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# Collective Risk Aversion

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## Abstract

In this paper we analyze the risk attitude of a group of heterogeneous agents and we develop a theory of comparative collective risk tolerance. In particular, we characterize how shifts in the distribution of individual levels of risk tolerance affect the representative agent's degree of risk tolerance. In a model with efficient risk-sharing and two agents (e.g. a household) with isoelastic preferences we show that an increase of the level of risk tolerance of one of the agents might have an ambiguous impact on the aggregate level of risk tolerance; the latter increases for some levels of aggregate wealth while it decreases for other levels of aggregate wealth. Specifically, there are two possible shapes for the aggregate risk tolerance as a function of the risk tolerance level of one of the agents: increasing curve or increasing then decreasing curve. For more general populations we characterize the effect of first-order like shifts (individual levels of risk tolerance more concentrated on high values) and second-order like shifts (more dispersion on individual levels of risk tolerance) on the collective level of risk tolerance. We also evaluate how shifts in the distribution of individual levels of risk tolerance impact the collective level of risk tolerance in a framework with exogenous egalitarian sharing rules. Our results permit to better characterize differences in risk taking behavior between groups and individuals and among groups with different distributions of risk preferences.

## 1 Introduction

Pratt (1964) and Arrow (1965) established the necessary and sufficient conditions for comparing attitudes towards risk. Agent A is more risk averse than agent B if, equivalently, 1) A rejects all lotteries that B rejects, 2) A's utility function is a concave transformation of B's utility function, 3) A's coefficient of absolute risk aversion is higher than B's coefficient for all wealth levels, and 4) the premium that A is willing to pay to escape a fair risk is higher than the premium that B is willing to pay.

These conditions apply not only to individuals but also to utility functions of representative agents. If markets are complete, then there exists such a representative agent and by comparing the degree of risk aversion of different

representative agents one can evaluate changes in aggregate behavior. For example, in an economy with a representative agent that is more risk averse the equity risk premium will be larger.

The interpretation of such comparative statics results, however, is clouded by the following fact. The risk aversion of the representative agent depends upon both the distribution of preferences among the agents and their optimal allocations. So a change in the distribution of preferences impacts the aggregate level of risk aversion through two channels. A direct one as well as an indirect one due to the fact that changes in the distribution of preferences lead, in turn, to changes in the allocation of wealth (risk sharing). Therefore, risk aversion has to be endogeneously determined.

There is one special case in which the problem greatly simplifies: if all consumers have a constant and common absolute cautiousness (the derivative of the reciprocal of absolute risk aversion) – e.g. under CARA or CRRA utility functions with a common level of relative risk aversion –, the representative agent has the same absolute cautiousness (Wilson 1968). Comparative statics of risk aversion at the aggregate level is then not different from comparative statics at the individual level. The assumption of homogeneity in individual preferences, however, does not have empirical support (e.g. Barsky et al. 1997) and, in fact, defeats the purpose of Arrow-Pratt’s theory of comparative risk aversion.

In this paper we consider an exchange economy with a single good, complete markets, and consumers with isoelastic but heterogeneous preferences. Hara et al. (2007) studied the properties of the representative agent’s preferences for a given distribution of individual preferences in this setup. We extend their analysis by evaluating how changes in the distribution of individual preferences affect the representative agent’s attitudes towards risk.

As underlined above, the difficulty in establishing a theory of comparative *collective* risk tolerance stems from the fact that the representative agent’s degree of risk tolerance is different from the mean of the individual degrees of risk tolerance (Wilson 1968) and is endogenous if risk is shared efficiently. Consider, for example, two groups which are identical except that in one group there is an individual that is more risk tolerant. Is it possible to conclude that the group with this individual will be more willing to take risks? Mazzocco (2004) shows on a numerical example and with two agents that an increase in the level of risk tolerance of one of the agents might have an ambiguous impact on the aggregate level of risk tolerance. It increases for some levels of aggregate wealth while it decreases for other levels of aggregate wealth. In fact, the aggregate level of risk tolerance is a wealth share weighted average of the individual levels of risk tolerance and increasing the risk tolerance level of one agent has two opposite effects: an increase of one of the terms of the average but a possible decrease of its relative weight in the average. Despite these difficulties, our objective is to provide a structure to the problem of comparing attitudes towards risk among heterogeneous groups.

We show, in the setting of Mazzocco (2004)’s paper, that there are two possible shapes for the aggregate risk tolerance as a function of the risk tolerance

level of one of the agents: increasing curve or increasing then decreasing curve. We clearly characterize these two different situations in terms of the size of the aggregate endowment relative to the endowment that corresponds to the fair efficient allocation. We also characterize, for the two-agent case and for more general populations, first-order like shifts (individual levels of risk tolerance more concentrated on high values) that have an unambiguous impact on the collective level of risk tolerance.

Since the key aspect of our economy is preference heterogeneity we also evaluate how more dispersion on the individual levels of risk tolerance (second-order shifts) affect the representative agent's preferences. We show that, for high levels of wealth (relative to the level that corresponds to the fair efficient allocation), more heterogeneity tends to increase aggregate risk tolerance, while the opposite is true for low levels of wealth.

We then extend our analysis to a framework in which all members of a group receive the same endowment (egalitarian groups). This is the setup used in a number of recent experiments that evaluate the degree of risk aversion in groups as compared to individuals (e.g. Masclet et al. 2009, Baker et al. 2008, Shupp and Williams 2008). This setup is also appropriate to analyze situations in which the members of a group derive utility from a public good and situations in which a private good is simultaneously consumed by many individuals. For example, many goods within a household are simultaneously consumed by all the members of a family. Within this framework, and under very general preferences, we establish the impact of first-order shifts and second-order shifts on the collective level of risk tolerance.

The paper proceeds as follows. In Section 2 we present the model with efficient risk-sharing and we establish a number of useful results about the efficient allocations of endowments and the representative agent's preferences. In Section 3 we briefly evaluate the case of CARA preferences, which serves as a useful benchmark. In Section 4 we analyze the case of CRRA preferences. After presenting general properties of the representative agent's preferences we evaluate shifts in the distribution of individual preferences, first in the case of two agents and then under more general populations. In Section 5 we evaluate collective risk preferences for the case of exogenous egalitarian sharing rules, while section 6 concludes.

## 2 The model

We consider a standard static model in which a group of heterogeneous agents consume a single good. The per capita endowment in the consumption good is defined by a random variable  $x$  on the probability space  $(\Omega, F, P)$ . Agents have a common belief over the probability space. In order to take into account finite as well as infinite sets of agents, the agent space is described by  $(I, \iota, Q)$ , where  $I = [0, \infty)$  and  $Q$  is a probability measure on  $I$ . Individuals are indexed by  $i \in I$  and we denote by  $E^Q$  the expectation with respect to  $Q$ .

In such an economy, we consider a social planner who wants to reach a Pareto

efficient allocation of risks and solves the following maximization program

$$U(x) = \max_{\int x_i dQ(i)=x} \int \lambda_i u_i(x_i) dQ(i). \quad (1)$$

where  $u_i$  is the utility function of agent  $i$ , where  $x_i$  is the consumption of agent  $i$  and where  $\lambda_i$  is the weight granted to agent  $i$  by the social planner. The aggregate utility function  $U(x)$  corresponds to the highest social utility level among all possible endowment distributions across agents.

Throughout the paper, we make the following assumption on the utility functions.

**Assumption (U)** For all  $i$ , the utility function  $u_i : [d_i, \infty) \rightarrow \mathbb{R} \cup \{-\infty\}$  is assumed to be infinitely differentiable on  $(d_i, \infty)$  with  $u'_i > 0$  and  $u''_i < 0$  and satisfies Inada's conditions, i.e.  $\lim_{x \rightarrow d_i} u'_i(x) = \infty$  and  $\lim_{x \rightarrow \infty} u'_i(x) = 0$ .

For a given agent  $i$  and a given consumption level  $x$ , the absolute (resp. relative) risk aversion  $A_i(x)$  (resp.  $R_i(x)$ ), the absolute (resp. relative) risk tolerance  $t_i(x)$  (resp.  $s_i(x)$ ) are given by

$$\begin{aligned} A_i(x) &= -\frac{u''_i(x)}{u'_i(x)}, & R_i(x) &= -x \frac{u''_i(x)}{u'_i(x)} = x A_i(x) \\ t_i(x) &= -\frac{u'_i(x)}{u''_i(x)} = \frac{1}{A_i(x)}, & s_i(x) &= -\frac{u'_i(x)}{x u''_i(x)} = \frac{1}{R_i(x)} = \frac{t_i(x)}{x}. \end{aligned}$$

Note that CARA and CRRA utility functions clearly satisfy assumption (U).

If we denote by  $v$  the function defined by  $v(x, i) = u'_i(x)$ , we will also make the following assumption.

**Assumption (LSPM)** The function  $v$  is log-supermodular in  $(x, i)$ , i.e.  $\frac{\partial \log v}{\partial x}(x, i)$  is nondecreasing in  $i$ .

Remark that the log-supermodularity of  $v(x, i)$  means that  $A(x, i) = A_i(x)$  is nonincreasing in  $i$  or that agent  $i$  is less risk averse (and more risk tolerant) than agent  $j$  when  $i \geq j$ .

We have then the following classical result

**Proposition 1** Under Assumption (U), there exists a family of functions  $(f_i)_{i \in [0, \infty]}$  such that

- $f_i : [d, \infty) \rightarrow [d_i, \infty)$  with  $d = \int d_i dQ(i)$  is infinitely differentiable and increasing
- $\int f_i(x) dQ(i) = x$  for all  $x \in [d, \infty)$
- $U(x) = \int \lambda_i u_i(f_i(x)) dQ(i)$ .

We will say that  $(f_i)_{i \in [0, \infty]}$  is an efficient sharing rule associated with the maximization program of Eq. (1).

We recall the following well known results that relate the aggregate utility risk aversion and risk tolerance to the individual ones through the efficient sharing rule.

**Proposition 2 (Wilson, 1968 and Hara et al. 2007)** *Let us assume that (U) is satisfied. Let  $x$  be a given aggregate wealth and let  $(f_i)_{i \in I}$  be the efficient sharing rules associated with the maximization program of Eq. (1). The aggregate absolute risk tolerance  $t(x) = -\frac{U'(x)}{U''(x)}$  and the aggregate relative risk tolerance  $s(x) = -\frac{U'(x)}{xU''(x)}$  are given by*

$$\begin{aligned} t(x) &= \int t_i(f_i(x))dQ(i), \\ s(x) &= \int \frac{f_i(x)}{x} s_i(f_i(x))dQ(i). \end{aligned}$$

The relative risk tolerance  $s(x)$  of the group is then an average of the individual levels of relative risk tolerances  $s_i(f_i(x))$  weighted by the optimal individual shares of consumption. Analogously, the degree of relative risk aversion of the representative agent is an average of the population degrees of relative risk aversion. The group is then less risk averse than the most risk averse agent and more risk averse than the least risk-averse one.

It is easy to show that  $s'(x)$  is positive and then that the aggregate relative risk aversion  $R(x) = -x \frac{U''(x)}{U'(x)}$  is decreasing in  $x$ . This fact has been underlined by Hara et al. (2007, Proposition 6). They further show (Corollary 7) that  $R(x)$  approaches the degree of relative risk aversion of the most (least) risk averse agent in the economy as  $x$  converges to zero (infinity). This result is interesting in light of Brunnermeier and Nagel (2008) who claim that "CRRA is a good description of microeconomic behavior" while models that lead to a representative agent with contracyclical relative risk aversion exhibit interesting macroeconomic properties.

At this stage we consider very general utility functions and we may assume, without loss of generality, that the central planner grants the same weight to all the agents (it suffices to replace the utility function  $u_i$  by  $\lambda_i u_i$ , note that the LSPM property is not impacted by this modification). In the next we consider then the equally weighted Pareto optimum. We also assume that there exists an efficient fair allocation. In other words, there exists  $x^*$  such that  $(x_i)_{i \in I}$ , with  $x_i = x^*$  for all  $i$ , is efficient.

The following proposition provides an analysis of how the aggregate consumption  $x$  is shared among the agents depending on the position of  $x$  relatively to the fair efficient allocation  $x^*$

**Proposition 3** *Under the Assumptions (U) and (LSPM), we have the following results.*

1. *For  $x \geq x^*$ , the optimal allocation  $(x_i)_{i \in I}$  associated to the aggregate wealth  $x^*$  is such that  $x_i \geq x^*$ , for all  $i$ , and  $x_i$  increases with  $i$ . Furthermore, if all the utility functions are DARA then  $t_i(x_i)$  increases with  $i$ .*
2. *For  $x \leq x^*$ , the optimal allocation  $(x_i)_{i \in I}$  associated to the aggregate wealth  $x^*$  is such that  $x_i \leq x^*$ , for all  $i$ , and  $x_i$  decreases with  $i$ .*

### 3 CARA utility functions

Let us consider constant absolute risk-aversion/tolerance utility functions of the form

$$u_i(x) = -\theta_i \exp\left(-\frac{x}{\theta_i}\right). \quad (2)$$

We have  $t_i(x) = \theta_i$  and  $t(x) = \int \theta_i dQ(i)$ . If the agents are indexed by their absolute levels of risk tolerance we have  $\theta_i = i$  and the log-supermodularity assumption is satisfied. We have then  $t(x) = E^Q [\tilde{\theta}]$  and the aggregate level of risk tolerance does not depend on the wealth allocation among the agents. It is immediate that FSD shifts on the distribution of the individual levels of risk tolerance lead to an increase of the aggregate level of absolute (and relative) risk tolerance. More heterogeneity, in the sense of shifts in the distribution of preferences that preserve the mean, have no effect on the representative agent's risk tolerance. These results will serve as a useful benchmark.

### 4 CRRA utility functions

Let us consider constant relative risk-aversion/tolerance utility functions of the form

$$u_i(x) = \frac{1}{1 - \frac{1}{b_i}} x^{1 - \frac{1}{b_i}}. \quad (3)$$

where  $b_i$  is the level of relative risk tolerance of individual  $i$  and  $\frac{1}{b_i}$  is his level of relative risk aversion. In such a setting, we have

$$A_i = \frac{1}{b_i x}, \quad R_i = \frac{1}{b_i}, \quad t_i = b_i x, \quad \text{and} \quad s_i = b_i.$$

Since the utility functions are anymore defined up to a multiplicative constant, we do not assume anymore that the  $\lambda_i = 1$  for all  $i$ . However, we still assume that there exists a wealth level  $x^*$  for which the fair allocation  $(x_i)_{i \in I}$  with  $x_i = x^*$  for all  $i$ , is efficient. Note that the existence of such a fair allocation can always be granted by the social planner through a judicious choice of the weights  $(\lambda_i)_{i \in I}$ . The first-order conditions for Pareto optimality give then that  $\lambda_i (x^*)^{1 - \frac{1}{b_i}}$  is independent of  $i$ . The Pareto problem can be rewritten as follows  $U(x) = \max \int \frac{x_i}{x^*} dQ(i) = \frac{x}{x^*} \int \frac{1}{1 - \frac{1}{b_i}} \left(\frac{x_i}{x^*}\right)^{1 - \frac{1}{b_i}} dQ(i)$ , up to a multiplicative constant. If we renormalize the individual consumptions in order to measure them in terms of multiples of  $x^*$ , we are led to analyze the situation where  $u_i(x) = \frac{1}{1 - \frac{1}{b_i}} x^{1 - \frac{1}{b_i}}$  and all the weights  $\lambda_i$  are equal to 1. Note that with this renormalization,  $x = 1$  corresponds to the efficient fair allocation.

In the next we consider then the equally weighted Pareto optimum. Since the agents differ by only one characteristic, namely their level  $b_i$  of relative risk

tolerance, we might index them by this characteristic or we may, in other words, assume that  $b_i = i$ . For a given function  $h$ , we may then write indifferently  $\int h(b_i)dQ(i)$  or  $\int h(b)dQ(b)$  or  $E^Q [h(\tilde{b})]$ .

Note that the level of relative risk-aversion is then decreasing with  $i$  and the log-supermodularity condition is immediately satisfied.

**Proposition 4** *In an economy made of agents with constant but heterogeneous levels of relative risk aversion, we have at the equally weighted Pareto optimum*

$$U(x) = \int \frac{b_i}{b_i - 1} e^{(1-b_i)\Phi^{-1}(x)} dQ(i)$$

with

$$\Phi(t) \equiv \int e^{-b_i t} dQ(i).$$

The representative agent's degree of relative risk aversion  $R(x)$  is given by

$$R(x) \equiv -x \frac{U''(x)}{U'(x)} = \frac{x}{\int b_i d\tilde{Q}(i)} = \frac{1}{s(x)}. \quad (4)$$

$$\text{with } \frac{d\tilde{Q}}{dQ} \equiv \frac{e^{-b\Phi^{-1}(x)}}{\int e^{-b_i\Phi^{-1}(x)} dQ(i)}.$$

As seen in the proof, the Lagrange multiplier of the Pareto optimum problem is given by  $q = \exp(\Phi^{-1}(x))$  and  $q$  is then the shadow price associated to the constraint  $\sum_{i \in I} x_i = x$ . We clearly have  $\Phi(0) = 1$ , which means that  $\Phi^{-1}(x^*) = 0$  and  $q(x^*) = 1$  for the efficient fair allocation  $x^* = 1$ . Since the agents are risk averse, high levels of aggregate wealth have a low shadow price and low levels of aggregate wealth have high shadow price and we can easily derive that  $q(x) < 1$  for  $x > x^*$  and  $q(x) > 1$  for  $x < x^*$ . This means, in particular, that we have  $\Phi^{-1}(x) < 0$  for  $x > 1$  and  $\Phi^{-1}(x) > 0$  for  $x < 1$ .

## 4.1 A model with two agents

Mazzocco (2004) shows that, in a model with two agents, an increase in the level of risk tolerance of one of the agents might have an ambiguous impact on the aggregate level of risk tolerance. It increases for some levels of aggregate wealth while it decreases for other levels of aggregate wealth. Since this is only stated on a numerical example in Mazzocco (2004), let us clearly express this result.

**Proposition 5** *In a model with two agents with  $b_1 < b_2$ , there exists  $\bar{x} \leq 1$  such that a small increase of  $b_2$  leads to an increase of the aggregate level of risk tolerance  $t(x)$  for all  $x \geq \bar{x}$  and to a decrease of the aggregate level of risk tolerance  $t(x)$  for all  $x \leq \bar{x}$ .*



Recall that after our normalization,  $x^* = 1$  corresponds to the fair efficient allocation. Proposition 5 means then that an increase of the risk tolerance level of the most risk tolerant agent increases (decreases) the aggregate level of risk tolerance for levels of wealth above (below) a given threshold that is below the fair efficient allocation. Note that the threshold  $\bar{x}$  depends on  $b_1$  and  $b_2$ . This means that for  $x$  above the fair allocation, any increase of  $b_2$  increases the aggregate risk tolerance. In fact, above the fair allocation, an increase of  $b_2$  also increases the weight granted to  $b_2$  leading to an increase of the aggregate level of risk tolerance. For a wealth level  $x$  below the fair allocation, the impact of an increase of  $b_2$  is less clear. Indeed, we showed in Proposition 3 that for low levels of aggregate wealth the least risk tolerant agent (the most risk averse) has a larger share of the total wealth and an increase of  $b_2$  leads to an increase of the weight granted to  $b_1$  (the share of total wealth of agent 1). The increase of  $b_2$  has then two effects in opposite directions: an increase of one of the terms of the average (namely the greatest one) and an increase of the weight of the smallest one. Since the second effect does not exist for  $x = 1$ , the first effect continues to dominate for  $x$  above a given threshold  $\bar{x} \leq 1$  while the second effect dominates for  $x \leq \bar{x}$ .

Figures 1 and 2 illustrate the impact of an increase of  $b_2$ .

The next Proposition analyzes more in detail how the aggregate level of risk tolerance evolves as a function of  $b_2$ .

**Proposition 6** *In a model with two agents with  $b_1 < b_2$ ,  $b_1$  being given, for  $x \geq \frac{1}{2}$ , the function  $b_2 \rightarrow t_x(b_2)$  is increasing on  $(b_1, \infty)$  with  $\lim_{b_2 \rightarrow b_1} t_x(b_2) = xb_1$  and  $t_x(b_2) \sim_{b_2 \rightarrow \infty} (x - \frac{1}{2})b_2$ . For  $x < \frac{1}{2}$ , there exists  $b^*(x, b_1) > b_1$  such that the function  $b_2 \rightarrow t_x(b_2)$  is increasing on  $(b_1, b^*(x, b_1))$  and decreasing on  $(b^*(x, b_1), \infty)$  with  $\lim_{b_2 \rightarrow b_1} t_x(b_2) = xb_1$  and  $\lim_{b_2 \rightarrow \infty} t_x(b_2) = xb_1$ .*

Figures 3, 4 and 5 illustrate the different possible shapes for  $b_2 \rightarrow t_x(b_2)$  (or equivalently of  $b_2 \rightarrow s_x(b_2)$ ). The asymptotic behavior of  $t_x(b_2)/b_2$  is illustrated in Figures 6 and 7. In summary, for (very) low levels of wealth, increasing the risk tolerance of the more risk tolerant agent has an ambiguous impact on the representative agent's attitude towards risk.

In the next proposition we show that the ambiguous impact disappears when we consider a uniform increase of risk tolerance across the agents.

**Proposition 7** *Let  $b_1 < b_2$  be given.*

1. *Let us consider a uniform increase of the individual levels of risk tolerance of the form  $b_1 + h$  and  $b_2 + h$  with  $h > 0$ . The associated level of aggregate risk tolerance  $t_x(h)$  increases with  $h$  for all  $x$ .*
2. *Let  $k > 0$  be given and let us consider an increase of the individual levels of risk tolerance of the form  $b_1 + kh$  and  $b_2 + h$  with  $h > 0$ . The associated level of aggregate risk tolerance  $t_x(h)$  increases with  $h$  for  $x \leq 1$  if  $k \geq \frac{b_1}{b_2}$  and increases with  $h$  for  $x \geq 1$  if  $k \leq 1$ . In particular,  $t_x(h)$  increases with  $h$  for all  $x$  if  $k \in \left[\frac{b_1}{b_2}, 1\right]$ .*

Let us illustrate the second point by two extreme situations. For  $k$  very small (near to 0), the shifts we are considering are almost of the form  $(b_1, b_2) \rightarrow (b_1, b_2 + \varepsilon)$  that have already been considered in Proposition 5. These shifts increase the risk tolerance level of the second agent and also increase its weight for  $x \geq 1$ . These shifts lead then to an unambiguous increase of the aggregate level of risk tolerance for  $x \geq 1$ . For  $k$  near infinity (and  $h$  very small), the shifts we are considering are almost of the form  $(b_1, b_2) \rightarrow (b_1 + \varepsilon, b_2)$  and such shifts increase the risk tolerance level of the first agent and also increase its weight when  $x \leq 1$ . These shifts lead then to an unambiguous increase of the aggregate level of risk tolerance for  $x \leq 1$ . The proposition shows that there is a range for  $k$  for which the shifts have an unambiguous impact without restrictions on  $x$ .

We are also interested in the impact of more heterogeneity among our 2 agents. The next result shows that more heterogeneity leads to a higher aggregate risk tolerance level for high wealth levels (above the fair efficient allocation) and to a lower aggregate risk tolerance level for low wealth levels (below the fair efficient allocation).

**Proposition 8** *Let  $b_1$  and  $b_2$  be given with  $b_1 < b_2$  and let us consider a shift of the form  $b_1 - h$  and  $b_2 + h$  with  $h > 0$ . The associated level of aggregate risk tolerance  $t_x(h)$  increases (resp. decreases) with  $h$  for  $x \geq 1$  (for  $x \leq 1$ ).*

This result is very intuitive. We have already seen that the aggregate level of risk tolerance is near the risk tolerance level of the most risk tolerant agent for high levels of wealth and is near the risk tolerance level of the least risk tolerant level agent for low levels of wealth. More heterogeneity leads to an increase of the risk tolerance level of the most risk tolerant agent and to a decrease of the risk tolerance level of the least risk tolerant agent. This leads then to an increase of the aggregate level of risk tolerance for high levels of wealth and to a decrease of the aggregate level of risk tolerance for low levels of wealth. Proposition 8 permits to give a precise meaning to high and low levels of wealth since the fair efficient allocation appears to be the relevant threshold. Figure 8 illustrates this result.

## 4.2 General populations

The different results of the previous section permit to see that first-order stochastic dominance shifts do not guarantee an increase in the aggregate risk tolerance. In this section we initially consider whether a stronger notion of first-order stochastic dominance leads to an unambiguous impact on the representative agent's degree of risk tolerance. Then we evaluate the effect of more heterogeneity in the underlying population. In order to treat the problem in a quite general setting, we consider from now on general populations described by a distribution on  $I = [0, \infty)$ . To relate the results in this more general setting to those obtained in the 2-agent framework, we will attach a specific attention to distributions with a 2-point support.

Since FSD is not a good candidate to obtain comparative static results, let us recall the following definition corresponding to a stronger notion of first-order dominance.

**Definition 1.** Monotone Likelihood Ratio Dominance (MLR). Let  $P$  and  $Q$  denote two probability measures on  $I = [0, \infty)$ . We say that  $P$  dominates  $Q$  in the sense of MLR ( $P \succ_{MLR} Q$ ) if there exist numbers  $0 \leq \alpha \leq \beta \leq \infty$  and a nondecreasing function  $h : [\alpha, \beta] \rightarrow [0, \infty]$  such that  $P([0, \alpha)) = Q((\beta, \infty)) = 0$  and  $dP(i) = h(i)dQ(i)$ .

In other words, an MLR dominated shift for a given probability measure puts less weight for higher values of  $i$ . This concept is widely used in the statistical literature and was first introduced in the context of portfolio problems by Landsberger and Meilijson (1990). MLR dominance is stronger than FSD and, in particular, an MLR dominated shift for a given distribution reduces the mean.

When the supports of  $P$  and  $Q$  are reduced to two points,  $(b_1^1, b_2^1)$  with  $b_1^1 < b_2^1$  for  $P$  and  $(b_1^2, b_2^2)$  with  $b_1^2 < b_2^2$  for  $Q$ , we necessarily have  $b_1^2 < b_2^2 \leq b_1^1 < b_2^1$  or  $b_1^2 = b_1^1$  and  $b_2^2 = b_2^1$ . In the first case, any average of the risk tolerance levels in the support of  $Q$  is smaller than any average of the risk tolerance levels in the support of  $P$  and the aggregate risk tolerance level is higher under  $P$ . The (most) interesting case is when both probability measures have the same support. We then have two populations with the same set of possible levels of individual level of risk tolerance  $b_1$  and  $b_2$  but with different proportions of agents in each category: a proportion  $p_1$  (resp.  $p_2 = 1 - p_1$ ) of agents that have an individual level  $b_1$  (resp.  $b_2$ ) of risk tolerance under  $P$  and a proportion  $q_1$  (resp.  $q_2 = 1 - q_1$ ) of agents that have an individual level  $b_1$  (resp.  $b_2$ ) of risk tolerance under  $Q$ . The MLR dominance is characterized in this setting by  $\frac{p_2}{p_1} \geq \frac{q_2}{q_1}$  (or equivalently  $q_1 \geq p_1$  or  $q_2 \leq p_2$ )

**Proposition 9** *Let us consider two populations. In the first one, we have proportions  $p_1$  (resp.  $p_2 = 1 - p_1$ ) of agents with an individual level  $b_1$  of risk tolerance (resp.  $b_2$ ). In the second one, we have proportions  $q_1$  (resp.  $q_2 = 1 - q_1$ ) of agents with an individual level  $b_1$  of risk tolerance (resp.  $b_2$ ). If we assume that  $\frac{p_2}{p_1} \geq \frac{q_2}{q_1}$ , then the aggregate level of risk tolerance is higher in the first population.*

In summary, when the support of the population is reduced to two points, an MLR dominant shift in the degree of risk tolerance increases the representative agent's risk tolerance. The following proposition generalizes the impact of MLR shifts for distributions with more general supports.

**Proposition 10** *Let us consider two populations characterized by two distributions  $P$  and  $Q$  of individual levels of risk tolerance. If  $P \succ_{MLR} Q$  then for all  $x \leq 1$ , we have  $t_x^P \geq t_x^Q$ . However,  $P \succ_{MLR} Q$  does not guarantee an increase of the aggregate level of risk tolerance when  $x > 1$ .*

MLR provides then a satisfying answer to the impact of shifts for low levels of wealth (when  $x \leq 1$ ), which corresponds to the case where the unilateral

increase of one of the individual levels of risk tolerance failed to guarantee an increase of the aggregate level of risk tolerance. In the following Proposition we show that when the density function (introduced in Definition 1)  $h = \frac{dP}{dQ}$  has an exponential growth rate, then we have an unambiguous impact on collective risk tolerance.

**Proposition 11** *Let us consider two populations characterized by two distributions  $P$  and  $Q$  on  $[0, \infty)$  of individual levels of risk tolerance such that  $P \succ_{MLR} Q$  with  $\frac{dP}{dQ}(b) = \lambda \exp(kb)$  for some positive  $k$  and  $\lambda$ . For all  $x$ , we have  $t_x^P \geq t_x^Q$ .*

We have seen in the 2-agent setting (Proposition 8) that more heterogeneity has a clear impact on the aggregate level of risk tolerance depending on the relative position of the aggregate wealth with respect to the fair efficient allocation. We are now interested in establishing the effect of "more heterogeneity" in the general setting. For this purpose, we introduce the following definition.

**Definition 2.** Portfolio Dominance (PD). Let  $Q_1$  and  $Q_2$  denote two probability measures on  $I = [0, \infty)$ . We say that  $Q_1$  dominates  $Q_2$  in the sense of PD ( $Q_1 \succ_{PD} Q_2$ ) if we have  $\int v(i - a)dQ_1(i) = 0 \Rightarrow \int v(i - a)dQ_2(i) \leq 0$  for any real number  $a$  and any nonnegative and nonincreasing function  $v$ .

This concept has been introduced in the context of portfolio problems by Landsberger and Meilijson (1990) and further studied by Gollier (1997). In the portfolio context it is related to the degree of riskiness of the asset returns. In our context, it is related to the level of individual heterogeneity in relative risk tolerance. In particular, a mean preserving PD dominated shift for a given distribution increases the variance (Jouini and Napp, 2008, Proposition 3).

The following proposition uses this concept to characterize the impact of "more heterogeneity" in the distribution of individual preferences.

**Proposition 12** *Let us consider two populations respectively characterized by distributions  $P$  and  $Q$  of individual levels of risk tolerance. If  $P$  and  $Q$  are symmetric with respect to some  $b^*$  with  $\frac{dQ}{dP}$  nonincreasing before  $b^*$  and nondecreasing after  $b^*$  then  $P \succ_{PD} Q$  and  $P \succ_{SSD} Q$  and the aggregate level of risk tolerance  $t_x^Q$  under  $Q$  is higher than (resp. lower than) the aggregate level of risk tolerance  $t_x^P$  under  $P$  for  $x \geq 1$  (for  $x \leq 1$ ).*

## 5 The case of *egalitarian* groups

It is interesting to analyze the aggregate behavior in a model where all the agents consume the total consumption  $x$ . This is the case when  $x$  is a public good. This also the case when  $x$  is a private good but simultaneously consumed by all the agents in the group. We may consider that both agents in a couple get utility from saving money or from holding consumption goods and consider these goods as owned by the couple and not shared among them through an efficient sharing rule. This is also the setup used in a number of recent experiments that compare the degree of risk aversion of groups with that of individuals (e.g. Masclet et al. 2009, Baker et al. 2008, Shupp and Williams 2008) and where the

rewards of the group are exogenously divided equally among its members. We may imagine, for example, that such experiments reflect the widely observed regularity of partnerships with equal sharing rules.

In the next, the endowment in the consumption good is defined by a random variable  $x$  on the probability space  $(\Omega, F, P)$  and the social utility function is given by

$$U(x) = \int \lambda_i u_i(x) dQ(i). \quad (5)$$

where  $u_i$  is the utility function of agent  $i$  and where  $\lambda_i$  is the weight granted to agent  $i$ .

We have then

$$R(x) \equiv -x \frac{U''(x)}{U'(x)} = -x \frac{\int \lambda_i u_i''(x) dQ(i)}{\int \lambda_i u_i'(x) dQ(i)} = \int R_i(x) dP^u(i) \quad (6)$$

where  $P^u$  is the probability measure defined by  $\frac{dP^u}{dQ}(i) = \frac{\lambda_i u_i'(x)}{\int \lambda_i u_i'(x) dQ(i)}$ .

Let us first analyze the case with 2 agents and CRRA functions. We have then  $u_i(x) = \frac{1}{1-\frac{1}{b_i}} x^{1-\frac{1}{b_i}}$ ,  $i = 1, 2$ , and we take  $\lambda_i = 1$ ,  $i = 1, 2$ , as in the previous section. Equation (6) can be rewritten as follows

$$R(x) = \frac{\frac{1}{b_1} x^{-\frac{1}{b_1}} + \frac{1}{b_2} x^{-\frac{1}{b_2}}}{x^{-\frac{1}{b_1}} + x^{-\frac{1}{b_2}}} = \frac{R_1 x^{-R_1} + R_2 x^{-R_2}}{x^{-R_1} + x^{-R_2}}$$

and the aggregate relative risk aversion is a weighted arithmetic average of the individual levels of relative risk aversion. As in the private good case, the aggregate level of relative risk aversion decreases with  $x$  and it approaches the degree of relative risk aversion of the most (least) risk averse agent in the economy as  $x$  converges to zero (infinity). Notice also that the weights are given by the individual marginal utilities and the highest weight is granted to the lowest (highest) individual level relative risk aversion for  $x > 1$  (for  $x < 1$ ). An increase of the individual level of risk tolerance of the most risk tolerant agent might then have an ambiguous impact. We have the following result

**Proposition 13** *In a model with a public good, two agents and CRRA functions with  $b_1 < b_2$ , there exists  $\bar{x} \leq 1$  such that a small increase of  $b_2$  leads to an increase of the aggregate level of risk tolerance  $t(x)$  for all  $x \geq \bar{x}$  and to a decrease of the aggregate level of risk tolerance  $t(x)$  for all  $x \leq \bar{x}$ .*

In particular, this means that FSD shifts of the individual levels of risk tolerance are not sufficient to increase the aggregate level of risk tolerance. The next result illustrates the impact of a mean preserving spread on the individual levels of risk aversion in a 2-agent setting.

**Proposition 14** *In a model with a public good, two agents and CRRA functions with  $b_1 < b_2$  (or equivalently  $R_1 > R_2$ ), a shift of the form  $R_1 - h$  and  $R_2 + h$  with  $h > 0$  increases (decreases) the aggregate level of relative risk aversion  $R(x)$  for  $x \leq 1$  (for  $x \geq 1$ ). It increases (decreases) the aggregate level of risk tolerance  $t(x)$  for  $x \geq 1$  (for  $x \leq 1$ ).*

In the next we characterize, in a general distribution setting, the impact of MLR shifts on the aggregate level of risk tolerance/aversion. Note that the following result is obtained for very general utility functions.

**Proposition 15** *Let us consider two populations respectively characterized by distributions  $P_1$  and  $P_2$  on  $(I, \iota)$ . Under Assumptions (U) and (LSPM) and if  $P_2 \succ_{MLR} P_1$  then the aggregate level of risk aversion (risk tolerance) under  $P_1$  is higher (lower) than under  $P_2$ .*

The result in Proposition 15 is quite powerful. Under the assumption that the members of the group consume the same endowment, and under weak restrictions on the individual utility functions, if the individual levels of risk aversion are more concentrated on high values (in the sense of MLR dominance) then the collective level of risk aversion will be higher for all endowment levels. In this case MLR dominance provides a clear characterization of comparative collective risk aversion<sup>1</sup>.

The next proposition analyzes the impact of more heterogeneity on the individual levels of risk aversion. For a given distribution  $P$  on  $(I, \iota)$  and for a given  $x$ , we denote by  $P^x$  the image measure of  $P$  by  $i \rightarrow -\frac{u'_i(x)}{u_i(x)}$ . The measure  $P^x$  describes the distribution of the individual levels of risk aversion at a given wealth level  $x$ .

**Proposition 16** *Let us consider two populations respectively characterized by distributions  $P_1$  and  $P_2$  on  $(I, \iota)$  and let us assume that (U) is satisfied. If, for a given  $x$ ,  $u'_i(x)$  is nondecreasing in  $i$ ,  $-\frac{u''_i(x)}{u_i(x)}$  is decreasing with  $i^2$  and  $P_2^x \succ_{PD} P_1^x$ , then the aggregate level of risk aversion (risk tolerance), at  $x$ , under  $P_2$  is higher (lower) than under  $P_1$ .*

In particular, if there exists  $x^*$  such all the individual marginal utilities  $u'_i(x^*)$  are equal, then  $u'_i(x)$  is nondecreasing in  $i$  for  $x \leq x^*$ . It suffices then to have that  $-\frac{u''_i(x)}{u_i(x)}$  is decreasing with  $i$  and  $P_2^x \succ_{PD} P_1^x$  to conclude that the aggregate level of risk aversion, at  $x$ , under  $P_2$  is higher than under  $P_1$ . In other words, under weak assumptions on the utility function, less heterogeneity in risk aversion, in the sense of PD dominance, implies a higher level of collective risk aversion when wealth is sufficiently low.

## 6 Conclusion

Mazzocco (2004) established the counter-intuitive result that an increase in the level of risk tolerance of one of the individuals in a couple may reduce their collective degree of risk tolerance. We studied precisely the conditions for this

<sup>1</sup>Note that Proposition 15 can easily be extended for higher order collective preferences towards risk. For example, if we assume that  $u''(x, i)$  is LSPM in  $(x, i)$ , then an MLR dominant shift decreases the collective degree of absolute (and relative) prudence.

<sup>2</sup>Note that this last condition is just a little bit stronger than the LSPM condition.

phenomenon to occur. More generally, we established conditions under which groups with individual levels of risk tolerance more concentrated on high values and groups that are more heterogeneous will display higher risk tolerance, both with efficient risk-sharing and with an exogenous egalitarian sharing rule. Our results permit to better characterize differences in risk taking behavior between groups and individuals and among groups with different distributions of risk preferences.

It should be possible to design experiments to evaluate if our results are consistent with elicited risk attitudes of groups and individuals. Shupp and Williams (2008) compare the willingness to pay for lotteries of small groups and individuals in a setup similar to that of Section 5. They conclude that, for most lotteries, group choices are significantly different from the mean of the individual choices (groups tend to be more risk averse than individuals for low-expected-value lotteries but less risk averse than individuals for high-expected-value lotteries). These results are consistent with a large number of studies in social psychology that show "risky" and "cautious" shifts in group risk-taking behavior relative to the mean of the individual choices (see e.g. Clark 1971). We have seen that there is no reason to believe that the group's willingness to pay (derived from the collective preferences), and more generally the willingness to take risks, should be the same as the mean of the individual members' willingness to pay. In particular, even with CRRA individual preferences, the fact that the representative agent's relative risk aversion decreases with  $x$  implies that "cautious" shifts should be more prevalent in low-expected-value (low-stakes) lotteries while "risky" shifts should be more prevalent in high-expected-value (high-stakes) lotteries, precisely what Shupp and Williams (2008) found. It would be interesting to further explore the experimental relevance of our results by proceeding to inter-groups comparisons. For instance, to analyze the difference in risk attitudes between two couples that only differ by the risk aversion level of one of the members, e.g. the man.

## 7 Appendix

**Proof of Proposition 1.** Let us denote by  $\varphi$  the function defined by  $\varphi(q) = \int (u'_i)^{-1} \left( \frac{q}{\lambda_i} \right) dQ(i)$ . By Inada's conditions and since  $u_i$  is increasing and strictly concave for all  $i$ ,  $\varphi$  is well defined on  $(0, \infty)$  and decreasing. Furthermore, from the monotone convergence Theorem we have  $\lim_{x \rightarrow 0} \varphi(q) = \infty$  and  $\lim_{x \rightarrow \infty} \varphi(q) = d$ . Let us then define  $f_i : [d, \infty) \rightarrow [d_i, \infty)$  by  $f_i(x) = (u'_i)^{-1} \left( \frac{\varphi^{-1}(x)}{\lambda_i} \right)$ , we clearly have  $x = \int f_i(x) dQ(i)$  and  $\lambda_i u'_i(f_i(x)) = \varphi^{-1}(x)$  for all  $i$  and is independent of  $i$ . The family  $(f_i(x))$  satisfies then the first-order conditions of the maximization program defined by Eq. (1) and since this program is concave we have  $U(x) = \int \lambda_i u_i(f_i(x)) dQ(i)$ . ■

**Proof of Proposition 3.** Since the fair allocation  $x_i = x^*$ ,  $i \in I$ , is efficient and since we granted the same weight to all the agents, the first-order conditions for Pareto optimality give us that  $u'_i(x)$  is independent of  $i$ . Since  $\frac{\partial u}{\partial x}(x, i)$  is

LSPM, integrating  $\frac{\partial}{\partial x} \log u'_i(x)$  between  $x \geq x^*$  and  $x^*$ , gives us that  $u'_i(x)$  increases with  $i$ . Let us consider the Pareto allocation  $(x_i)_{i \in I}$  associated to the aggregate wealth  $x \geq x^*$ . We have that  $u'_i(x_i)$  is independent of  $i$ . We also necessarily have  $x_{i_0} \geq x^*$  for some  $i_0$  and then  $u'_{i_0}(x_{i_0}) \leq u'_{i_0}(x^*)$  by concavity of the utility functions. We have then  $u'_i(x_i) \leq u'_i(x^*)$  for all  $i$  and consequently  $x_i \geq x^*$  for all  $i$ . Since  $u'_i(x)$  increases with  $i$  and  $u'_i(x)$  is independent of  $i$  and since  $u'_i(x)$  is decreasing in  $x$ , we have that  $x_i$  increases with  $i$ . By the LSPM property  $t(x, i)$  is increasing in  $i$  and by the DARA property, it is increasing in  $x$ . We have then that  $t(x_i, i)$  is increasing in  $i$ .

For  $x \leq x^*$ , integrating  $\frac{\partial}{\partial x} \log u'_i(x)$  between  $x$  and  $x^*$ , gives us that  $u'_i(x)$  decreases with  $i$ . The same kind of arguments as above give that  $x_i$  decreases with  $i$ . ■

**Proof of Proposition 4.** The first-order condition gives us the optimal allocation of agent  $i$ ,  $x_i = q^{-b_i}$ , where  $q$  is the Lagrange multiplier. Using the resource constraint we obtain  $\int q^{-b_i} dQ(i) = x$  or  $\int \exp(-b_i \ln q) dQ(i) = x$ . We obtain that  $q = \exp(\Phi^{-1}(x))$  hence  $x_i = e^{-b_i \Phi^{-1}(x)}$ . Plugging this back into the utility function we obtain the representative agent's utility.

The degree of relative risk aversion of the representative agent can be derived directly from the expression of the representative agent utility function or using Proposition 2. We have  $s = \int s_i \frac{x_i}{x} dQ(i)$ . Since  $x_i = e^{-b_i \Phi^{-1}(x)}$  the result follows. ■

**Proof of Proposition 5.** Let us consider  $b_1$  as given and denote by  $t_x(b_2)$  the aggregate risk tolerance level when the risk tolerance level of agent 2 is given by  $b_2 \geq b_1$ . We have

$$t_x(b_2) = \frac{1}{2} b_1 \exp(-b_1 a_x(b_2)) + \frac{1}{2} b_2 \exp(-b_2 a_x(b_2))$$

where  $a_x(b_2)$  is the solution of

$$\psi(a_x(b_2), b_1, b_2) = x$$

with

$$\psi(a, b_1, b_2) = \frac{1}{2} \exp(-b_1 a) + \frac{1}{2} \exp(-b_2 a).$$

We get after computations

$$\frac{d}{db_2} t_x(b_2) = \frac{1}{2} \frac{\exp(-(b_1 + b_2) a_x(b_2))}{b_1 \exp(-b_1 a_x(b_2)) + b_2 \exp(-b_2 a_x(b_2))} \varphi(a_x(b_2), b_1, b_2)$$

with

$$\varphi(a, b_1, b_2) = b_1 + b_2 \exp((b_1 - b_2) a) - a (b_2 - b_1) b_1.$$

For  $x \geq 1$ , we have  $a_x(b_2) \leq 0$  and  $\frac{d}{db_2} t_x(b_2) > 0$ . The aggregate level of risk tolerance is an increasing function of  $b_2$ .

Let us now focus on the case  $x \leq 1$ . It is easy to check that  $a_x(b_2)$  is always positive for  $x \in (0, 1)$  and decreases with  $x$  from  $\infty$  to 0. It is also easy to see that  $\varphi(a, b_1, b_2)$  is decreasing in  $a$ , positive for  $a = 0$  and converges to  $-\infty$  when



$a$  converges to  $\infty$ . There exists then a level  $x^* < 1$  such that  $\frac{d}{db_2}t_x(b_2) = 0$ . For  $x < x^*$  (resp.  $x > x^*$ ), we have  $\frac{d}{db_2}t_x(b_2) < 0$  (resp.  $\frac{d}{db_2}t_x(b_2) > 0$ ). ■

**Proof of Proposition 6.** We have

$$t_x(b_2) = \frac{1}{2}b_1 \exp(-b_1 a_x(b_2)) + \frac{1}{2}b_2 \exp(-b_2 a_x(b_2))$$

where  $a_x(b_2)$  is the solution of

$$\frac{1}{2} \exp(-b_1 a_x(b_2)) + \frac{1}{2} \exp(-b_2 a_x(b_2)) = x.$$

We know that

$$\frac{d}{db_2}t_x(b_2) = \varphi(a_x(b_2), b_1, b_2) = b_1 + b_2 \exp((b_1 - b_2) a_x(b_2)) - a_x(b_2) (b_2 - b_1) b_1.$$

We have already seen that  $\varphi(a, b_1, b_2)$  is decreasing in  $a$ , positive for  $a = 0$  and converges to  $-\infty$  when  $a$  converges to  $\infty$ . Let us denote by  $a(b_1, b_2)$  the solution of  $\varphi(a, b_1, b_2) = 0$ . Let us first consider the case  $x \leq \frac{1}{2}$ . The function  $a_x(b_2)$  is clearly decreasing with  $b_2$  and we have  $\lim_{b_2 \rightarrow b_1} a_x(b_2) = -\frac{\ln x}{b_1}$  and  $\lim_{b_2 \rightarrow \infty} a_x(b_2) = -\frac{\ln 2x}{b_1}$ . Furthermore, since  $\frac{\partial \varphi}{\partial a}$  is decreasing,  $\frac{\partial a}{\partial b_2}$  has the same sign as  $\frac{\partial \varphi}{\partial b_2}$ . Direct computations give  $\frac{\partial \varphi}{\partial b_2} = u - ab_1 - ab_2 u$  with  $u = \exp((b_1 - b_2) a)$  and  $a$  such that  $b_1 + b_2 u - b_1 a (b_2 - b_1) = 0$ . Substituting  $a$  in  $\frac{\partial \varphi}{\partial b_2}$  gives  $\frac{\partial \varphi}{\partial b_2} = (b_1 - b_2)^{-1} b_1^{-1} (b_2 b_1 u + b_1^2 + b_1^2 u + b_2^2 u^2) < 0$ . The function  $a(b_1, b_2)$  decreases then with  $b_2$  and we have  $\lim_{b_2 \rightarrow b_1} a(b_1, b_2) = \infty$  and  $\lim_{b_2 \rightarrow \infty} a_x(b_2) = 0$ . There exists then  $b_2^*$  such that  $a_x(b_2) = a(b_1, b_2)$  and  $\frac{d}{db_2}t_x(b_2) = 0$ .

It is immediate that for  $a_x(b_2) > a(b_1, b_2)$  we have  $\frac{d}{db_2}t_x(b_2) < 0$  and for  $a_x(b_2) < a(b_1, b_2)$  we have  $\frac{d}{db_2}t_x(b_2) > 0$ . It suffices to show that  $a_x(b_2)$  and  $a(b_1, b_2)$  cross only once to establish the result. Let us consider  $b_2^*$  such that  $a_x(b_2) = a(b_1, b_2)$  and let us compute  $\frac{d}{db_2}(a_x(b_2) - a(b_1, b_2))$  at  $b_2^*$ . By definition, we have  $a(b_1, b_2^*) = a_x(b_2^*)$  and we denote it by  $a^*$ . Direct computations give

$$\begin{aligned} & \frac{d}{db_2} (a_x(b_2^*) - a(b_1, b_2^*)) \\ &= \frac{\frac{\partial \varphi}{\partial b_2}}{\frac{\partial \varphi}{\partial a}}(a^*, b_1, b_2^*) - \frac{\frac{\partial \psi}{\partial b_2}}{\frac{\partial \psi}{\partial a}}(a^*, b_1, b_2^*) \\ &= \frac{A}{(b_1 e^{-a^* b_1} + b_2^* e^{-a^* b_2^*}) (b_1 + b_2^* e^{a^*(b_1 - b_2^*)}) (b_2^* - b_1)}. \end{aligned}$$

with

$$A = \left( a^* b_1^2 e^{-a^* b_1} + a^* b_1^2 e^{-a^* b_2^*} + (a^* b_1 b_2^* e^{-a^* b_1} + a^* b_1 b_2^* e^{-a^* b_2^*} - b_1 e^{-a^* b_1} - b_2^* e^{-a^* b_2^*}) e^{a^*(b_1 - b_2^*)} \right).$$

By definition, we have  $\varphi(a^*, b_1, b_2^*) = 0$ . Replacing  $b_2^* e^{a^*(b_1 - b_2^*)}$  by  $a^*(b_2^* - b_1) b_1 - b_1$ , we get

$$\frac{d}{db_2} (a_x(b_2^*) - a(b_1, b_2^*)) = \frac{b_1^2 a^* (e^{-a^* b_1} + e^{-a^* b_2^*} + a^*(b_2^* - b_1) e^{-a^* b_2^*})}{(b_1 e^{-a^* b_1} + b_2^* e^{-a^* b_2^*}) (b_1 + b_2^* e^{a^*(b_1 - b_2^*)}) (b_2^* - b_1)} > 0.$$

We have then that  $\frac{d}{db_2}a_x(b_2) > \frac{d}{db_2}a(b_1, b_2)$  each time these two functions cross. This means that they can cross only once.

For  $x \geq 1$ , we have already shown that  $\frac{d}{db_2}t_x(b_2)$  is positive for all  $b_2$  and  $t_x(b_2)$  is then increasing.

For  $1 \geq x \geq \frac{1}{2}$ , let us show that there is no crossing between  $a_x(b_2)$  and  $a(b_1, b_2)$ . For that purpose let us consider  $a = a_x(b_2) = a(b_1, b_2)$ . We have then

$$\begin{aligned} \frac{1}{2} \exp(-b_1 a) + \frac{1}{2} \exp(-b_2 a) &= x \\ \text{and } b_1 + b_2 \exp((b_1 - b_2) a) - a(b_2 - b_1) b_1 &= 0 \end{aligned}$$

which can be rewritten as follows

$$\begin{aligned} \exp((b_1 - b_2) a) &= 2x \exp(b_1 a) - 1 \\ \text{and } \exp((b_1 - b_2) a) &= \frac{a(b_2 - b_1) b_1 - b_1}{b_2} \end{aligned}$$

which gives

$$\begin{aligned} 2x \exp(b_1 a) - 1 &= \frac{a(b_2 - b_1) b_1 - b_1}{b_2} \\ \text{or } 2xb_2 \exp(b_1 a) &= (ab_1 + 1)(b_2 - b_1) \end{aligned}$$

Remark that  $\exp(b_1 a) \geq 1 + b_1 a$  and we have then  $2xb_2 \leq (b_2 - b_1)$  or  $(2x - 1)b_2 \leq -b_1$  which is impossible since we assumed  $2x - 1 \geq 0$ .

For  $x \geq \frac{1}{2}$ , it is easy to check that  $\lim_{b_2 \rightarrow b_1} a_x(b_2) = -\frac{\ln x}{b_1}$  and  $a_x(b_2) \sim_{b_2 \rightarrow \infty} -\frac{\ln(2x-1)}{b_2}$ . The limits of  $t_x$  derive from there. ■

**Proof of Proposition 7.** It suffices to prove directly the second point. We have

$$t_x(h) = \frac{1}{2}(b_1 + kh) \exp(-(b_1 + kh)a_x(h)) + \frac{1}{2}(b_2 + h) \exp(-(b_2 + h)a_x(h))$$

where  $a_x(h)$  is the solution of  $\psi(a_x(h), b_1, b_1 + kh) = x$ .

We get after computations

$$\frac{d}{dh}t_x(0) \equiv \phi(a_x(0), b_1, b_2)$$

where

$$\phi(a, b_1, b_2) = kb_1 \exp((b_2 - b_1) a) + b_2 \exp((b_1 - b_2) a) + b_1 + kb_2 + a(b_2 - b_1)(kb_2 - b_1).$$

For  $x \leq 1$ , we have  $a_x(0) \geq 0$  and it suffices to impose  $k \geq \frac{b_1}{b_2}$  to have  $\frac{d}{dh}t_x(0) \geq 0$  for all  $x$ .

For  $x \geq 1$ , we have  $a_x(0) \leq 0$  and it suffices to remark that  $\frac{\partial^2 \phi}{\partial a^2}$  is positive, that  $\frac{\partial \phi}{\partial a}(0, b_1, b_2) = (b_2 - b_1)(b_2 + b_1)(k - 1)$  and  $\phi(0, b_1, b_2) = (k + 1)(b_1 + b_2) > 0$ . It suffices to impose  $k \leq 1$  to have  $\frac{\partial \phi}{\partial a}(0, b_1, b_2) \leq 0$  and then  $\phi(a, b_1, b_2) \geq 0$  for

$a \leq 0$  and hence  $\frac{d}{dh} t_x(0) \geq 0$ . ■

**Proof of Proposition 8.** We have

$$t_x(h) = \frac{1}{2} (b_1 - h) \exp(-(b_1 - h)a_x(h)) + \frac{1}{2} (b_2 + h) \exp(-(b_2 + h)a_x(h))$$

where  $a_x(h)$  is the solution of  $\psi(a_x(h), b_1 - h, b_2 + h) = x$ . We want to show that for  $x < 1$ , we have  $\frac{dt_x}{dh}(0) < 0$  and for  $x > 1$  we have  $\frac{dt_x}{dh}(0) < 0$ .

We have  $\frac{da_x}{dh}(0) = \frac{a(\exp(-ab_1) - \exp(-ab_2))}{b_1 \exp(-ab_1) + b_2 \exp(-ab_2)}$  where  $a = a_x(0)$ , and

$$\begin{aligned} \frac{dt_x}{dh}(0) &= -\frac{1}{2} \exp(-ab_1) + \frac{1}{2} \exp(-ab_2) + \frac{1}{2} ab_1 \exp(-ab_1) - \frac{1}{2} ab_2 \exp(-ab_2) \\ &\quad - \left( \frac{1}{2} b_1^2 \exp(-ab_1) + \frac{1}{2} b_2^2 \exp(-ab_2) \right) \left( \frac{a(\exp(-ab_1) - \exp(-ab_2))}{b_1 \exp(-ab_1) + b_2 \exp(-ab_2)} \right) \end{aligned}$$

which is of the same sign as  $g(a)$  with

$$\begin{aligned} g(a) &= (b_1 \exp(-ab_1) + b_2 \exp(-ab_2)) (\exp(-ab_2) - \exp(-ab_1) + ab_1 \exp(-ab_1) - ab_2 \exp(-ab_2)) \\ &\quad - (a(\exp(-ab_1) - \exp(-ab_2))) (b_1^2 \exp(-ab_1) + b_2^2 \exp(-ab_2)) \\ &= -b_1 \exp(-2ab_1) + b_2 \exp(-2ab_2) - (b_2 - b_1 - ab_1^2 + ab_2^2) \exp(-a(b_1 + b_2)) \end{aligned}$$

which has the same sign as  $-\ell(a)$  with

$$\ell(a) = b_1 \exp(-a(b_1 - b_2)) - b_2 \exp(-a(b_2 - b_1)) + (b_2 - b_1 - ab_1^2 + ab_2^2).$$

We have  $\lim_{a \rightarrow \infty} \ell(a) = \infty$ ,  $\ell(0) = 0$  and  $\lim_{a \rightarrow -\infty} \ell(a) = -\infty$ . We also have

$$\ell'(a) = (b_1 - b_2)(-b_2 - b_1 - b_1 \exp(a(b_2 - b_1)) - b_2 \exp(a(b_1 - b_2))) > 0.$$

The function  $\ell$  is then negative on  $\mathbb{R}_-$  and positive on  $\mathbb{R}_+$ . Since  $a > 0$  for  $x < 1$  and  $a < 0$  for  $x > 1$  this gives the result. ■

**Proof of Proposition 9.** We denote by  $t_x^P$  (resp.  $t_x^Q$ ) the aggregate level of risk tolerance in the first (resp. second) population when the aggregate wealth is  $x$ . We have

$$t_x^P = p_1 b_1 \exp(-b_1 a_x^P) + p_2 b_2 \exp(-b_2 a_x^P)$$

where  $a_x^P$  is the solution of

$$p_1 \exp(-b_1 a_x^P) + p_2 \exp(-b_2 a_x^P) = x.$$

We have similar formulas for  $t_x^Q$  and we have that  $t_x^P \geq t_x^Q$  if and only

$$\frac{q_1 \exp(-b_1 a_x^Q)}{p_1 \exp(-b_1 a_x^P)} \geq \frac{q_2 \exp(-b_2 a_x^Q)}{p_2 \exp(-b_2 a_x^P)} \text{ or equivalently } p_1 \exp(-b_1 a_x^P) \leq q_1 \exp(-b_1 a_x^Q) \quad (7)$$

If  $x \leq 1$ , we have  $a_x^P \leq a_x^Q$  and the result is immediate. Let us now consider the case  $x \geq 1$ . Note that  $\exp(-b_1 a_x^Q)$  and  $\exp(-b_2 a_x^Q)$  correspond to the Pareto optimal allocations  $x_1^Q$  and  $x_2^Q$  in population  $Q$  while  $\exp(-b_1 a_x^P)$  and  $\exp(-b_2 a_x^P)$

correspond to the Pareto optimal allocation  $x_1^P$  and  $x_2^P$  in population  $P$ . We want to prove that  $\frac{p_1}{q_1}x_1^P \leq x_1^Q$ . Remark that the allocation  $(\frac{p_1}{q_1}x_1^P, \frac{p_2}{q_2}x_2^P)$  is feasible in population  $Q$ . Let us compare  $u_1' \left( \frac{p_1}{q_1}x_1^P \right)$  and  $u_2' \left( \frac{p_2}{q_2}x_2^P \right)$  or  $\left( \frac{p_1}{q_1}x_1^P \right)^{-\frac{1}{b_1}}$  and  $\left( \frac{p_2}{q_2}x_2^P \right)^{-\frac{1}{b_2}}$ . By construction we have  $(x_1^P)^{-\frac{1}{b_1}} = (x_2^P)^{-\frac{1}{b_2}}$  and since  $\frac{p_1}{q_1} \leq 1$  and  $\frac{p_2}{q_2} \geq 1$  we have  $u_1' \left( \frac{p_1}{q_1}x_1^P \right) \geq u_2' \left( \frac{p_2}{q_2}x_2^P \right)$ . Since Pareto optimal allocations are characterized by the condition  $u_1' \left( x_1^Q \right) = u_2' \left( x_2^Q \right)$  and since the allocation  $(\frac{p_1}{q_1}x_1^P, \frac{p_2}{q_2}x_2^P)$  is feasible, we necessarily have, by concavity of  $u_1$  and  $u_2$ ,  $\frac{p_1}{q_1}x_1^P \leq x_1^Q$ . ■

**Proof of Proposition 10.** We first prove that for  $x \leq 1$  an MLR shift is sufficient to increase the aggregate level of risk tolerance. Suppose that  $P \succ_{MLR} Q$ . Since the MLR order is stronger than the FSD order, we have for all nondecreasing function  $h$ ,  $E^P \left[ h(\tilde{b}) \right] \geq E^Q \left[ h(\tilde{b}) \right]$ , hence  $\Phi_Q(t) \geq \Phi_P(t)$  for  $t \geq 0$ . Since  $\Phi_Q$  and  $\Phi_P$  are decreasing, then for all  $x \leq 1$ ,  $\Phi_P^{-1}(x) \leq \Phi_Q^{-1}(x)$ . Since

$$\frac{d\tilde{Q}}{d\tilde{P}} = \frac{dQ}{dP} e^{-b(\Phi_Q^{-1}(x) - \Phi_P^{-1}(x))} \frac{E^P \left[ \frac{e^{-b\Phi_P^{-1}(x)}}{e^{-b\Phi_Q^{-1}(x)}} \right]}{E^Q \left[ \frac{e^{-b\Phi_Q^{-1}(x)}}{e^{-b\Phi_Q^{-1}(x)}} \right]},$$

we obtain that  $\frac{d\tilde{Q}}{d\tilde{P}}$  is the product (modulo a constant) of the decreasing function  $e^{-b(\Phi_Q^{-1}(x) - \Phi_P^{-1}(x))}$  with the decreasing function  $\frac{dQ}{dP}$  (both of them being positive) and is then decreasing and  $\tilde{P} \succeq_{MLR} \tilde{Q}$  which gives  $E^{\tilde{Q}} \left[ \tilde{b} \right] \leq E^{\tilde{P}} \left[ \tilde{b} \right]$  and  $t_x^{\tilde{P}} \geq t_x^{\tilde{Q}}$  or  $R_x^{\tilde{P}} \leq R_x^{\tilde{Q}}$ .

However, MLR does not guarantee an increase of the aggregate level of risk tolerance when  $x > 1$  as shown in the next counter-example.

Let us consider a model in which the distribution of the risk tolerances is given by  $\exp(-b)1_{b \geq 0}$  and let us consider a shift obtained through multiplication of the exponential density by  $1 + \varepsilon 1_{b > b^*}$  (for some  $b^* > 0$  and some  $\varepsilon > 0$ ) and by renormalization. The function  $b \rightarrow 1 + \varepsilon 1_{b > b^*}$  is nondecreasing and the shift is MLR. The aggregate level of risk tolerance in the initial population is given by  $t(x) = \int_0^\infty b \exp(-b) \exp(-ab) db$  where  $a$  solves  $\int_0^\infty \exp(-b) \exp(-ab) db = x$ . We obtain  $a = \frac{1}{x} - 1$  and  $t(x) = \frac{1}{(a+1)^2} = x^2$ .

After the shift, the aggregate level of risk tolerance is given by

$$\begin{aligned} t_\varepsilon(x) &= \frac{\int_0^\infty b \exp(-b) \exp(-a_\varepsilon b) db + \varepsilon \int_{b^*}^\infty b \exp(-b) \exp(-a_\varepsilon b) db}{\int_0^\infty \exp(-b) db + \varepsilon \int_{b^*}^\infty \exp(-b) db} \\ &= \frac{\frac{1}{(a_\varepsilon+1)^2} + \varepsilon \left( \frac{1}{a_\varepsilon+1} b^* \exp(-(a_\varepsilon+1)b^*) + \frac{1}{(a_\varepsilon+1)^2} \exp(-(a_\varepsilon+1)b^*) \right)}{1 + \varepsilon \exp(-b^*)} \end{aligned}$$

where  $a_\varepsilon$  solves  $\frac{\int_0^\infty \exp(-b) \exp(-a_\varepsilon b) db + \varepsilon \int_{b^*}^\infty \exp(-b) \exp(-a_\varepsilon b) db}{\int_0^\infty \exp(-b) db + \varepsilon \int_{b^*}^\infty \exp(-b) db} = x$  or  $\frac{\frac{1}{a_\varepsilon+1} + \varepsilon \frac{1}{a_\varepsilon+1} \exp(-(a_\varepsilon+1)b^*)}{1 + \varepsilon \exp(-b^*)} = x$ . We have then  $\frac{1}{a+1} = \frac{\frac{1}{a_\varepsilon+1} + \varepsilon \frac{1}{a_\varepsilon+1} \exp(-(a_\varepsilon+1)b^*)}{1 + \varepsilon \exp(-b^*)}$ . Let us consider the difference

$t_\varepsilon(x) - t(x)$ . It is positively proportional to

$$\Delta = \left( \frac{1}{\beta^2} + \varepsilon \left( \frac{1}{\beta} y^* \exp(-\beta y^*) + \frac{1}{\beta^2} \exp(-\beta y^*) \right) \right) (1 + \varepsilon \exp(-y^*)) - \left( \frac{1}{\beta} + \varepsilon \frac{1}{\beta} \exp(-\beta y^*) \right)^2$$

where  $\beta = a_\varepsilon + 1$  and where  $\frac{1}{a+1}$  has been replaced by its value as a function of  $\beta$ . This quantity is of the form  $\mu\varepsilon + \nu\varepsilon^2$  and since we want it to be positive for all  $\varepsilon > 0$  we have to check if  $\mu$  is positive. But  $\mu = \frac{(\exp(\beta b^*) - \exp(b^*) + \beta b^* \exp(b^*))}{\exp(\beta b^*) \exp(b^*) \beta^2}$  and is positively proportional to  $\exp(\beta b^*) - \exp(b^*) + \beta b^* \exp(b^*)$ . It is easy to remark that for  $\beta = \frac{1}{b^{*2}}$  and  $b^*$  sufficiently large, this quantity is negative. Let us chose then a pair  $(\beta, b^*)$  for which this quantity is negative and let us take  $\varepsilon$  sufficiently small such that the quantity  $\Delta$  itself is negative and let us finally take  $x = \frac{\frac{1}{(a_\varepsilon+1)^2} + \varepsilon \left( \frac{1}{a_\varepsilon+1} b^* \exp(-(a_\varepsilon+1)b^*) + \frac{1}{(a_\varepsilon+1)^2} \exp(-(a_\varepsilon+1)b^*) \right)}{1 + \varepsilon \exp(-b^*)}$ . The resulting shift leads then to a decrease of the collective level of risk tolerance at  $x$ . ■

**Proof of Proposition 11.** We just have to consider the case where  $x \geq 1$ . We have

$$\begin{aligned} \frac{d\tilde{Q}}{d\tilde{P}} &= \frac{dQ}{dP} e^{-b(\Phi_Q^{-1}(x) - \Phi_P^{-1}(x))} \frac{E^P \left[ e^{-b\Phi_P^{-1}(x)} \right]}{E^Q \left[ e^{-b\Phi_Q^{-1}(x)} \right]} \\ &= \exp(-kb - b(\Phi_Q^{-1}(x) - \Phi_P^{-1}(x))). \end{aligned}$$

To conclude, it is sufficient to show that  $k + (\Phi_Q^{-1}(x) - \Phi_P^{-1}(x)) > 0$ . We have

$$E^P \left[ e^{-b(\Phi_Q^{-1}(x) + k)} \right] = \frac{E^Q \left[ e^{-b\Phi_Q^{-1}(x)} \right]}{E^Q [\exp(kb)]} < x \text{ and since } \Phi_P^{-1}(x) \text{ is characterized by } E^P \left[ e^{-b\Phi_P^{-1}(x)} \right] = x \text{ we have } k + (\Phi_Q^{-1}(x) - \Phi_P^{-1}(x)) > 0. \blacksquare$$

**Proof of Proposition 12.** Assume that  $P$  and  $Q$  are symmetric with respect to some  $b^*$  with  $\frac{dQ}{dP}$  nonincreasing before  $b^*$  and nondecreasing after  $b^*$  then  $P \succ_{PD} Q$  and  $P \succ_{SSD} Q$  (Jouini-Napp, 2008). Let us denote by  $\Phi_P$  and  $\Phi_Q$  the functions respectively defined by  $\Phi_P(t) = \int e^{-bt} dP(b)$  and  $\Phi_Q(t) = \int e^{-bt} dQ(b)$ . Since  $e^{-bt}$  is decreasing and convex for  $t \geq 0$ , we have by SSD,  $\Phi_Q(t) \geq \Phi_P(t)$  for all  $t \geq 0$ . For  $x \leq 1$ ,  $\Phi_P^{-1}(x)$  and  $\Phi_Q^{-1}(x)$  are positive. Furthermore, both  $\Phi_Q$  and  $\Phi_P$  are decreasing and we have then  $\Phi_Q^{-1}(x) \geq \Phi_P^{-1}(x)$  for all  $x \leq 1$ .

Since  $b \rightarrow e^{-b\Phi_P^{-1}(x)}$  is decreasing and positive, we have by PD,  $\frac{E^P \left[ b e^{-b\Phi_P^{-1}(x)} \right]}{E^P \left[ e^{-b\Phi_P^{-1}(x)} \right]} \geq$

$$\frac{E^Q \left[ b e^{-b\Phi_P^{-1}(x)} \right]}{E^Q \left[ e^{-b\Phi_P^{-1}(x)} \right]} = E^{P_1} [b] \text{ with } \frac{dP_1}{dQ} = \frac{e^{-b\Phi_P^{-1}(x)}}{E^Q \left[ e^{-b\Phi_P^{-1}(x)} \right]}. \text{ Let us consider } Q_1 \text{ defined by } \frac{dQ_1}{dQ} = \frac{e^{-b\Phi_Q^{-1}(x)}}{E^Q \left[ e^{-b\Phi_Q^{-1}(x)} \right]}. \text{ We have } \frac{dQ_1}{dP_1} = \frac{E^Q \left[ e^{-b\Phi_P^{-1}(x)} \right]}{E^Q \left[ e^{-b\Phi_Q^{-1}(x)} \right]} e^{-b(\Phi_Q^{-1}(x) - \Phi_P^{-1}(x))} \text{ and is}$$

decreasing. Hence  $E^{P_1}[b] \geq E^{Q_1}[b] = \frac{E^Q \left[ \frac{be^{-b\Phi_Q^{-1}(x)}}{e^{-b\Phi_Q^{-1}(x)}} \right]}{E^Q \left[ \frac{e^{-b\Phi_Q^{-1}(x)}}{e^{-b\Phi_Q^{-1}(x)}} \right]}$ . We have then the result

for  $x \leq 1$ . For  $x \geq 1$ , since both distributions are symmetric with respect to  $b^*$ , we have  $\frac{E^P \left[ \frac{be^{-b\Phi_P^{-1}(x)}}{e^{-b\Phi_P^{-1}(x)}} \right]}{E^P \left[ \frac{e^{-b\Phi_P^{-1}(x)}}{e^{-b\Phi_P^{-1}(x)}} \right]} = \frac{E^P \left[ \frac{(2b^* - b)e^{-(2b^* - b)\Phi_P^{-1}(x)}}{e^{-(2b^* - b)\Phi_P^{-1}(x)}} \right]}{E^P \left[ \frac{e^{-(2b^* - b)\Phi_P^{-1}(x)}}{e^{-(2b^* - b)\Phi_P^{-1}(x)}} \right]} = 2b^* - \frac{E^P \left[ \frac{be^{b\Phi_P^{-1}(x)}}{e^{b\Phi_P^{-1}(x)}} \right]}{E^P \left[ \frac{e^{b\Phi_P^{-1}(x)}}{e^{b\Phi_P^{-1}(x)}} \right]}$

and  $\frac{E^Q \left[ \frac{be^{-b\Phi_Q^{-1}(x)}}{e^{-b\Phi_Q^{-1}(x)}} \right]}{E^Q \left[ \frac{e^{-b\Phi_Q^{-1}(x)}}{e^{-b\Phi_Q^{-1}(x)}} \right]} = 2b^* - \frac{E^Q \left[ \frac{be^{b\Phi_Q^{-1}(x)}}{e^{b\Phi_Q^{-1}(x)}} \right]}{E^Q \left[ \frac{e^{b\Phi_Q^{-1}(x)}}{e^{b\Phi_Q^{-1}(x)}} \right]}$ . Since  $\Phi_P^{-1}(x)$  is negative for  $x \geq 1$ ,

the function  $b \rightarrow e^{b\Phi_P^{-1}(x)}$  is decreasing and positive, and we have by PD that  $\frac{E^P \left[ \frac{be^{b\Phi_P^{-1}(x)}}{e^{b\Phi_P^{-1}(x)}} \right]}{E^P \left[ \frac{e^{b\Phi_P^{-1}(x)}}{e^{b\Phi_P^{-1}(x)}} \right]} \geq \frac{E^Q \left[ \frac{be^{b\Phi_Q^{-1}(x)}}{e^{b\Phi_Q^{-1}(x)}} \right]}{E^Q \left[ \frac{e^{b\Phi_Q^{-1}(x)}}{e^{b\Phi_Q^{-1}(x)}} \right]}$  or  $E^{P_1}[b] = \frac{E^Q \left[ \frac{be^{-b\Phi_P^{-1}(x)}}{e^{-b\Phi_P^{-1}(x)}} \right]}{E^Q \left[ \frac{e^{-b\Phi_P^{-1}(x)}}{e^{-b\Phi_P^{-1}(x)}} \right]} \geq \frac{E^P \left[ \frac{be^{-b\Phi_P^{-1}(x)}}{e^{-b\Phi_P^{-1}(x)}} \right]}{E^P \left[ \frac{e^{-b\Phi_P^{-1}(x)}}{e^{-b\Phi_P^{-1}(x)}} \right]}$ .

Since  $\frac{dQ_1}{dP_1} = \frac{E^Q \left[ \frac{e^{-b\Phi_P^{-1}(x)}}{e^{-b\Phi_Q^{-1}(x)}} \right]}{E^Q \left[ \frac{e^{-b\Phi_Q^{-1}(x)}}{e^{-b\Phi_Q^{-1}(x)}} \right]} e^{-b(\Phi_Q^{-1}(x) - \Phi_P^{-1}(x))}$ , it is now increasing, we have then

$E^{P_1}[b] \leq E^{Q_1}[b]$  which gives the result. ■

**Proof of Proposition 13.** Let us denote by  $t(x, h)$  the aggregate level of risk tolerance at  $x$  when the individual levels of risk tolerance are given by  $b_1$  and  $b_2 + h$ , we have

$$t(x, h) = \frac{\exp\left(-\frac{\ln x}{b_1}\right) + \exp\left(-\frac{\ln x}{b_2+h}\right)}{\frac{1}{b_1} \exp\left(-\frac{\ln x}{b_1}\right) + \frac{1}{b_2+h} \exp\left(-\frac{\ln x}{b_2+h}\right)}$$

and

$$\frac{\partial t}{\partial h}(x, 0) = \frac{b_1 \left( (b_2 - b_1) (\ln x) e^{-\frac{\ln x}{b_1}} + b_1 b_2 \left( e^{-\frac{\ln x}{b_1}} + e^{-\frac{\ln x}{b_2}} \right) \right) e^{-\frac{\ln x}{b_2}}}{b_2 \left( b_1 e^{-\frac{\ln x}{b_2}} + b_2 e^{-\frac{\ln x}{b_1}} \right)^2}$$

and we clearly have  $\frac{\partial t}{\partial h}(x, 0) > 0$  for  $x > 1$ . Furthermore, if we denote by  $L(x)$  the quantity  $L(x) = (b_2 - b_1) (\ln x) + b_1 b_2 \left( 1 + e^{\ln x \left( \frac{1}{b_1} - \frac{1}{b_2} \right)} \right)$ , we have  $\frac{dL}{d \ln x} = (b_2 - b_1) \left( 1 + e^{\ln x \left( \frac{1}{b_1} - \frac{1}{b_2} \right)} \right) > 0$  and  $\lim_{x \rightarrow 0} L(x) = -\infty$  and  $L(1) = 2b_1 b_2 > 0$ .

There exists then  $x^* < 1$  such that  $\frac{\partial t}{\partial h}(x, 0) < 0$  for  $x < x^*$  and  $\frac{\partial t}{\partial h}(x, 0) > 0$  for  $x > x^*$ . ■

**Proof of Proposition 14.** Let us denote by  $R(x, h)$  the aggregate level of risk aversion at  $x$  when the individual levels of risk aversion are given by  $R_1 - h$  and  $R_2 + h$ , we have  $R(x, h) = \frac{(R_1 - h) \exp(-\ln x (R_1 - h)) + (R_2 + h) \exp(-\ln x (R_2 + h))}{\exp(-\ln x (R_1 - h)) + \exp(-\ln x (R_2 + h))}$  and  $\frac{\partial R}{\partial h}(x, 0) = -\frac{e^{-2 \ln x R_1} - e^{-2 \ln x R_2} + 2 \ln x (R_2 - R_1) e^{-\ln x R_1} e^{-\ln x R_2}}{(e^{-\ln x R_1} + e^{-\ln x R_2})^2}$  which is clearly negative for  $x > 1$  and positive for  $x < 1$ . ■

**Proof of Proposition 15.** Let us denote by  $U(x, 1)$  and  $U(x, 2)$  the social utility functions respectively associated to  $P_1$  and  $P_2$ . We denote respectively

by  $F(i, 1)$  and  $F(i, 2)$  the cumulative distributions of  $P_1$  and  $P_2$ . We have

$$U'(x, j) = \int \frac{\partial u}{\partial x}(x, i) F'(i, j) di, \quad j = 1, 2.$$

By assumption,  $\frac{\partial u}{\partial x}(x, i)$  is log-supermodular. Furthermore, since  $P_2 \succ_{MLR} P_1$ ,  $F'(i, j)$  is also LSPM. By Karlin's Theorem  $U'(x, j)$  is log-supermodular. Therefore,  $\frac{\partial \ln U'(x, j)}{\partial x}$  increases with  $j$  or in other words

$$-\frac{U''(x, 1)}{U'(x, 1)} \geq -\frac{U''(x, 2)}{U'(x, 2)}$$

which gives  $R_1(x) \geq R_2(x)$ . ■

**Proof of Proposition 16.** Let us denote by  $U(x, 1)$  and  $U(x, 2)$  the social utility functions respectively associated to  $P_1$  and  $P_2$ . Since  $\frac{\partial u}{\partial x}(x, i)$  is increasing in  $i$  and  $-\frac{u''_i(x)}{u'_i(x)}$  is decreasing in  $i$ , there exists a nonnegative and decreasing function  $\Psi^x$  such that  $E \left[ u'_i(x) \left| -\frac{u''_i(x)}{u'_i(x)} \right. \right] = \Psi^x \left( -\frac{u''_i(x)}{u'_i(x)} \right)$  for  $i \in I$ . We have then

$$\begin{aligned} -\frac{U''(x, j)}{U'(x, j)} &= \frac{\int -\frac{u''_i(x)}{u'_i(x)} u'_i(x) dP_j(i)}{\int u'_i(x, j) dP_j(i)}, \quad j = 1, 2 \\ &= \frac{E^{P_j^x} [X \Psi^x(X)]}{E^{P_j^x} [\Psi^x(X)]}. \end{aligned}$$

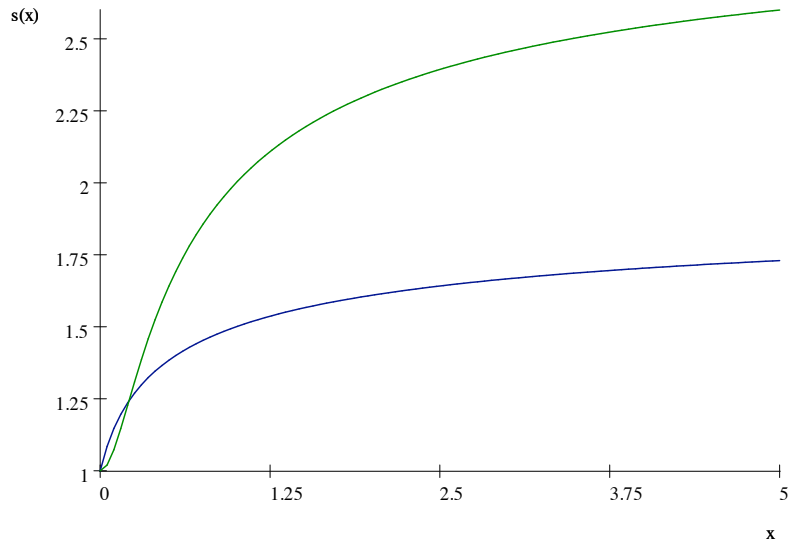
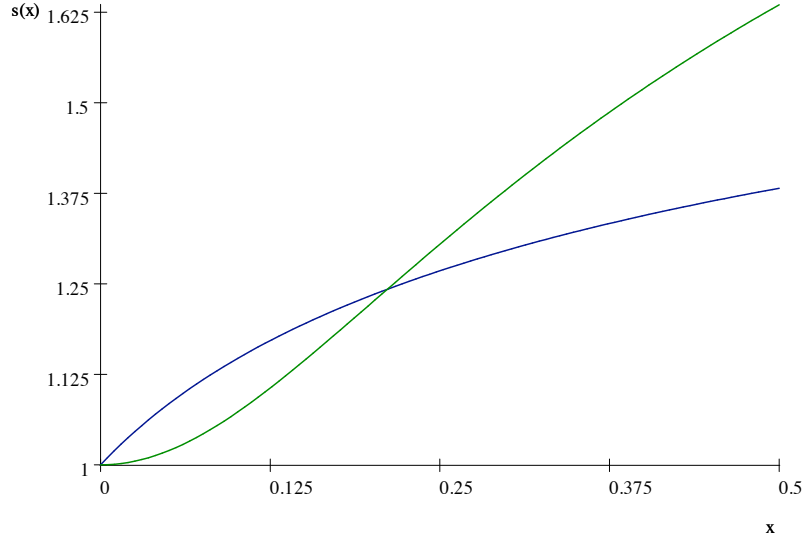
By Portfolio Dominance we immediately have  $\frac{E^{P_2^x} [X \Psi^x(X)]}{E^{P_2^x} [\Psi^x(X)]} \geq \frac{E^{P_1^x} [X \Psi^x(X)]}{E^{P_1^x} [\Psi^x(X)]}$ . ■

## References

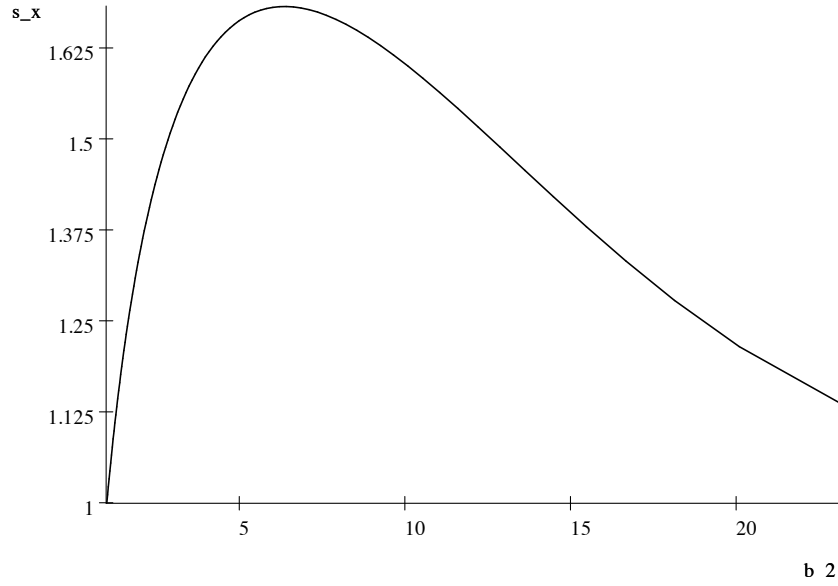
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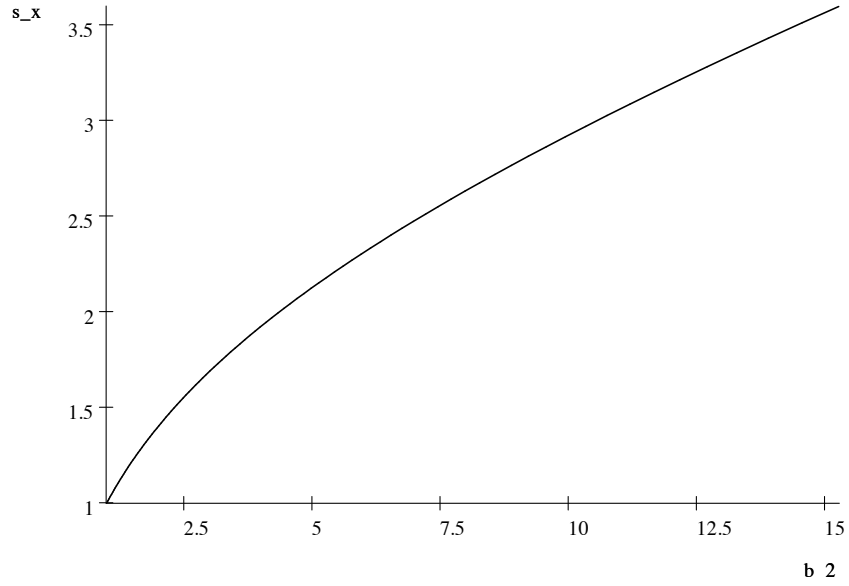




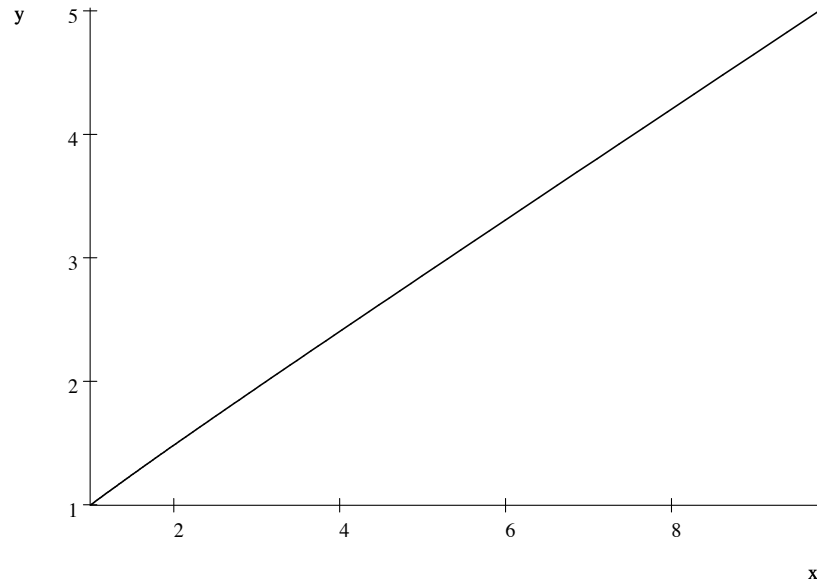
**Figure 1 & 2:** These figures represent, at two different scales, the aggregate relative risk tolerance as a function of the total wealth with  $b_1 = 1$  and  $b_2 = 2$  for the blue curve and  $b'_1 = 1$  and  $b'_2 = 3$  for the green curve. Both curves converge slowly to the associated level of relative risk tolerance as can be shown on the second figure. An increase of the risk tolerance level of the most risk tolerant agent leads to an increase (decrease) of the aggregate level of risk tolerance above (below) a given threshold  $x^* \leq 1$ . With our parameters, we have  $x^* = 0.21$ . When  $b_2 = 2$  and  $b'_2$  is in the neighborhood of  $b_2$ , we have  $x^* = 0.14$ . When  $b'_2 = 3$  and  $b_2$  is in the neighborhood of  $b'_2$ , we have  $x^* = 0.27$ .



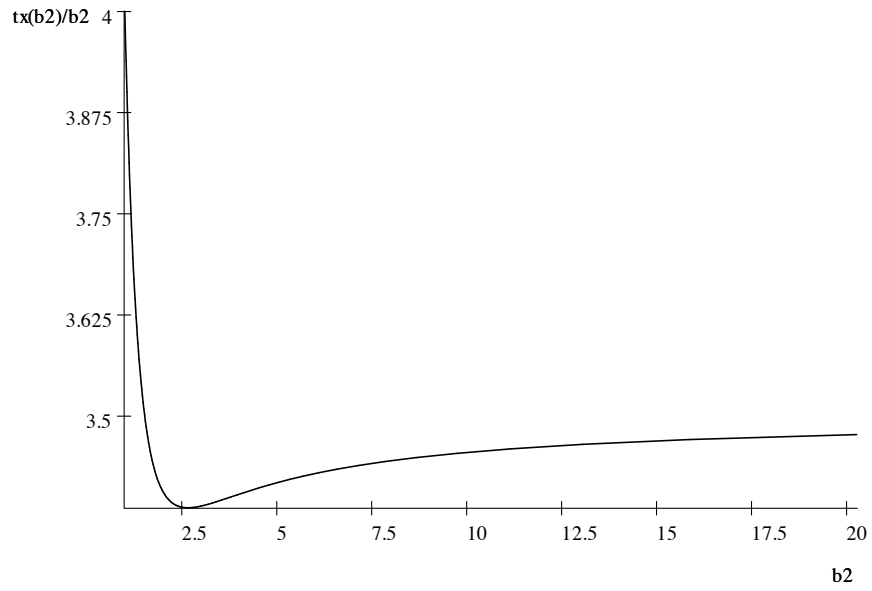
**Figure 3:** In a setting with two agents with levels of relative risk tolerance  $b_1$  and  $b_2$ , we represent the aggregate relative risk tolerance  $s_x(b_2)$  as a function of  $b_2$  for  $b_1 = 1$  and for  $x = 0.4$ . For  $b_2 = b_1 = 1$ ,  $s_x$  is equal to 1. The aggregate relative risk tolerance increases then decreases with  $b_2$  and  $\lim_{b_2 \rightarrow \infty} s_x(b_2) = b_1$



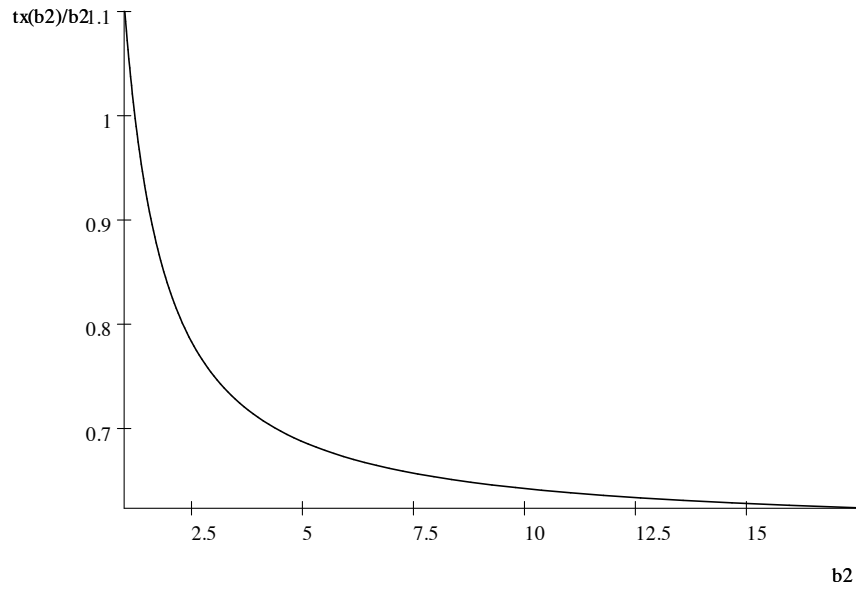
**Figure 4:** In a setting with two agents with levels of relative risk tolerance  $b_1$  and  $b_2$ , we represent the aggregate relative risk tolerance  $s_x(b_2)$  as a function of  $b_2$  for  $b_1 = 1$  and for  $x = 0.55$ . For  $b_2 = b_1 = 1$ ,  $s_x$  is equal to 1. The aggregate relative risk tolerance increases with  $b_2$ .



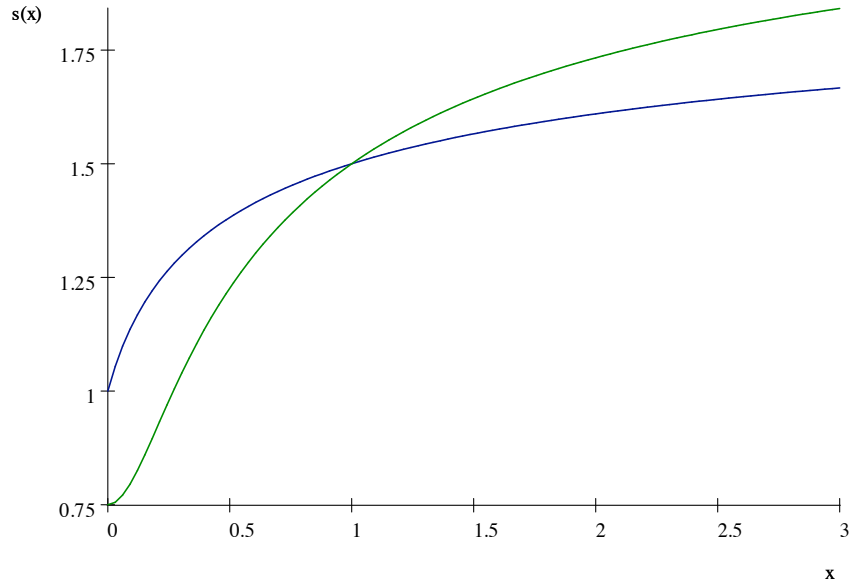
**Figure 5:** In a setting with two agents with levels of relative risk tolerance  $b_1$  and  $b_2$ , we represent the aggregate relative risk tolerance  $s_x(b_2)$  as a function of  $b_2$  for  $b_1 = 1$  and for  $x = 0.9$ . For  $b_2 = b_1 = 1$ ,  $s_x$  is equal to 1. The aggregate relative risk tolerance increases with  $b_2$  and is almost linear.



**Figure 6:** In a setting with two agents with levels of relative risk tolerance  $b_1$  and  $b_2$ , we represent the ratio  $t_x(b_2)/b_2$  for  $x = 4$  and  $b_1 = 1$ . For  $b_2 = b_1 = 1$  it is immediate that  $t_x(b_2)/b_2 = t_x(b_2) = x$ . The ratio converges to  $x - 0.5 = 3.5$  when  $b_2$  converges to  $\infty$ .



**Figure 7:** In a setting with two agents with levels of relative risk tolerance  $b_1$  and  $b_2$ , we represent the ratio  $t_x(b_2)/b_2$  for  $x = 1.1$  and  $b_1 = 1$ . For  $b_2 = b_1 = 1$  it is immediate that  $t_x(b_2)/b_2 = t_x(b_2) = x$ . The ratio converges to  $x - 0.5 = 0.6$  when  $b_2$  converges to  $\infty$ .



**Figure 8:** In a setting with two agents with levels of relative risk tolerance  $b_1$  and  $b_2$ , we represent the aggregate relative risk tolerance as a function of the total wealth with  $b_1 = 1$  and  $b_2 = 2$  for the blue curve and  $b'_1 = \frac{3}{4}$  and  $b'_2 = \frac{9}{4}$  for the green curve. The shift from  $(b_1, b_2)$  to  $(b'_1, b'_2)$  is a Mean Preserving Spread as in Proposition 8 and the two curves cross at  $x^* = 1$ .