation which was first found [32] in 1983; it was rediscovered later in several publications, in particular [7].

The second order Peregrine-like solution was first constructed in [7]. Other Peregrine-like solutions were found for reduced Maxwell-Bloch integrable system [25, 26]. In [19], the n-phase quasi-periodic modulations of the plane waves solutions were constructed via appropriate degeneration of the finite gap periodic solutions to the NLS equation. These results have provided a way to construct multi-parametric families of multi-Peregrine solutions.

Interesting families of higher order were constructed using Darboux transformations in a series of articles by Akhmediev et al. [1, 2, 3, 4, 5, 6, 8].

Rational solutions of NLS equation are given in [31] in a determinantal representation as degenerate solutions of Davey-Stewartson equations. Another approach using Crum transformation is given in [18] to get sequences of rational solutions of NLS equation. Very recently, it is shown in [12] that rational solutions can be written as a quotient of two wronskians using modified version of [13], with some different reasoning; moreover, the link between quasi-rational solutions to the focusing NLS equation and the rational solution of the KP-I equation is established.

Here, we give another representation in terms of wronskians with simple generating functions, using central ideas of A. Its from finite gaps solutions in terms of theta functions.

With this formulation we recover as particular case, Akhmediev's quasi-rational solutions of NLS equation.

2 Expression of solutions of NLS equation in terms of Fredholm determinant

2.1 Solutions of NLS equation in terms of θ functions

The solution of the NLS equation

$$iv_t + v_{xx} + 2|v|^2v = 0, \quad (1)$$

is given in terms of theta function by (see [19])

$$v(x, t) = \frac{\theta_3(x, t)}{\theta_1(x, t)} \exp(2it - i\varphi), \quad (2)$$

where

$$\begin{aligned} \theta_r(x, t) = & \sum_{k \in \{0;1\}^{2N}} \exp\left\{ \sum_{\mu > \nu, \mu, \nu=1}^{2N} \ln \left(\frac{\gamma_\nu - \gamma_\mu}{\gamma_\nu + \gamma_\mu} \right)^2 k_\mu k_\nu \right. \\ & \left. + \left(\sum_{\nu=1}^{2N} i\kappa_\nu(x - x_{0\nu}) - 2\delta_\nu(t - t_{0\nu}) + (r-1) \ln \frac{\gamma_\nu - i}{\gamma_\nu + i} + \sum_{\mu=1, \mu \neq \nu}^{2N} \ln \left| \frac{\gamma_\nu + \gamma_\mu}{\gamma_\nu - \gamma_\mu} \right| + \pi i \epsilon_\nu \right) k_\nu \right\}, \end{aligned} \quad (3) \quad r = 1, 3.$$

In this formula, the quantities κ_ν , δ_ν , γ_ν are functions of the parameters λ_ν , $\nu = 1, \dots, 2N$, N being a positive integer, $N \geq 1$ and they are given by the following equations,

$$\kappa_\nu = 2\sqrt{1 - \lambda_\nu^2}, \quad \delta_\nu = \kappa_\nu \lambda_\nu, \quad \gamma_\nu = \sqrt{\frac{1 - \lambda_\nu}{1 + \lambda_\nu}}, \quad \epsilon_\nu \in \{0; 1\}. \quad (4)$$

φ , $x_{0\nu}$, $t_{0\nu}$, λ_ν , and κ_ν , δ_ν , γ_ν , $\nu = 1 \dots 2N$ are real numbers satisfying the conditions

$$\begin{aligned} x_{0N+j} &= x_{0j}, & t_{0N+j} &= t_{0j}, & 0 < \lambda_j < 1, \\ \lambda_{N+j} &= -\lambda_j, & \kappa_{N+j} &= \kappa_j, & \delta_{N+j} &= -\delta_j, & \gamma_{N+j} &= 1/\gamma_j, & j &= 1 \dots N. \end{aligned}$$

2.2 Relation between θ and Fredholm determinant

The function θ_r defined in (3) can be rewritten with a summation in terms of subsets of $[1, \dots, 2N]$

$$\begin{aligned} \theta_r(x, t) = & \sum_{J \subset \{1, \dots, 2N\}} \prod_{\nu \in J} (-1)^{\epsilon_\nu} \prod_{\nu \in J, \mu \notin J} \left| \frac{\gamma_\nu + \gamma_\mu}{\gamma_\nu - \gamma_\mu} \right| \\ & \times \exp\left\{ \sum_{\nu \in J} i\kappa_\nu(x - x_{0\nu}) - 2\delta_\nu(t - t_{0\nu}) + x_{r,\nu} \right\}, \end{aligned}$$

with

$$x_{r,\nu} = (r-1) \ln \frac{\gamma_\nu - i}{\gamma_\nu + i}, \quad 1 \leq j \leq 2N, \quad (5)$$

in particular

$$\begin{aligned} x_{r,j} &= (r-1) \ln \frac{\gamma_j - i}{\gamma_j + i}, & 1 \leq j \leq N, \\ x_{r,N+j} &= -(r-1) \ln \frac{\gamma_j - i}{\gamma_j + i} - (r-1)i\pi, & 1 \leq j \leq N. \end{aligned} \quad (6)$$

In the preceding formula, ϵ_j can be chosen arbitrary, for example it can be fixed in the following way :

$\epsilon_j = j$ for $1 \leq j \leq N$, and $\epsilon_j = j + 1$ for $N + 1 \leq j \leq 2N$.

The link between theta representation of solutions of NLS equation and Fredholm determinant formulation was already presented in [19]. In more details, it is given in a recent paper of Kirillov and Van Diejen [37]. In particular, they compute $\det(I + A)$, where I is the unit matrix and $A = (a_{jk})_{1 \leq j, k \leq 2N}$ the matrix defined as :

$$a_{jk} = \frac{2\epsilon_j K_j}{K_j + K_k} \prod_{l \neq j} \left| \frac{K_j + K_l}{K_j - K_l} \right| \exp(-2K_j x), \quad (7)$$

where $\epsilon_j \in \{-1; +1\}$.

Then it is shown that $\det(I + A)$ has the following form

$$\det(I + A) = \sum_{J \subset \{1, \dots, N\}} \prod_{j \in J} \epsilon_j \prod_{j \in J, k \notin J} \left| \frac{K_j + K_k}{K_j - K_k} \right| \exp(-2x \sum_{j \in J} K_j). \quad (8)$$

Using the same strategy, we can compute $\det(I + A_r)$ where $A_r = (a_{\nu\mu})_{1 \leq \nu, \mu \leq 2N}$ is the matrix defined as :

$$a_{\nu\mu} = \frac{2(-1)^{\epsilon_\nu} \gamma_\nu}{\gamma_\nu + \gamma_\mu} \prod_{\eta \neq \nu} \left| \frac{\gamma_\nu + \gamma_\eta}{\gamma_\nu - \gamma_\eta} \right| \exp(iK_\nu(x - x_{0\nu}) - 2\delta_\nu(t - t_{0\nu}) + x_{r,\nu}). \quad (9)$$

Then $\det(I + A_r)$ has the following form

$$\det(I + A_r) = \sum_{J \subset \{1, \dots, 2N\}} \prod_{\nu \in J} (-1)^{\epsilon_\nu} \prod_{\nu \in J, \mu \notin J} \left| \frac{\gamma_\nu + \gamma_\mu}{\gamma_\nu - \gamma_\mu} \right| \exp(iK_\nu(x - x_{0\nu}) - 2\delta_\nu(t - t_{0\nu}) + x_{r,\nu}). \quad (10)$$

From the beginning of this section, $\tilde{\theta}$ has the same expression as in (10) so, we have clearly the equality

$$\theta_r = \det(I + A_r). \quad (11)$$

Then the solution of NLS equation takes the form

$$v(x, t) = \frac{\det(I + A_3(x, t))}{\det(I + A_1(x, t))} \exp(2it - i\varphi). \quad (12)$$

2.3 Example

If we consider the case where $N = 1$ we get :

$$\begin{aligned} \theta_3(x, t) &= 1 - \exp\left(-2 \ln \frac{\gamma_1 - i}{\gamma_1 + i} + \ln \frac{1 + \gamma_1^2}{1 - \gamma_1^2} + iK_1(x - x_{01}) + 2\delta_1(t - t_{01})\right) \\ &\quad - \exp\left(-2 \ln \frac{\gamma_1 - i}{\gamma_1 + i} + \ln \frac{1 + \gamma_1^2}{1 - \gamma_1^2} + iK_1(x - x_{01}) - 2\delta_1(t - t_{01})\right) \\ &\quad + \exp(iK_1(x - x_{01}) + 2\delta_1(t - t_{01})). \end{aligned}$$

It is clearly the expression of $\det(I + A_3)$.

$$\begin{aligned} \theta_1(x, t) &= 1 - \exp\left(\ln \frac{1 + \gamma_1^2}{1 - \gamma_1^2} + iK_1(x - x_{01}) + 2\delta_1(t - t_{01})\right) \\ &\quad - \exp\left(\ln \frac{1 + \gamma_1^2}{1 - \gamma_1^2} + iK_1(x - x_{01}) - 2\delta_1(t - t_{01})\right) \\ &\quad + \exp(iK_1(x - x_{01}) + 2\delta_1(t - t_{01})). \end{aligned}$$

We rekind the expression of $\det(I + A_1)$.

Then we can write clearly the solution of NLS equation in the form

$$v(x, t) = \frac{\det(I + A_3(x, t))}{\det(I + A_1(x, t))} \exp(2it - i\varphi). \quad (13)$$

3 Expression of solutions of NLS equation in terms of wronkian determinant

3.1 Link between Fredholm determinants and wronskians

We consider the following functions

$$\begin{aligned} \phi_\nu(y) &= \sin(K_\nu(x - x_{0\nu})/2 + i\delta_\nu(t - t_{0\nu}) - ix_{r,\nu}/2 + \gamma_\nu y), \quad 1 \leq \nu \leq N, \\ \phi_\nu(y) &= \cos(K_\nu(x - x_{0\nu})/2 + i\delta_\nu(t - t_{0\nu}) - ix_{r,\nu}/2 + \gamma_\nu y), \quad N + 1 \leq \nu \leq 2N. \end{aligned} \quad (14)$$

These functions depend on r . We must denote them $\phi_\nu^r(y)$. For simplicity, in this section we denote them $\phi_\nu(y)$.

We use the following notations :

$\Theta_\nu = K_\nu(x - x_{0\nu})/2 + i\delta_\nu(t - t_{0\nu}) - ix_{r,\nu}/2 + \gamma_\nu y$, $1 \leq j \leq 2N$.
 $W_r(y) = W(\phi_1, \dots, \phi_{2N})$ is the wronskian $W_r(y) = \det[(\partial_y^{\mu-1} \phi_\nu)_{\nu, \mu \in [1, \dots, 2N]}]$.
We consider the matrix $D_r = (d_{\nu\mu})_{\nu, \mu \in [1, \dots, 2N]}$ defined by

$$d_{\nu\mu} = \frac{2(-1)^{\epsilon_\nu} \gamma_\nu}{\gamma_\nu + \gamma_\mu} \prod_{\eta \neq \nu} \left| \frac{\gamma_\nu + \gamma_\mu}{\gamma_\nu - \gamma_\mu} \right| \exp(i\kappa_\nu(x - x_{0\nu}) - 2\delta_\nu(t - t_{0\nu}) + x_{r,\nu}),$$

$$1 \leq \nu \leq 2N, \quad 1 \leq \mu \leq 2N,$$

with

$$x_{r,\nu} = (r - 1) \ln \frac{\gamma_\nu - i}{\gamma_\nu + i}.$$

Then we have the following statement

Theorem 3.1

$$\det(I + D_r) = k_r(0) \times W_r(\phi_1, \dots, \phi_{2N})(0), \quad (15)$$

where

$$k_r(y) = \frac{2^{2N} \exp(i \sum_{\nu=1}^{2N} \Theta_\nu)}{\prod_{\nu=2}^{2N} \prod_{\mu=1}^{\nu-1} (\gamma_\nu - \gamma_\mu)}.$$

Proof : We start to remove the factor $(2i)^{-1} e^{i\Theta_\nu}$ in each row ν in the wronskian $W_r(y)$ for $1 \leq \nu \leq 2N$.

Then

$$W_r = \prod_{\nu=1}^{2N} e^{i\Theta_\nu} (2i)^{-N} (2)^{-N} \times W_{r1}, \quad (16)$$

with

$$W_{r1} = \begin{vmatrix} (1 - e^{-2i\Theta_1}) & i\gamma_1(1 + e^{-2i\Theta_1}) & \dots & (i\gamma_1)^{2N-1}(1 + (-1)^{2N} e^{-2i\Theta_1}) \\ (1 - e^{-2i\Theta_2}) & i\gamma_2(1 + e^{-2i\Theta_2}) & \dots & (i\gamma_2)^{2N-1}(1 + (-1)^{2N} e^{-2i\Theta_2}) \\ \vdots & \vdots & \vdots & \vdots \\ (1 - e^{-2i\Theta_{2N}}) & i\gamma_{2N}(1 + e^{-2i\Theta_{2N}}) & \dots & (i\gamma_{2N})^{2N-1}(1 + (-1)^{2N} e^{-2i\Theta_{2N}}) \end{vmatrix}$$

The determinant W_{r1} can be written as

$$W_{r1} = \det(\alpha_{jk} e_j + \beta_{jk}),$$

where $\alpha_{jk} = (-1)^k (i\gamma_j)^{k-1}$, $e_j = e^{-2i\theta_j}$, and $\beta_{jk} = (i\gamma_j)^{k-1}$, $1 \leq j \leq N$, $1 \leq k \leq 2N$,

$\alpha_{jk} = (-1)^{k-1}(i\gamma_j)^{k-1}$, $e_j = e^{-2i\theta_j}$, and $\beta_{jk} = (i\gamma_j)^{k-1}$, $N+1 \leq j \leq 2N$, $1 \leq k \leq 2N$.

Denoting $U = (\alpha_{ij})_{i,j \in [1, \dots, 2N]}$, $V = (\beta_{ij})_{i,j \in [1, \dots, 2N]}$, the determinant of U is clearly equal to

$$\det(U) = (i)^{\frac{2N(2N-1)}{2}} \prod_{2N \geq l > m \geq 1} (\gamma_l - \gamma_m). \quad (17)$$

Then we use the following Lemma

Lemma 3.1 *Let $A = (a_{ij})_{i,j \in [1, \dots, N]}$, $B = (b_{ij})_{i,j \in [1, \dots, N]}$, $(H_{ij})_{i,j \in [1, \dots, N]}$, the matrix formed by replacing the j th row of A by the i th row of B
Then*

$$\det(a_{ij}x_i + b_{ij}) = \det(a_{ij}) \times \det\left(\delta_{ij}x_i + \frac{\det(H_{ij})}{\det(a_{ij})}\right) \quad (18)$$

Proof : For $\tilde{A} = (\tilde{a}_{ji})_{i,j \in [1, \dots, N]}$ the transposed matrix in cofactors of A , we have the well known formula $A \times {}^t \tilde{A} = \det A \times I$.

So it is clear that $\det(\tilde{A}) = (\det(A))^{N-1}$.

The general term of the product $(c_{ij})_{i,j \in [1, \dots, N]} = (a_{ij}x_i + b_{ij})_{i,j \in [1, \dots, N]} \times (\tilde{a}_{ji})_{i,j \in [1, \dots, N]}$ can be written as

$$\begin{aligned} c_{ij} &= \sum_{s=1}^N (a_{is}x_i + b_{is}) \times \tilde{a}_{js} \\ &= x_i \sum_{s=1}^N a_{is} \tilde{a}_{js} + \sum_{s=1}^N b_{is} \tilde{a}_{js} \\ &= \delta_{ij} \det(A) x_i + \det(H_{ij}). \end{aligned}$$

We get

$$\det(c_{ij}) = \det(a_{ij}x_i + b_{ij}) \times (\det(A))^{N-1} = (\det(A))^N \times \det\left(\delta_{ij}x_i + \frac{\det(H_{ij})}{\det(A)}\right).$$

Thus $\det(a_{ij}x_i + b_{ij}) = \det(A) \times \det\left(\delta_{ij}x_i + \frac{\det(H_{ij})}{\det(A)}\right)$.

□

Using the previous lemma (18), we get :

$$\det(\alpha_{ij}e_i + \beta_{ij}) = \det(\alpha_{ij}) \times \det\left(\delta_{ij}e_i + \frac{\det(H_{ij})}{\det(\alpha_{ij})}\right),$$

where $(H_{ij})_{i,j \in [1, \dots, N]}$ is the matrix formed by replacing the j th row of U by the i th row of V defined previously.

We compute $\det(H_{ij})$ and we get

$$\det(H_{ij}) = (-1)^{\frac{2N(2N+1)}{2}+1} (i)^{\frac{2N(2N-1)}{2}} \prod_{2N \geq l > m \geq 1, l \neq j, m \neq j} (\gamma_l - \gamma_m) \prod_{l < j} (\gamma_k - \gamma_l) \prod_{l > j} (\gamma_k - \gamma_l). \quad (19)$$

We can simplify the quotient $q = \frac{\det(H_{ij})}{\det(\alpha_{ij})}$:

$$\begin{aligned}
q &= \frac{(-1)^{\epsilon(j)} \prod_{l \neq j} (\gamma_l + \gamma_k)}{(-1)^{k-1} \prod_{l \neq k} (\gamma_l - \gamma_k)} \\
&= \frac{(-1)^{\epsilon(j)-k+1} 2\gamma_k \prod_{l \neq k} (\gamma_l + \gamma_k)}{(\gamma_j + \gamma_k) \prod_{l \neq k} (\gamma_l - \gamma_k)} \\
&= \frac{(-1)^{\epsilon(j)} 2\gamma_k}{(\gamma_j + \gamma_k)} \prod_{l \neq k} \left| \frac{\gamma_l + \gamma_k}{\gamma_l - \gamma_k} \right|. \tag{20}
\end{aligned}$$

So $\det(\delta_{jk}e_j + \frac{\det(H_{jk})}{\det(\alpha_{jk})})$ can be expressed as

$$\det(\delta_{jk}e_j + \frac{\det(H_{jk})}{\det(\alpha_{jk})}) = \prod_{j=1}^{2N} e^{-2i\Theta_j} \det(\delta_{jk} + \frac{(-1)^{\epsilon(j)} 2\gamma_k}{(\gamma_j + \gamma_k)} \prod_{l \neq k} \left| \frac{\gamma_l + \gamma_k}{\gamma_l - \gamma_k} \right| e^{2i\Theta_j}).$$

Then dividing each column k by γ_k , $1 \leq k \leq N$ and multiplying each row j by γ_j , $1 \leq j \leq N$, we get

$$\det(\delta_{jk}e_j + \frac{\det(H_{jk})}{\det(\alpha_{jk})}) = \prod_{j=1}^{2N} e^{-2i\Theta_j} \det(\delta_{jk} + \frac{(-1)^{\epsilon(j)} 2\gamma_j}{(\gamma_j + \gamma_k)} \prod_{l \neq k} \left| \frac{\gamma_l + \gamma_k}{\gamma_l - \gamma_k} \right| e^{2i\Theta_j}),$$

and therefore the wronskian

$$W_r(\phi_1, \dots, \phi_N)(0)$$

can be written as

$$\prod_{j=1}^{2N} e^{i\Theta_j|_{y=0}} (2)^{-2N} (i)^{\frac{2N(2N-2)}{2}} \prod_{j=2}^{2N} \prod_{i=1}^{j-1} (\gamma_j - \gamma_i) \prod_{j=1}^{2N} e^{-2i\Theta_j|_{y=0}} \det(I + D_r)$$

It follows that

$$\det(I + D_r) = \frac{e^{i \sum_{j=1}^{2N} \Theta_j|_{y=0}} 2^{2N}}{\prod_{j=2}^{2N} \prod_{i=1}^{j-1} (\gamma_j - \gamma_i)} W_r(\phi_1, \dots, \phi_{2N})(0) = k_r(0) W_r(\phi_1, \dots, \phi_{2N})(0). \tag{21}$$

□

So, the solution of NLS equation takes the form

$$v(x, t) = \frac{k_3(0)W_3(0)}{k_1(0)W_1(0)} \exp(2it - i\varphi) \tag{22}$$

and we get the following result

Theorem 3.2 *The function v defined by*

$$v(x, t) = \frac{W_3(\phi_1, \dots, \phi_{2N})(0)}{W_1(\phi_1, \dots, \phi_{2N})(0)} \exp(2it - i\varphi).$$

is solution of the NLS equation (1)

$$iv_t + v_{xx} + 2|v|^2v = 0,$$

where

$W_r(y) = W(\phi_1, \dots, \phi_{2N})$ *is the wronskian* $W_r(y) = \det[(\partial_y^{\mu-1}\phi_\nu)_{\nu, \mu \in [1, \dots, 2N]}]$, *and* ϕ_ν *the functions given by*

$$\phi_\nu(y) = \sin(K_\nu(x - x_{0\nu})/2 + i\delta_\nu(t - t_{0\nu}) - ix_{r,\nu}/2 + \gamma_\nu y), \quad 1 \leq \nu \leq N,$$

$$\phi_\nu(y) = \cos(K_\nu(x - x_{0\nu})/2 + i\delta_\nu(t - t_{0\nu}) - ix_{r,\nu}/2 + \gamma_\nu y), \quad N + 1 \leq \nu \leq 2N,$$

$\kappa_\nu, \delta_\nu, \gamma_\nu$ *parameters defined by (4) for* $\nu \in [1, \dots, 2N]$

$$\kappa_\nu = 2\sqrt{1 - \lambda_\nu^2}, \quad \delta_\nu = \kappa_\nu \lambda_\nu, \quad \gamma_\nu = \sqrt{\frac{1 - \lambda_\nu}{1 + \lambda_\nu}}, \quad \epsilon_\nu \in \{0; 1\},$$

where λ_ν *is an arbitrary parameter such that* $0 < \lambda_j < 1$ *and* $\lambda_{N+j} = -\lambda_j$ *for* $j \in [1, \dots, N]$.

3.2 Example

If we consider the simplest case where $N = 1$, we get the following results. The wronskian $W_3(0)$ is equal to

$$W_3(0) = \frac{1}{\gamma_1(1 + \lambda_1)} \times [\lambda_1 \cos(K_1(x - x_{01})) - \cosh(2\delta_1(t - t_{01}) - x_1)].$$

The Fredholm determinant $F_3(0) = \det(I + A_3)$ takes the form

$$F_3(0) = \frac{2e^{iK_1x}}{\lambda_1} \times [\lambda_1 \cos(K_1(x - x_{01})) - \cosh(2\delta_1(t - t_{01}) - x_1)].$$

So we have the equality

$$\det(I + A_3) = C \times W_3,$$

with

$$C = \frac{K_1 e^{iK_1 x}}{\lambda_1}.$$

If we compute

$$k_3(0) = \frac{2^{2N} \exp(i \sum_{\nu=1}^{2N} \Theta_\nu|_{y=0})}{\prod_{\nu=2}^{2N} \prod_{\mu=1}^{\nu-1} (\gamma_\nu - \gamma_\mu)}$$

we get :

$$\exp(i \sum_{\nu=1}^{2N} \Theta_\nu|_{y=0}) = e^{iK_1 x};$$

$$\prod_{\nu=2}^{2N} \prod_{\mu=1}^{\nu-1} (\gamma_\nu - \gamma_\mu) = \gamma_2 - \gamma_1 = \frac{1}{\gamma_1} - \gamma_1 = \sqrt{\frac{1+\lambda_1}{1-\lambda_1}} - \sqrt{\frac{1-\lambda_1}{1+\lambda_1}} = \frac{2\lambda_1}{\sqrt{1-\lambda_1^2}} = \frac{2\lambda_1}{K_1/2} = \frac{4\lambda_1}{K_1};$$

So,

$$k_3(0) = \frac{2^{2N} \exp(i \sum_{\nu=1}^{2N} \Theta_\nu)}{\prod_{\nu=2}^{2N} \prod_{\mu=1}^{\nu-1} (\gamma_\nu - \gamma_\mu)} = \frac{K_1 e^{iK_1 x}}{\lambda_1}$$

and we have the equality

$$\det(I + A_3) = k_3(0) \times W_3.$$

4 Construction of quasi-rational solutions of NLS equation

4.1 Taking the limit when the parameters $\lambda_j \rightarrow 1$ for $1 \leq j \leq N$ and $\lambda_j \rightarrow -1$ for $N+1 \leq j \leq 2N$

In the following, we show how we can obtain quasi-rational solutions of NLS equation by a rough estimation.

We consider the parameter λ_j written in the form $1 - \epsilon^2 c_j^2$. For simplicity, we denote d_j the term $\frac{c_j}{\sqrt{2}}$.

When ϵ goes to 0, we realize limited expansions at order $2N - 1$ of κ_j , γ_j , δ_j , $x_{r,j}$, κ_{N+j} , γ_{N+j} , δ_{N+j} , $x_{r,N+j}$, for $1 \leq j \leq N$. For example, in the case where $N = 1$, it gives :

$$\begin{aligned}
\kappa_j &= 4d_j\epsilon + O(\epsilon^2), & \gamma_j &= d_j\epsilon + O(\epsilon^2), & \delta_j &= 4d_j\epsilon + O(\epsilon^2), \\
x_{r,j} &= (r-1)(2id_j\epsilon + O(\epsilon^2)), \\
\kappa_{N+j} &= 4d_j\epsilon + O(\epsilon^2), & \gamma_{N+j} &= 1/(d_j\epsilon) - (d_j\epsilon)/2 + O(\epsilon^2), & \delta_{N+j} &= -4d_j\epsilon + O(\epsilon^2), \\
x_{r,N+j} &= -(r-1)(2id_j\epsilon + O(\epsilon^2)), \\
1 &\leq j \leq N.
\end{aligned}$$

Then we realize limited expansions at order $2N - 1$ of the functions $\phi_j^r(0)$ and $\phi_{N+j}^r(0)$, for $1 \leq j \leq N$:

$$\begin{aligned}
\phi_j^1(0) &= \sin E_j, \\
\phi_j^3(0) &= \sin D_j, \\
\phi_{N+j}^1(0) &= \sin E'_j, \\
\phi_{N+j}^3(0) &= \sin D'_j,
\end{aligned}$$

with

$$\begin{aligned}
D_j &= K_j(x - x_{0j})/2 + i\delta_j(t - t_{0j}) - ix_j/2, \\
E_j &= K_j(x - x_{0j})/2 + i\delta_j(t - t_{0j}), \\
D'_j &= K_j(x - x_{0j})/2 - i\delta_j(t - t_{0j}) + ix_j/2, \\
E'_j &= K_j(x - x_{0j})/2 - i\delta_j(t - t_{0j}).
\end{aligned}$$

For example, in the case where $N = 1$, it gives :

$$\begin{aligned}
D_j &= 2\epsilon d_j(x - x_{0j} + 2i(t - t_{0j}) + 1) + O(\epsilon^2), \\
E_j &= 2\epsilon d_j(x - x_{0j} + 2i(t - t_{0j})) + O(\epsilon^2), \\
D'_j &= 2\epsilon d_j(x - x_{0j} - 2i(t - t_{0j}) - 1) + O(\epsilon^2), \\
E'_j &= 2\epsilon d_j((x - x_{0j} - 2i(t - t_{0j}))) + O(\epsilon^2),
\end{aligned}$$

and

$$\begin{aligned}
\phi_j^1(0) &= 2\epsilon d_j(x - x_{0j} + 2i(t - t_{0j})) + O(\epsilon^2), \\
\phi_j^3(0) &= 2\epsilon d_j(x - x_{0j} + 2i(t - t_{0j}) + 1) + O(\epsilon^2), \\
\phi_{N+j}^1(0) &= 1 + O(\epsilon^2), \\
\phi_{N+j}^3(0) &= 1 + O(\epsilon^2).
\end{aligned}$$

Then according to the relation

$$v(x, t) = \frac{W_3(\phi_1, \dots, \phi_N)(0)}{W_1(\phi_1, \dots, \phi_N)(0)} \exp(2it - i\varphi).$$

the solution of NLS equation takes the form :

$$v(x, t) = \exp(2it - i\varphi) \frac{N(x, t, \epsilon)}{D(x, t, \epsilon)}$$

$$= \exp(2it - i\varphi) \frac{\begin{array}{c|cccccc} \sin D_1 & \dots & \sin D_N & \cos D'_1 & \dots & \cos D'_N \\ \gamma_1 \cos D_1 & \dots & \gamma_N \cos D_N & -\frac{1}{\gamma_1} \sin D'_1 & \dots & -\frac{1}{\gamma_N} \sin D'_N \\ \gamma_1^2 \sin D_1 & \dots & \gamma_N^2 \sin D_N & \frac{1}{\gamma_1} \cos D'_1 & \dots & \frac{1}{\gamma_N} \cos D'_N \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \gamma_1^{2N-1} \cos D_1 & \dots & \gamma_N^{2N-1} \cos D_N & -\frac{1}{\gamma_1^{2N-1}} \sin D'_1 & \dots & -\frac{1}{\gamma_N^{2N-1}} \sin D'_N \end{array}}{\begin{array}{c|cccccc} \sin E_1 & \dots & \sin E_N & \cos E'_1 & \dots & \cos E'_N \\ \gamma_1 \cos E_1 & \dots & \gamma_N \cos E_N & -\frac{1}{\gamma_1} \sin E'_1 & \dots & -\frac{1}{\gamma_N} \sin E'_N \\ \gamma_1^2 \sin E_1 & \dots & \gamma_N^2 \sin E_N & \frac{1}{\gamma_1} \cos E'_1 & \dots & \frac{1}{\gamma_N} \cos E'_N \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \gamma_1^{2N-1} \cos E_1 & \dots & \gamma_N^{2N-1} \cos E_N & -\frac{1}{\gamma_1^{2N-1}} \sin E'_1 & \dots & -\frac{1}{\gamma_N^{2N-1}} \sin E'_N \end{array}}$$

We use the classical developments

$$\sin z = \sum_{j=1}^N \frac{(-1)^{j-1} z^{2j-1}}{(2j-1)!} + O(z^{2N}) \text{ and } \cos z = \sum_{j=0}^{2N} \frac{(-1)^j z^{2j}}{(2j)!} + O(z^{2N+1}),$$

for z in a neighborhood of 0.

Using the fact that $D_j = E_j + \alpha_j(\epsilon)$, $D'_j = E'_j - \alpha_j(\epsilon)$, and addition formulas for trigonometric functions, we can expand the numerator $N(x, t, \epsilon)$ along the columns; it takes the form of

$$\prod_{j=1}^N \cos^2(\alpha_j(\epsilon)) D(x, t, \epsilon) + A.$$

If we divide each term of this sum by $D(x, t)$, when ϵ tends to 0, the first term tends to 1; in each other term of the sum of A, we can multiply each fraction by some power of γ_j to get a finite limit when ϵ goes to 0.

So we get really a non trivial and non degenerate quotient of two polynomials in x and t by an exponential.

Practically, we keep only the terms of lower degree in ϵ , in numerator and denominator which expressed as powers of x and t

So we get in this way a quasi-rational solution of the NLS equation.

So we get the following result

Theorem 4.1 *The function v defined by*

$$v(x, t) = \lim_{\epsilon \rightarrow 0} \frac{W_3(\phi_1, \dots, \phi_{2N})(0)}{W_1(\phi_1, \dots, \phi_{2N})(0)} \exp(2it - i\varphi).$$

is a quasi-rational solution of the NLS equation (1)

$$iv_t + v_{xx} + 2|v|^2v = 0,$$

where

$W_r(y) = W(\phi_1, \dots, \phi_{2N})$ is the wronskian $W_r(y) = \det[(\partial_y^{\mu-1} \phi_\nu)_{\nu, \mu \in [1, \dots, 2N]}]$, and ϕ_ν the functions given by

$$\phi_\nu(y) = \sin(K_\nu(x - x_{0\nu})/2 + i\delta_\nu(t - t_{0\nu}) - ix_{r,\nu}/2 + \gamma_\nu y), \quad 1 \leq \nu \leq N,$$

$$\phi_\nu(y) = \cos(K_\nu(x - x_{0\nu})/2 + i\delta_\nu(t - t_{0\nu}) - ix_{r,\nu}/2 + \gamma_\nu y), \quad N + 1 \leq \nu \leq 2N,$$

$\kappa_\nu, \delta_\nu, \gamma_\nu$ parameters defined by (4) for $\nu \in [1, \dots, 2N]$

$$\kappa_\nu = 2\sqrt{1 - \lambda_\nu^2}, \quad \delta_\nu = \kappa_\nu \lambda_\nu, \quad \gamma_\nu = \sqrt{\frac{1 - \lambda_\nu}{1 + \lambda_\nu}}, \quad \epsilon_\nu \in \{0; 1\},$$

where $\lambda_j = 1 - \epsilon^2 c_j^2$ is an arbitrary parameter such that $0 < \lambda_j < 1$ and $\lambda_{N+j} = -\lambda_j, c_j$ an arbitrary real number, for $j \in [1, \dots, N]$,

If we want to recover exact solution of Akhmediev type, we must more precise in the estimations.

4.2 Examples

For brevity, in the following we choose $x_{0i} = 0, t_{0i} = 0$ and $\varphi = 0$ for $1 \leq i \leq 2N$.

4.2.1 Case $N=1$

If we consider the case $N = 1$, from (22), we realize a development at order 1 of W_3 and W_1 in ϵ . The solution of NLS equation can be written as

$$v(x, t) = -\frac{(-4x^2 - 16t^2 + 16it + 3)}{4x^2 + 16t^2 + 1} \exp(2it).$$

Then v takes the form

$$v(x, t) = 1 - 4 \frac{1 + 4it}{1 + 16t^2 + 4x^2} \exp(2it).$$

If we make the following changes of variables $t \rightarrow X/2$ and $x \rightarrow T$, we get exactly first order Akhmediev's solution (see [1]), solution of $iu_x + \frac{1}{2}u_{tt} + u|u|^2 = 0$,

$$v(x, t) = \left(1 - \frac{4(1 + 2ix)}{1 + 4x^2 + 4t^2}\right) \exp(ix).$$

We represent here the modulus of v in function of $x \in [-5; 5]$ and $t \in [-5; 5]$; we get the following graphic :

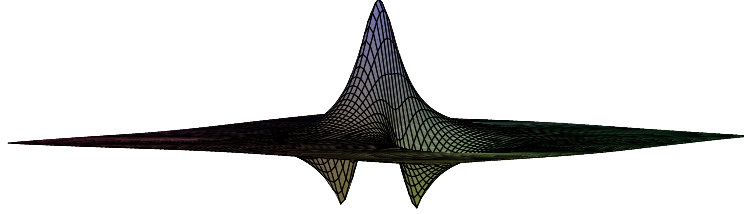


Figure 1: Solution of NLS, M=2.

4.2.2 Case N=2

In the case $N = 2$, we must realize a development at order 3 in ϵ . From (22), the solution of NLS equation can be written as

$$v(x, t) = \frac{N(x, t)}{D(x, t)} \exp(2it),$$

with

$$N(x, t) = 5760x^2t^2 - 3072x^2t^4 + 1872t^2 - 64x^6 - 45 - 4096t^6 + 180x^2 + 144x^4 - 768x^4t^2 \\ + 768ix^4t + 8448t^4 - 1152ix^2t + 1536it^3 - 720it + 12288it^5 + 6144it^3x^2,$$

and

$$D(x, t) = -1152x^2t^2 + 3072x^2t^4 + 1584t^2 + 64x^6 + 9 + 4096t^6 + 108x^2 + 48x^4 + 768x^4t^2 + 6912t^4.$$

If we make the following changes of variables $t \rightarrow X/2$ and $x \rightarrow T$, we get

$$v(x, t) = \frac{N_1(x, t)}{D_1(x, t)} \exp(ix),$$

with

$$N_1(x, t) = 1440x^2t^2 - 192x^4t^2 + 468x^2 - 64t^6 - 45 - 64x^6 + 180t^2 + 144t^4 - 192x^2t^4 - 576it^2x \\ + 528x^4 + 384ix^5 + 192ix^3 - 360ix + 384it^4x + 768ix^3t^2,$$

and

$$D_1(x, t) = -288x^2t^2 + 192x^4t^2 + 396x^2 + 64t^6 + 9 + 64x^6 + 108t^2 + 48t^4 + 192x^2t^4 + 432x^4.$$

It can be reduced exactly at the second order Akhmediev's solution (solution of $iu_x + \frac{1}{2}u_{tt} + u|u|^2 = 0$) (see [1]).

$$v(x, t) = 1 + \frac{3/8 - 3t^2 - 2t^4 - 9x^2 - 10x^4 - 12t^2x^2 + ix(15/4 + 6t^2 - 4t^4 - 2x^2 - 4x^4 - 8t^2x^2)}{3/32 + 9/8t^2 + 1/2t^4 + 2/3t^6 + 33/8x^2 + 9/2x^4 + 2/3x^6 - 3t^2x^2 + 2t^4x^2 + 2t^2x^4} \exp(ix).$$

We represent the modulus of v in function of $x \in [-5; 5]$ and $t \in [-5; 5]$; we get the following graphic :

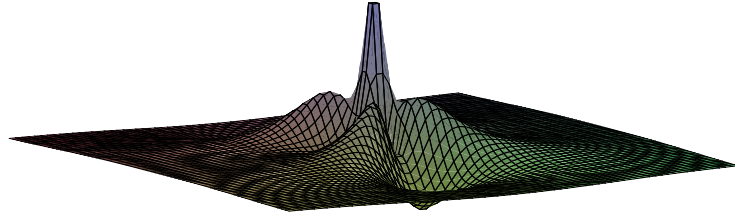


Figure 2: Solution of NLS, M=4.

4.2.3 Case N=3

In the case $N = 3$, we must realize a development at order 5 in ϵ . If we make the following changes of variables $t \rightarrow X/2$ and $x \rightarrow T$, from (22), the solution of NLS equation in the formulation of Akhmediev $iu_x + \frac{1}{2}u_{tt} + u|u|^2 = 0$ can be written as

$$v(x, t) = \frac{N(x, t)}{D(x, t)} \exp(ix),$$

with

$$\begin{aligned}
N(x, t) = & 14175 + 49152i t^{10} x - 737280i t^2 x^7 + 5253120i t^2 x^5 + 907200i t^2 x - 1720320i t^6 x^3 - 184320i t^8 x \\
& + 5356800x^4 t^2 - 576000x^4 t^4 - 875520x^6 t^2 - 172800t^4 x^2 - 4600800x^2 t^2 + 829440x^8 t^2 - 24576x^{10} t^2 - 61440x^8 t^4 \\
& + 460800t^8 x^2 - 506880x^2 t^6 + 1658880x^6 t^4 - 4838400i x^3 t^2 - 81920x^6 t^6 + 1413120x^4 t^6 - 61440x^4 t^8 - 24576x^2 t^{10} \\
& - 5238000x^4 - 567000x^2 - 226800t^4 + 2384640x^6 + 172800t^6 + 2085120x^8 + 141312x^{10} + 57600t^8 \\
& - 113400t^2 - 4096x^{12} - 4096t^{12} + 18432t^{10} - 1382400i t^4 x^3 - 1036800i t^4 x - 460800i t^6 x - 2580480i t^4 x^5 \\
& + 491520i t^6 x^5 + 491520i t^4 x^7 + 245760i t^8 x^3 + 245760i t^2 x^9 + 307200i x^9 - 475200i x^3 - 2672640i x^7 + 49152i x^{11} \\
& - 7395840i x^5 + 226800i x
\end{aligned}$$

and

$$\begin{aligned}
D(x, t) = & 2025 + 5011200x^4 t^2 - 345600x^4 t^4 + 3456000x^6 t^2 + 864000t^4 x^2 - 583200x^2 t^2 + 276480x^8 t^2 + 24576x^{10} t^2 \\
& + 61440x^8 t^4 - 92160t^8 x^2 + 138240x^2 t^6 + 61440x^6 t^4 + 81920x^6 t^6 - 184320x^4 t^6 + 61440x^4 t^8 + 24576x^2 t^{10} \\
& + 2300400x^4 + 372600x^2 + 54000t^4 + 979200x^6 + 149760t^6 + 956160x^8 + 129024x^{10} + 34560t^8 \\
& + 48600t^2 + 4096x^{12} + 4096t^{12} + 6144t^{10}.
\end{aligned}$$

We get the following graphic for the modulus of v in function of $x \in [-5; 5]$ and $t \in [-5; 5]$:

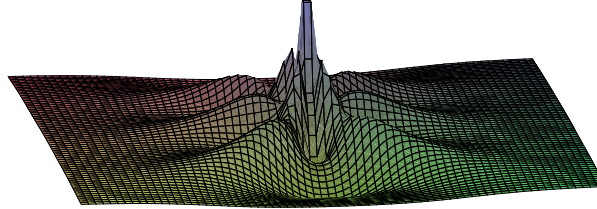


Figure 3: Solution of NLS, M=6.

This method can be extended to get quasi-rational solutions of NLS equation at any order.

5 Conclusion

Here is given a new formulation of solutions of NLS equation that gives as particular case, Akhmediev's solutions. These solutions expressed as a quotient of wronskians are similar to a previous result of Eleonski, Krichever and Kulagin [13]. Their formulation can also be seen as a quotient of two wronkians but is more complicated because it needs recursion operations. However, it can be used to get solutions of NLS equation (see [12]).

This new formulation can give explicitly a set of non singular solutions of NLS equation which can be obtained by a simpler way than with the Krichever method.

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