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Quasi-rational solutions of the NLS equation and rogue waves

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Abstract

We degenerate solutions of the NLS equation from the general formulation in terms of Fredholm determinant given in [10], to get quasi-rational solutions of NLS equations. For this we establish a link between Fredholm determinants and wronskians. We give solutions of the focusing NLS equation as a quotient of two wronskian determinants of a system of sine functions. By appropriate passage to the limit when the periods tend to infinity, we recover exact expression for higher Peregrine-Akhmediev breathers obtained in [13] by Peregrine, and by Eleonskii, Krichever, Kulagin and Akhmediev [8, 2, 1]. It gives a new approach to get an analytical description of the rogue waves.

1 Introduction

The nonlinear Schrödinger equation (NLS) equation plays a fundamental role in hydrodynamics, in particular for waves in deep water in the context of the rogue waves or in non-linear optics.

In this paper, we will study solutions of NLS equation from the theta formulation given in [10] and give a new representation in terms of wronskians. The solutions take the form of a quotient of two wronskians of even order $2N$ of elementary functions depending on a certain number of parameters. We will call these related solutions, solutions of NLS of order N .

To get from this form, quasi-rational solutions of NLS equation, we take the limit when some parameter goes to 0.

For $N = 1$, we recover the well known Peregrine's solution of the focusing NLS equation first found [13] in 1983; it was rediscovered later in several publications, in particular [3].

The second order Peregrine-like solution was first constructed in [3]. Other Peregrine-like solutions were found for reduced self-induced transparency (SIT) integrable system [11]. In [10], the n-phase quasi-periodic modulations of the plane waves solutions were constructed via appropriate degeneration of the finite gap periodic solutions to the NLS equation. These results have provided a way to construct multi-parametric families of multi-Peregrine solutions.

Same families of higher order were constructed in a series of articles by Akhmediev et al. [1, 2].

Rational solutions of NLS equation are given in [12] in a determinantal representation as degenerate solutions of Davey-Stewartson equations. Another approach using Crum transformation is given in [9] to get sequences of rational solutions of NLS equation. Very recently, it is shown in [6] that rational solutions can be written as a quotient of two wronskians using modified version of [8], with some different reasoning; moreover, the link between quasi-rational solutions of the focusing NLS equation and the rational solution of the KP-I equation is established.

With this formulation we recover as particular case, Akhmediev's quasi-rational solutions of NLS equation.

2 Expression of solutions of NLS equation in terms of Fredholm determinant

2.1 Solutions of NLS equation in terms of θ functions

The solution of the NLS equation

$$iv_t + v_{xx} + 2|v|^2v = 0, \quad (1)$$

is given in terms of truncated theta function by (see [10])

$$v(x, t) = \frac{\theta_3(x, t)}{\theta_1(x, t)} \exp(2it - i\varphi), \quad (2)$$

where

$$\begin{aligned} \theta_r(x, t) = & \sum_{k \in \{0;1\}^{2N}} \exp\left\{ \sum_{\mu > \nu, \mu, \nu=1}^{2N} \ln \left(\frac{\gamma_\nu - \gamma_\mu}{\gamma_\nu + \gamma_\mu} \right)^2 k_\mu k_\nu \right. \\ & \left. + \left(\sum_{\nu=1}^{2N} i\kappa_\nu(x - x_{0\nu}) - 2\delta_\nu(t - t_{0\nu}) + (r-1) \ln \frac{\gamma_\nu - i}{\gamma_\nu + i} + \sum_{\mu=1, \mu \neq \nu}^{2N} \ln \left| \frac{\gamma_\nu + \gamma_\mu}{\gamma_\nu - \gamma_\mu} \right| + \pi i \epsilon_\nu \right) k_\nu \right\}, \end{aligned} \quad (3) \quad r = 1, 3.$$

In this formula, the quantities κ_ν , δ_ν , γ_ν are functions of the parameters λ_ν , $\nu = 1, \dots, 2N$, N being a positive integer, $N \geq 1$ and they are given by the following equations,

$$\kappa_\nu = 2\sqrt{1 - \lambda_\nu^2}, \quad \delta_\nu = \kappa_\nu \lambda_\nu, \quad \gamma_\nu = \sqrt{\frac{1 - \lambda_\nu}{1 + \lambda_\nu}}, \quad \epsilon_\nu \in \{0;1\}. \quad (4)$$

φ , $x_{0\nu}$, $t_{0\nu}$, $\nu = 1 \dots 2N$ are arbitrary real numbers.

We also note that

$$\lambda_{N+j} = -\lambda_j, \quad \kappa_{N+j} = \kappa_j, \quad \delta_{N+j} = -\delta_j, \quad \gamma_{N+j} = 1/\gamma_j, \quad j = 1 \dots N. \quad (5)$$

2.2 Relation between θ and Fredholm determinant

The function θ_r defined in (3) can be rewritten with a summation in terms of subsets of $[1, \dots, 2N]$

$$\begin{aligned} \theta_r(x, t) = & \sum_{J \subset \{1, \dots, 2N\}} \prod_{\nu \in J} (-1)^{\epsilon_\nu} \prod_{\nu \in J, \mu \notin J} \left| \frac{\gamma_\nu + \gamma_\mu}{\gamma_\nu - \gamma_\mu} \right| \\ & \times \exp\left\{ \sum_{\nu \in J} i\kappa_\nu(x - x_{0\nu}) - 2\delta_\nu(t - t_{0\nu}) + x_{r,\nu} \right\}, \end{aligned}$$

with

$$x_{r,\nu} = (r-1) \ln \frac{\gamma_\nu - i}{\gamma_\nu + i}, \quad 1 \leq j \leq 2N, \quad (6)$$

in particular

$$\begin{aligned} x_{r,j} &= (r-1) \ln \frac{\gamma_j - i}{\gamma_j + i}, \quad 1 \leq j \leq N, \\ x_{r,N+j} &= -(r-1) \ln \frac{\gamma_j - i}{\gamma_j + i} - (r-1)i\pi, \quad 1 \leq j \leq N. \end{aligned} \quad (7)$$

In formula (3), ϵ_j can be chosen arbitrary, for example it can be fixed in the following way :

$\epsilon_j = j$ for $1 \leq j \leq N$, and $\epsilon_j = j + 1$ for $N + 1 \leq j \leq 2N$.

The link between theta representation of solutions of NLS equation and Fredholm determinant formulation was already presented in [10]. In more details, it is given in a recent paper of Kirillov and Van Diejen [14]. In particular, they compute $\det(I + A)$, where I is the unit matrix and $A = (a_{jk})_{1 \leq j, k \leq 2N}$ the matrix defined as :

$$a_{jk} = \frac{2\epsilon_j \kappa_j}{\kappa_j + \kappa_k} \prod_{l \neq j} \left| \frac{\kappa_j + \kappa_l}{\kappa_j - \kappa_l} \right| \exp(-2\kappa_j x), \quad (8)$$

where $\epsilon_j \in \{-1; +1\}$.

Then it is shown that $\det(I + A)$ has the following form (see [14], p 84-87)

$$\det(I + A) = \sum_{J \subset \{1, \dots, N\}} \prod_{j \in J} \epsilon_j \prod_{j \in J, k \notin J} \left| \frac{\kappa_j + \kappa_k}{\kappa_j - \kappa_k} \right| \exp(-2x \sum_{j \in J} \kappa_j). \quad (9)$$

Using the same strategy, we can compute $\det(I + A_r)$ where $A_r = (a_{\nu\mu})_{1 \leq \nu, \mu \leq 2N}$ is the matrix defined as :

$$a_{\nu\mu} = \frac{2(-1)^{\epsilon_\nu} \gamma_\nu}{\gamma_\nu + \gamma_\mu} \prod_{\eta \neq \nu} \left| \frac{\gamma_\nu + \gamma_\eta}{\gamma_\nu - \gamma_\eta} \right| \exp(i\kappa_\nu(x - x_{0\nu}) - 2\delta_\nu(t - t_{0\nu}) + x_{r,\nu}). \quad (10)$$

Then $\det(I + A_r)$ has the following form

$$\det(I + A_r) = \sum_{J \subset \{1, \dots, 2N\}} \prod_{\nu \in J} (-1)^{\epsilon_\nu} \prod_{\nu \in J, \mu \notin J} \left| \frac{\gamma_\nu + \gamma_\mu}{\gamma_\nu - \gamma_\mu} \right| \exp(i\kappa_\nu(x - x_{0\nu}) - 2\delta_\nu(t - t_{0\nu}) + x_{r,\nu}). \quad (11)$$

From the beginning of this section, $\tilde{\theta}$ has the same expression as in (11) so, we have clearly the equality

$$\theta_r = \det(I + A_r). \quad (12)$$

Then the solution of NLS equation takes the form

$$v(x, t) = \frac{\det(I + A_3(x, t))}{\det(I + A_1(x, t))} \exp(2it - i\varphi). \quad (13)$$

It must be noted that the formulae (13) already appeared in [10].

3 Expression of solutions of NLS equation in terms of wronkian determinant

3.1 Link between Fredholm determinants and wronskians

We consider the following functions

$$\begin{aligned}\phi_\nu(y) &= \sin(\kappa_\nu(x - x_{0\nu})/2 + i\delta_\nu(t - t_{0\nu}) - ix_{r,\nu}/2 + \gamma_\nu y), & 1 \leq \nu \leq N, \\ \phi_\nu(y) &= \cos(\kappa_\nu(x - x_{0\nu})/2 + i\delta_\nu(t - t_{0\nu}) - ix_{r,\nu}/2 + \gamma_\nu y), & N + 1 \leq \nu \leq 2N.\end{aligned}\quad (14)$$

These functions depend on r . We must denote them $\phi_\nu^r(y)$. For simplicity, in this section we denote them $\phi_\nu(y)$.

We use the following notations :

$$\Theta_\nu = \kappa_\nu(x - x_{0\nu})/2 + i\delta_\nu(t - t_{0\nu}) - ix_{r,\nu}/2 + \gamma_\nu y, \quad 1 \leq \nu \leq 2N.$$

$W_r(y) = W(\phi_1, \dots, \phi_{2N})$ is the wronskian $W_r(y) = \det[(\partial_y^{\mu-1} \phi_\nu)_{\nu, \mu \in [1, \dots, 2N]}]$.

We consider the matrix $D_r = (d_{\nu\mu})_{\nu, \mu \in [1, \dots, 2N]}$ defined by

$$\begin{aligned}d_{\nu\mu} &= \frac{2(-1)^{\epsilon_\nu} \gamma_\nu}{\gamma_\nu + \gamma_\mu} \prod_{\eta \neq \nu} \left| \frac{\gamma_\nu + \gamma_\mu}{\gamma_\nu - \gamma_\mu} \right| \exp(i\kappa_\nu(x - x_{0\nu}) - 2\delta_\nu(t - t_{0\nu}) + x_{r,\nu}), \\ &1 \leq \nu \leq 2N, \quad 1 \leq \mu \leq 2N,\end{aligned}$$

with

$$x_{r,\nu} = (r - 1) \ln \frac{\gamma_\nu - i}{\gamma_\nu + i}.$$

Then we have the following statement

Theorem 3.1

$$\det(I + D_r) = k_r(0) \times W_r(\phi_1, \dots, \phi_{2N})(0), \quad (15)$$

where

$$k_r(y) = \frac{2^{2N} \exp(i \sum_{\nu=1}^{2N} \Theta_\nu)}{\prod_{\nu=2}^{2N} \prod_{\mu=1}^{\nu-1} (\gamma_\nu - \gamma_\mu)}.$$

Proof : We start to remove the factor $(2i)^{-1} e^{i\Theta_\nu}$ in each row ν in the wronskian $W_r(y)$ for $1 \leq \nu \leq 2N$.

Then

$$W_r = \prod_{\nu=1}^{2N} e^{i\Theta_\nu} (2i)^{-N} (2)^{-N} \times W_{r1}, \quad (16)$$

with

$$W_{r1} = \begin{vmatrix} (1 - e^{-2i\Theta_1}) & i\gamma_1(1 + e^{-2i\Theta_1}) & \dots & (i\gamma_1)^{2N-1}(1 + (-1)^{2N}e^{-2i\Theta_1}) \\ (1 - e^{-2i\Theta_2}) & i\gamma_2(1 + e^{-2i\Theta_2}) & \dots & (i\gamma_2)^{2N-1}(1 + (-1)^{2N}e^{-2i\Theta_2}) \\ \vdots & \vdots & \vdots & \vdots \\ (1 - e^{-2i\Theta_{2N}}) & i\gamma_{2N}(1 + e^{-2i\Theta_{2N}}) & \dots & (i\gamma_{2N})^{2N-1}(1 + (-1)^{2N}e^{-2i\Theta_{2N}}) \end{vmatrix}$$

The determinant W_{r1} can be written as

$$W_{r1} = \det(\alpha_{jk}e_j + \beta_{jk}),$$

where $\alpha_{jk} = (-1)^k(i\gamma_j)^{k-1}$, $e_j = e^{-2i\theta_j}$, and $\beta_{jk} = (i\gamma_j)^{k-1}$, $1 \leq j \leq N$, $1 \leq k \leq 2N$,

$\alpha_{jk} = (-1)^{k-1}(i\gamma_j)^{k-1}$, $e_j = e^{-2i\theta_j}$, and $\beta_{jk} = (i\gamma_j)^{k-1}$, $N+1 \leq j \leq 2N$, $1 \leq k \leq 2N$.

Denoting $U = (\alpha_{ij})_{i,j \in [1, \dots, 2N]}$, $V = (\beta_{ij})_{i,j \in [1, \dots, 2N]}$, the determinant of U is clearly equal to

$$\det(U) = (i)^{\frac{2N(2N-1)}{2}} \prod_{2N \geq l > m \geq 1} (\gamma_l - \gamma_m). \quad (17)$$

Then we use the following Lemma

Lemma 3.1 *Let $A = (a_{ij})_{i,j \in [1, \dots, N]}$, $B = (b_{ij})_{i,j \in [1, \dots, N]}$, $(H_{ij})_{i,j \in [1, \dots, N]}$, the matrix formed by replacing the j th row of A by the i th row of B*
Then

$$\det(a_{ij}x_i + b_{ij}) = \det(a_{ij}) \times \det(\delta_{ij}x_i + \frac{\det(H_{ij})}{\det(a_{ij})}) \quad (18)$$

Proof : For $\tilde{A} = (\tilde{a}_{ji})_{i,j \in [1, \dots, N]}$ the transposed matrix in cofactors of A , we have the well known formula $A \times^t \tilde{A} = \det A \times I$.

So it is clear that $\det(\tilde{A}) = (\det(A))^{N-1}$.

The general term of the product $(c_{ij})_{i,j \in [1, \dots, N]} = (a_{ij}x_i + b_{ij})_{i,j \in [1, \dots, N]} \times (\tilde{a}_{ji})_{i,j \in [1, \dots, N]}$ can be written as

$$\begin{aligned} c_{ij} &= \sum_{s=1}^N (a_{is}x_i + b_{is}) \times \tilde{a}_{js} \\ &= x_i \sum_{s=1}^N a_{is}\tilde{a}_{js} + \sum_{s=1}^N b_{is}\tilde{a}_{js} \\ &= \delta_{ij} \det(A)x_i + \det(H_{ij}). \end{aligned}$$

We get

$$\det(c_{ij}) = \det(a_{ij}x_i + b_{ij}) \times (\det(A))^{N-1} = (\det(A))^N \times \det(\delta_{ij}x_i + \frac{\det(H_{ij})}{\det(A)}).$$

$$\text{Thus } \det(a_{ij}x_i + b_{ij}) = \det(A) \times \det(\delta_{ij}x_i + \frac{\det(H_{ij})}{\det(A)}).$$

□

Using the previous lemma (18), we get :

$$\det(\alpha_{ij}e_i + \beta_{ij}) = \det(\alpha_{ij}) \times \det(\delta_{ij}e_i + \frac{\det(H_{ij})}{\det(\alpha_{ij})}),$$

where $(H_{ij})_{i,j \in [1, \dots, N]}$ is the matrix formed by replacing the j th row of U by the i th row of V defined previously.

We compute $\det(H_{ij})$ and we get

$$\det(H_{ij}) = (-1)^{\frac{2N(2N+1)}{2}+1} (i)^{\frac{2N(2N-1)}{2}} \prod_{2N \geq l > m \geq 1, l \neq j, m \neq j} (\gamma_l - \gamma_m) \prod_{l < j} (\gamma_k - \gamma_l) \prod_{l > j} (\gamma_k - \gamma_l). \quad (19)$$

We can simplify the quotient $q = \frac{\det(H_{ij})}{\det(\alpha_{ij})}$:

$$\begin{aligned} q &= \frac{(-1)^{\epsilon(j)} \prod_{l \neq j} (\gamma_l + \gamma_k)}{(-1)^{k-1} \prod_{l \neq k} (\gamma_l - \gamma_k)} \\ &= \frac{(-1)^{\epsilon(j)-k+1} 2\gamma_k \prod_{l \neq k} (\gamma_l + \gamma_k)}{(\gamma_j + \gamma_k) \prod_{l \neq k} (\gamma_l - \gamma_k)} \\ &= \frac{(-1)^{\epsilon(j)} 2\gamma_k}{(\gamma_j + \gamma_k)} \prod_{l \neq k} \left| \frac{\gamma_l + \gamma_k}{\gamma_l - \gamma_k} \right|. \end{aligned} \quad (20)$$

So $\det(\delta_{jk}e_j + \frac{\det(H_{jk})}{\det(\alpha_{jk})})$ can be expressed as

$$\det(\delta_{jk}e_j + \frac{\det(H_{jk})}{\det(\alpha_{jk})}) = \prod_{j=1}^{2N} e^{-2i\Theta_j} \det(\delta_{jk} + \frac{(-1)^{\epsilon(j)} 2\gamma_k}{(\gamma_j + \gamma_k)} \prod_{l \neq k} \left| \frac{\gamma_l + \gamma_k}{\gamma_l - \gamma_k} \right| e^{2i\Theta_j}).$$

Then dividing each column k by γ_k , $1 \leq k \leq N$ and multiplying each row j by γ_j , $1 \leq j \leq N$, we get

$$\det(\delta_{jk}e_j + \frac{\det(H_{jk})}{\det(\alpha_{jk})}) = \prod_{j=1}^{2N} e^{-2i\Theta_j} \det(\delta_{jk} + \frac{(-1)^{\epsilon(j)} 2\gamma_j}{(\gamma_j + \gamma_k)} \prod_{l \neq k} \left| \frac{\gamma_l + \gamma_k}{\gamma_l - \gamma_k} \right| e^{2i\Theta_j}),$$

and therefore the wronskian

$$W_r(\phi_1, \dots, \phi_N)(0)$$

can be written as

$$\prod_{j=1}^{2N} e^{i\Theta_j|_{y=0}} (2)^{-2N} (i)^{\frac{2N(2N-2)}{2}} \prod_{j=2}^{2N} \prod_{i=1}^{j-1} (\gamma_j - \gamma_i) \prod_{j=1}^{2N} e^{-2i\Theta_j|_{y=0}} \det(I + D_r)$$

It follows that

$$\det(I + D_r) = \frac{e^{i \sum_{j=1}^{2N} \Theta_j|_{y=0}} 2^{2N}}{\prod_{j=2}^{2N} \prod_{i=1}^{j-1} (\gamma_j - \gamma_i)} W_r(\phi_1, \dots, \phi_{2N})(0) = k_r(0) W_r(\phi_1, \dots, \phi_{2N})(0). \quad (21)$$

□

So, the solution of NLS equation takes the form

$$v(x, t) = \frac{k_3(0)W_3(0)}{k_1(0)W_1(0)} \exp(2it - i\varphi) \quad (22)$$

3.2 Wronskian representation of solutions of NLS equation

From the previous section, we get the following result :

Theorem 3.2 *The function v defined by*

$$v(x, t) = \frac{W_3(\phi_1, \dots, \phi_{2N})(0)}{W_1(\phi_1, \dots, \phi_{2N})(0)} \exp(2it - i\varphi).$$

is solution of the NLS equation (1)

$$iv_t + v_{xx} + 2|v|^2v = 0,$$

where

$W_r(y) = W(\phi_1, \dots, \phi_{2N})$ is the wronskian $W_r(y) = \det[(\partial_y^{\mu-1} \phi_\nu)_{\nu, \mu \in [1, \dots, 2N]}]$, and ϕ_ν the functions given by

$$\phi_\nu(y) = \sin(\kappa_\nu(x - x_{0\nu})/2 + i\delta_\nu(t - t_{0\nu}) - ix_{r,\nu}/2 + \gamma_\nu y), \quad 1 \leq \nu \leq N,$$

$$\phi_\nu(y) = \cos(\kappa_\nu(x - x_{0\nu})/2 + i\delta_\nu(t - t_{0\nu}) - ix_{r,\nu}/2 + \gamma_\nu y), \quad N + 1 \leq \nu \leq 2N,$$

$\kappa_\nu, \delta_\nu, \gamma_\nu$ parameters defined by (4) for $\nu \in [1, \dots, 2N]$

$$\kappa_\nu = 2\sqrt{1 - \lambda_\nu^2}, \quad \delta_\nu = \kappa_\nu \lambda_\nu, \quad \gamma_\nu = \sqrt{\frac{1 - \lambda_\nu}{1 + \lambda_\nu}}, \quad \epsilon_\nu \in \{0; 1\},$$

where λ_ν is an arbitrary parameter such that $0 < \lambda_j < 1$ and $\lambda_{N+j} = -\lambda_j$ for $j \in [1, \dots, N]$.

4 Construction of quasi-rational solutions of NLS equation

4.1 Taking the limit when the parameters $\lambda_j \rightarrow 1$ for $1 \leq j \leq N$ and $\lambda_j \rightarrow -1$ for $N+1 \leq j \leq 2N$

In the following, we show how we can obtain quasi-rational solutions of NLS equation by a simple limiting procedure.

We consider the parameter λ_j written in the form $1 - \epsilon^2 c_j^2$. For simplicity, we denote d_j the term $\frac{c_j}{\sqrt{2}}$.

When ϵ goes to 0, we realize limited expansions at order p , for $1 \leq j \leq N$, of the terms

$$\begin{aligned}\kappa_j &= 4d_j\epsilon(1 - \epsilon^2 d_j^2)^{1/2}, \quad \delta_j = 4d_j\epsilon(1 - 2\epsilon^2 d_j^2)(1 - \epsilon^2 d_j^2)^{1/2}, \\ \gamma_j &= d_j\epsilon(1 - 2\epsilon^2 d_j^2)^{-1/2}, \quad x_{r,j} = (r-1)\ln(1 - 2d_j^2\epsilon^2 - 2id_j\epsilon(1 - \epsilon^2 d_j^2)^{1/2}), \\ \kappa_{N+j} &= 4d_j\epsilon(1 - \epsilon^2 d_j^2)^{1/2}, \quad \delta_{N+j} = -4d_j\epsilon(1 - 2\epsilon^2 d_j^2)(1 - \epsilon^2 d_j^2)^{1/2}, \\ \gamma_{N+j} &= 1/(d_j\epsilon)(1 - 2\epsilon^2 d_j^2)^{1/2}, \quad x_{r,N+j} = (r-1)\ln(1 - 2d_j^2\epsilon^2 + 2id_j\epsilon(1 - \epsilon^2 d_j^2)^{1/2}),\end{aligned}$$

For example, the expansions at order 1 gives :

$$\begin{aligned}\kappa_j &= 4d_j\epsilon + O(\epsilon^2), \quad \gamma_j = d_j\epsilon + O(\epsilon^2), \quad \delta_j = 4d_j\epsilon + O(\epsilon^2), \\ x_{r,j} &= (r-1)(2id_j\epsilon + O(\epsilon^2)), \\ \kappa_{N+j} &= 4d_j\epsilon + O(\epsilon^2), \quad \gamma_{N+j} = 1/(d_j\epsilon) - (d_j\epsilon)/2 + O(\epsilon^2), \quad \delta_{N+j} = -4d_j\epsilon + O(\epsilon^2), \\ x_{r,N+j} &= -(r-1)(2id_j\epsilon + O(\epsilon^2)), \\ 1 &\leq j \leq N.\end{aligned}$$

Then, we realize limited expansions at order p in ϵ of the functions $\phi_j^r(0)$ and $\phi_{N+j}^r(0)$, for $1 \leq j \leq N$:

$$\begin{aligned}\phi_j^1(0) &= \sin E_j = A_j + O(\epsilon^{p+1}), \\ \phi_j^3(0) &= \sin D_j = B_j + O(\epsilon^{p+1}), \\ \phi_{N+j}^1(0) &= \sin E'_j = A'_j + O(\epsilon^{p+1}), \\ \phi_{N+j}^3(0) &= \sin D'_j = B'_j + O(\epsilon^{p+1}),\end{aligned}$$

with

$$\begin{aligned}D_j &= \kappa_j(x - x_{0j})/2 + i\delta_j(t - t_{0j}) - ix_j/2, \\ E_j &= \kappa_j(x - x_{0j})/2 + i\delta_j(t - t_{0j}), \\ D'_j &= \kappa_j(x - x_{0j})/2 - i\delta_j(t - t_{0j}) + ix_j/2, \\ E'_j &= \kappa_j(x - x_{0j})/2 - i\delta_j(t - t_{0j}).\end{aligned}$$

Then according to the relation

$$v(x, t) = \frac{W_3(\phi_1, \dots, \phi_N)(0)}{W_1(\phi_1, \dots, \phi_N)(0)} \exp(2it - i\varphi).$$

the solution of NLS equation takes the form :

$$v(x, t) = \exp(2it - i\varphi) \frac{N(x, t, \epsilon)}{D(x, t, \epsilon)}$$

$$= \exp(2it - i\varphi) \frac{\begin{vmatrix} \sin D_1 & \dots & \sin D_N & \cos D'_1 & \dots & \cos D'_N \\ \gamma_1 \cos D_1 & \dots & \gamma_N \cos D_N & -\frac{1}{\gamma_1} \sin D'_1 & \dots & -\frac{1}{\gamma_N} \sin D'_N \\ \gamma_1^2 \sin D_1 & \dots & \gamma_N^2 \sin D_N & \frac{1}{\gamma_1^2} \cos D'_1 & \dots & \frac{1}{\gamma_N^2} \cos D'_N \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \gamma_1^{2N-1} \cos D_1 & \dots & \gamma_N^{2N-1} \cos D_N & -\frac{1}{\gamma_1^{2N-1}} \sin D'_1 & \dots & -\frac{1}{\gamma_N^{2N-1}} \sin D'_N \end{vmatrix}}{\begin{vmatrix} \sin E_1 & \dots & \sin E_N & \cos E'_1 & \dots & \cos E'_N \\ \gamma_1 \cos E_1 & \dots & \gamma_N \cos E_N & -\frac{1}{\gamma_1} \sin E'_1 & \dots & -\frac{1}{\gamma_N} \sin E'_N \\ \gamma_1^2 \sin E_1 & \dots & \gamma_N^2 \sin E_N & \frac{1}{\gamma_1^2} \cos E'_1 & \dots & \frac{1}{\gamma_N^2} \cos E'_N \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \gamma_1^{2N-1} \cos E_1 & \dots & \gamma_N^{2N-1} \cos E_N & -\frac{1}{\gamma_1^{2N-1}} \sin E'_1 & \dots & -\frac{1}{\gamma_N^{2N-1}} \sin E'_N \end{vmatrix}}$$

It can be rewritten as

$$\exp(2it - i\varphi) \frac{\begin{vmatrix} \sin D_1 & \dots & \sin D_N & \gamma_1^{2N-1} \cos D'_1 & \dots & \gamma_N^{2N-1} \cos D'_N \\ \gamma_1 \cos D_1 & \dots & \gamma_N \cos D_N & -\gamma_1^{2N-2} \sin D'_1 & \dots & -\gamma_N^{2N-2} \sin D'_N \\ \gamma_1^2 \sin D_1 & \dots & \gamma_N^2 \sin D_N & \gamma_1^{2N-3} \cos D'_1 & \dots & \gamma_N^{2N-3} \cos D'_N \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \gamma_1^{2N-1} \cos D_1 & \dots & \gamma_N^{2N-1} \cos D_N & -\sin D'_1 & \dots & -\sin D'_N \end{vmatrix}}{\begin{vmatrix} \sin E_1 & \dots & \sin E_N & \gamma_1^{2N-1} \cos E'_1 & \dots & \gamma_N^{2N-1} \cos E'_N \\ \gamma_1 \cos E_1 & \dots & \gamma_N \cos E_N & -\gamma_1^{2N-2} \sin E'_1 & \dots & -\gamma_N^{2N-2} \sin E'_N \\ \gamma_1^2 \sin E_1 & \dots & \gamma_N^2 \sin E_N & \gamma_1^{2N-3} \cos E'_1 & \dots & \gamma_N^{2N-3} \cos E'_N \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \gamma_1^{2N-1} \cos E_1 & \dots & \gamma_N^{2N-1} \cos E_N & -\sin E'_1 & \dots & -\sin E'_N \end{vmatrix}}$$

We can replace each term in these determinants by their limited expansion at order p . The coefficients of each power of ϵ of each development is a polynomial in x and t . The relations $D_j = E_j - ix_j/2$ and $D'_j = E'_j + ix_j/2$,

and the addition formulas for sine and cosine functions, for $1 \leq j \leq N$, proves that after simplifications, we have same smallest power of ϵ . So it is clear that after simplifications, the limit is quotient of two polynomials when ϵ goes to 0.

Remark 4.1 *Using the fact that $D_j = E_j + \alpha_j(\epsilon)$, $D'_j = E'_j - \alpha_j(\epsilon)$, and addition formulas for trigonometric functions, we can expand the numerator $N(x, t, \epsilon)$ along the columns; it takes the form of*

$$\prod_{j=1}^N \cos^2(\alpha_j(\epsilon)) D(x, t, \epsilon) + A.$$

We divide each term of this sum by $D(x, t, \epsilon)$.

We denote this expression by S .

When ϵ tends to 0 the first term tends to 1; in each other term of the sum, we can simplify each fraction by some power of ϵ to get a finite limit when ϵ goes to 0.

In fact, in the development of $N(x, t, \epsilon)$ (along the columns), we have a sum of 2^M quotients of determinants of order M . We don't have place here to give all the details of the proof. To have an idea of the proof, we treat the case $M = 2$.

In this case, the sum S takes the form

$$S = \frac{\cos^2(\alpha_1)}{D(x, t, \epsilon)} \begin{vmatrix} \sin E_1 & \cos E'_1 \\ \gamma_1 \cos E_1 & -\frac{1}{\gamma_1} \sin E'_1 \end{vmatrix} + \frac{\sin(\alpha_1) \cos(\alpha_1)}{D(x, t, \epsilon)} \begin{vmatrix} \sin E_1 & \sin E'_1 \\ \gamma_1 \cos E_1 & \frac{1}{\gamma_1} \cos E'_1 \end{vmatrix} \\ + \frac{\sin(\alpha_1) \cos(\alpha_1)}{D(x, t, \epsilon)} \begin{vmatrix} \cos E_1 & \cos E'_1 \\ -\gamma_1 \sin E_1 & -\frac{1}{\gamma_1} \sin E'_1 \end{vmatrix} + \frac{\sin^2(\alpha_1)}{D(x, t, \epsilon)} \begin{vmatrix} \cos E_1 & \sin E'_1 \\ -\gamma_1 \sin E_1 & \frac{1}{\gamma_1} \cos E'_1 \end{vmatrix},$$

with

$$D(x, t, \epsilon) = \begin{vmatrix} \sin E_1 & \cos E'_1 \\ \gamma_1 \cos E_1 & -\frac{1}{\gamma_1} \sin E'_1 \end{vmatrix}.$$

Using expansions in ϵ at order 1 of all the terms, we get

$$S = 1 - \frac{4(x + 2it)}{1 + 4x^2 + 16t^2} + \frac{4(x - 2it)}{1 + 4x^2 + 16t^2} - \frac{4}{1 + 4x^2 + 16t^2} = 1 - \frac{4(1 + 4it)}{1 + 4x^2 + 16t^2}.$$

The development of $N(x, t, \epsilon)$ to get the sum S gives directly the representation of Akhmediev for v under the form of 1 plus a quotient of two polynomials in x and t .

So we get the following result

Theorem 4.1 *The function v defined by*

$$v(x, t) = \lim_{\epsilon \rightarrow 0} \frac{W_3(\phi_1, \dots, \phi_{2N})(0)}{W_1(\phi_1, \dots, \phi_{2N})(0)} \exp(2it - i\varphi).$$

is a quasi-rational solution of the NLS equation (1)

$$iv_t + v_{xx} + 2|v|^2v = 0.$$

$W_r(y) = W(\phi_1, \dots, \phi_{2N})$ *is the wronskian*

$$W_r(y) = \det[(\partial_y^{\mu-1} \phi_\nu)_{\nu, \mu \in [1, \dots, 2N]}].$$

ϕ_ν *are the functions given by*

$$\begin{aligned} \phi_\nu(y) &= \sin(\kappa_\nu(x - x_{0\nu})/2 + i\delta_\nu(t - t_{0\nu}) - ix_{r,\nu}/2 + \gamma_\nu y), & 1 \leq \nu \leq N. \\ \phi_\nu(y) &= \cos(\kappa_\nu(x - x_{0\nu})/2 + i\delta_\nu(t - t_{0\nu}) - ix_{r,\nu}/2 + \gamma_\nu y), & N + 1 \leq \nu \leq 2N. \end{aligned}$$

$\kappa_\nu, \delta_\nu, \gamma_\nu$ *are parameters defined by (4) for $\nu \in [1, \dots, 2N]$*

$$\kappa_\nu = 2\sqrt{1 - \lambda_\nu^2}, \quad \delta_\nu = \kappa_\nu \lambda_\nu, \quad \gamma_\nu = \sqrt{\frac{1 - \lambda_\nu}{1 + \lambda_\nu}}, \quad \epsilon_\nu \in \{0; 1\}.$$

$\lambda_j = 1 - \epsilon^2 c_j^2$ *is an arbitrary parameter such that $0 < \lambda_j < 1$ and $\lambda_{N+j} = -\lambda_j$, c_j an arbitrary real number, for $j \in [1, \dots, N]$.*

4.2 Quasi-rational solutions of order N

For brevity, in the following we choose $x_{0i} = 0$, $t_{0i} = 0$ and $\varphi = 0$ for $1 \leq i \leq N$.

4.2.1 Case $N=1$

If we consider the case $N = 1$, from (22), we realize a development at order 1 of W_3 and W_1 in ϵ . The solution of NLS equation can be written as

$$v(x, t) = -\frac{(-4x^2 - 16t^2 + 16it + 3)}{4x^2 + 16t^2 + 1} \exp(2it).$$

Then v takes the form

$$v(x, t) = 1 - 4 \frac{1 + 4it}{1 + 16t^2 + 4x^2} \exp(2it).$$

If we make the following changes of variables $t \rightarrow X/2$ and $x \rightarrow T$, we get exactly first order Akhmediev's solution (see [1]), solution of $iu_x + \frac{1}{2}u_{tt} + u|u|^2 = 0$,

$$v(x, t) = \left(1 - \frac{4(1 + 2ix)}{1 + 4x^2 + 4t^2}\right) \exp(ix).$$

We represent here the modulus of v in function of $x \in [-5; 5]$ and $t \in [-5; 5]$; we get the following graphic :

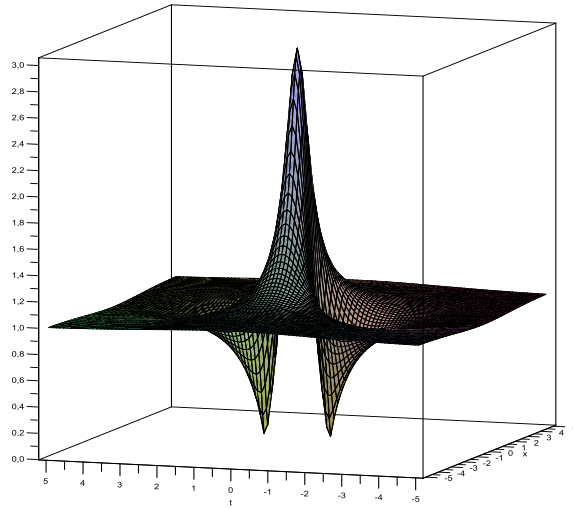


Figure 1: Solution of NLS, N=1.

4.2.2 Case N=2

In the case $N = 2$, we must realize a development at order 3 in ϵ . From (22), the solution of NLS equation can be written as

$$v(x, t) = \frac{N(x, t)}{D(x, t)} \exp(2it),$$

with

$$\begin{aligned} N(x, t) = & 5760x^2t^2 - 3072x^2t^4 + 1872t^2 - 64x^6 - 45 - 4096t^6 + 180x^2 + 144x^4 - 768x^4t^2 \\ & + 768ix^4t + 8448t^4 - 1152ix^2t + 1536it^3 - 720it + 12288it^5 + 6144it^3x^2, \end{aligned}$$

and

$$D(x, t) = -1152x^2t^2 + 3072x^2t^4 + 1584t^2 + 64x^6 + 9 + 4096t^6 + 108x^2 + 48x^4 + 768x^4t^2 + 6912t^4.$$

If we make the following changes of variables $t \rightarrow X/2$ and $x \rightarrow T$, we get

$$v(x, t) = \frac{N_1(x, t)}{D_1(x, t)} \exp(ix),$$

with

$$\begin{aligned} N_1(x, t) = & 1440x^2t^2 - 192x^4t^2 + 468x^2 - 64t^6 - 45 - 64x^6 + 180t^2 + 144t^4 - 192x^2t^4 - 576it^2x \\ & + 528x^4 + 384ix^5 + 192ix^3 - 360ix + 384it^4x + 768ix^3t^2, \end{aligned}$$

and

$$D_1(x, t) = -288x^2t^2 + 192x^4t^2 + 396x^2 + 64t^6 + 9 + 64x^6 + 108t^2 + 48t^4 + 192x^2t^4 + 432x^4.$$

It can be reduced exactly at the second order Akhmediev's solution (solution of $iu_x + \frac{1}{2}u_{tt} + u|u|^2 = 0$) (see [1]).

$$v(x, t) = 1 + \frac{3/8 - 3t^2 - 2t^4 - 9x^2 - 10x^4 - 12t^2x^2 + ix(15/4 + 6t^2 - 4t^4 - 2x^2 - 4x^4 - 8t^2x^2)}{3/32 + 9/8t^2 + 1/2t^4 + 2/3t^6 + 33/8x^2 + 9/2x^4 + 2/3x^6 - 3t^2x^2 + 2t^4x^2 + 2t^2x^4} \exp(ix).$$

We represent the modulus of v in function of $x \in [-5; 5]$ and $t \in [-5; 5]$; we get the following graphic :

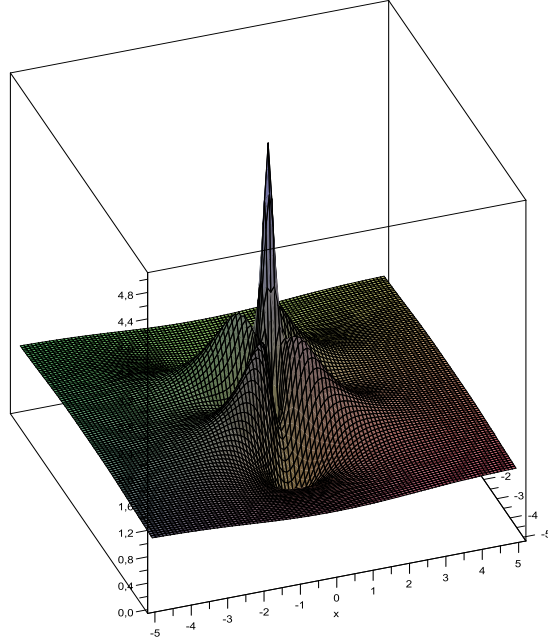


Figure 2: Solution of NLS, N=2.

4.2.3 Case N=3

In the case $N = 3$, we must realize a development at order 5 in ϵ . If we make the following changes of variables $t \rightarrow X/2$ and $x \rightarrow T$, from (22), the solution of NLS equation in the formulation of Akhmediev $iv_x + \frac{1}{2}v_{tt} + v|v|^2 = 0$ can be written as

$$v(x, t) = \frac{N(x, t)}{D(x, t)} \exp(ix),$$

with

$$\begin{aligned} N(x, t) = & 14175 + 49152i t^{10} x - 737280i t^2 x^7 + 5253120i t^2 x^5 + 907200i t^2 x - 1720320i t^6 x^3 - 184320i t^8 x \\ & + 5356800x^4 t^2 - 576000x^4 t^4 - 875520x^6 t^2 - 172800t^4 x^2 - 4600800x^2 t^2 + 829440x^8 t^2 - 24576x^{10} t^2 - 61440x^8 t^4 \\ & + 460800t^8 x^2 - 506880x^2 t^6 + 1658880x^6 t^4 - 4838400i x^3 t^2 - 81920x^6 t^6 + 1413120x^4 t^6 - 61440x^4 t^8 - 24576x^2 t^{10} \\ & - 5238000x^4 - 567000x^2 - 226800t^4 + 2384640x^6 + 172800t^6 + 2085120x^8 + 141312x^{10} + 57600t^8 \\ & - 113400t^2 - 4096x^{12} - 4096t^{12} + 18432t^{10} - 1382400i t^4 x^3 - 1036800i t^4 x - 460800i t^6 x - 2580480i t^4 x^5 \end{aligned}$$

$$+491520i t^6 x^5 + 491520i t^4 x^7 + 245760i t^8 x^3 + 245760i t^2 x^9 + 307200i x^9 - 475200i x^3 - 2672640i x^7 + 49152i x^{11} \\ - 7395840i x^5 + 226800i x$$

and

$$D(x, t) = 2025 + 5011200x^4 t^2 - 345600x^4 t^4 + 3456000x^6 t^2 + 864000t^4 x^2 - 583200x^2 t^2 + 276480x^8 t^2 + 24576x^{10} t^2 \\ + 61440x^8 t^4 - 92160t^8 x^2 + 138240x^2 t^6 + 61440x^6 t^4 + 81920x^6 t^6 - 184320x^4 t^6 + 61440x^4 t^8 + 24576x^2 t^{10} \\ + 2300400x^4 + 372600x^2 + 54000t^4 + 979200x^6 + 149760t^6 + 956160x^8 + 129024x^{10} + 34560t^8 \\ + 48600t^2 + 4096x^{12} + 4096t^{12} + 6144t^{10}.$$

We get the following graphic for the modulus of v in function of $x \in [-5; 5]$ and $t \in [-5; 5]$:

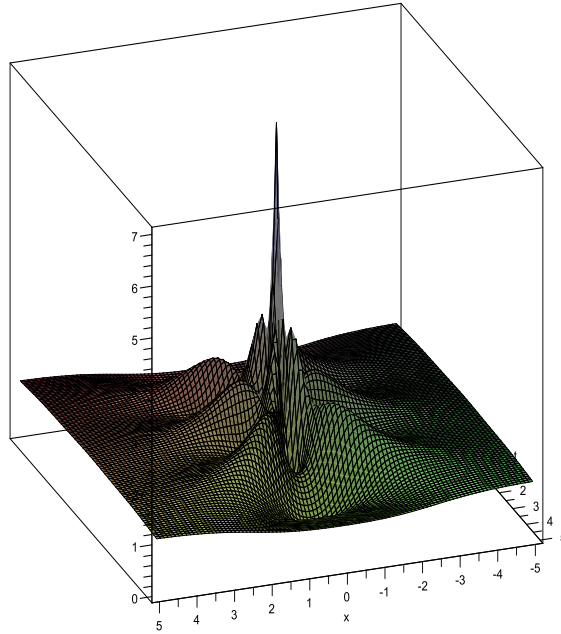


Figure 3: Solution of NLS, N=3.

An another type of solution different from this one was given recently by Akhmediev [1].

The results presented here are in accordance with the preceding work [6];

here we give moreover an explicit expression of the solutions of NLS equation for the order 3.

4.3 Case N=4

The solution of NLS equation $iv_x + \frac{1}{2}v_{tt} + v|v|^2 = 0$ can be written as

$$v(x, t) = \frac{N(x, t)}{D(x, t)} \exp(ix),$$

with N and D polynomials both of degree 20. Their expressions are too long to be reproduced here.

We get the following graphic for the modulus of v in function of $x \in [-5; 5]$ and $t \in [-5; 5]$:

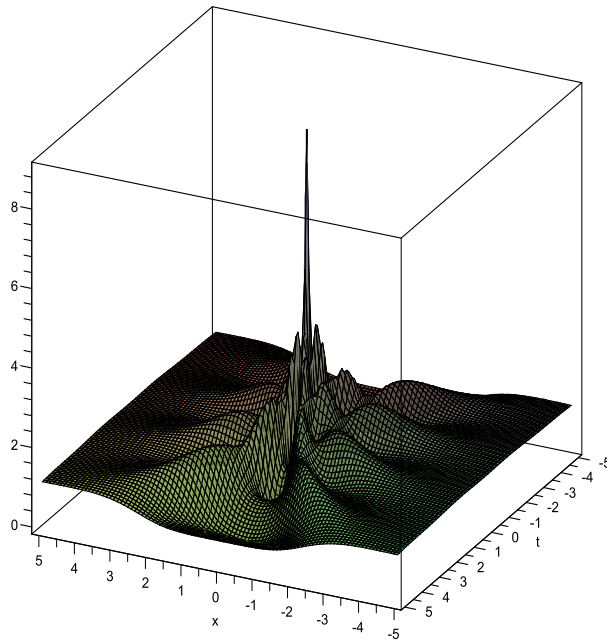


Figure 4: Solution of NLS, N=4.

We recover a result of Akhmediev formulated in [1] in the case where $x = 0$. Here we give the complete solution in x and t . As my knowledge, it is the first time that this solution is given.

This method can be extended to get an infinite family of quasi-rational solutions of NLS equation at any order.

5 Conclusion

Here is given a new formulation of solutions of NLS equation that gives as particular case, Akhmediev's solutions. These solutions expressed as a quotient of wronskians are similar to a previous result of Eleonski, Krichever and Kulagin [8]. It has been rewritten recently and their formulation can also be seen as a quotient of two wronkians (see [6]); the method describe in [6] can be used to get solutions of NLS equation, but it is more complicated to find explicit solution in x and t ; the computation gives complicated results. This method described in the present paper provides a powerfull tool to get explicitly solutions in analytical form. This new formulation gives an infinite set of non singular solution of NLS equation.

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