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# Quasi-rational solutions of the NLS equation and rogue waves

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## Abstract

This article contains two basic results. First we derive a wronskian representation for the well known multi phase  $N$ -periodic solutions to the focusing NLS equation. Next we perform a special passage to the limit when all the periods tend to infinity.

We claim that this provides a new systematic approach for constructing the so called higher Peregrine breathers labeled by the positive integer  $N$ . We give explicit representations of all higher Peregrine breathers for  $N = 1, 2, 3, 4$  and speculate about further extensions.

## 1 Introduction

The nonlinear Schrödinger equation (NLS) equation plays a fundamental role in hydrodynamics, in particular for waves in deep water in the context of the rogue waves or in non-linear optics.

The basic work on the subject is due to Zakharov in 1969 [12]. The first solution, actually called breather, was discovered by Peregrine in 1983 [11]; it was the simplest solution of NLS founded.

The second order Peregrine-like solution was first constructed by Akhmediev et al. in [3].

Same families of higher order were constructed in a series of articles by Akhmediev et al. [1, 2] using Darboux transformations.

Other Peregrine-like solutions were found for reduced self-induced transparency (SIT) integrable systems [10]. In [9], the  $N$ -phase quasi-periodic

modulations of the plane waves solutions were constructed via appropriate degeneration of the finite gap periodic solutions to the NLS equation. These results in principle provided a way to construct multi-parametric families of multi-Peregrine solutions.

Very recently, it is shown in [6] that rational solutions of NLS equation can be written as a quotient of two wronskians using modified version of [8], with some different reasoning; moreover, the link between quasi-rational solutions of the focusing NLS equation and the rational solution of the KP-I equation is established.

With this formulation we recover as particular case, Akhmediev's quasi-rational solutions of NLS equation.

In this paper, we will give a new representation of the solutions of the NLS equation in terms of wronskians. The solutions take the form of a quotient of two wronskians of even order  $2N$  of elementary functions depending on a certain number of parameters.

We will call these related solutions, solutions of NLS of order  $N$ .

Then, to get quasi-rational solutions of NLS equation, we take the limit when some parameter goes to 0.

For  $N = 1$ , we recover the well known Peregrine's solution [11] of the focusing NLS equation. For  $N = 2, 3$ , we recover Akhmediev's breathers. In the case  $N = 4$ , we give explicit analytic solution of NLS with corresponding graphic of the modulus of the solution in the  $(x, t)$  coordinates.

Here we give a new approach to get higher order Peregrine solutions different from all previous works.

## 2 Expression of solutions of NLS equation in terms of Fredholm determinant

### 2.1 Solutions of NLS equation in terms of $\theta$ functions

The solution of the NLS equation

$$iv_t + v_{xx} + 2|v|^2v = 0, \tag{1}$$

is given in terms of truncated theta function by (see [9])

$$v(x, t) = \frac{\theta_3(x, t)}{\theta_1(x, t)} \exp(2it - i\varphi). \tag{2}$$

The functions  $\theta_r(x, t)$ , ( $r = 1, 3$ ) are the functions defined by

$$\theta_r(x, t) = \sum_{k \in \{0,1\}^{2N}} g_{r,k} \quad (3)$$

with  $g_{r,k}$  given by

$$g_{r,k} = \exp \left\{ \sum_{\mu > \nu, \mu, \nu=1}^{2N} \ln \left( \frac{\gamma_\nu - \gamma_\mu}{\gamma_\nu + \gamma_\mu} \right)^2 k_\mu k_\nu \right. \\ \left. + \left( \sum_{\nu=1}^{2N} i \kappa_\nu x - 2\delta_\nu t + (r-1) \ln \frac{\gamma_\nu - i}{\gamma_\nu + i} + \sum_{\mu=1, \mu \neq \nu}^{2N} \ln \left| \frac{\gamma_\nu + \gamma_\mu}{\gamma_\nu - \gamma_\mu} \right| + \pi i \epsilon_\nu \right) k_\nu \right\}. \quad (4)$$

The solutions depend on a certain number of parameters :

$\varphi$ ;

$N$  parameters  $\lambda_j$ , satisfying the relations

$$0 < \lambda_j < 1, \quad \lambda_{N+j} = -\lambda_j, \quad 1 \leq j \leq N; \quad (5)$$

The terms  $\epsilon_\nu$ ,  $1 \leq \nu \leq 2N$  are arbitrary numbers equal to 0 or 1.

In the preceding formula, the terms  $\kappa_\nu$ ,  $\delta_\nu$ ,  $\gamma_\nu$  are functions of the parameters  $\lambda_\nu$ ,  $\nu = 1, \dots, 2N$ , and they are given by the following equations,

$$\kappa_\nu = 2\sqrt{1 - \lambda_\nu^2}, \quad \delta_\nu = \kappa_\nu \lambda_\nu, \quad \gamma_\nu = \sqrt{\frac{1 - \lambda_\nu}{1 + \lambda_\nu}}. \quad (6)$$

We also note that

$$\kappa_{N+j} = \kappa_j, \quad \delta_{N+j} = -\delta_j, \quad \gamma_{N+j} = 1/\gamma_j, \quad j = 1 \dots N. \quad (7)$$

## 2.2 Relation between $\theta$ and Fredholm determinant

It was already mentioned in [9] that the function  $\theta_r$  defined in (3) can be rewritten as a Fredholm determinant.

The expression given in [9] is different from which we need in the following. We need different choices of  $\epsilon_\nu$ . It is the crucial point to get quasi rational solutions of NLS.

Here, we make the following choices :

$$\epsilon_\nu = 0, \quad 1 \leq \nu \leq N \\ \epsilon_\nu = 1, \quad N + 1 \leq \nu \leq 2N. \quad (8)$$

The function  $\theta_r$  defined in (3) can be rewritten with a summation in terms of subsets of  $[1, \dots, 2N]$

$$\begin{aligned} \theta_r(x, t) = & \sum_{J \subset \{1, \dots, 2N\}} \prod_{\nu \in J} (-1)^{\epsilon_\nu} \prod_{\nu \in J, \mu \notin J} \left| \frac{\gamma_\nu + \gamma_\mu}{\gamma_\nu - \gamma_\mu} \right| \\ & \times \exp\left\{ \sum_{\nu \in J} i\kappa_\nu x - 2\delta_\nu t + x_{r,\nu} \right\}, \end{aligned}$$

with

$$x_{r,\nu} = (r-1) \ln \frac{\gamma_\nu - i}{\gamma_\nu + i}, \quad 1 \leq j \leq 2N, \quad (9)$$

in particular

$$\begin{aligned} x_{r,j} &= (r-1) \ln \frac{\gamma_j - i}{\gamma_j + i}, \quad 1 \leq j \leq N, \\ x_{r,N+j} &= -(r-1) \ln \frac{\gamma_j - i}{\gamma_j + i} - (r-1)i\pi, \quad 1 \leq j \leq N. \end{aligned} \quad (10)$$

We consider  $A_r = (a_{\nu\mu})_{1 \leq \nu, \mu \leq 2N}$  the matrix defined by

$$a_{\nu\mu} = (-1)^{\epsilon_\nu} \prod_{\lambda \neq \mu} \left| \frac{\gamma_\lambda + \gamma_\nu}{\gamma_\lambda - \gamma_\mu} \right| \exp(i\kappa_\nu x - 2\delta_\nu t + x_{r,\nu}). \quad (11)$$

Then  $\det(I + A_r)$  has the following form

$$\det(I + A_r) = \sum_{J \subset \{1, \dots, 2N\}} \prod_{\nu \in J} (-1)^{\epsilon_\nu} \prod_{\nu \in J, \mu \notin J} \left| \frac{\gamma_\nu + \gamma_\mu}{\gamma_\nu - \gamma_\mu} \right| \exp(i\kappa_\nu x - 2\delta_\nu t + x_{r,\nu}). \quad (12)$$

From the beginning of this section,  $\tilde{\theta}$  has the same expression as in (12) so, we have clearly the equality

$$\theta_r = \det(I + A_r). \quad (13)$$

Then the solution of NLS equation takes the form

$$v(x, t) = \frac{\det(I + A_3(x, t))}{\det(I + A_1(x, t))} \exp(2it - i\varphi). \quad (14)$$

### 3 Expression of solutions of NLS equation in terms of wronkian determinant

#### 3.1 Link between Fredholm determinants and wronskians

We consider the following functions

$$\begin{aligned}\phi_\nu^r(y) &= \sin(\kappa_\nu x/2 + i\delta_\nu t - ix_{r,\nu}/2 + \gamma_\nu y), & 1 \leq \nu \leq N, \\ \phi_\nu^r(y) &= \cos(\kappa_\nu x/2 + i\delta_\nu t - ix_{r,\nu}/2 + \gamma_\nu y), & N+1 \leq \nu \leq 2N.\end{aligned}\quad (15)$$

For simplicity, in this section we denote them  $\phi_\nu(y)$ .

We use the following notations :

$$\Theta_\nu = \kappa_\nu x/2 + i\delta_\nu t - ix_{r,\nu}/2 + \gamma_\nu y, \quad 1 \leq \nu \leq 2N.$$

$W_r(y) = W(\phi_1, \dots, \phi_{2N})$  is the wronkian

$$W_r(y) = \det[(\partial_y^{\mu-1} \phi_\nu)_{\nu, \mu \in [1, \dots, 2N]}]. \quad (16)$$

We consider the matrix  $D_r = (d_{\nu\mu})_{\nu, \mu \in [1, \dots, 2N]}$  defined by

$$\begin{aligned}d_{\nu\mu} &= (-1)^{\epsilon_\nu} \prod_{\lambda \neq \mu} \left| \frac{\gamma_\lambda + \gamma_\nu}{\gamma_\lambda - \gamma_\mu} \right| \exp(i\kappa_\nu x - 2\delta_\nu t + x_{r,\nu}), \\ &1 \leq \nu \leq 2N, \quad 1 \leq \mu \leq 2N,\end{aligned}$$

with

$$x_{r,\nu} = (r-1) \ln \frac{\gamma_\nu - i}{\gamma_\nu + i}.$$

Then we have the following statement

**Theorem 3.1**

$$\det(I + D_r) = k_r(0) \times W_r(\phi_1, \dots, \phi_{2N})(0), \quad (17)$$

where

$$k_r(y) = \frac{2^{2N} \exp(i \sum_{\nu=1}^{2N} \Theta_\nu)}{\prod_{\nu=2}^{2N} \prod_{\mu=1}^{\nu-1} (\gamma_\nu - \gamma_\mu)}.$$

**Proof :** We start to remove the factor  $(2i)^{-1} e^{i\Theta_\nu}$  in each row  $\nu$  in the wronkian  $W_r(y)$  for  $1 \leq \nu \leq 2N$ .

Then

$$W_r = \prod_{\nu=1}^{2N} e^{i\Theta_\nu} (2i)^{-N} (2)^{-N} \times W_{r1}, \quad (18)$$

with

$$W_{r1} = \begin{vmatrix} (1 - e^{-2i\Theta_1}) & i\gamma_1(1 + e^{-2i\Theta_1}) & \dots & (i\gamma_1)^{2N-1}(1 + (-1)^{2N}e^{-2i\Theta_1}) \\ (1 - e^{-2i\Theta_2}) & i\gamma_2(1 + e^{-2i\Theta_2}) & \dots & (i\gamma_2)^{2N-1}(1 + (-1)^{2N}e^{-2i\Theta_2}) \\ \vdots & \vdots & \vdots & \vdots \\ (1 - e^{-2i\Theta_{2N}}) & i\gamma_{2N}(1 + e^{-2i\Theta_{2N}}) & \dots & (i\gamma_{2N})^{2N-1}(1 + (-1)^{2N}e^{-2i\Theta_{2N}}) \end{vmatrix}$$

The determinant  $W_{r1}$  can be written as

$$W_{r1} = \det(\alpha_{jk}e_j + \beta_{jk}),$$

where  $\alpha_{jk} = (-1)^k(i\gamma_j)^{k-1}$ ,  $e_j = e^{-2i\Theta_j}$ , and  $\beta_{jk} = (i\gamma_j)^{k-1}$ ,  $1 \leq j \leq N$ ,  $1 \leq k \leq 2N$ ,

$\alpha_{jk} = (-1)^{k-1}(i\gamma_j)^{k-1}$ ,  $e_j = e^{-2i\Theta_j}$ , and  $\beta_{jk} = (i\gamma_j)^{k-1}$ ,  $N+1 \leq j \leq 2N$ ,  $1 \leq k \leq 2N$ .

Denoting  $U = (\alpha_{ij})_{i,j \in [1, \dots, 2N]}$ ,  $V = (\beta_{ij})_{i,j \in [1, \dots, 2N]}$ , the determinant of  $U$  is clearly equal to

$$\det(U) = (i)^{\frac{2N(2N-1)}{2}} \prod_{2N \geq l > m \geq 1} (\gamma_l - \gamma_m). \quad (19)$$

Then we use the following Lemma

**Lemma 3.1** *Let  $A = (a_{ij})_{i,j \in [1, \dots, N]}$ ,  $B = (b_{ij})_{i,j \in [1, \dots, N]}$ ,  $(H_{ij})_{i,j \in [1, \dots, N]}$ , the matrix formed by replacing the  $j$ th row of  $A$  by the  $i$ th row of  $B$*   
Then

$$\det(a_{ij}x_i + b_{ij}) = \det(a_{ij}) \times \det(\delta_{ij}x_i + \frac{\det(H_{ij})}{\det(a_{ij})}) \quad (20)$$

**Proof :** For  $\tilde{A} = (\tilde{a}_{ji})_{i,j \in [1, \dots, N]}$  the transposed matrix in cofactors of  $A$ , we have the well known formula  $A \times^t \tilde{A} = \det A \times I$ .

So it is clear that  $\det(\tilde{A}) = (\det(A))^{N-1}$ .

The general term of the product  $(c_{ij})_{i,j \in [1, \dots, N]} = (a_{ij}x_i + b_{ij})_{i,j \in [1, \dots, N]} \times (\tilde{a}_{ji})_{i,j \in [1, \dots, N]}$  can be written as

$$\begin{aligned} c_{ij} &= \sum_{s=1}^N (a_{is}x_i + b_{is}) \times \tilde{a}_{js} \\ &= x_i \sum_{s=1}^N a_{is}\tilde{a}_{js} + \sum_{s=1}^N b_{is}\tilde{a}_{js} \\ &= \delta_{ij} \det(A)x_i + \det(H_{ij}). \end{aligned}$$

We get

$$\det(c_{ij}) = \det(a_{ij}x_i + b_{ij}) \times (\det(A))^{N-1} = (\det(A))^N \times \det(\delta_{ij}x_i + \frac{\det(H_{ij})}{\det(A)}).$$

$$\text{Thus } \det(a_{ij}x_i + b_{ij}) = \det(A) \times \det(\delta_{ij}x_i + \frac{\det(H_{ij})}{\det(A)}).$$

□

Using the previous lemma (20), we get :

$$\det(\alpha_{ij}e_i + \beta_{ij}) = \det(\alpha_{ij}) \times \det(\delta_{ij}e_i + \frac{\det(H_{ij})}{\det(\alpha_{ij})}),$$

where  $(H_{ij})_{i,j \in [1, \dots, N]}$  is the matrix formed by replacing the  $j$ th row of  $U$  by the  $i$ th row of  $V$  defined previously.

We compute  $\det(H_{ij})$  and we get

$$\det(H_{jk}) = (-1)^{E_k(i)} \frac{2^{N(2N-1)}}{2} \prod_{2N \geq l > m \geq 1, l \neq k, m \neq k} (\gamma_l - \gamma_m) \prod_{l \neq k} (\gamma_l + \gamma_j), \quad (21)$$

with  $E_k = k - 1$  if  $1 \leq k \leq N$  and  $E_k = k$  if  $N + 1 \leq k \leq 2N$ .

We can simplify the quotient  $q = \frac{\det(H_{jk})}{\det(\alpha_{jk})}$  :

$$q = \frac{(-1)^{\epsilon(k)} \prod_{l \neq k} (\gamma_l + \gamma_j)}{\prod_{l \neq k} (\gamma_l - \gamma_k)}. \quad (22)$$

So  $\det(\delta_{jk}e_j + \frac{\det(H_{jk})}{\det(\alpha_{jk})})$  can be expressed as

$$\det(\delta_{jk}e_j + \frac{\det(H_{jk})}{\det(\alpha_{jk})}) = \prod_{j=1}^{2N} e^{-2i\Theta_j} \det(\delta_{jk} + (-1)^{\epsilon(k)} \prod_{l \neq k} \left| \frac{\gamma_l + \gamma_j}{\gamma_l - \gamma_k} \right| e^{2i\Theta_j}).$$

Then the wronskian

$$W_r(\phi_1, \dots, \phi_N)(0)$$

can be written as

$$\prod_{j=1}^{2N} e^{i\Theta_j|_{y=0}} (2)^{-2N} (i)^{\frac{2N(2N-2)}{2}} \prod_{j=2}^{2N} \prod_{i=1}^{j-1} (\gamma_j - \gamma_i) \prod_{j=1}^{2N} e^{-2i\Theta_j|_{y=0}} \det(I + D_r)$$

It follows that

$$\det(I + D_r) = \frac{e^{i \sum_{j=1}^{2N} \Theta_j|_{y=0}} 2^{2N}}{\prod_{j=2}^{2N} \prod_{i=1}^{j-1} (\gamma_j - \gamma_i)} W_r(\phi_1, \dots, \phi_{2N})(0) = k_r(0) W_r(\phi_1, \dots, \phi_{2N})(0). \quad (23)$$

□

So, the solution of NLS equation takes the form

$$v(x, t) = \frac{W_3(0)}{W_1(0)} \exp(2it - i\varphi) \quad (24)$$



## 3.2 Wronskian representation of solutions of NLS equation

From the previous section, we get the following result :

**Theorem 3.2** *The function  $v$  defined by*

$$v(x, t) = \frac{W_3(0)}{W_1(0)} \exp(2it - i\varphi). \quad (25)$$

*is solution of the NLS equation (1)*

$$iv_t + v_{xx} + 2|v|^2v = 0.$$

**Remark 3.1** *In formula (25),  $W_r(y)$  is the wronskian defined in (16); the functions  $\phi_\nu^r$  are given by (15);  $\kappa_\nu$ ,  $\delta_\nu$ ,  $\gamma_\nu$  are defined by (6);  $\lambda_\nu$  are arbitrary parameters given by (5).*

## 4 Construction of quasi-rational solutions of NLS equation

### 4.1 Taking the limit when the parameters $\lambda_j \rightarrow 1$ for $1 \leq j \leq N$ and $\lambda_j \rightarrow -1$ for $N + 1 \leq j \leq 2N$

In the following, we show how we can obtain quasi-rational solutions of NLS equation by a rough estimation.

We consider the parameter  $\lambda_j$  written in the form  $1 - \epsilon^2 c_j^2$ . For simplicity, we denote  $d_j$  the term  $\frac{c_j}{\sqrt{2}}$ .

When  $\epsilon$  goes to 0, we realize limited expansions at order  $2N - 1$  of  $\kappa_j$ ,  $\gamma_j$ ,  $\delta_j$ ,  $x_{r,j}$ ,  $\kappa_{N+j}$ ,  $\gamma_{N+j}$ ,  $\delta_{N+j}$ ,  $x_{r,N+j}$ , for  $1 \leq j \leq N$ . For example, in the case where  $N = 1$ , it gives :

$$\begin{aligned} \kappa_j &= 4d_j\epsilon + O(\epsilon^2), & \gamma_j &= d_j\epsilon + O(\epsilon^2), & \delta_j &= 4d_j\epsilon + O(\epsilon^2), \\ x_{r,j} &= (r-1)(2id_j\epsilon + O(\epsilon^2)), \\ \kappa_{N+j} &= 4d_j\epsilon + O(\epsilon^2), & \gamma_{N+j} &= 1/(d_j\epsilon) - (d_j\epsilon)/2 + O(\epsilon^2), & \delta_{N+j} &= -4d_j\epsilon + O(\epsilon^2), \\ x_{r,N+j} &= -(r-1)(2id_j\epsilon + O(\epsilon^2)), \\ & 1 \leq j \leq N. \end{aligned}$$

Then we realize limited expansions at order  $2N - 1$  of the functions  $\phi_j^r(0)$  and  $\phi_{N+j}^r(0)$ , for  $1 \leq j \leq N$  :

$$\begin{aligned}\phi_j^1(0) &= \sin E_j, \\ \phi_j^3(0) &= \sin D_j, \\ \phi_{N+j}^1(0) &= \sin E'_j, \\ \phi_{N+j}^3(0) &= \sin D'_j,\end{aligned}$$

with

$$\begin{aligned}D_j &= K_j(x - x_{0j})/2 + i\delta_j(t - t_{0j}) - ix_j/2, \\ E_j &= K_j(x - x_{0j})/2 + i\delta_j(t - t_{0j}), \\ D'_j &= K_j(x - x_{0j})/2 - i\delta_j(t - t_{0j}) + ix_j/2, \\ E'_j &= K_j(x - x_{0j})/2 - i\delta_j(t - t_{0j}).\end{aligned}$$

For example, in the case where  $N = 1$ , it gives :

$$\begin{aligned}D_j &= 2\epsilon d_j(x - x_{0j} + 2i(t - t_{0j}) + 1) + O(\epsilon^2), \\ E_j &= 2\epsilon d_j(x - x_{0j} + 2i(t - t_{0j})) + O(\epsilon^2), \\ D'_j &= 2\epsilon d_j(x - x_{0j} - 2i(t - t_{0j}) - 1) + O(\epsilon^2), \\ E'_j &= 2\epsilon d_j((x - x_{0j} - 2i(t - t_{0j})) + O(\epsilon^2)),\end{aligned}$$

and

$$\begin{aligned}\phi_j^1(0) &= 2\epsilon d_j(x - x_{0j} + 2i(t - t_{0j})) + O(\epsilon^2), \\ \phi_j^3(0) &= 2\epsilon d_j(x - x_{0j} + 2i(t - t_{0j}) + 1) + O(\epsilon^2), \\ \phi_{N+j}^1(0) &= 1 + O(\epsilon^2), \\ \phi_{N+j}^3(0) &= 1 + O(\epsilon^2).\end{aligned}$$

Then according to the relation

$$v(x, t) = \frac{W_3(\phi_1, \dots, \phi_N)(0)}{W_1(\phi_1, \dots, \phi_N)(0)} \exp(2it - i\varphi).$$

the solution of NLS equation takes the form :

$$v(x, t) = \exp(2it - i\varphi) \frac{N(x, t, \epsilon)}{D(x, t, \epsilon)}$$

$$= \exp(2it - i\varphi) \begin{array}{c} \left| \begin{array}{cccccc} \sin D_1 & \dots & \sin D_N & \cos D'_1 & \dots & \cos D'_N \\ \gamma_1 \cos D_1 & \dots & \gamma_N \cos D_N & -\frac{1}{\gamma_1} \sin D'_1 & \dots & -\frac{1}{\gamma_N} \sin D'_N \\ \gamma_1^2 \sin D_1 & \dots & \gamma_N^2 \sin D_N & \frac{1}{\gamma_1^2} \cos D'_1 & \dots & \frac{1}{\gamma_N^2} \cos D'_N \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \gamma_1^{2N-1} \cos D_1 & \dots & \gamma_N^{2N-1} \cos D_N & -\frac{1}{\gamma_1^{2N-1}} \sin D'_1 & \dots & -\frac{1}{\gamma_N^{2N-1}} \sin D'_N \end{array} \right| \\ \hline \left| \begin{array}{cccccc} \sin E_1 & \dots & \sin E_N & \cos E'_1 & \dots & \cos E'_N \\ \gamma_1 \cos E_1 & \dots & \gamma_N \cos E_N & -\frac{1}{\gamma_1} \sin E'_1 & \dots & -\frac{1}{\gamma_N} \sin E'_N \\ \gamma_1^2 \sin E_1 & \dots & \gamma_N^2 \sin E_N & \frac{1}{\gamma_1^2} \cos E'_1 & \dots & \frac{1}{\gamma_N^2} \cos E'_N \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \gamma_1^{2N-1} \cos E_1 & \dots & \gamma_N^{2N-1} \cos E_N & -\frac{1}{\gamma_1^{2N-1}} \sin E'_1 & \dots & -\frac{1}{\gamma_N^{2N-1}} \sin E'_N \end{array} \right| \end{array}$$

We use the classical developments

$$\sin z = \sum_{j=1}^N \frac{(-1)^{j-1} z^{2j-1}}{(2j-1)!} + O(z^{2N}) \text{ and } \cos z = \sum_{j=0}^{2N} \frac{(-1)^j z^{2j}}{(2j)!} + O(z^{2N+1}),$$

for  $z$  in a neighborhood of 0.

Using the fact that  $D_j = E_j + \alpha_j(\epsilon)$ ,  $D'_j = E'_j - \alpha_j(\epsilon)$ , and addition formulas for trigonometric functions, we can expand the numerator  $N(x, t, \epsilon)$  along the columns; it takes the form of

$$\prod_{j=1}^N \cos^2(\alpha_j(\epsilon)) D(x, t, \epsilon) + A.$$

If we divide each term of this sum by  $D(x, t)$ , when  $\epsilon$  tends to 0, the first term tends to 1; in each other term of the sum of A, we can multiply each fraction by some power of  $\gamma_j$  to get a finite limit when  $\epsilon$  goes to 0.

So we get really a non trivial and non degenerate quotient of two polynomials in  $x$  and  $t$  by an exponential.

Practically, we keep only the terms of lower degree in  $\epsilon$ , in numerator and denominator which expressed as powers of  $x$  and  $t$

So we get in this way a quasi-rational solution of the NLS equation.

So we get the following result

**Theorem 4.1** *The function  $v$  defined by*

$$v(x, t) = \lim_{\epsilon \rightarrow 0} \frac{W_3(\phi_1, \dots, \phi_{2N})(0)}{W_1(\phi_1, \dots, \phi_{2N})(0)} \exp(2it - i\varphi).$$

is a quasi-rational solution of the NLS equation (1)

$$iv_t + v_{xx} + 2|v|^2v = 0,$$

where

$W_r(y) = W(\phi_1, \dots, \phi_{2N})$  is the wronskian  $W_r(y) = \det[(\partial_y^{\mu-1} \phi_\nu)_{\nu, \mu \in [1, \dots, 2N]}]$ , and  $\phi_\nu$  the functions given by

$$\phi_\nu(y) = \sin(K_\nu(x - x_{0\nu})/2 + i\delta_\nu(t - t_{0\nu}) - ix_{r,\nu}/2 + \gamma_\nu y), \quad 1 \leq \nu \leq N,$$

$$\phi_\nu(y) = \cos(K_\nu(x - x_{0\nu})/2 + i\delta_\nu(t - t_{0\nu}) - ix_{r,\nu}/2 + \gamma_\nu y), \quad N+1 \leq \nu \leq 2N,$$

$\kappa_\nu, \delta_\nu, \gamma_\nu$  parameters defined by (6) for  $\nu \in [1, \dots, 2N]$

$$\kappa_\nu = 2\sqrt{1 - \lambda_\nu^2}, \quad \delta_\nu = \kappa_\nu \lambda_\nu, \quad \gamma_\nu = \sqrt{\frac{1 - \lambda_\nu}{1 + \lambda_\nu}}, \quad \epsilon_\nu \in \{0; 1\},$$

where  $\lambda_j = 1 - \epsilon^2 c_j^2$  is an arbitrary parameter such that  $0 < \lambda_j < 1$  and  $\lambda_{N+j} = -\lambda_j$ ,  $c_j$  an arbitrary real number, for  $j \in [1, \dots, N]$ ,

If we want to recover exact solution of Akhmediev type, we must more precise in the estimations.

## 4.2 Quasi-rational solutions of order $N$

For brevity, in the following we choose  $x_{0i} = 0$ ,  $t_{0i} = 0$  for  $1 \leq i \leq 2N$ , and  $\varphi = 0$ .

To compare our solutions of NLS equation written in the context of hydrodynamic (1) with recent studies in fiber optics, we can make the following changes of variables

$$\begin{aligned} t &\rightarrow X/2 \\ x &\rightarrow T. \end{aligned} \tag{26}$$

The equation (1) become

$$iu_x + \frac{1}{2}u_{tt} + u|u|^2 = 0. \tag{27}$$

We give all the solutions for (1), but the reader can easily get the analogues for (27) by using the formulas (26).

### 4.2.1 Case N=1

If we consider the case  $N = 1$ , from (??), we realize an expansion at order 1 of  $W_3$  and  $W_1$  in  $\epsilon$ . The solution of NLS equation can be written as

$$v(x, t) = \frac{-4x^2 - 16t^2 + 16it + 3}{4x^2 + 16t^2 + 1} \exp(2it).$$

If we make the preceding change of variable (26), we get exactly first order Akhmediev's solution (see [1]).

In particular,

$$v(x, 0) = \frac{3 - 4x^2}{1 + 4x^2}.$$

We represent here the modulus of  $v$  in function of  $x \in [-5; 5]$  and  $t \in [-5; 5]$  by the figure 1. The maximum of modulus of  $v$  is equal to 3.

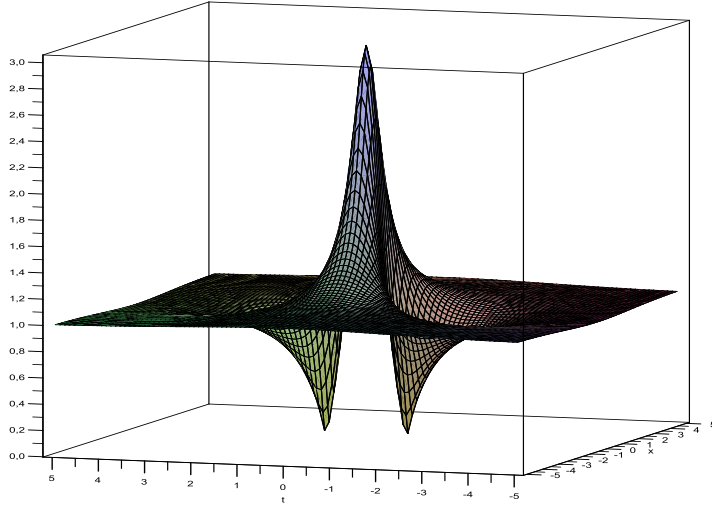


Figure 1: Solution to NLS equation for N=1.

#### 4.2.2 Case N=2

In the case  $N = 2$ , we realize an expansion at order 3 in  $\epsilon$ . From (??), the solution of NLS equation can be written as

$$v(x, t) = \frac{N(x, t)}{D(x, t)} \exp(2it),$$

with

$$N(x, t) = 64x^6 - (144 + 768it - 768t^2)x^4 - (180 + 6144it^3 + 5760t^2 - 1152it - 3072t^4)x^2 + 45 + 720it - 1536it^3 - 12288it^5 + 4096t^6 - 8448t^4 - 1872t^2$$

and

$$D(x, t) = 64x^6 + (48 + 768t^2)x^4 + (108 - 1152t^2 + 3072t^4)x^2 + 9 + 4096t^6 + 6912t^4 + 1584t^2.$$

If we make the preceding changes of variables defined by (26), it can be reduced exactly at the second order Akhmediev's solution (see [1]).

In particular,

$$v(x, 0) = \frac{45 - 180x^2 - 144x^4 + 64x^6}{9 + 108x^2 + 48x^4 + 64x^6}.$$

We represent the modulus of  $v$  in function of  $x \in [-5; 5]$  and  $t \in [-5; 5]$  in figure 2. The maximum of modulus of  $v$  is equal to 5.

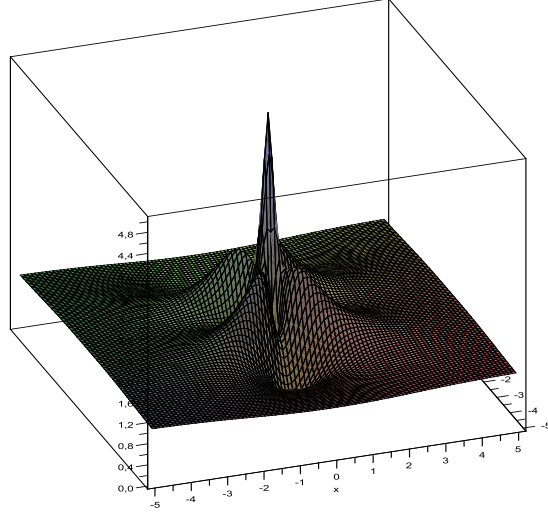


Figure 2: Solution to NLS equation for  $N=2$ .

### 4.2.3 Case $N=3$

In the case  $N = 3$ , we make an expansion at order 5 in  $\epsilon$ . From (24), the solution of NLS equation (27) can be written as

$$v(x, t) = \frac{N(x, t)}{D(x, t)} \exp(2it),$$

with

$$\begin{aligned} N(x, t) = & -4096x^{12} + (18432 + 98304it - 98304t^2)x^{10} + (1843200t^2 + 1966080it^3 - 368640it - 983040t^4 + 57600)x^8 \\ & + (22609920t^4 - 13762560it^3 - 5242880t^6 - 921600it + 172800 - 2027520t^2 + 15728640it^5)x^6 \\ & + (-226800 + 62914560it^7 - 691200t^2 - 2073600it - 15728640t^8 \\ & - 9216000t^4 - 82575360it^5 + 106168320t^6 - 11059200it^3)x^4 \\ & + (-18403200t^2 + 212336640t^8 - 25165824t^{10} + 85708800t^4 - 38707200it^3 - 56033280t^6 \end{aligned}$$

$$\begin{aligned}
& +125829120it^9 + 1814400it + 168099840it^5 - 94371840it^7 - 113400)x^2 \\
& +14175-83808000t^4-3801600it^3+100663296it^{11}+157286400it^9+144703488t^{10}-2268000t^2-342097920it^7 \\
& -16777216t^{12} + 533790720t^8 - 236666880it^5 + 152616960t^6 + 453600it
\end{aligned}$$

and

$$\begin{aligned}
D(x, t) = & 4096x^{12} + (98304t^2 + 6144)x^{10} + (-368640t^2 + 34560 + 983040t^4)x^8 \\
& +(-2949120t^4 + 149760 + 5242880t^6 + 552960t^2)x^6 \\
& +(3456000t^2 + 15728640t^8 + 54000 - 5529600t^4 + 3932160t^6)x^4 \\
& +(25165824t^{10} + 80179200t^4 + 221184000t^6 + 70778880t^8 - 2332800t^2 + 48600)x^2 \\
& +2025 + 36806400t^4 + 1490400t^2 + 62668800t^6 + 244776960t^8 + 132120576t^{10} + 16777216t^{12}
\end{aligned}$$

The solution with initial condition  $t = 0$  takes the form

$$v(x, 0) = \frac{14175 - 113400x^2 - 226800x^4 + 172800x^6 + 57600x^8 + 18432x^{10} - 4096x^{12}}{2025 + 48600x^2 + 54000x^4 + 149760x^6 + 34560x^8 + 6144x^{10} + 4096x^{12}}.$$

We get the following graphic for the modulus of  $v$  in function of  $x \in [-5; 5]$  and  $t \in [-5; 5]$ , given by figure 3 : the maximum of modulus of  $v$  is equal to 7.

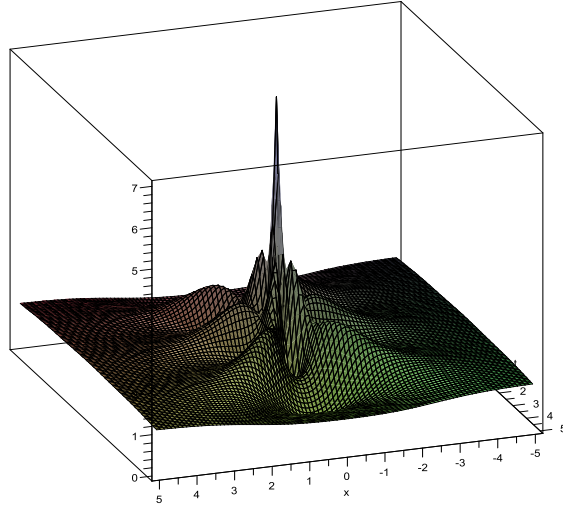


Figure 3: Solution to NLS equation for N=3.



If we make the preceding changes of variables defined by (26), we still recover the solution given recently by Akhmediev [1].

The results presented here are in accordance with the preceding work [6]; here we give moreover an explicit expression of the solutions of NLS equation for the order 3.

### 4.3 Case N=4

For this case, we realize an expansion at order 7 in  $\epsilon$ . The solution of NLS equation (1) can be written as

$$v(x, t) = \frac{N(x, t)}{D(x, t)} \exp(2it).$$

The polynomials  $N$  and  $D$  are defined by

$$\begin{aligned} N(x, t) = & -1048576x^{20} + (7864320 + 41943040it - 41943040t^2)x^{18} \\ & + (44236800 - 283115520it + 1509949440it^3 - 754974720t^4 + 1415577600t^2)x^{16} \\ & + (324403200 - 4246732800t^2 + 24159191040it^5 + 37748736000t^4 - 1415577600it - 25165824000it^3 - 8053063680t^6)x^{14} \\ & + (13212057600it^3 + 225485783040it^7 - 493250150400it^5 - 9083289600it \\ & + 465064427520t^6 - 10734796800t^2 - 56371445760t^8 - 235615027200t^4 + 1354752000)x^{12} \\ & + (-3223742054400t^6 + 1352914698240it^9 - 4396972769280it^7 - 138726604800it^3 - 74008166400t^2 \\ & - 6096384000 + 9909043200t^4 + 3297729576960t^8 + 1088673546240it^5 - 270582939648t^{10} - 32514048000it)x^{10} \\ & + (14543833006080t^{10} - 402554880000it^3 - 9398592000 - 804722688000t^2 + 5411658792960it^{11} \\ & + 10477161676800t^6 + 12049396531200it^7 - 1086898176000t^4 - 901943132160t^{12} - 18232639488000t^8 \\ & - 20857434931200it^9 - 5588700364800it^5 + 121927680000it)x^8 \\ & + (48197586124800it^9 + 16161649459200t^6 - 2061584302080t^{14} - 19920273408000t^4 + 47933344972800t^8 \\ & + 14431090114560it^{13} - 19074908160000it^5 + 7153090560000it^3 - 36923296972800t^{10} + 150377472000it \\ & - 11430720000 + 40587440947200t^{12} - 52312701665280it^{11} + 678730752000t^2 + 12472182374400it^7)x^6 \\ & + (-98297708544000it^9 + 273331047628800t^{10} + 24739011624960it^{15} - 207309570048000t^6 + 350542080000t^2 \\ & + 70490456064000it^5 + 52425444556800t^{12} + 291375415296000t^8 - 3092376453120t^{16} + 137168640000it \\ & + 9644670000 - 54116587929600it^{13} - 344240160768000it^7 + 37205154201600it^{11} + 69578470195200t^{14} \\ & + 3170119680000it^3 + 3292047360000t^4)x^4 + (24739011624960it^{17} + 1181263645900800t^{12} + 66486093742080t^{16} \\ & - 1542243483648000t^{10} - 1290950148096000it^9 + 20615843020800it^{15} + 330497733427200t^{14} \\ & + 1311675120000t^2 - 19203609600000t^4 - 73156608000000it^5 - 77157360000it - 202937204736000it^{13} \\ & + 546681913344000it^7 - 2748779069440t^{18} + 6958006272000t^6 - 2693991392870400it^{11} \end{aligned}$$

$$\begin{aligned}
& -1756650405888000t^8 + 4511324160000it^3 + 2679075000)x^2 \\
& -200930625+1099511627776t^{20}+1883933717299200it^{13}-26800595927040t^{18}+569274531840000it^9-379138238054400t^{16} \\
& -395779571712000it^7-796770120499200t^{14}-10995116277760it^{19}+832835262873600t^{12}-53014155264000it^5 \\
& +523098390528000t^{10}-63909113364480it^{17}+963558457344000t^8+369538986147840it^{15}+48039505920000t^6 \\
& +10716300000it-11341179360000t^4-240045120000it^3-87873660000t^2+806992478208000it^{11},
\end{aligned}$$

$$\begin{aligned}
D(x, t) = & 1048576x^{20} + (41943040t^2 + 2621440)x^{18} + (-283115520t^2 + 754974720t^4 + 26542080)x^{16} \\
& + (471859200t^2 + 8053063680t^6 + 265420800 - 7549747200t^4)x^{14} \\
& + (6606028800t^4 + 7431782400t^2 + 56371445760t^8 - 70464307200t^6 + 1741824000)x^{12} \\
& + (270582939648t^{10} + 49235558400t^2 + 264241152000t^6 + 1683763200 + 188271820800t^4 - 253671505920t^8)x^{10} \\
& + (4280706662400t^8 + 338228674560t^{10} - 128894976000t^2 - 594542592000t^6 + 4463424000 + 1532805120000t^4 + 901943132160t^{12})x^8 \\
& + (8382528000 + 134120448000t^2 + 68966940672000t^8 + 2061584302080t^{14} - 2536095744000t^4 \\
& + 42658430976000t^6 + 6313601925120t^{12} + 31962609745920t^{10})x^6 \\
& + (161269678080000t^8 + 46039891968000t^6 - 365783040000t^4 + 332943851520000t^{10} + 123453466214400t^{12} \\
& + 1786050000 + 23192823398400t^{14} + 320060160000t^2 + 3092376453120t^{16})x^4 \\
& + (15545779200000t^4 + 158993694720000t^6 + 241591910400000t^{14} - 417368899584000t^{10} - 58785398784000t^8 \\
& + 893025000 + 2748779069440t^{18} + 38654705664000t^{16} - 122880240000t^2 + 296795661926400t^{12})x^2 \\
& + 22325625 + 46437300000t^2 + 1053529473024000t^{12} + 1099511627776t^{20} + 190374425395200t^{16} + 25426206392320t^{18} \\
& + 3594961440000t^4 + 435233046528000t^8 + 17801441280000t^6 + 592866548121600t^{14} + 727066135756800t^{10}
\end{aligned}$$

Moreover,

$$v(x, 0) = \frac{N(x, 0)}{D(x, 0)},$$

$$\begin{aligned}
N(x, 0) = & -200930625 + 2679075000x^2 + 9644670000x^4 - 11430720000x^6 - 9398592000x^8 - 6096384000x^{10} \\
& + 1354752000x^{12} + 324403200x^{14} + 44236800x^{16} + 7864320x^{18} - 1048576x^{20}, \\
D(x, 0) = & 22325625 + 893025000x^2 + 1786050000x^4 + 8382528000x^6 + 4463424000x^8 + 1683763200x^{10} \\
& + 1741824000x^{12} + 265420800x^{14} + 26542080x^{16} + 2621440x^{18} + 1048576x^{20}.
\end{aligned}$$

We recover a result of Akhmediev formulated in [1] in the case where  $x = 0$ . Here we give the complete solution in  $x$  and  $t$ .

We can make the following conjecture : at the order  $N$ , we get with this method a quasi rational solution  $v$  quotient of two polynomials of degree  $N(N + 1)$ .

There is  $N(N + 1) - 1$  local maximum for the modulus and the global maximum modulus is equal to  $2N + 1$ .

We get the following graphic for the modulus of  $v$  in function of  $x \in [-5; 5]$  and  $t \in [-5; 5]$  : the maximum of modulus of  $v$  is equal to 9. It is given in figure 4.

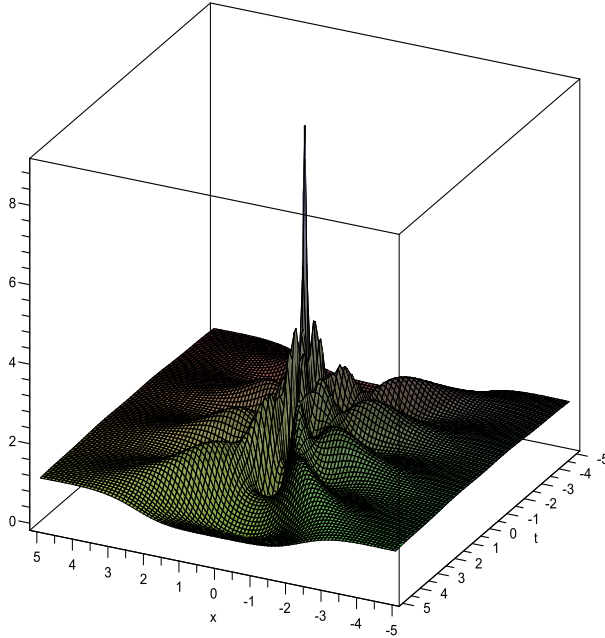


Figure 4: Solution to NLS equation for  $N=4$ .

This method can be extended to get an infinite family of quasi-rational solutions of NLS equation at any order.

## 5 Conclusion

Here is given a new formulation of solutions of NLS equation that provides as particular case, Akhmediev's solutions. These solutions expressed as a quotient of wronskians are similar to a previous result of Eleonski, Krichever and Kulagin [8]. It has been rewritten recently and their formulation can also be seen as a quotient of two wronkians (see [6]); the method describe in [6] can be also used to get solutions of NLS equation, but it is more complicated to find explicit solution in  $x$  and  $t$ ; the computation gives complicated results. This method described in the present paper provides a powerful tool to get explicitly solutions in analytical form. This new formulation gives an infinite set of non singular solution of NLS equation.

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