



**HAL**  
open science

# Quasi-rational solutions of the NLS equation and rogue waves

Pierre Gaillard

► **To cite this version:**

Pierre Gaillard. Quasi-rational solutions of the NLS equation and rogue waves. 2010. halshs-00536287v4

**HAL Id: halshs-00536287**

**<https://shs.hal.science/halshs-00536287v4>**

Preprint submitted on 10 Feb 2011 (v4), last revised 16 Sep 2012 (v6)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Quasi-rational solutions of the NLS equation and rogue waves

<sup>+</sup>Pierre Gaillard, <sup>+</sup> Université de Bourgogne, Dijon, France :  
e-mail: Pierre.Gaillard@u-bourgogne.fr,

November 4, 2010

## Abstract

We degenerate solutions of the NLS equation from the general formulation in terms of Fredholm determinant given in [10], to get quasi-rational solutions of NLS equations. For this we establish a link between Fredholm determinants and wronskians. We give solutions of the focusing NLS equation as a quotient of two wronskian determinants of a system of sine functions. By appropriate passage to the limit when the periods tend to infinity, we recover exact expression for higher Peregrine-Akhmediev breathers obtained in [13] by Peregrine, and by Eleonskii, Krichever, Kulagin and Akhmediev [8, 2, 1]. It gives a new approach to get an analytical description of the rogue waves.

## 1 Introduction

The nonlinear Schrödinger equation (NLS) equation plays a fundamental role in hydrodynamics, in particular for waves in deep water in the context of the rogue waves or in non-linear optics.

In this paper, we will study solutions of NLS equation from the theta formulation given in [10] and give a new representation in terms of wronskians. The solutions take the form of a quotient of two wronskians of even order  $2N$  of elementary functions depending on a certain number of parameters. We will call these related solutions, solutions of NLS of order  $N$ .

To get from this form, quasi-rational solutions of NLS equation, we take the limit when some parameter goes to 0.

For  $N = 1$ , we recover the well known Peregrine's solution of the focusing NLS equation first found [13] in 1983; it was rediscovered later in several publications, in particular [3].

The second order Peregrine-like solution was first constructed in [3]. Other Peregrine-like solutions were found for reduced self-induced transparency (SIT) integrable system [11]. In [10], the n-phase quasi-periodic modulations of the plane waves solutions were constructed via appropriate degeneration of the finite gap periodic solutions to the NLS equation. These results have provided a way to construct multi-parametric families of multi-Peregrine solutions.

Same families of higher order were constructed in a series of articles by Akhmediev et al. [1, 2].

Rational solutions of NLS equation are given in [12] in a determinantal representation as degenerate solutions of Davey-Stewartson equations. Another approach using Crum transformation is given in [9] to get sequences of rational solutions of NLS equation. Very recently, it is shown in [6] that rational solutions can be written as a quotient of two wronskians using modified version of [8], with some different reasoning; moreover, the link between quasi-rational solutions of the focusing NLS equation and the rational solution of the KP-I equation is established.

With this formulation we recover as particular case, Akhmediev's quasi-rational solutions of NLS equation.

## 2 Expression of solutions of NLS equation in terms of Fredholm determinant

### 2.1 Solutions of NLS equation in terms of $\theta$ functions

The solution of the NLS equation

$$iv_t + v_{xx} + 2|v|^2v = 0, \quad (1)$$

is given in terms of truncated theta function by (see [10])

$$v(x, t) = \frac{\theta_3(x, t)}{\theta_1(x, t)} \exp(2it - i\varphi), \quad (2)$$

where

$$\begin{aligned} \theta_r(x, t) = & \sum_{k \in \{0;1\}^{2N}} \exp\left\{ \sum_{\mu > \nu, \mu, \nu=1}^{2N} \ln \left( \frac{\gamma_\nu - \gamma_\mu}{\gamma_\nu + \gamma_\mu} \right)^2 k_\mu k_\nu \right. \\ & \left. + \left( \sum_{\nu=1}^{2N} i\kappa_\nu(x - x_{0\nu}) - 2\delta_\nu(t - t_{0\nu}) + (r-1) \ln \frac{\gamma_\nu - i}{\gamma_\nu + i} + \sum_{\mu=1, \mu \neq \nu}^{2N} \ln \left| \frac{\gamma_\nu + \gamma_\mu}{\gamma_\nu - \gamma_\mu} \right| + \pi i \epsilon_\nu \right) k_\nu \right\}, \end{aligned} \quad (3) \quad r = 1, 3.$$

In this formula, the quantities  $\kappa_\nu$ ,  $\delta_\nu$ ,  $\gamma_\nu$  are functions of the parameters  $\lambda_\nu$ ,  $\nu = 1, \dots, 2N$ ,  $N$  being a positive integer,  $N \geq 1$  and they are given by the following equations,

$$\kappa_\nu = 2\sqrt{1 - \lambda_\nu^2}, \quad \delta_\nu = \kappa_\nu \lambda_\nu, \quad \gamma_\nu = \sqrt{\frac{1 - \lambda_\nu}{1 + \lambda_\nu}}, \quad \epsilon_\nu \in \{0;1\}. \quad (4)$$

$\varphi$ ,  $x_{0\nu}$ ,  $t_{0\nu}$ ,  $\nu = 1 \dots 2N$  are arbitrary real numbers.

We also note that

$$\lambda_{N+j} = -\lambda_j, \quad \kappa_{N+j} = \kappa_j, \quad \delta_{N+j} = -\delta_j, \quad \gamma_{N+j} = 1/\gamma_j, \quad j = 1 \dots N. \quad (5)$$

## 2.2 Relation between $\theta$ and Fredholm determinant

The function  $\theta_r$  defined in (3) can be rewritten with a summation in terms of subsets of  $[1, \dots, 2N]$

$$\begin{aligned} \theta_r(x, t) = & \sum_{J \subset \{1, \dots, 2N\}} \prod_{\nu \in J} (-1)^{\epsilon_\nu} \prod_{\nu \in J, \mu \notin J} \left| \frac{\gamma_\nu + \gamma_\mu}{\gamma_\nu - \gamma_\mu} \right| \\ & \times \exp\left\{ \sum_{\nu \in J} i\kappa_\nu(x - x_{0\nu}) - 2\delta_\nu(t - t_{0\nu}) + x_{r,\nu} \right\}, \end{aligned}$$

with

$$x_{r,\nu} = (r-1) \ln \frac{\gamma_\nu - i}{\gamma_\nu + i}, \quad 1 \leq j \leq 2N, \quad (6)$$

in particular

$$\begin{aligned} x_{r,j} &= (r-1) \ln \frac{\gamma_j - i}{\gamma_j + i}, \quad 1 \leq j \leq N, \\ x_{r,N+j} &= -(r-1) \ln \frac{\gamma_j - i}{\gamma_j + i} - (r-1)i\pi, \quad 1 \leq j \leq N. \end{aligned} \quad (7)$$

In formula (3),  $\epsilon_j$  can be chosen arbitrary, for example it can be fixed in the following way :

$\epsilon_j = j$  for  $1 \leq j \leq N$ , and  $\epsilon_j = j + 1$  for  $N + 1 \leq j \leq 2N$ .

The link between theta representation of solutions of NLS equation and Fredholm determinant formulation was already presented in [10]. In more details, it is given in a recent paper of Kirillov and Van Diejen [14]. In particular, they compute  $\det(I + A)$ , where  $I$  is the unit matrix and  $A = (a_{jk})_{1 \leq j, k \leq 2N}$  the matrix defined as :

$$a_{jk} = \frac{2\epsilon_j \kappa_j}{\kappa_j + \kappa_k} \prod_{l \neq j} \left| \frac{\kappa_j + \kappa_l}{\kappa_j - \kappa_l} \right| \exp(-2\kappa_j x), \quad (8)$$

where  $\epsilon_j \in \{-1; +1\}$ .

Then it is shown that  $\det(I + A)$  has the following form (see [14], p 84-87)

$$\det(I + A) = \sum_{J \subset \{1, \dots, N\}} \prod_{j \in J} \epsilon_j \prod_{j \in J, k \notin J} \left| \frac{\kappa_j + \kappa_k}{\kappa_j - \kappa_k} \right| \exp(-2x \sum_{j \in J} \kappa_j). \quad (9)$$

Using the same strategy, we can compute  $\det(I + A_r)$  where  $A_r = (a_{\nu\mu})_{1 \leq \nu, \mu \leq 2N}$  is the matrix defined as :

$$a_{\nu\mu} = \frac{2(-1)^{\epsilon_\nu} \gamma_\nu}{\gamma_\nu + \gamma_\mu} \prod_{\eta \neq \nu} \left| \frac{\gamma_\nu + \gamma_\eta}{\gamma_\nu - \gamma_\eta} \right| \exp(i\kappa_\nu(x - x_{0\nu}) - 2\delta_\nu(t - t_{0\nu}) + x_{r,\nu}). \quad (10)$$

Then  $\det(I + A_r)$  has the following form

$$\det(I + A_r) = \sum_{J \subset \{1, \dots, 2N\}} \prod_{\nu \in J} (-1)^{\epsilon_\nu} \prod_{\nu \in J, \mu \notin J} \left| \frac{\gamma_\nu + \gamma_\mu}{\gamma_\nu - \gamma_\mu} \right| \exp(i\kappa_\nu(x - x_{0\nu}) - 2\delta_\nu(t - t_{0\nu}) + x_{r,\nu}). \quad (11)$$

From the beginning of this section,  $\tilde{\theta}$  has the same expression as in (11) so, we have clearly the equality

$$\theta_r = \det(I + A_r). \quad (12)$$

Then the solution of NLS equation takes the form

$$v(x, t) = \frac{\det(I + A_3(x, t))}{\det(I + A_1(x, t))} \exp(2it - i\varphi). \quad (13)$$

It must be noted that the formulae (13) already appeared in [10].

### 3 Expression of solutions of NLS equation in terms of wronkian determinant

#### 3.1 Link between Fredholm determinants and wronskians

We consider the following functions

$$\begin{aligned}\phi_\nu(y) &= \sin(\kappa_\nu(x - x_{0\nu})/2 + i\delta_\nu(t - t_{0\nu}) - ix_{r,\nu}/2 + \gamma_\nu y), & 1 \leq \nu \leq N, \\ \phi_\nu(y) &= \cos(\kappa_\nu(x - x_{0\nu})/2 + i\delta_\nu(t - t_{0\nu}) - ix_{r,\nu}/2 + \gamma_\nu y), & N + 1 \leq \nu \leq 2N.\end{aligned}\quad (14)$$

These functions depend on  $r$ . We must denote them  $\phi_\nu^r(y)$ . For simplicity, in this section we denote them  $\phi_\nu(y)$ .

We use the following notations :

$$\Theta_\nu = \kappa_\nu(x - x_{0\nu})/2 + i\delta_\nu(t - t_{0\nu}) - ix_{r,\nu}/2 + \gamma_\nu y, \quad 1 \leq \nu \leq 2N.$$

$W_r(y) = W(\phi_1, \dots, \phi_{2N})$  is the wronskian  $W_r(y) = \det[(\partial_y^{\mu-1} \phi_\nu)_{\nu, \mu \in [1, \dots, 2N]}]$ .

We consider the matrix  $D_r = (d_{\nu\mu})_{\nu, \mu \in [1, \dots, 2N]}$  defined by

$$\begin{aligned}d_{\nu\mu} &= \frac{2(-1)^{\epsilon_\nu} \gamma_\nu}{\gamma_\nu + \gamma_\mu} \prod_{\eta \neq \nu} \left| \frac{\gamma_\nu + \gamma_\mu}{\gamma_\nu - \gamma_\mu} \right| \exp(i\kappa_\nu(x - x_{0\nu}) - 2\delta_\nu(t - t_{0\nu}) + x_{r,\nu}), \\ &1 \leq \nu \leq 2N, \quad 1 \leq \mu \leq 2N,\end{aligned}$$

with

$$x_{r,\nu} = (r - 1) \ln \frac{\gamma_\nu - i}{\gamma_\nu + i}.$$

Then we have the following statement

#### Theorem 3.1

$$\det(I + D_r) = k_r(0) \times W_r(\phi_1, \dots, \phi_{2N})(0), \quad (15)$$

where

$$k_r(y) = \frac{2^{2N} \exp(i \sum_{\nu=1}^{2N} \Theta_\nu)}{\prod_{\nu=2}^{2N} \prod_{\mu=1}^{\nu-1} (\gamma_\nu - \gamma_\mu)}.$$

**Proof :** We start to remove the factor  $(2i)^{-1} e^{i\Theta_\nu}$  in each row  $\nu$  in the wronskian  $W_r(y)$  for  $1 \leq \nu \leq 2N$ .

Then

$$W_r = \prod_{\nu=1}^{2N} e^{i\Theta_\nu} (2i)^{-N} (2)^{-N} \times W_{r1}, \quad (16)$$

with

$$W_{r1} = \begin{vmatrix} (1 - e^{-2i\Theta_1}) & i\gamma_1(1 + e^{-2i\Theta_1}) & \dots & (i\gamma_1)^{2N-1}(1 + (-1)^{2N}e^{-2i\Theta_1}) \\ (1 - e^{-2i\Theta_2}) & i\gamma_2(1 + e^{-2i\Theta_2}) & \dots & (i\gamma_2)^{2N-1}(1 + (-1)^{2N}e^{-2i\Theta_2}) \\ \vdots & \vdots & \vdots & \vdots \\ (1 - e^{-2i\Theta_{2N}}) & i\gamma_{2N}(1 + e^{-2i\Theta_{2N}}) & \dots & (i\gamma_{2N})^{2N-1}(1 + (-1)^{2N}e^{-2i\Theta_{2N}}) \end{vmatrix}$$

The determinant  $W_{r1}$  can be written as

$$W_{r1} = \det(\alpha_{jk}e_j + \beta_{jk}),$$

where  $\alpha_{jk} = (-1)^k(i\gamma_j)^{k-1}$ ,  $e_j = e^{-2i\theta_j}$ , and  $\beta_{jk} = (i\gamma_j)^{k-1}$ ,  $1 \leq j \leq N$ ,  $1 \leq k \leq 2N$ ,

$\alpha_{jk} = (-1)^{k-1}(i\gamma_j)^{k-1}$ ,  $e_j = e^{-2i\theta_j}$ , and  $\beta_{jk} = (i\gamma_j)^{k-1}$ ,  $N+1 \leq j \leq 2N$ ,  $1 \leq k \leq 2N$ .

Denoting  $U = (\alpha_{ij})_{i,j \in [1, \dots, 2N]}$ ,  $V = (\beta_{ij})_{i,j \in [1, \dots, 2N]}$ , the determinant of  $U$  is clearly equal to

$$\det(U) = (i)^{\frac{2N(2N-1)}{2}} \prod_{2N \geq l > m \geq 1} (\gamma_l - \gamma_m). \quad (17)$$

Then we use the following Lemma

**Lemma 3.1** *Let  $A = (a_{ij})_{i,j \in [1, \dots, N]}$ ,  $B = (b_{ij})_{i,j \in [1, \dots, N]}$ ,  $(H_{ij})_{i,j \in [1, \dots, N]}$ , the matrix formed by replacing the  $j$ th row of  $A$  by the  $i$ th row of  $B$*   
Then

$$\det(a_{ij}x_i + b_{ij}) = \det(a_{ij}) \times \det(\delta_{ij}x_i + \frac{\det(H_{ij})}{\det(a_{ij})}) \quad (18)$$

**Proof :** For  $\tilde{A} = (\tilde{a}_{ji})_{i,j \in [1, \dots, N]}$  the transposed matrix in cofactors of  $A$ , we have the well known formula  $A \times^t \tilde{A} = \det A \times I$ .

So it is clear that  $\det(\tilde{A}) = (\det(A))^{N-1}$ .

The general term of the product  $(c_{ij})_{i,j \in [1, \dots, N]} = (a_{ij}x_i + b_{ij})_{i,j \in [1, \dots, N]} \times (\tilde{a}_{ji})_{i,j \in [1, \dots, N]}$  can be written as

$$\begin{aligned} c_{ij} &= \sum_{s=1}^N (a_{is}x_i + b_{is}) \times \tilde{a}_{js} \\ &= x_i \sum_{s=1}^N a_{is}\tilde{a}_{js} + \sum_{s=1}^N b_{is}\tilde{a}_{js} \\ &= \delta_{ij} \det(A)x_i + \det(H_{ij}). \end{aligned}$$

We get

$$\det(c_{ij}) = \det(a_{ij}x_i + b_{ij}) \times (\det(A))^{N-1} = (\det(A))^N \times \det(\delta_{ij}x_i + \frac{\det(H_{ij})}{\det(A)}).$$

$$\text{Thus } \det(a_{ij}x_i + b_{ij}) = \det(A) \times \det(\delta_{ij}x_i + \frac{\det(H_{ij})}{\det(A)}).$$

□

Using the previous lemma (18), we get :

$$\det(\alpha_{ij}e_i + \beta_{ij}) = \det(\alpha_{ij}) \times \det(\delta_{ij}e_i + \frac{\det(H_{ij})}{\det(\alpha_{ij})}),$$

where  $(H_{ij})_{i,j \in [1, \dots, N]}$  is the matrix formed by replacing the  $j$ th row of  $U$  by the  $i$ th row of  $V$  defined previously.

We compute  $\det(H_{ij})$  and we get

$$\det(H_{ij}) = (-1)^{\frac{2N(2N+1)}{2}+1} (i)^{\frac{2N(2N-1)}{2}} \prod_{2N \geq l > m \geq 1, l \neq j, m \neq j} (\gamma_l - \gamma_m) \prod_{l < j} (\gamma_k - \gamma_l) \prod_{l > j} (\gamma_k - \gamma_l). \quad (19)$$

We can simplify the quotient  $q = \frac{\det(H_{ij})}{\det(\alpha_{ij})}$  :

$$\begin{aligned} q &= \frac{(-1)^{\epsilon(j)} \prod_{l \neq j} (\gamma_l + \gamma_k)}{(-1)^{k-1} \prod_{l \neq k} (\gamma_l - \gamma_k)} \\ &= \frac{(-1)^{\epsilon(j)-k+1} 2\gamma_k \prod_{l \neq k} (\gamma_l + \gamma_k)}{(\gamma_j + \gamma_k) \prod_{l \neq k} (\gamma_l - \gamma_k)} \\ &= \frac{(-1)^{\epsilon(j)} 2\gamma_k}{(\gamma_j + \gamma_k)} \prod_{l \neq k} \left| \frac{\gamma_l + \gamma_k}{\gamma_l - \gamma_k} \right|. \end{aligned} \quad (20)$$

So  $\det(\delta_{jk}e_j + \frac{\det(H_{jk})}{\det(\alpha_{jk})})$  can be expressed as

$$\det(\delta_{jk}e_j + \frac{\det(H_{jk})}{\det(\alpha_{jk})}) = \prod_{j=1}^{2N} e^{-2i\Theta_j} \det(\delta_{jk} + \frac{(-1)^{\epsilon(j)} 2\gamma_k}{(\gamma_j + \gamma_k)} \prod_{l \neq k} \left| \frac{\gamma_l + \gamma_k}{\gamma_l - \gamma_k} \right| e^{2i\Theta_j}).$$

Then dividing each column  $k$  by  $\gamma_k$ ,  $1 \leq k \leq N$  and multiplying each row  $j$  by  $\gamma_j$ ,  $1 \leq j \leq N$ , we get

$$\det(\delta_{jk}e_j + \frac{\det(H_{jk})}{\det(\alpha_{jk})}) = \prod_{j=1}^{2N} e^{-2i\Theta_j} \det(\delta_{jk} + \frac{(-1)^{\epsilon(j)} 2\gamma_j}{(\gamma_j + \gamma_k)} \prod_{l \neq k} \left| \frac{\gamma_l + \gamma_k}{\gamma_l - \gamma_k} \right| e^{2i\Theta_j}),$$

and therefore the wronskian

$$W_r(\phi_1, \dots, \phi_N)(0)$$



can be written as

$$\prod_{j=1}^{2N} e^{i\Theta_j|_{y=0}} (2)^{-2N} (i)^{\frac{2N(2N-2)}{2}} \prod_{j=2}^{2N} \prod_{i=1}^{j-1} (\gamma_j - \gamma_i) \prod_{j=1}^{2N} e^{-2i\Theta_j|_{y=0}} \det(I + D_r)$$

It follows that

$$\det(I + D_r) = \frac{e^{i \sum_{j=1}^{2N} \Theta_j|_{y=0}} 2^{2N}}{\prod_{j=2}^{2N} \prod_{i=1}^{j-1} (\gamma_j - \gamma_i)} W_r(\phi_1, \dots, \phi_{2N})(0) = k_r(0) W_r(\phi_1, \dots, \phi_{2N})(0). \quad (21)$$

□

So, the solution of NLS equation takes the form

$$v(x, t) = \frac{k_3(0)W_3(0)}{k_1(0)W_1(0)} \exp(2it - i\varphi) \quad (22)$$

### 3.2 Wronskian representation of solutions of NLS equation

From the previous section, we get the following result :

**Theorem 3.2** *The function  $v$  defined by*

$$v(x, t) = \frac{W_3(\phi_1, \dots, \phi_{2N})(0)}{W_1(\phi_1, \dots, \phi_{2N})(0)} \exp(2it - i\varphi).$$

*is solution of the NLS equation (1)*

$$iv_t + v_{xx} + 2|v|^2v = 0,$$

where

$W_r(y) = W(\phi_1, \dots, \phi_{2N})$  is the wronskian  $W_r(y) = \det[(\partial_y^{\mu-1} \phi_\nu)_{\nu, \mu \in [1, \dots, 2N]}]$ , and  $\phi_\nu$  the functions given by

$$\phi_\nu(y) = \sin(\kappa_\nu(x - x_{0\nu})/2 + i\delta_\nu(t - t_{0\nu}) - ix_{r,\nu}/2 + \gamma_\nu y), \quad 1 \leq \nu \leq N,$$

$$\phi_\nu(y) = \cos(\kappa_\nu(x - x_{0\nu})/2 + i\delta_\nu(t - t_{0\nu}) - ix_{r,\nu}/2 + \gamma_\nu y), \quad N + 1 \leq \nu \leq 2N,$$

$\kappa_\nu, \delta_\nu, \gamma_\nu$  parameters defined by (4) for  $\nu \in [1, \dots, 2N]$

$$\kappa_\nu = 2\sqrt{1 - \lambda_\nu^2}, \quad \delta_\nu = \kappa_\nu \lambda_\nu, \quad \gamma_\nu = \sqrt{\frac{1 - \lambda_\nu}{1 + \lambda_\nu}}, \quad \epsilon_\nu \in \{0; 1\},$$

where  $\lambda_\nu$  is an arbitrary parameter such that  $0 < \lambda_j < 1$  and  $\lambda_{N+j} = -\lambda_j$  for  $j \in [1, \dots, N]$ .

## 4 Construction of quasi-rational solutions of NLS equation

### 4.1 Taking the limit when the parameters $\lambda_j \rightarrow 1$ for $1 \leq j \leq N$ and $\lambda_j \rightarrow -1$ for $N+1 \leq j \leq 2N$

In the following, we show how we can obtain quasi-rational solutions of NLS equation by a simple limiting procedure.

We consider the parameter  $\lambda_j$  written in the form  $1 - \epsilon^2 c_j^2$ . For simplicity, we denote  $d_j$  the term  $\frac{c_j}{\sqrt{2}}$ .

When  $\epsilon$  goes to 0, we realize limited expansions at order  $p$ , for  $1 \leq j \leq N$ , of the terms

$$\begin{aligned}\kappa_j &= 4d_j\epsilon(1 - \epsilon^2 d_j^2)^{1/2}, \quad \delta_j = 4d_j\epsilon(1 - 2\epsilon^2 d_j^2)(1 - \epsilon^2 d_j^2)^{1/2}, \\ \gamma_j &= d_j\epsilon(1 - 2\epsilon^2 d_j^2)^{-1/2}, \quad x_{r,j} = (r-1)\ln(1 - 2d_j^2\epsilon^2 - 2id_j\epsilon(1 - \epsilon^2 d_j^2)^{1/2}), \\ \kappa_{N+j} &= 4d_j\epsilon(1 - \epsilon^2 d_j^2)^{1/2}, \quad \delta_{N+j} = -4d_j\epsilon(1 - 2\epsilon^2 d_j^2)(1 - \epsilon^2 d_j^2)^{1/2}, \\ \gamma_{N+j} &= 1/(d_j\epsilon)(1 - 2\epsilon^2 d_j^2)^{1/2}, \quad x_{r,N+j} = (r-1)\ln(1 - 2d_j^2\epsilon^2 + 2id_j\epsilon(1 - \epsilon^2 d_j^2)^{1/2}),\end{aligned}$$

For example, the expansions at order 1 gives :

$$\begin{aligned}\kappa_j &= 4d_j\epsilon + O(\epsilon^2), \quad \gamma_j = d_j\epsilon + O(\epsilon^2), \quad \delta_j = 4d_j\epsilon + O(\epsilon^2), \\ x_{r,j} &= (r-1)(2id_j\epsilon + O(\epsilon^2)), \\ \kappa_{N+j} &= 4d_j\epsilon + O(\epsilon^2), \quad \gamma_{N+j} = 1/(d_j\epsilon) - (d_j\epsilon)/2 + O(\epsilon^2), \quad \delta_{N+j} = -4d_j\epsilon + O(\epsilon^2), \\ x_{r,N+j} &= -(r-1)(2id_j\epsilon + O(\epsilon^2)), \\ 1 &\leq j \leq N.\end{aligned}$$

Then, we realize limited expansions at order  $p$  in  $\epsilon$  of the functions  $\phi_j^r(0)$  and  $\phi_{N+j}^r(0)$ , for  $1 \leq j \leq N$  :

$$\begin{aligned}\phi_j^1(0) &= \sin E_j = A_j + O(\epsilon^{p+1}), \\ \phi_j^3(0) &= \sin D_j = B_j + O(\epsilon^{p+1}), \\ \phi_{N+j}^1(0) &= \sin E'_j = A'_j + O(\epsilon^{p+1}), \\ \phi_{N+j}^3(0) &= \sin D'_j = B'_j + O(\epsilon^{p+1}),\end{aligned}$$

with

$$\begin{aligned}D_j &= \kappa_j(x - x_{0j})/2 + i\delta_j(t - t_{0j}) - ix_j/2, \\ E_j &= \kappa_j(x - x_{0j})/2 + i\delta_j(t - t_{0j}), \\ D'_j &= \kappa_j(x - x_{0j})/2 - i\delta_j(t - t_{0j}) + ix_j/2, \\ E'_j &= \kappa_j(x - x_{0j})/2 - i\delta_j(t - t_{0j}).\end{aligned}$$

Then according to the relation

$$v(x, t) = \frac{W_3(\phi_1, \dots, \phi_N)(0)}{W_1(\phi_1, \dots, \phi_N)(0)} \exp(2it - i\varphi).$$

the solution of NLS equation takes the form :

$$v(x, t) = \exp(2it - i\varphi) \frac{N(x, t, \epsilon)}{D(x, t, \epsilon)}$$

$$= \exp(2it - i\varphi) \frac{\begin{vmatrix} \sin D_1 & \dots & \sin D_N & \cos D'_1 & \dots & \cos D'_N \\ \gamma_1 \cos D_1 & \dots & \gamma_N \cos D_N & -\frac{1}{\gamma_1} \sin D'_1 & \dots & -\frac{1}{\gamma_N} \sin D'_N \\ \gamma_1^2 \sin D_1 & \dots & \gamma_N^2 \sin D_N & \frac{1}{\gamma_1^2} \cos D'_1 & \dots & \frac{1}{\gamma_N^2} \cos D'_N \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \gamma_1^{2N-1} \cos D_1 & \dots & \gamma_N^{2N-1} \cos D_N & -\frac{1}{\gamma_1^{2N-1}} \sin D'_1 & \dots & -\frac{1}{\gamma_N^{2N-1}} \sin D'_N \end{vmatrix}}{\begin{vmatrix} \sin E_1 & \dots & \sin E_N & \cos E'_1 & \dots & \cos E'_N \\ \gamma_1 \cos E_1 & \dots & \gamma_N \cos E_N & -\frac{1}{\gamma_1} \sin E'_1 & \dots & -\frac{1}{\gamma_N} \sin E'_N \\ \gamma_1^2 \sin E_1 & \dots & \gamma_N^2 \sin E_N & \frac{1}{\gamma_1^2} \cos E'_1 & \dots & \frac{1}{\gamma_N^2} \cos E'_N \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \gamma_1^{2N-1} \cos E_1 & \dots & \gamma_N^{2N-1} \cos E_N & -\frac{1}{\gamma_1^{2N-1}} \sin E'_1 & \dots & -\frac{1}{\gamma_N^{2N-1}} \sin E'_N \end{vmatrix}}$$

It can be rewritten as

$$\exp(2it - i\varphi) \frac{\begin{vmatrix} \sin D_1 & \dots & \sin D_N & \gamma_1^{2N-1} \cos D'_1 & \dots & \gamma_N^{2N-1} \cos D'_N \\ \gamma_1 \cos D_1 & \dots & \gamma_N \cos D_N & -\gamma_1^{2N-2} \sin D'_1 & \dots & -\gamma_N^{2N-2} \sin D'_N \\ \gamma_1^2 \sin D_1 & \dots & \gamma_N^2 \sin D_N & \gamma_1^{2N-3} \cos D'_1 & \dots & \gamma_N^{2N-3} \cos D'_N \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \gamma_1^{2N-1} \cos D_1 & \dots & \gamma_N^{2N-1} \cos D_N & -\sin D'_1 & \dots & -\sin D'_N \end{vmatrix}}{\begin{vmatrix} \sin E_1 & \dots & \sin E_N & \gamma_1^{2N-1} \cos E'_1 & \dots & \gamma_N^{2N-1} \cos E'_N \\ \gamma_1 \cos E_1 & \dots & \gamma_N \cos E_N & -\gamma_1^{2N-2} \sin E'_1 & \dots & -\gamma_N^{2N-2} \sin E'_N \\ \gamma_1^2 \sin E_1 & \dots & \gamma_N^2 \sin E_N & \gamma_1^{2N-3} \cos E'_1 & \dots & \gamma_N^{2N-3} \cos E'_N \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \gamma_1^{2N-1} \cos E_1 & \dots & \gamma_N^{2N-1} \cos E_N & -\sin E'_1 & \dots & -\sin E'_N \end{vmatrix}}$$

We can replace each term in these determinants by their limited expansion at order  $p$ . The coefficients of each power of  $\epsilon$  of each development is a polynomial in  $x$  and  $t$ . The relations  $D_j = E_j - ix_j/2$  and  $D'_j = E'_j + ix_j/2$ ,

and the addition formulas for sine and cosine functions, for  $1 \leq j \leq N$ , proves that after simplifications, we have same smallest power of  $\epsilon$ . So it is clear that after simplifications, the limit is quotient of two polynomials when  $\epsilon$  goes to 0.

**Remark 4.1** *Using the fact that  $D_j = E_j + \alpha_j(\epsilon)$ ,  $D'_j = E'_j - \alpha_j(\epsilon)$ , and addition formulas for trigonometric functions, we can expand the numerator  $N(x, t, \epsilon)$  along the columns; it takes the form of*

$$\prod_{j=1}^N \cos^2(\alpha_j(\epsilon)) D(x, t, \epsilon) + A.$$

We divide each term of this sum by  $D(x, t, \epsilon)$ .

We denote this expression by  $S$ .

When  $\epsilon$  tends to 0 the first term tends to 1; in each other term of the sum, we can simplify each fraction by some power of  $\epsilon$  to get a finite limit when  $\epsilon$  goes to 0.

In fact, in the development of  $N(x, t, \epsilon)$  (along the columns), we have a sum of  $2^M$  quotients of determinants of order  $M$ . We don't have place here to give all the details of the proof. To have an idea of the proof, we treat the case  $M = 2$ .

In this case, the sum  $S$  takes the form

$$S = \frac{\cos^2(\alpha_1)}{D(x, t, \epsilon)} \begin{vmatrix} \sin E_1 & \cos E'_1 \\ \gamma_1 \cos E_1 & -\frac{1}{\gamma_1} \sin E'_1 \end{vmatrix} + \frac{\sin(\alpha_1) \cos(\alpha_1)}{D(x, t, \epsilon)} \begin{vmatrix} \sin E_1 & \sin E'_1 \\ \gamma_1 \cos E_1 & \frac{1}{\gamma_1} \cos E'_1 \end{vmatrix} \\ + \frac{\sin(\alpha_1) \cos(\alpha_1)}{D(x, t, \epsilon)} \begin{vmatrix} \cos E_1 & \cos E'_1 \\ -\gamma_1 \sin E_1 & -\frac{1}{\gamma_1} \sin E'_1 \end{vmatrix} + \frac{\sin^2(\alpha_1)}{D(x, t, \epsilon)} \begin{vmatrix} \cos E_1 & \sin E'_1 \\ -\gamma_1 \sin E_1 & \frac{1}{\gamma_1} \cos E'_1 \end{vmatrix},$$

with

$$D(x, t, \epsilon) = \begin{vmatrix} \sin E_1 & \cos E'_1 \\ \gamma_1 \cos E_1 & -\frac{1}{\gamma_1} \sin E'_1 \end{vmatrix}.$$

Using expansions in  $\epsilon$  at order 1 of all the terms, we get

$$S = 1 - \frac{4(x + 2it)}{1 + 4x^2 + 16t^2} + \frac{4(x - 2it)}{1 + 4x^2 + 16t^2} - \frac{4}{1 + 4x^2 + 16t^2} = 1 - \frac{4(1 + 4it)}{1 + 4x^2 + 16t^2}.$$

The development of  $N(x, t, \epsilon)$  to get the sum  $S$  gives directly the representation of Akhmediev for  $v$  under the form of 1 plus a quotient of two polynomials in  $x$  and  $t$ .

So we get the following result

**Theorem 4.1** *The function  $v$  defined by*

$$v(x, t) = \lim_{\epsilon \rightarrow 0} \frac{W_3(\phi_1, \dots, \phi_{2N})(0)}{W_1(\phi_1, \dots, \phi_{2N})(0)} \exp(2it - i\varphi).$$

*is a quasi-rational solution of the NLS equation (1)*

$$iv_t + v_{xx} + 2|v|^2v = 0.$$

$W_r(y) = W(\phi_1, \dots, \phi_{2N})$  *is the wronskian*

$$W_r(y) = \det[(\partial_y^{\mu-1} \phi_\nu)_{\nu, \mu \in [1, \dots, 2N]}].$$

$\phi_\nu$  *are the functions given by*

$$\begin{aligned} \phi_\nu(y) &= \sin(\kappa_\nu(x - x_{0\nu})/2 + i\delta_\nu(t - t_{0\nu}) - ix_{r,\nu}/2 + \gamma_\nu y), & 1 \leq \nu \leq N. \\ \phi_\nu(y) &= \cos(\kappa_\nu(x - x_{0\nu})/2 + i\delta_\nu(t - t_{0\nu}) - ix_{r,\nu}/2 + \gamma_\nu y), & N + 1 \leq \nu \leq 2N. \end{aligned}$$

$\kappa_\nu, \delta_\nu, \gamma_\nu$  *are parameters defined by (4) for  $\nu \in [1, \dots, 2N]$*

$$\kappa_\nu = 2\sqrt{1 - \lambda_\nu^2}, \quad \delta_\nu = \kappa_\nu \lambda_\nu, \quad \gamma_\nu = \sqrt{\frac{1 - \lambda_\nu}{1 + \lambda_\nu}}, \quad \epsilon_\nu \in \{0; 1\}.$$

$\lambda_j = 1 - \epsilon^2 c_j^2$  *is an arbitrary parameter such that  $0 < \lambda_j < 1$  and  $\lambda_{N+j} = -\lambda_j$ ,  $c_j$  an arbitrary real number, for  $j \in [1, \dots, N]$ .*

## 4.2 Quasi-rational solutions of order $N$

For brevity, in the following we choose  $x_{0i} = 0$ ,  $t_{0i} = 0$  and  $\varphi = 0$  for  $1 \leq i \leq N$ .

### 4.2.1 Case $N=1$

If we consider the case  $N = 1$ , from (22), we realize a development at order 1 of  $W_3$  and  $W_1$  in  $\epsilon$ . The solution of NLS equation can be written as

$$v(x, t) = -\frac{(-4x^2 - 16t^2 + 16it + 3)}{4x^2 + 16t^2 + 1} \exp(2it).$$

Then  $v$  takes the form

$$v(x, t) = 1 - 4 \frac{1 + 4it}{1 + 16t^2 + 4x^2} \exp(2it).$$

If we make the following changes of variables  $t \rightarrow X/2$  and  $x \rightarrow T$ , we get exactly first order Akhmediev's solution (see [1]), solution of  $iu_x + \frac{1}{2}u_{tt} + u|u|^2 = 0$ ,

$$v(x, t) = \left(1 - \frac{4(1 + 2ix)}{1 + 4x^2 + 4t^2}\right) \exp(ix).$$

We represent here the modulus of  $v$  in function of  $x \in [-5; 5]$  and  $t \in [-5; 5]$ ; we get the following graphic :

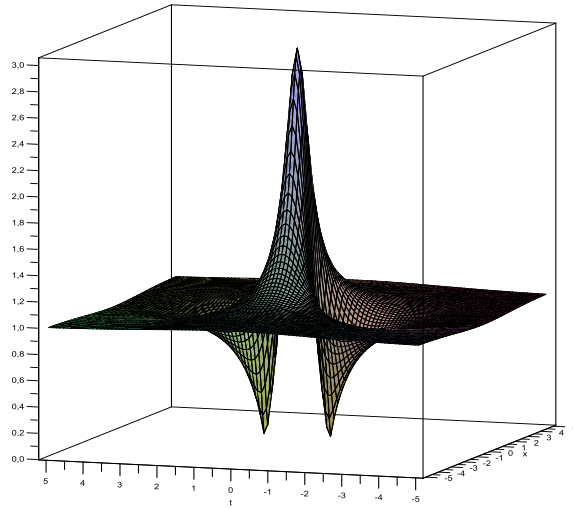


Figure 1: Solution of NLS, N=1.

### 4.2.2 Case N=2

In the case  $N = 2$ , we must realize a development at order 3 in  $\epsilon$ . From (22), the solution of NLS equation can be written as

$$v(x, t) = \frac{N(x, t)}{D(x, t)} \exp(2it),$$

with

$$N(x, t) = 5760x^2t^2 - 3072x^2t^4 + 1872t^2 - 64x^6 - 45 - 4096t^6 + 180x^2 + 144x^4 - 768x^4t^2 \\ + 768ix^4t + 8448t^4 - 1152ix^2t + 1536it^3 - 720it + 12288it^5 + 6144it^3x^2,$$

and

$$D(x, t) = -1152x^2t^2 + 3072x^2t^4 + 1584t^2 + 64x^6 + 9 + 4096t^6 + 108x^2 + 48x^4 + 768x^4t^2 + 6912t^4.$$

If we make the following changes of variables  $t \rightarrow X/2$  and  $x \rightarrow T$ , we get

$$v(x, t) = \frac{N_1(x, t)}{D_1(x, t)} \exp(ix),$$

with

$$N_1(x, t) = 1440x^2t^2 - 192x^4t^2 + 468x^2 - 64t^6 - 45 - 64x^6 + 180t^2 + 144t^4 - 192x^2t^4 - 576it^2x \\ + 528x^4 + 384ix^5 + 192ix^3 - 360ix + 384it^4x + 768ix^3t^2,$$

and

$$D_1(x, t) = -288x^2t^2 + 192x^4t^2 + 396x^2 + 64t^6 + 9 + 64x^6 + 108t^2 + 48t^4 + 192x^2t^4 + 432x^4.$$

It can be reduced exactly at the second order Akhmediev's solution (solution of  $iu_x + \frac{1}{2}u_{tt} + u|u|^2 = 0$ ) (see [1]).

$$v(x, t) = 1 + \frac{3/8 - 3t^2 - 2t^4 - 9x^2 - 10x^4 - 12t^2x^2 + ix(15/4 + 6t^2 - 4t^4 - 2x^2 - 4x^4 - 8t^2x^2)}{3/32 + 9/8t^2 + 1/2t^4 + 2/3t^6 + 33/8x^2 + 9/2x^4 + 2/3x^6 - 3t^2x^2 + 2t^4x^2 + 2t^2x^4} \exp(ix).$$

We represent the modulus of  $v$  in function of  $x \in [-5; 5]$  and  $t \in [-5; 5]$ ; we get the following graphic :

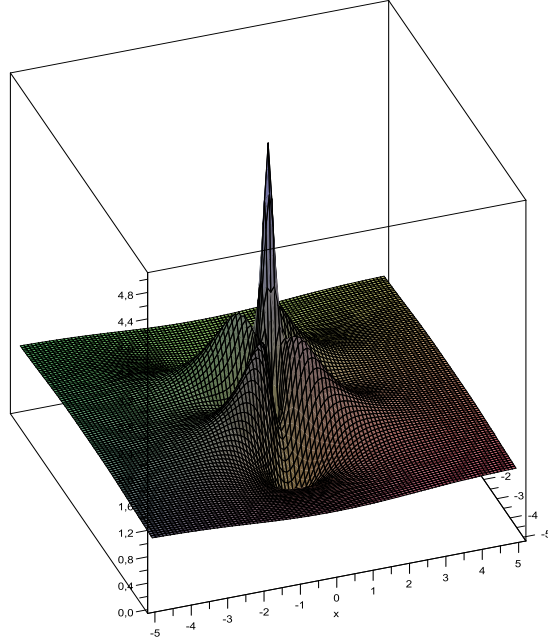


Figure 2: Solution of NLS, N=2.

### 4.2.3 Case N=3

In the case  $N = 3$ , we must realize a development at order 5 in  $\epsilon$ . If we make the following changes of variables  $t \rightarrow X/2$  and  $x \rightarrow T$ , from (22), the solution of NLS equation in the formulation of Akhmediev  $iv_x + \frac{1}{2}v_{tt} + v|v|^2 = 0$  can be written as

$$v(x, t) = \frac{N(x, t)}{D(x, t)} \exp(ix),$$

with

$$\begin{aligned} N(x, t) = & 14175 + 49152i t^{10} x - 737280i t^2 x^7 + 5253120i t^2 x^5 + 907200i t^2 x - 1720320i t^6 x^3 - 184320i t^8 x \\ & + 5356800x^4 t^2 - 576000x^4 t^4 - 875520x^6 t^2 - 172800t^4 x^2 - 4600800x^2 t^2 + 829440x^8 t^2 - 24576x^{10} t^2 - 61440x^8 t^4 \\ & + 460800t^8 x^2 - 506880x^2 t^6 + 1658880x^6 t^4 - 4838400i x^3 t^2 - 81920x^6 t^6 + 1413120x^4 t^6 - 61440x^4 t^8 - 24576x^2 t^{10} \\ & - 5238000x^4 - 567000x^2 - 226800t^4 + 2384640x^6 + 172800t^6 + 2085120x^8 + 141312x^{10} + 57600t^8 \\ & - 113400t^2 - 4096x^{12} - 4096t^{12} + 18432t^{10} - 1382400i t^4 x^3 - 1036800i t^4 x - 460800i t^6 x - 2580480i t^4 x^5 \end{aligned}$$



$$+491520i t^6 x^5 + 491520i t^4 x^7 + 245760i t^8 x^3 + 245760i t^2 x^9 + 307200i x^9 - 475200i x^3 - 2672640i x^7 + 49152i x^{11} \\ - 7395840i x^5 + 226800i x$$

and

$$D(x, t) = 2025 + 5011200x^4 t^2 - 345600x^4 t^4 + 3456000x^6 t^2 + 864000t^4 x^2 - 583200x^2 t^2 + 276480x^8 t^2 + 24576x^{10} t^2 \\ + 61440x^8 t^4 - 92160t^8 x^2 + 138240x^2 t^6 + 61440x^6 t^4 + 81920x^6 t^6 - 184320x^4 t^6 + 61440x^4 t^8 + 24576x^2 t^{10} \\ + 2300400x^4 + 372600x^2 + 54000t^4 + 979200x^6 + 149760t^6 + 956160x^8 + 129024x^{10} + 34560t^8 \\ + 48600t^2 + 4096x^{12} + 4096t^{12} + 6144t^{10}.$$

We get the following graphic for the modulus of  $v$  in function of  $x \in [-5; 5]$  and  $t \in [-5; 5]$  :

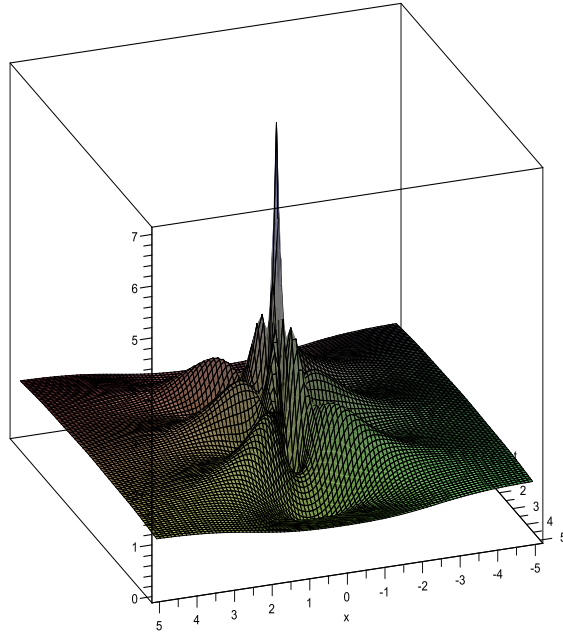


Figure 3: Solution of NLS, N=3.

An another type of solution different from this one was given recently by Akhmediev [1].

The results presented here are in accordance with the preceding work [6];

here we give moreover an explicit expression of the solutions of NLS equation for the order 3.

### 4.3 Case N=4

The solution of NLS equation  $iv_x + \frac{1}{2}v_{tt} + v|v|^2 = 0$  can be written as

$$v(x, t) = \frac{N(x, t)}{D(x, t)} \exp(ix),$$

with  $N$  and  $D$  polynomials both of degree 20. Their expressions are too long to be reproduced here.

We get the following graphic for the modulus of  $v$  in function of  $x \in [-5; 5]$  and  $t \in [-5; 5]$  :

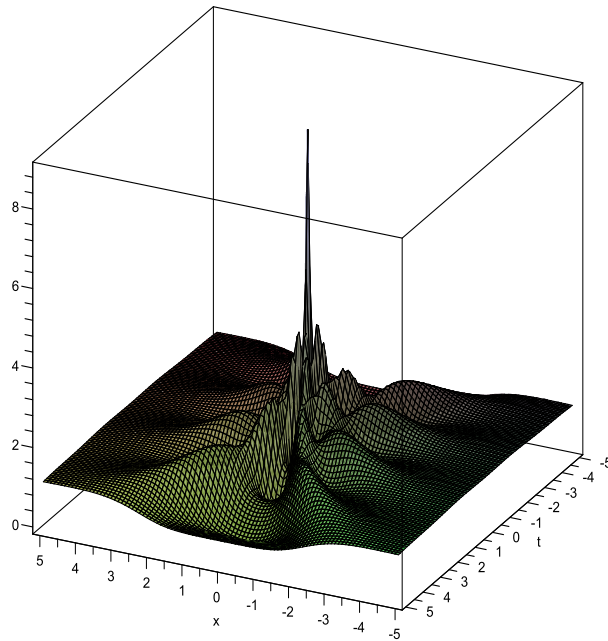


Figure 4: Solution of NLS, N=4.

We recover a result of Akhmediev formulated in [1] in the case where  $x = 0$ . Here we give the complete solution in  $x$  and  $t$ . As my knowledge, it is the first time that this solution is given.

This method can be extended to get an infinite family of quasi-rational solutions of NLS equation at any order.

## 5 Conclusion

Here is given a new formulation of solutions of NLS equation that gives as particular case, Akhmediev's solutions. These solutions expressed as a quotient of wronskians are similar to a previous result of Eleonski, Krichever and Kulagin [8]. It has been rewritten recently and their formulation can also be seen as a quotient of two wronkians (see [6]); the method describe in [6] can be used to get solutions of NLS equation, but it is more complicated to find explicit solution in  $x$  and  $t$ ; the computation gives complicated results. This method described in the present paper provides a powerfull tool to get explicitly solutions in analytical form. This new formulation gives an infinite set of non singular solution of NLS equation.

## References

- [1] N. Akhmediev, A. Ankiewicz, J.M. Soto-Crespo, Rogue waves and rational solutions of nonlinear Schrödinger equation, *Physical Review E*, V. **80**, N. 026601, (2009).
- [2] N. Akhmediev, V. Eleonskii, N. Kulagin, Exact first order solutions of the nonlinear Schrödinger equation, *Th. Math. Phys.*, V. **72**, N. 2, 183-196, (1987).
- [3] N. Akhmediev, V. Eleonsky, N. Kulagin, Generation of periodic trains of picosecond pulses in an optical fiber : exact solutions, *Sov. Phys. J.E.T.P.*, V. **62**, 894-899, (1985).
- [4] N. Akhmediev, A. Ankiewicz, P.A. Clarkson, Rogue waves, rational solutions, the patterns of their zeros and integral relations, *J. Phys. A : Math. Theor.*, V. **43**, 122002, 1-9, (2010).

- [5] E.D. Belokolos, A.i. Bobenko, A.R. Its, V.Z. Enolskij and V.B. Matveev, *Algebro-geometric approach to nonlinear integrable equations*, Springer series in nonlinear dynamics, Springer Verlag, 1-360, (1994).
- [6] P. Dubard, P. Gaillard, C. Klein, V.B. Matveev, On multi-rogue waves solutions of the NLS equation and positon solutions of the KdV equation, *Eur. Phys. J. Special Topics*, V. **185**, 247-258, (2010).
- [7] P. Dubard, V.B. Matveev, Multi-rogue waves solutions of the focusing NLS equation and the KP-I equation, accepted for publication in *Nat. Hazards Earth Syst. Sci.*, V. 10, Nov. (2010).
- [8] V. Eleonskii, I. Krichever, N. Kulagin, Rational multisoliton solutions to the NLS equation, *Soviet Doklady 1986 sect. Math. Phys.*, **287**, 606-610, (1986).
- [9] A.N.W Hone, Crum transformation and rational solutions of the non-focusing nonlinear Schrödinger equation, *J. Phys. A : Math. Gen.*, V. **30**, 7473-7483, (1997).
- [10] A.R. Its, A.V. Rybin, M.A. Salle, Exact integration of nonlinear Schrödinger equation, *Teore. i Mat. Fiz.*, V. **74.**, N. 1, 29-45, (1988).
- [11] V.B. Matveev, M.A. Salle, Darboux transformations and solitons, *Series in Nonlinear Dynamics*, Springer Verlag, Berlin, (1991).
- [12] D. Pelinovsky, On a structure of the explicit solutions of the Davey-Stewartson equations, *Phys. D*, V. **87**, 115-122, (1995).
- [13] D. Peregrine, Water waves, nonlinear Schrödinger equations and their solutions, *J. Austral. Math. Soc. Ser. B*, V. **25**, 16-43, (1983).
- [14] J.F. Van Diejen, A.N. Kirillov, A combinatorial formula for the associated Legendre functions of integer degree, *Adv. in Math.*, V. **149**, 61-88, (2000).