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Collective attention and ranking methods

Gabrielle DEMANGE¹

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Abstract

Ranking systems are becoming increasingly important in many areas, in the Web environment and academic life for instance. Presumably a ranking helps individuals to make decisions by providing them with relevant information. In a world with a tremendous amount of choices, a ranking plays also the crucial role of influencing the attention that is devoted to the various alternatives. In recurrent situations, attention will, in turn, alter the new statements on which subsequent rankings will be based. The paper proposes an analysis of this feedback by studying some reasonable dynamics that a ranking method may induce. The feedback is shown to depend strongly on the used ranking method. Two main families of methods are investigated, one based on the notion of 'handicaps', the other one on the notion of peers' rankings.

Keywords ranking, scoring, invariant method, peers' method, attention, handicap, scaling matrix, dynamics through influence.

1 Introduction

The use of rankings is becoming pervasive in many areas, including the Web environment for ranking pages and academia for ranking researchers, journals, and universities. The public good aspect of information explains the use of rankings. Rankings are based on a costly process of gathering and summarizing some relevant information on the alternatives in a particular topic. When such information is relevant to anyone, the publication of rankings avoids each individual to pay the search and processing costs. For that very reason, rankings have some influence on the attention that is devoted to the various alternatives. In recurrent situations, attention will, in turn, alter the new statements on which subsequent rankings will be based. This paper proposes an analysis of the feedback between rankings, attention intensities, and statements by studying some reasonable dynamics.

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A ranking problem is described by a set of items to be ranked and a set of 'experts' who provide statements on the items on which the ranking will be based. A ranking is cardinal up to a multiplicative scalar, meaning that a ranking assigns relative scores to items. Let us describe some prominent problems. In a ranking of journals based on citations, journals are both the items to be ranked and the experts; a journal's statement is derived from the number of citations made by its articles towards articles published in the different journals. In a ranking of Web pages based on the link structure, a page's statements is given by its links towards other pages. Here also the items to be ranked -the Web pages- coincide with the experts. This is not the case in our third example, an apportionment problem. The problem is to allocate the seats in an assembly to parties as a result of the votes of various electoral bodies, regional for example. Here the items to be ranked are the parties, the experts are the regions, and a region's statement is given by the number of votes gathered by each party in the region. Voting by committee in which voters are asked to weigh the candidates is also a ranking problem. However, such situations are plagued with plain strategic issues that are not the subject of this paper. The focus is on recurrent situations in which the influence of experts is 'diffuse', channelled through the impact of rankings on attention.

The analysis bears on ranking methods that satisfy two important properties. The first property is *intensity invariance*. The property has been introduced for dealing with the situations in which the 'intensity' of statements is not controlled, as in the two first examples above, where neither the number of citations per article nor the number of links from a page are restricted *a priori*. To avoid an expert to increase its impact on the final ranking by an inflation in its statements, experts' statements are adjusted so as to obtain an 'intensity invariant' ranking method (there are other justifications, as explained in the paper). In the journals' example, the cites of a journal are deflated by the average number of cites per article, so that the ranking depends on the proportions of the cites allocated by journals to the different journals. In the Web environment, one deflates a link from a site by the total number of links from that site. This is what is performed by one of the most well known methods, the 'invariant' method, which serves as a basis to PageRank of Google (Brin and Page 1998). Factoring out the intensity of outward links avoids pages to increase their score by inflating the number of links.

The second property, that of *supporting weights*, views a method as simultaneously assigning scores to the items and weights to the experts. The ranking writes as a weighted combination of the experts' statements and, furthermore, the scores and the weights form some equilibrium relationship. The property is satisfied by most current methods although it has not been made explicit so far. The counting method, which ranks items according to their received totals, simply assigns identical weights to the experts whatever their statements. The invariant method, alluded to above,

determines which pages are influential on the basis that a page is influential if it is heavily cited by other influential pages. This generates a loopback effect between the score as an item and the weight as an expert. By its very definition the invariant method assigns scores and weights that form an equilibrium for the process of equalization of the weights to the scores. The property of supporting weights is useful for various reasons. In particular, meaningful methods can be introduced by considering new equilibrium relationships. Also it allows us to give a definition to peers' methods, which does not simply require the sets of experts and items coincide but also that scores and weights be strongly related, as in the invariant method.

The first part of the paper considers static problems, in which the experts' statements are given. I introduce a new ranking method that is both intensity-invariant and supported by equilibrium weights. The equilibrium is based on the notion of handicaps. There are indeed strong relationships between rankings and handicaps. Since the purpose of handicaps is to adjust the items' marks so as to equalize their 'strength', rankings and handicaps are inversely related to each other: the handicap of i is half that of ℓ if i can be said to be twice as good as ℓ , that is if i 's score is the double of that of ℓ . The method, called the *handicap-based method*, is characterized by simple properties. The computation of the handicap-based ranking relies on a well-known procedure of matrix scaling, called RAS method or iterative proportional fitting procedure.²

The second part of the paper studies a recurrent framework to analyze the influence of rankings. This influence is driven by their impact on attention intensities. In a context in which the number of alternatives to consider is huge, experts cannot carefully assess each one and tend to pay more attention to those whose score is higher. For example, while working on a paper, a researcher who uses rankings tends to read more the journals whose ranks are higher. Voters tend to pay more attention to parties with large scores in the past because these parties have more chances of winning or they have more resources to spend on the media. An 'influence function' describes how the current ranking modifies attention intensities. This generates a joint dynamics on rankings and statements because statements depend on both preferences and attention: the current ranking modifies attention intensities, hence the next statements on which next ranking is based. An intuition is that, as past statements have an impact on future statements through rankings computation, we might expect 'the rich to get richer'. For example journals with a lot of past citations are more likely to be cited again, which may result in an improvement in their scores. However, the impact of such self-enforcing mechanism may differ according to the ranking method. Our aim is to investigate more precisely this link between a ranking method and the dynamics.

The analysis is carried out on intensity-invariant methods but the main insights remain, and are

²The procedure is used in various areas, in statistics for adjusting contingency tables, in economics for balancing international trade accounts (Bacharach 1965) or for filling missing accounting data as in cross-banking relationships.

most likely to be reinforced, when intensity is not factored out. The influence function is first assumed to take a simple linear form. Contrasted results are obtained for two different classes of methods. The first class, called the generalized handicap-based methods, is obtained from the handicap-based method by modifying the experts' weights. The class includes both the handicap-based and the counting methods. These methods guarantee stability in the sense that, given preferences for the experts, the sequence of rankings converges towards a unique rest point (see more precisely Proposition 3).

The second class is the class of peers' methods. The rationale behind a peers' method is that the ability of an individual to perform (measured by his score) is correlated with his ability to judge others' performance. In particular, for a method supported by weights, a minimal requirement is that an individual who obtains a small score is assigned a small expert's weight. This defines a peers' method. The invariant method is a peers' method since scores and experts' weights are equalized. In Demange (2009) I show that the dynamics for the invariant method may admit multiple limit points. According to Propositions 4 and 5, such multiplicity is bound to happen: Whatever peers' method, the dynamics may admit multiple limit points for some preferences, each one corresponding to a different support (the support is the subset of items that keep a positive score). Furthermore, the supports of the limit points are independent of the peers' method. Such result illustrates the self-sustaining aspect of a peers' method. Self-sustainability here is not obtained through plain manipulation but through the coordination device induced by the influence of the ranking.

An influence function is a crude but simple way of modeling some form of bounded rationality or myopic behavior due to search costs or persuasion bias for example. An alternative framework is to base the influence through a Bayesian reasoning, which requires to make explicit the type of information individuals are searching for. Not only a Bayesian framework is much more complex in the problems we are interested in because information is typically multi-dimensional, but also there is evidence that people might not be Bayesian. In particular, the marketing literature puts forward a persuasion bias, according to which the repetition of the same information has an effect (see deMarzo, Vayanos, and Zwiebel (2003) for a discussion and references on this aspect). An influence function may be interpreted as modeling a persuasion bias, in which a higher score is akin to a high repetition of the same piece of information.

This paper is about the convergence of behaviors and statements. This is also the concern of the large literature that analyzes the influence of opinions channelled by 'neighbors' in a network, as deMarzo et al. (2003), the book of Goyal (2005), or Golub and Jackson (2008) for instance. The analyzed situation is one in which individuals receive private signals about a state of the world and repeatedly communicate with their neighbors. The influence of communicated opinions may be

driven by bayesian revision or specified by some up-dating rule. A main question is whether (non-strategic) communication will lead opinions to converge to a common belief and, if they converge, how the limit belief relates to the initial opinions and the network structure. In our setting instead, information, which is embodied in the ranking, is made public and influences all experts in an identical way. The impact however differs across experts because they differ in their preferences. The analysis shows that the interplay of preferences and the ranking method may induce a variety of different outcomes.

Researchers in computer science have also concerns about the influence of the rankings provided by search engines. The main criticism is that rankings are biased towards already popular Web pages, thus preventing the rise in popularity of recently created high quality pages. Proposals to correct the bias, such as introducing some randomness in the rankings (Pandey et al. 2005), or to account of the date of creation of a page in the computation of the ranking (Cho, Roy, and Adams 2005) have been advanced.

Finally, the paper is also related to social choice theory, especially to the axiomatization approach that aims to characterize methods through properties or 'axioms'. The most relevant papers for our work are the axiomatizations of the invariant method (Palacios-Huerta and Volij 2004, Slutzki and Volij 2006, and Altman and Tennenholtz 2005). Here the handicap-based method is introduced and an axiomatization is provided.

The paper is organized as follows. Next section presents ranking methods, gives examples, and defines some properties. Section 3 introduces the handicap-based method and provides a characterization with some simple axioms. Section 4 is devoted to the dynamics under a linear influence function for the generalized handicap-based methods and the peers' methods. Section 5 presents some extensions and concluding remarks. Some technical proofs are given in Section 6.

2 Ranking methods

This section describes the framework, recalls some well known methods, and introduces standard properties.

2.1 The framework

Let N be the set of items to be ranked. Items can be individuals, journals, articles, political parties. Let M be the set of 'judges'. Judges can be experts, voters. In the judgment by 'peers', the two sets coincide. In the following, an element of N is called an *item* and an element of M an *expert*, keeping in mind the different interpretations. The cardinality of N is denoted by n and that of M

by m .

Experts provide some statements on which the ranking of the items will be based. Experts' statements are described by a $n \times m$ *statement matrix* $\pi = (\pi_{i,j})$, in which j 's column represents j 's statement over N . Given these statements, one seeks for a ranking that assigns to each item i a non-negative number r_i , called the *score* of i . The aim of the ranking is to provide the relative strength of the n items. This means that not only the order of the item scores matters but also their values up to a multiplicative constant. Thus, normalizing the sum of the scores to 1, a ranking of N is given by a vector r in the simplex Δ_N : $\Delta_N = \{r = (r_i) \in \mathbb{R}^n, r_i \geq 0, \sum_i r_i = 1\}$.

A method assigns a ranking to each possible statement matrix. Statement matrices are first restricted to be positive, that is $\pi_{i,j}$ to be all positive. Formally, given N and M , a *ranking or scoring* method F assigns to each positive $n \times m$ matrix π a positive ranking $r = F(\pi)$ in Δ_N .

The three examples described in the introduction are cast into this framework.

For ranking journals based on citations, N and M are both given by the set of journals to be compared, statements are the number of citations by articles in journal j towards articles published in journal i . To be more precise, let $C_{i,j}$ be the total number of cites from j to i in a relevant period. Cites are normalized to account for the total number n_j of articles in j : this gives matrix π in which the value $\pi_{i,j} = \frac{C_{i,j}}{n_j}$ is the average number of references of an article from j to i .

For ranking Web pages based on the link structure, the two sets of items and experts N and M coincide with the set of 'relevant' pages and the method defines a ranking of the pages that is based on the links within N . Hence the statement matrix π is the adjacent matrix of the Web network: it has $\pi_{i,j}$ equal to 1 if page j points to i and 0 otherwise. The matrix has many zeros because many pages are not pointing to each other. A perturbation technic makes the matrix positive.

In an apportionment problem, N is the set of parties, M is the set of constituencies, say regions, and statements are the number of votes from region j to party i . One wants to allocate a given number of seats to the parties, under possible constraints on the representation of regions, i.e. not simply counting the total of the votes obtained by each party). A first task amounts to assign a ranking to parties based on π (this leaves aside the allocation of the seats to parties within regions; for a detailed analysis of this problem, see Balinski and Demange 1989-a and 1989-b).

Let us start by describing natural properties that one may want any method to satisfy. Intensity invariance, uniformity, and exactness appear in the literature under various names.

Basic properties Intensity invariance requires the ranking not to be affected by the intensity of an expert's statements, which can be for example scaled so that their total is 1. We use the notation:

$$\pi_{*j} = \sum_{i \in N} \pi_{i,j}.$$

Definition 1 A method F is **intensity invariant** if $F(\pi') = F(\pi)$ for π' the matrix obtained from

π by multiplying a column by any positive μ , or equivalently

$$F([\pi]) = F(\pi) \text{ where } [\pi]_{i,j} = \frac{\pi_{i,j}}{\pi_{*j}}, \text{ each } i, j.$$

The equivalence is obtained by scaling iteratively each column of a matrix π so as to obtain matrix $[\pi]$, in which intensity is 'factored out'. The value $[\pi]_{i,j}$ is the *share* assigned by j to i (or received by i from j). Intensity invariant methods are easily derived from any method as follows. The *intensity invariant version* G of a method F is defined by setting $G(\pi) = F([\pi])$ for each π , that is by applying F after scaling each expert's statement. A matrix whose columns sums are equal to 1 is called *normalized*.

The intensity invariance property can be justified differently depending on the context. Let us give two different justifications.

A first justification, as alluded to in the introduction, is that one does not want an expert to 'weigh' more because of an inflation in its statements. In the case of journals, factoring out reference intensity avoids to introduce a bias due to the fact that the average number of cites per article differs across journals.³ In the case of Web pages, intensity invariance avoids that a page improves its score simply by multiplying the pages it points to.

A second justification is that one wants to adjust *a priori* the experts' statements, possibly in a biased way. Intensity invariance basically assigns *a priori* a sum to each expert's statements, where *a priori* means independently of the statement matrix π . Requiring an equal sum for each expert's statements does not imply some form of equal treatment property if the method is not anonymous with respect to the experts (namely the ranking is not necessarily independent of a permutation of the columns). To illustrate this point consider an apportionment problem, in which an expert represents an electoral regional body and statements are the number of votes in each region. One may want to over-represent a region with small population and under-represent one with large population so that each region has the same expected impact on the final result. This requirement is achieved by assigning the same total of 'adjusted' votes in each region by adjusting for the sizes of the population. More generally, in some problems, one may want to treat experts in a different way. This feature can easily be accommodated by assigning different experts' totals and by adjusting each column sum to its assigned total.

To state the next properties, we introduce balanced matrices. Balanced matrices constitute a benchmark in which there is no rationale for distinguishing between items if experts are not

³Recall that $\pi_{i,j}$ is the average number of references of an article from j to i . Factoring out intensity implies that one works with $[\pi]_{i,j} = \pi_{i,j}/\pi_{*j}$. The j 's statement vector in $[\pi]$ describes in which proportion the cites made by an article in j are received by i on average. See Palacios-Huerta and Volij (2004) for an analysis of the impact of cite intensities on the ranking of economic journals.

discriminated *a priori*: rows obtain identical totals or, put equivalently, rows obtain equal scores under the counting method (the counting method simply assigns to each item a score equal to the average experts statements). Formally, a matrix is said to be (a, b) -balanced if each row receives the same total as well as each column: $\pi_{i*} = a$ each i , $\pi_{*j} = b$ each j (where a and b are linked by $na = mb$). When there is no need to specify the values for a and b , we simply say that the matrix is balanced. For a square matrix, n equal to m , the matrix is proportional to a bi-stochastic matrix. Let \mathbf{e}_N denotes the 'flat' ranking with all its components equal: $\mathbf{e}_N = (\frac{1}{n})$.

Definition 2 *A method F is uniform if $F(\pi) = \mathbf{e}_N$ for all balanced π . A method is exact if the reverse is true for normalized matrices: $F([\pi]) = \mathbf{e}_N$ implies that $[\pi]$ is balanced.*

A method is uniform if it assigns equal scores to each balanced statement matrix. Exactness asks conversely that items obtain equal scores only if they receive the same total of normalized experts' statements. Note that requiring that equal scores are assigned only to balanced matrices (i.e., that $F(\pi) = \mathbf{e}_N$ implies that π is balanced) is too strong: no intensity invariant and uniform method satisfies such a requirement. To see this, start with a balanced matrix. Its ranking is e_N by uniformity. Multiply the columns by distinct numbers. The obtained matrix is not balanced but intensity invariance requires its ranking to be e_N .

Homogeneity The homogeneity property that we introduce now is very natural but has not yet be considered in the literature, as far as we know.⁴ For cardinal rankings, the values taken by the scores, and not only the order, matter. In fact, due to the normalization, a ranking is characterized by the relative scores. Saying that the score of i is twice the score of k should mean that i is 'twice as good' as k . In this interpretation, if we start with a matrix and all the shares of i are multiplied by a factor, i 's relative position should be multiplied by the same factor. This is the homogeneity property.

Definition 3 *A method is homogeneous if multiplying row i by a positive scalar λ multiplies its rank relative to other rows by the same λ .*

Uniformity and exactness are inherited by a method when factoring out intensities. This is not true for the homogeneity property, as illustrated by the counting method.

Three methods: The counting, invariant, and Hits methods We define the methods for any matrix π . The intensity invariant version is obtained by applying the method to the normalized matrices $[\pi]$.

⁴The homogeneity axiom introduced in Palacios-Huerta and Volij (2004) differs from ours: their axiom bears on a given matrix that has two proportional rows whereas ours bears on two distinct matrices.

The counting method simply assigns scores proportional to the total number of received statements:

$$r_i = \frac{\pi_{i*}}{\sum_{\ell} \pi_{\ell*}} \text{ where } \pi_{i*} = \sum_{j \in M} \pi_{i,j}. \quad (1)$$

The counting method defined by (1) is homogeneous but the intensity invariant version is not. Consider the following example

$$\pi = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \text{ and } \pi' = \begin{pmatrix} 4 & 2 \\ 1 & 2 \end{pmatrix},$$

in which matrix π' is obtained by multiplying the first row of π by 2. The intensity invariant version of the counting method assigns equal scores, $(1/2, 1/2)$, to π but $(13/20, 7/20)$ to π' instead of $(2/3, 1/3)$ as required by homogeneity. The reason is that expert 1 likes relatively more item 1 than expert 2 does. Hence the total of expert 1 increases more than that of expert 2 so that when normalizing π' , the adjustment on 1's statements is larger than on 2's. This explains why the score of 1 is less than doubled relative to that of 2.

The counting method treats experts equally. Instead, the next methods - the Liebowitz-Palmer⁵ and the invariant method- treat experts differently. The sets of items and experts coincide ($N = M$). The methods are 'peers' methods based on the premise that the statements made by a peer as an expert should be weighed by the received score as an item. This induces a loopback effect: a score of an item is defined as proportional to the sum of the received shares weighted by the experts' scores. Specifically the method looks for r in Δ_N that satisfies

$$r_i = \lambda \sum_{j \in N} \pi_{i,j} r_j \text{ for each } i \text{ for some positive } \lambda. \quad (2)$$

Equations (2) say that the positive vector r is an eigenvector of matrix π . By Perron-Frobenius theorem on matrices with positive elements, such an eigenvector exists, and is unique up to a scalar. Hence there is a unique r in the simplex that satisfies⁶ (2): the method is well defined.

The invariant method is the intensity invariant version of the Liebowitz-Palmer method. For a

⁵This terminology refers to the work of Liebowitz and Palmer (1984), who use an approximation of the method for ranking economic journals. The methods have been (re)defined and used in various contexts: in sociology by Katz (1953) and Bonacich (1987), in academics for ranking journals by Pinski and Narin (1976), and on the Web for ranking pages using the link structure between pages.

⁶This is more generally true for irreducible matrices but not for any non-negative matrix: which may admit distinct eigenvectors in the simplex. This explains why the method cannot be extended by continuity to the set of all matrices with null elements.

normalized matrix, the largest eigenvalue is one⁷ so that the invariant score of π satisfies

$$r_i = \sum_{j \in N} [\pi]_{i,j} r_j \text{ for each } i. \quad (3)$$

The method is uniform, exact, but not homogeneous.

The Hits method, introduced by Kleinberg (1999), also ranks Web pages on the basis of their link structure. Given a relevant set of pages, N , the Hits method defines a ranking of these pages, based on the links within N . Thus, as for the invariant method, the two sets of items and experts coincide. The method distinguishes two weights for each 'page', one associated with the relevance or authority of a page, the other with the adequacy of a page to point towards the relevant pages. The first set of weights defines the ranking, which should help users to find the relevant pages. The second set of weights identifies the pages -called 'hubs'- that are important because they point to relevant pages (but might be not useful to Internet users). Specifically the method assigns the ranking r and the experts weights q in Δ_N that satisfy

$$r_i = \sum_j \pi_{i,j} q_j \text{ each } i \text{ and } q_j = \lambda \sum_i \pi_{i,j} r_i \text{ each } j \quad (4)$$

for some positive λ . As argued by Kleinberg (1999), hubs and authorities exhibit a *mutually reinforcing relationship*: a good authority is a page that is pointed to by many good hubs, a good hub is one that points to many good authorities.

In matrix form, (4) writes as $r = \pi q$ and $q = \lambda \tilde{\pi} r$ where $\tilde{\pi}$ is the transpose of π . Thus the 'authority' weights r and the 'hub' weights q are well defined as respectively the normalized principal eigenvectors of the positive matrices $\pi \tilde{\pi}$ and $\tilde{\pi} \pi$. The method is uniform, exact, but not homogeneous.

Although the two sets of items and experts coincide, the Hits method is not qualified as a peers' method according to the definition introduced in Section 4.3. The reason is that item scores and expert's weights may widely differ and a main purpose of the Hits method is precisely to allow this distinction.

2.2 Supporting Weights and peers' methods

The invariant and the Hits methods, by their very definitions, assign not only a ranking but also 'weights' to experts in such a way that the ranking writes as the combination of the statements according to these weights (as can be seen from (3) or (4)). This is also true for the counting method: the experts' weights are all equal, equal to $1/m$, whatever the statements. We see these weights as supporting the methods. Such a property is formally defined as follows.

⁷Recall that the eigenvalues of a matrix are identical to those of its transpose. The set of equations $\sum_i \pi_{i,j} = 1$ for each j implies that $\mathbf{1}_N$, the n -vector with components equal to 1, is a positive eigenvector of the transpose of π with eigenvalue 1.

Definition 4 The method F is supported by the weight functions $Q^F = (Q_j^F)$ where the Q_j^F are positive functions defined over the set of positive matrices if for each π

$$F_i(\pi) = \sum_{j \in M} \frac{\pi_{i,j}}{\sum_{\ell \in N} \pi_{\ell,j}} Q_j^F(\pi) \text{ each } i. \quad (5)$$

The vector $Q^F(\pi)$ surely belongs to Δ_M . For an intensity invariant method, the weights are intensity invariant : $Q^F(\pi) = Q^F([\pi])$.

According to (5) the ranking $F(\pi)$ is a combination of the experts' shares. The fact that the weights $Q_j^F(\pi)$ necessarily sum to 1 hold because (5) is written in terms of shares (this is not linked to intensity invariance).⁸ We have chosen to write in terms of shares so as to work only with normalized vectors. Thus when we write $r \propto x$ or $q \propto y$, r or q are uniquely defined by x or y .

In the counting method the weight vector is $\mathbf{e}_M = (1/m)$, in the invariant method it is the normalized principal eigenvector of π , and in the Hits method it is the normalized principal eigenvector of $\tilde{\pi}\pi$. An interesting situation is when the ranking and the supporting weights form an equilibrium relationship. These equilibrium relationships are the basis of the invariant method, which seeks to equalize the items' scores to the experts' weights, and the Hits method. As for the counting method, the relationship is trivial, based on the premise that no distinction should be made between experts whatsoever. In line with this interpretation, even when there are multiple ways to write the ranking as a combination of the columns, the function Q^F is well specified.⁹ For the counting method for example, the columns are linearly dependent for m larger than n but the weights are well defined, set to $1/m$.

The property of supporting weights is useful for various reasons. It provides a tool to build meaningful new methods based on alternative equilibrium relationships, as is the case for the handicap-based method introduced in Section 3. The property is useful to define new methods by adjusting the weights and to define properly what a peers' method is, as we explain now.

Transformed methods A method supported by weights can be transformed into another method by transforming the weights. Let F be supported by Q^F and g be a positive scalar function defined

⁸To see this, let $r = F(\pi)$ be in Δ_N and $q = Q^F(\pi)$ satisfy the relationships $r_i = \sum_{j \in M} [\pi_{i,j}] q_j$ each i in N . Summing over i , exchanging sums, and using that columns' sums are equal to 1 yields:

$$\sum_i r_i = \sum_i \sum_j [\pi_{i,j}] q_j = \sum_j (\sum_i [\pi_{i,j}]) q_j = \sum_j q_j.$$

Since $\sum_{i \in N} r_i = 1$, we obtain $\sum_{j \in M} q_j = 1$.

⁹In some situations, the fact that a ranking writes as a convex combination of the experts' statements can be interpreted as an efficiency criterion. Let each expert represent a well defined person, and interpret j 's column as the bliss ranking of j with preferences decreasing in the euclidean norm to the bliss point. Then a ranking outside the convex hull of the bliss points is Pareto-dominated: its projection on the convex hull is preferred by every expert. Pareto efficiency however does not pin down the weights when m is larger than n .

over $[0, 1]$. F is transformed by g into method G by adjusting the weights as follows: for each π , experts' weights are proportional to $g(Q_j^F(\pi))$. This gives

$$\text{for each } i \in N, G_i(\pi) = \sum_{j \in M} \pi_{i,j} \frac{g(q_j)}{\sum_{k \in M} g(q_k)} \text{ where } q = Q^F(\pi). \quad (6)$$

One may want to put restrictions on g , as we will see in next section. An interesting family of methods is obtained from F by letting g to be homogeneous and decreasing: $g(x) = x^{1-\gamma}$ for some γ positive. This family contains the method F ($\gamma = 0$) and the counting method ($\gamma = 1$).

Peers' methods A peers' method should not be defined simply by the fact that the two sets of items and experts coincide. A precise definition to what a peers' method means can be easily given for methods supported by weights. The main idea underlying a peers' method is that the ability to provide correct expertise is positively related with the performance as an item. This makes sense in a setting in which items are ordered by a single 'ability' parameter that drives their capacity both to perform and to judge others. In such a situation, the experts' weights should be correlated with items' scores. A minimal requirement is that an item which is assigned a small score is also assigned a small expert's weight and vice-versa. We define a peers' method as one for which the weight as an expert is bounded relative to the score as an item.

Definition 5 *Let $N = M$. A method F supported by Q^F is a peers' method if the ratio Q_i/F_i is bounded: there are positive k and k' such that*

$$k' \leq \frac{Q_i}{F_i}(\pi) \leq k \text{ for each positive } \pi.$$

Since the weights are bounded above by 1, the upper bound on the ratio - which writes equivalently as $Q_i(\pi) \leq kF_i(\pi)$ - bears when we consider matrices π with arbitrarily small scores for item i . Similarly, the lower bound bears for matrices with arbitrarily small weights for expert i . The invariant method is a peers' method since F and Q coincide. The counting method (when applied to $N = M$) is clearly not a peers' method since an item score can be arbitrarily small while its expert's weight is constant. The Hits method is not a peers' method, as the following example illustrates. Let the matrix

$$\pi(\epsilon) = \begin{pmatrix} 2\epsilon & \epsilon & \epsilon \\ 1/2 - \epsilon & \epsilon & 1 - 2\epsilon \\ 1/2 - \epsilon & 1 - 2\epsilon & \epsilon \end{pmatrix}.$$

As ϵ tends to 0, the ranking assigned by the Hits method converges to $(0, 1/2, 1/2)$ and the experts' weights converge to $(1/3, 1/3, 1/3)$: while the score of item 1 vanishes, the weight does not.¹⁰ For sake of comparison, the same limit ranking $(0, 1/2, 1/2)$ is obtained for the invariant method, which is also, by definition, the limit of the experts' weights.

Notation For a vector x , $x \gg 0$ means that all components are strictly positive, $x > 0$ all are nonnegative and one at least is positive. The support of x , $x > 0$ is the set of indices that are strictly positive.

For a vector x , $x \gg 0$ $r \propto x$ indicates r is 'proportional to' x .

Given a finite set I , $\mathbf{1}_I$ denotes the vector in \mathfrak{R}^I whose components are equal to 1, $\Delta_I = \{x = (x_i) \in \mathfrak{R}^I, x_i \geq 0, \sum_{i \in I} x_i = 1\}$ the simplex in \mathfrak{R}^I , i.e. the set of possible rankings of I , and $\mathbf{e}_I = \frac{1}{|I|} \mathbf{1}_I$ denotes the ranking with equal components.

Given a vector x in \mathfrak{R}^I , $dg(x)$ denotes the diagonal $I \times I$ matrix with x_i as the i -th element on the diagonal.

Given I and J two non-empty subsets of N and M respectively, let $\pi_{I \times J}$ denote the matrix obtained from π by keeping the rows indexed by i in I and the columns indexed by j in J .

3 The handicap-based method

This section introduces a new method based on the notion of handicaps. The purpose of handicaps is to equalize the strengths between items. Handicaps and scores may be seen as inversely related: saying that the handicap of i is twice that of ℓ means that the score of i is half that of ℓ . The handicap-based method is based on this idea: it looks for items' handicaps and experts' weights that form an equilibrium relationship and assigns rankings inversely related to the equilibrium handicaps.

To motivate the equilibrium relationships between handicaps and experts' weights, let us consider an iterative process.¹¹ (Similarly, the rankings and experts' weights assigned by the invariant and the Hits methods can be obtained as the equilibrium values of an iterative process, see respectively Amir 2002 and Kleinberg 1999.) Starting with equal weights for the experts, compute the handicaps

¹⁰

$$\tilde{\pi}(\epsilon)\pi(\epsilon) = \begin{pmatrix} 0 & 1/2 & 1/2 \\ \epsilon & \epsilon & 1-2\epsilon \\ \epsilon & 1-2\epsilon & \epsilon \end{pmatrix} \begin{pmatrix} 0 & \epsilon & \epsilon \\ 1/2 & \epsilon & 1-2\epsilon \\ 1/2 & 1-2\epsilon & \epsilon \end{pmatrix} \text{ converges to } \begin{pmatrix} 1/2 & 1/2 & 1/2 \\ 1/2 & 1 & 0 \\ 1/2 & 0 & 1 \end{pmatrix}.$$

The limit matrix has three independent positive eigenvectors: $(1/3, 1/3, 1/3)$, $(0, 1, 0)$, and $(0, 0, 1)$. Since, by symmetry, the principal eigenvector of $\tilde{\pi}(\epsilon)\pi(\epsilon)$ puts equal weights on 2 and 3, it converges to $(1/3, 1/3, 1/3)$, the eigenvector of the limit matrix that has this property.

¹¹Note that this iterative processes entirely differs from the one considered in Section 4. Here the matrix is kept fixed whereas in Section 4 it is modified along the process.

so as to equalize the scores between items. Now, the advice of an expert who assigns large shares to items with high handicaps may be considered as unduly represented. This leads to a reassessment of experts' weights so as to equalize the total of their statements weighted by handicaps. But then the handicaps have to be recomputed with the just defined experts' weights and so on. The process of adjustment in experts' weights and items' handicaps is shown to converge. The precise algorithm is the following.

Let us start by assigning handicaps to items that equalize their total adjusted count to a given value, say 1: i 's handicap h_i^0 is the value by which i 's total count must be multiplied so as to obtain 1

$$\left(\sum_j \pi_{i,j}\right)h_i^0 = 1.$$

Let us assign weights to experts so as to equalize the distributed handicaps across the experts. The total number of handicaps distributed by expert j is evaluated to $\sum_i \pi_{i,j}h_i^0$ so that the weight x_j^1 assigned to expert j is

$$x_j^1 = \frac{n}{m} \frac{1}{\sum_i \pi_{i,j}h_i^0}.$$

(The factor $\frac{n}{m}$ is chosen so as to keep the overall sum of the terms in the matrix $(\pi_{i,j}h_i^0x_j^1)$ constant, here equal to n , so the weights may not sum to 1.) Now, the items' counts are adjusted to the weighted sums of their shares computed with the just defined experts' weights, which, in turn, give new values h^1 for the handicaps: $(\sum_j \pi_{i,j}x_j^1)h_i^1 = 1$, each i . Iterating the operations, the process defines two sequences $h^\tau, x^\tau, \tau = 1, \dots$ by the following equations

$$\left(\sum_j \pi_{i,j}x_j^\tau\right)h_i^\tau = 1 \text{ each } i \text{ and } \left(\sum_i \pi_{i,j}h_i^\tau\right)x_j^{\tau+1} = \frac{n}{m} \text{ each } j. \quad (7)$$

The process (7) can be seen as alternatively scaling the rows and the columns of matrix π so as to have rows' sums equal to 1 and columns' sums equal to n/m . This process is known as the 'RAS' method¹² and the sequences (h^τ, x^τ) can be shown to converge to a positive vector (h, x) . Taking the limit in (7), (h, x) satisfies

$$\left(\sum_j \pi_{i,j}x_j\right)h_i = 1 \text{ each } i \text{ and } \left(\sum_i \pi_{i,j}h_i\right)x_j = \frac{n}{m} \text{ each } j. \quad (8)$$

Thus handicaps and experts' weights exhibit a mutual relationship: handicaps equalize item weighted counts and experts' weights equalize the distributed handicaps.

¹²The procedure has been used in various areas, in statistics for adjusting contingencies tables, in economics for balancing international trade accounts as developed by Bacharach (1965). The aim of the RAS method is to adjust a given matrix so as to satisfy constraints on the rows' and columns' sums, here respectively 1 and n/m . Hence the object of interest is the final adjusted matrix P of general element $p_{i,j} = \pi_{i,j}h_ix_j$. Instead, we are here interested in the vectors of adjustment, on which the handicap-based method is build.

The handicap-based method assigns a ranking inversely related to the handicap vector. This is a well defined method, which furthermore satisfies properties listed above, as stated in the following proposition.

Proposition 1 *Given a positive matrix π , there is a unique vector $r = (r_i)$ in Δ_N such that*

$$r_i = \sum_j \pi_{i,j} q_j \text{ for each } i \text{ where } \frac{1}{q_j} = \frac{m}{n} \sum_i \frac{\pi_{i,j}}{r_i} \text{ for each } j. \quad (9)$$

or, equivalently, there is a unique pair $r = (r_i), q = (q_j)$, r in Δ_N such that the matrix P of general element $(p_{i,j} = \pi_{i,j} \frac{q_j}{r_i})$ is balanced with rows' sums equal to 1.

The **handicap-based method** H assigns to each matrix this unique ranking r and is supported by the weights Q^H proportional to q : $Q_j^H(\pi) = q_j \pi_{*j}$, as defined by (9). H is intensity invariant, uniform, exact, and homogeneous.

Hence the ranking and the experts' weights assigned by the handicap-based method can be seen as the (unique normalized) vectors of adjustments of the statements matrix that are necessary to obtain a balanced matrix in which each item receives equal scores.

Proof of Proposition 1. First denote by P the matrix with general element $p_{i,j} = \pi_{i,j} h_i x_j$ where h and x satisfies (8). The matrix P has its rows' sums equal to 1 and its columns' sum equal to n/m so P is $(1, n/m)$ -balanced. One can show that P is the unique $(1, n/m)$ -balanced matrix that is obtained from π by multiplication of its rows and its columns by some numbers (see for example Balinski and Demange 1989-b). But the values of h and x for which (8) are satisfied are not unique: given a solution h, x , multiplying h by a positive λ and dividing x by the same λ yields another solution. Hence, we can choose the pair so that the ranking r defined by $r_i = 1/h_i$ is in Δ_N . Furthermore, denoting by q the corresponding value of x , we have $p_{i,j} = \pi_{i,j} \frac{q_j}{r_i}$, hence equations (9) are satisfied.

To show that r is unique, taking the log of the expression $p_{i,j} = \pi_{i,j} \frac{q_j}{r_i}$ yields

$$\log(r_i) - \log(q_j) = \log(\pi_{i,j}) - \log(p_{i,j}) \text{ for each } i, j.$$

This is a linear system in $a_i = \log(r_i), b_j = \log(q_j)$ (the right hand side is uniquely defined since P is unique). The kernel is defined by $a_i - b_j = 0$ for each i, j , hence the a_i and b_j are all equal to some constant. This implies that the r and the q that satisfy (9) are all obtained by multiplication by a constant, hence there is a unique r that belongs to Δ_N . This proves that H is a well defined method.

Let us show that H satisfies all properties.

H is intensity invariant. Let π' be obtained from π by multiplying column j of π by μ . Letting

q' be the vector obtained from q by dividing q_j by μ , the vectors r and q' satisfy (9) for π' . Hence, by the uniqueness result, $H(\pi')$ is equal to r .

H is exact. Let $H(\pi) = \mathbf{e}_N$ for a normalized matrix π . We show that the rows totals of matrix π are equal. This implies that π is balanced (because its columns sums are all equal to 1). Since by assumption $H(\pi) = \mathbf{e}_N = (1/n)$, the second equation in (9) yields that the weight vector q satisfies $1/q_j = m(\sum_i \pi_{i,j})$ for each j , which gives $q_j = 1/m$ since π is normalized. Plugging $r_i = 1/n$ and $q_j = 1/m$ for each i, j into the first set of equations of (9) we obtain that each row's total is equal to m/n : the matrix π is balanced.

H is homogeneous. For $\pi' = dg(\rho)\pi$ let $s_j = \sum_i \pi_{i,j} \rho_i$ denote the columns' sums of π' . We have $[\pi'_{i,j}] = \frac{\rho_i \pi_{i,j}}{s_j} = \frac{\rho_i r_i \pi_{i,j}}{q_j s_j}$. Hence, the matrix P is obtained from $[\pi']$ by multiplying each column j by $q_j s_j$ and dividing each row i by $\rho_i r_i$: $(p_{i,j} = [\pi'_{i,j}] \frac{q_j s_j}{\rho_i r_i})$. The uniqueness of the normalized adjustment vectors to a $(1, n/m)$ -balanced matrix implies that the handicap-based ranking r' for $[\pi']$ is the normalized vector proportional to $(\rho_i r_i)$: $r' = (\frac{\rho_i r_i}{\sum_i \rho_i r_i})$. Since intensity invariance implies $H(dg(\rho)\pi) = H([\pi'])$, we have $H(dg(\rho)\pi) \propto (\rho_i r_i)$: this proves homogeneity.

We can also derive the value of the weights assigned to $dg(\rho)\pi$. From $(p_{i,j} = [\pi'_{i,j}] \frac{q_j s_j}{\rho_i r_i})$ and the value of the ranking r' , the weights assigned to $[\pi']$ are $(\frac{q_j s_j}{\sum_i \rho_i r_i})$.¹³ These expressions will be useful in the sequel and we write them down:

$$\begin{aligned} H_i(dg(\rho)\pi) &= \frac{\rho_i r_i}{\sum_i \rho_i r_i} \text{ each } i, Q_j^H(dg(\rho)\pi) = \frac{q_j s_j}{\sum_i \rho_i r_i} \text{ each } j \\ \text{where } r &= H(\pi), q = Q^H(\pi), s_j = \sum_i \pi_{i,j} \rho_i. \end{aligned} \quad (10)$$

■

The next proposition provides a characterization of the handicap-based method.

Proposition 2 *The handicap-based method is the unique ranking method that is intensity invariant, exact, and homogeneous.*

Proof of Proposition 2. Let method F satisfy the properties listed in Proposition 2.

Given $r = F(\pi)$, divide each row i by r_i so as to obtain matrix $\pi' = dg(1/r_1, \dots, 1/r_n)\pi$. We show that the ratio of the score of i over that of j is multiplied by r_j/r_i so that the scores are equalized: $F(\pi') = \mathbf{e}_N$. This is proved by applying iteratively the homogeneity property of F . Start by dividing row 1 by r_1 so as to obtain matrix $\pi_1 = dg(1/r_1, 1, \dots, 1)\pi$. Homogeneity implies that $F(\pi_1)$ is the ranking in Δ_N proportional to $(1, r_2, \dots, r_n)$. Then divide row 2 of π_1 by r_2 so as to obtain matrix $\pi_2 = dg(1/r_1, 1/r_2, 1, \dots, 1)\pi$. Homogeneity implies that $F(\pi_2)$ proportional to

¹³One checks that these experts weights are indeed in the simplex since $\sum_j q_j s_j = \sum_j q_j \sum_i \pi_{i,j} \rho_i = \sum_i \rho_i \sum_j q_j \pi_{i,j} = \sum_i \rho_i r_i$.

$(1, 1, r_3, \dots, r_n)$. Iterating up to n we finally obtain the matrix $\pi' = dg(1/r_1, \dots, 1/r_n)\pi$ and that its ranking has all its components equal: $F(\pi') = \mathbf{e}_N$.

By intensity invariance of F we have $F([\pi']) = F(\pi') = \mathbf{e}_N$. Exactness implies that $[\pi']$ is balanced, i.e., that its rows' sums are equal to m/n . Since $[\pi'_{i,j}] = \frac{\pi_{i,j}}{r_i \sum_{\ell} \pi_{\ell,j}/r_{\ell}}$, this writes

$$\sum_j \frac{\pi_{i,j}}{r_i \sum_{\ell} \pi_{\ell,j}/r_{\ell}} = \frac{m}{n} \text{ each } i, \text{ or } r_i = \sum_j \pi_{i,j} q_j \text{ each } i \text{ where } \frac{1}{q_j} = \frac{m}{n} \sum_{\ell} \frac{\pi_{\ell,j}}{r_{\ell}} \text{ each } j.$$

Thus r and q satisfy (9). Since r is in Δ_N , r is equal to $H(\pi)$. ■

4 Dynamics

The statement matrix so far has been taken as given, implicitly reflecting preferences. Indeed, a premise of ranking methods is that statements are related to preferences. Citations or links are considered as positive votes. Even so, the absence of a citation to an article is not necessarily a negative vote simply because the paper might not have been read. In a context with many alternatives (potentially many relevant papers to read, many sites to visit) experts are not considering each alternative. Or they are not devoting the same amount of attention to each one. As a result, statements depend on both preferences and attention. This induces a channel through which rankings have some influence because they modify attention intensities hence the statements. We examine this influence by specifying how rankings modify attention.

4.1 The influence model

Attention intensities are described by a positive n -vector (b_i) , where b_i represents the intensity spent on i . In the context of journals for example, attention represents the selection of the journals that are read in statistical terms. When attention differs across two journals, the articles in the journal with the higher value for attention intensity have more chances to be read, everything equal. In an apportionment problem in which statements are the results of an election, attention represents the time spent by voters on listening to the parties.

Attention intensity modifies experts' statements. Let us interpret $\pi_{\cdot,j} = (\pi_{i,j})_{i \in N}$ as j 's 'true' preferences, that is the probability for j to state a positive vote on i when j evaluates each i with equal attention. Bias in attention b results in statements proportional to $(\pi_{i,j} b_i)$.

The influence of a ranking is described by an 'influence function' that assigns attention intensities to a ranking in Δ_N , where attention to i is increasing in the score of i . Natural influence functions are build on a scalar function B defined on $[0, 1]$, positive, and increasing: Given a public ranking

r , attention intensities are proportional to $(B(r_i))$. This yields the statement matrix

$$\pi'_{i,j} = \pi_{i,j} B(r_i).$$

(Since we consider intensity invariant methods we do not need to normalize the matrix at this stage). Examples of functions B are $B(x) = x^\alpha$ or $B(x) = e^{\alpha x}$ with $\alpha > 0$; the lower α , the less an expert is influenced by the ranking.

The statement matrix π' leads to an adjustment in the ranking, which, when made public, will modify the attention intensities, hence the next statements. This generates a joint dynamics on statements and rankings in which current statements determine the ranking which will influence subsequent statements through the impact on attention intensities. Our aim is to investigate this dynamics, and how it relates to the ranking method and the influence functions. In this section, we investigate the case where the influence function is linear, $B(x) = x$.

At the beginning of period $t + 1$, the ranking $r^{(t+1)}$ is published on the basis of the matrix $\pi^{(t)}$ given by

$$\pi_{i,j}^{(t)} = \pi_{i,j} r_i^{(t)} \text{ or in matrix notation } \pi^{(t)} = dg(r^{(t)})\pi.$$

Matrix $\pi^{(t)}$ is obtained from π by multiplying each row i by $r_i^{(t)}$. The joint dynamical process followed by the statement matrix and the ranking is defined by

$$\pi^{(t)} = dg(r^{(t)})\pi, r^{(t+1)} = F(\pi^{(t)}). \quad (11)$$

Consider a method supported by weights Q^F . Given π , the statement matrices along the process are all of the form $dg(\rho)\pi$ for some positive ρ . Denoting $\mathbf{q}_j(\rho)$ the weight of $dg(\rho)\pi$ to simplify notation: $\mathbf{q}_j(\rho) = Q_j^F(dg(\rho)\pi)$, the dynamics followed by $r^{(t)}$ writes

$$r_i^{(t+1)} = \sum_j \frac{\pi_{i,j} r_i^{(t)}}{\sum_{\ell \in N} \pi_{\ell,j} r_\ell^{(t)}} \mathbf{q}_j(r^{(t)}) \text{ each } i \text{ in } N. \quad (12)$$

To study the dynamics, we need some continuity assumptions. A minimal one is that the scores and the weights are continuous functions over the set of positive matrices. This is condition (a) below, a natural requirement satisfied by all methods introduced so far. Under this condition, any positive limit point r^* of the dynamics satisfies the fixed point condition

$$r_i^* = \sum_j \frac{\pi_{i,j} r_i^*}{\sum_{\ell} \pi_{\ell,j} r_\ell^*} \mathbf{q}_j(r^*) \text{ for each } i. \quad (13)$$

Observe that the condition for i is surely met if i 's score is null, $r_i^* = 0$, since the weights are bounded by 1. Thus we cannot exclude fixed points with some null components. But for these points, the limit matrix $dg(r^*)\pi$ has null rows so that the method and the weights $\mathbf{q}_j(r^*)$ may not be defined. To analyze whether such a boundary point r^* is indeed a limit point of the dynamics, we need to

consider the behavior of the ranking and the weights associated to statement matrices $dg(\rho)\pi$ for ρ in a neighborhood of r^* . This is why we need the additional continuity assumption (b), which bears on this behavior for any r^* in the simplex, possibly at the boundary.

Continuity assumption (C)

(a) F and Q^F are continuous functions over the set of positive matrices.

(b) $\mathbf{q}(\rho) = Q^F(dg(\rho)\pi)$ has a well defined limit when ρ tends to r^* where r^* is any vector in Δ_N . The limit is denoted by $\mathbf{q}(r^*)$.

Condition (a) does not deserve any comment. Condition (b) bears on the boundary of the set of positive matrices. It is clearly satisfied by the counting method since experts' weights are constant. It is also satisfied by the invariant and the handicap-based methods as we now show.

For the invariant method, given r^* with some null components, the limit $\mathbf{q}(r^*)$ is derived as follows. Denoting by I the support of r^* , consider $(dg(r^*)\pi)_{|I \times I}$ the $I \times I$ matrix formed by deleting the rows and the columns of $dg(r^*)\pi$ indexed by I . As ρ converges to r^* , the ranking and the weight vector converge on I towards the normalized principal eigenvector of that restricted matrix and to zero outside I . More generally, for a peers' method, the weight of an expert whose score converges to zero converges to zero as well. Condition (b) is thus satisfied if the weights on I converge, which is a mild assumption.

For the handicap-based method, Condition (b) follows straightforwardly by taking the limit in (10)

$$\lim_{\rho \rightarrow r^*} H_i(dg(\rho)\pi) = \frac{r_i^* r_i}{\sum_i r_i^* r_i} \text{ each } i, \quad \lim_{\rho \rightarrow r^*} Q_j^H(dg(\rho)\pi) = \frac{q_j s_j^*}{\sum_i r_i^* r_i} \text{ each } j \quad (14)$$

where $r = H(\pi)$, $q = Q^H(\pi)$, $s_j^* = \sum_i \pi_{i,j} r_i^*$.

In contrast to the invariant method, the limits of the scores and the weights are not obtained by applying the handicap-based method to the restricted matrix $(dg(r^*)\pi)_{|I \times I}$ (still denoting by I the support of r^*). As can be seen from (14), the limits depend on the statements of experts outside I through the values taken by H and Q^H on π . As a result, these limits may differ for two matrices π and π' which have identical rows on I , hence for which the matrices $dg(r^*)\pi$ and $dg(r^*)\pi'$ coincide. This implies that the handicap-based method cannot be extended into a continuous function over the whole set of non-negative matrices.

Condition (b) bears on some subset of the set of positive matrices. Of course, it would be simpler to consider methods that are continuous over the whole set of all nonnegative normalized matrices (these matrices have at least one positive element in each column). But very few methods are continuous over this larger set. The invariant method is not because a non-negative matrix may admit distinct principal eigenvectors in Δ_N , and continuity would require to deal with a multi-valued

function. Similar difficulties arise for the handicap-based method, as we have just seen.

From now on, we assume the continuity assumption (C) to be satisfied.

Rest points and their support A fixed point of the dynamics satisfies (13), and may have null components. This reflects a self-enforcing mechanism: an item the score of which is null is not assessed, hence not cited, which in turn justifies a null score. But such a mechanism may not be robust unless the point enjoys a minimum of stability. Recall that a fixed point r^* is locally asymptotically stable if the process converges to r^* for an open set of initial values for r^0 around r^* . Dividing (12) by $r_i^{(t)}$ gives an expression for the growth rate of i 's score. If the sequence converges to a limit point with null i component, the limit growth rate must be less than 1. This gives the following necessary conditions for a point to be stable.

Definition 6 *Necessary conditions for r^* to be stable for the dynamics (12) are*

$$\sum_{j \in M} \frac{\pi_{i,j}}{\sum_{\ell \in M} \pi_{\ell,j} r_\ell^*} \mathbf{q}_j(r^*) \leq 1 \text{ for each } i \text{ with an equality if } r_i^* > 0. \quad (15)$$

*A point that satisfies (15) is called a **rest point**.*

The next section displays a family of methods under which convergence is guaranteed, meaning that, given a matrix π , the sequence converges towards a unique rest point whatever the initial r^0 . In contrast, in section 4.3, we show that no peers' method guarantees the uniqueness of a rest point nor convergence.

4.2 On the convergence of generalized handicap-based methods

This section shows that the convergence towards a unique point is guaranteed for a family of methods based on the handicap-based methods. A generalized handicap-based method is defined as a transform of the handicap-based method by an homogeneous and concave function: $g(x) = x^{1-\gamma}$ for γ positive. From (6) such a method G writes as

$$G_i(\pi) = \sum_{j \in M} [\pi]_{i,j} \frac{q_j^{1-\gamma}}{\sum_{k \in M} q_k^{1-\gamma}} \text{ for each } i \in N \text{ where } q = Q^H(\pi). \quad (16)$$

The handicap-based method obtains for $\gamma = 0$ and the counting method for $\gamma = 1$. We prove a general convergence result for any method in the family obtained for positive γ . (When γ is larger than 1, there is an important change in the transformation since the weights are in the reverse order of the handicap weights. One may want to exclude these values. The next proposition is however valid.)

Proposition 3 Consider a generalized handicap-based method as defined by (16) with γ strictly positive. There is a unique rest point, which is furthermore globally stable: the dynamics (12) converges to it for any initial value r^0 .

This proves in particular the convergence for the counting method ($\gamma = 1$). The handicap-based method, which obtains for γ equal to zero, is not covered by Proposition 3. The case for γ null or negative are considered at the end of this section.

Proposition 3 is proved by showing that the following function L is a Lyapounov function for the dynamics:

$$\begin{aligned} \text{for } \gamma > 0, \gamma \neq 1 \quad L(\rho) &= \frac{1}{1-\gamma} \sum_j Q_j^H(\pi) \left(\sum_i \pi_{i,j} \rho_i \right)^{1-\gamma} \\ \text{for } \gamma = 1 \quad L(\rho) &= \sum_j Q_j^H(\pi) \ln \left(\sum_i \pi_{i,j} \rho_i \right). \end{aligned}$$

A technical lemma Recall that the matrices that appear in the process are all of the form $dg(\rho)\pi$ where π is the preferences matrix. The next lemma states conditions on the experts' weights on these matrices under which convergence is guaranteed.

Lemma 1 Let a method be supported by Q . Given a column-normalized matrix π , assume that there are some functions ψ_k , k in M , defined from the set of positive scalar numbers, \mathbb{R}_+ , to itself, continuous and decreasing such that

$$Q_j(dg(\rho)\pi) = \frac{s_j \psi_j(s_j)}{\sum_{k \in M} s_k \psi_k(s_k)} \text{ for each } j \in M \text{ where } s_j = \sum_{i \in N} \pi_{i,j} \rho_i. \quad (17)$$

Then there is a unique rest point, which is globally stable.

Expression (17) states how the modification of the statement matrix by ρ affects the experts' weights. As each row i is multiplied by ρ_i , the columns' sums are modified into the s_j . Conditions (17) require the experts' weights to depend only on these sums and in a separable way (up to the normalization).

The proof of Lemma 1 is given in the appendix. It goes as follows. Let function L be defined by $L(\rho) = \sum_j \Psi_j(\sum_i \pi_{i,j} \rho_i)$ where Ψ_j is a primitive of ψ_j . If expression (17) holds, then the conditions (15) on a rest point exactly coincide with the first order conditions of the maximization of L over r in Δ_N . This immediately implies that there is a unique rest point when L is strictly concave, which¹⁴ is the case when the ψ_j are strictly decreasing. Furthermore, in that case, L is shown to be

¹⁴Denoting the function on the left-hand side of (15) by $\phi_i(r)$, the conditions (15) write as $\phi_i(r) \leq 1$, with an equality for $r_i = 0$. Since we consider the maximization of L over the closed set Δ_N , the ϕ_i are not the partial derivatives of L with respect to r_i , hence they do not have to satisfy the integrability conditions $\partial \phi_i / \partial r_\ell = \partial \phi_\ell / \partial r_i$, for each i, ℓ .

a Lyapounov function for the process, which ensures the convergence to the rest point. If the ψ_j are not decreasing, L is not concave. Then the points satisfying the first order conditions associated to the program still coincide with the rest points but several rest points are possible. Furthermore L is no longer a Lyapounov function (see the two examples after the proof of the proposition).

Proof of Proposition 3. To apply Lemma 1, let us compute the weights $Q^G(dg(\rho)\pi)$ for the method transformed by $x^{1-\gamma}$ as ρ varies. By definition, the weights Q^G are proportional to $[Q^H]^{1-\gamma}$. In particular we have $Q^G(dg(\rho)\pi) = [Q^H(dg(\rho)\pi)]^{1-\gamma}$. From (10) the weight vector $Q^H(dg(\rho)\pi)$ for the handicapped-based method is the normalized vector proportional to $(Q_j^H(\pi)s_j)$. Hence

$$(Q_j^G(dg(\rho)\pi)) \propto (Q_j^H(\pi)^{1-\gamma} s_j^{1-\gamma}). \quad (18)$$

Thus (17) holds for functions ψ_j given by

$$\psi_j(s_j) = Q_j^H(\pi)^{1-\gamma} s_j^{-\gamma}. \quad (19)$$

For γ positive, the functions ψ_j are decreasing. Convergence follows by application of Lemma 1. ■

The handicap-based method, which obtains for γ equal to zero, is not covered by Proposition 3. We show multiple rest points are possible but in degenerate situations. When γ is negative, multiplicity is more severe, as will be illustrated with the case $\gamma = -1$.

The handicapped-based method For the handicapped-based method, the function L is linear so that the uniqueness of a rest point is not guaranteed. We show that indeed several rest points may occur but only in degenerate cases.

Let $r = H(\pi)$ and $q = Q^H(\pi)$, the values assigned by the handicapped-based method to π . Denote by I the set of items for which the handicap-based score is maximal. We show that the rest points are all the rankings whose support is included in I , i.e. with null components outside I . Hence when I is a singleton -which is the general case- there is a unique rest point. Otherwise, any ranking that charges the maximal handicap-based scores is a rest point.

Recall that $r^{(t+1)}$ and $q^{(t+1)}$ are the rankings and the weights assigned by the handicapped-based method to $\pi^{(t)} = dg(r^{(t)})\pi$. Using (10) we have for each $t \geq 0$

$$r_i^{(t+1)} = \lambda^{(t+1)} r_i^{(t)} r_i \text{ and } q_j^{(t+1)} = \lambda^{(t+1)} \left(\sum_{\ell} \pi_{\ell,j} r_{\ell}^{(t)} \right) q_j \quad (20)$$

for some $\lambda^{(t+1)}$. Consider the growth rate of score i , $\gamma_i^{(t+1)} = r_i^{(t+1)} / r_i^{(t)}$. (20) implies

$$\gamma_i^{(t+1)} / \gamma_{\ell}^{(t+1)} = \frac{r_i}{r_{\ell}},$$

that is the ratio of the growth rate of i over that of ℓ , stays constant equal to the ratio of their handicap-based scores.

Let i be not in I : i 's handicap-based score is not maximal, so that $r_i < r_\ell$ for some ℓ . Since rankings are in the simplex and i 's growth rate is strictly less than ℓ 's, i 's score $r_i^{(t+1)}$ must converge to 0. Thus any rest point has null components on $N - I$.

For i and ℓ both in I , the ratio r_i/r_ℓ is equal to one, thus the growth rates of the scores of i and ℓ are equal. This implies that the ratio $r_i^{(t)}/r_\ell^{(t)}$ stays constant equal to their initial values $r_i^{(0)}/r_\ell^{(0)}$. Thus $r^{(t)}$ converges towards the ranking whose components are proportional on I to their initial values and are null outside. This shows that any ranking with null components on $N - I$ is a rest point. \blacksquare

Example of multiplicity Consider a generalized handicap-based method G with $g(x) = x^2$ (which corresponds to $\gamma = -1$). The computation performed in the proof of proposition are still valid so that (17) is satisfied for $\psi_j(s_j) = Q_j^H(\pi)s_j$. However since these functions are increasing, Lemma 1 does not apply. We show that multiple rest points are possible. Let the 2×2 matrix

$$\pi = \begin{pmatrix} 1-a & b \\ a & 1-b \end{pmatrix}.$$

Easy computation gives that the handicap-based ranking and weights are

$$H(\pi)r \propto (\sqrt{(1-a)b}, \sqrt{a(1-b)}), Q^H(\pi) = q \propto (\sqrt{(1-b)b}, \sqrt{a(1-a)}). \quad (21)$$

From expression (18) the weights $w = Q^G(dg(\rho)\pi)$ satisfy $w \propto ((q_1s_1)^2, (q_2s_2)^2)$. We look for conditions under which the ranking (1,0) is a rest point, i.e. satisfies (15). The conditions write

$$(1-a)\frac{w_1}{s_1} + b\frac{w_2}{s_2} = 1 \text{ and } \frac{a}{(1-a)}\frac{w_1}{s_1} + \frac{(1-b)}{b}\frac{w_2}{s_2} \leq 1$$

where $s_1 = 1 - a$ and $s_2 = b$. The first condition writes $w_1 + w_2 = 1$ so it is surely satisfied. The second condition writes

$$\frac{a(1-a)(q_1)^2}{(q_1(1-a))^2 + (q_2b)^2} + \frac{(1-b)b(q_2)^2}{(q_1(1-a))^2 + (q_2b)^2} \leq 1.$$

Plugging the expression of q given in (21) and rearranging,¹⁵ the inequality can be rewritten as $3a + b \leq 1 + 4ab$. Exchanging the role of a and b , the ranking (0,1) is a rest point if $a + 3b \leq 1 + 4ab$. Clearly, the set of values for which both rankings (1,0) and (0,1) are rest points is an open set: multiplicity is a robust phenomena.

¹⁵Since (q_1, q_2) is proportional to $(\sqrt{(1-b)b}, \sqrt{a(1-a)})$ the inequality is equivalent to

$$2a(1-a)(1-b)b \leq (1-b)b(1-a)^2 + (a(1-a)b)^2,$$

that is $2a(1-b) \leq (1-b)(1-a) + ab$ and finally $0 \leq 1 - b - 3a + 4ab$.

4.3 Peers' methods

This section analyzes the dynamics under a peers' method. Recall that the ratio of an expert's weight to his score (as an item) admit a positive lower bound and a finite upper bound. This implies that the weight $\mathbf{q}_j(r)$ is null if and only if r_j is null (thanks to the continuity assumption (C)). Quite strong results are obtained using only this property. These results extend the analysis on the invariant method (Demange 2009) in which I show that the dynamics may have multiple rest points, or even multiple locally stable rest points. It turns out that this multiplicity is bound to occur with peers' methods.

Let us first characterize the supports of the rest points. As shown by next proposition, the characterization is independent of the peers' method under consideration.

Proposition 4 *Consider a peers' method. Given π , a subset I of N is the support of a rest point if and only if*

$$\text{there is } x \text{ in } \mathfrak{R}^I, x \gg 0, \pi_{I \times I} x = \mathbf{1}_I, \pi_{N-I \times I} x \leq \mathbf{1}_{N-I}. \quad (22)$$

That (22) is necessary for I to be the support of a rest point follows from the conditions (15) on a rest point: $\sum_j \frac{\pi_{i,j}}{\sum_{\ell \in N} \pi_{\ell,j} r_i^*} \mathbf{q}_j(r^*) \leq 1$ for each i with an equality if $r_i^* > 0$. With a peers' method, $\mathbf{q}_i(r^*)$ is null whenever r_i^* is null. Hence the vector x in \mathfrak{R}^I defined by

$$x_j = \frac{\mathbf{q}_j(r^*)}{\sum_i \pi_{i,j} r_i^*}, \text{ each } j \in I \quad (23)$$

is positive and satisfies $\sum_{j \in I} \pi_{i,j} x_j = 1$ for each $i \in I$ and $\sum_{j \in I} \pi_{i,j} x_j \leq 1$ for each $i \notin I$, i.e. x satisfies the system of linear inequalities (22).¹⁶

To show that conversely (22) guarantees the existence of a rest point with support in I , we need to find r^* in Δ_I that satisfies (23). We build a correspondence whose fixed points are solutions to (23) (see the details in the section proof). We want to find r that equalizes the ratios $\frac{\mathbf{q}_j(r)}{x_j \sum_i \pi_{i,j} r_i}$. Given r with a j 's ratio $\frac{\mathbf{q}_j(r)}{x_j \sum_i \pi_{i,j} r_i}$ that is not minimum, we assign to j a null score. The behavior of a peers' system ensures that a fixed point is strictly positive, hence that the ratios are equalized on I . Condition (22) then implies that the ratios are equalized to 1.

Proposition 4 can be interpreted as follows. Consider first the whole set N . Conditions (22) state the existence of a positive vector x for which $\pi x = \mathbf{1}_N$, or $\sum_j \pi_{i,j} x_j = 1$ for each i . In words, there is a set of experts' weights that equalize the expected weighted totals across items for the true preferences. (Of course the same interpretation holds for the normalized weightse). Thus, N is

¹⁶Observe that the same argument can be used for any method and gives $\pi_{I \times M} x = \mathbf{1}_I$. For $N = M$ and a method that is not a peers' method, the vector x is not required to be null outside the support of r^* . As a result, for I strict subset of M the system $\pi_{I \times M} x = \mathbf{1}_I$ has more unknown than equations, hence is not much informative.

the support of a rest point when experts' preferences are not in a clear way in favor of some items. (When the matrix π is invertible, the condition simply writes as $\pi^{-1}\mathbf{1}_N \gg 0$.) Consider now a subset I of N and interpret similarly x as a weight vector on I . Conditions (22) require the existence of a weight vector on I that equalizes the weighted score of each item in I , and in addition, that gives a lower weighted score to items not in I .

From Proposition 4, the supports of the rest points are independent of the peers' method. The rest points, that is the precise values assumed by the scores on such a support, are not. We now study the convergence.

Proposition 5 *Consider a peers' method. There are matrices π for which the dynamics (12) admit several locally stable points.*

It is easy to understand why a problem may admit multiple stable rest points. Let preferences be sufficiently antagonistic in the following way. Take I a subset and choose preferences from I to $N - I$ small enough and the same for $N - I$ to I , that is the $\pi_{i,j}$ small enough for i in I and j not in I or the reverse. First I or a subset of I is the support of a rest point, and similarly for $N - I$. The key point is that the stability of a rest point null on $N - I$ is independent of the values of $\pi_{i,j}$ for j not in I , that is of the preferences of $N - I$.

5 Extensions and concluding remarks

We first analyze the impact of non linear influence functions.

5.1 Non linear influence functions

So far we have assumed a specific form, linear, for the influence of rankings. This section considers more general influence functions B , $B(r) = r^\alpha$, as described in Section 4.1. Specifically, given an announced ranking, j 's statements are now proportional to $\pi_{i,j}B(r_i)$. Thus given $r^{(t)}$, the adjusted citation matrix $\pi^{(t)}$ is

$$\pi_{i,j}^{(t)} = \pi_{i,j}B(r_i^{(t)}) = \pi_{i,j}r_i^{(t)\alpha}.$$

With a slight abuse of notation let r^α denote the vector (r_k^α) . Using as previously the notation $\mathbf{q}_j(\rho) = Q_j^F(dg(\rho)\pi)$, the process followed by the rankings is

$$r_i^{(t+1)} = \sum_j \frac{\pi_{i,j}r_i^{(t)\alpha}}{\sum_\ell \pi_{\ell,j}r_\ell^{(t)\alpha}} \mathbf{q}_j(r^{(t)\alpha}) \text{ each } i. \quad (24)$$

A rest point r^* satisfies

$$\sum_j \frac{\pi_{i,j}r_i^{*1-\alpha}}{\sum_\ell \pi_{\ell,j}r_\ell^{*\alpha}} \mathbf{q}_j(r^{*\alpha}) \leq 1 \text{ for each } i \text{ with an equality if } r_i^* > 0. \quad (25)$$

Intuitively, as α increases, the influence becomes more discriminating. The analysis differs sensibly depending on the value of α with respect to 1, the boundary case $\alpha = 1$ being the case studied in the previous sections.

Diminishing marginal impact ($\alpha < 1$) For $\alpha < 1$, (24) implies that the growth rate of i 's score is strictly larger than 1 for r_i small enough. As a result, the possible rest points are necessarily strictly positive and satisfy

$$\sum_j \frac{\pi_{i,j} r_i^{*1-\alpha}}{\sum_\ell \pi_{\ell,j} r_\ell^{*\alpha}} \mathbf{q}_j(r^{*\alpha}) = 1 \text{ for each } i. \quad (26)$$

More generally, even without convergence, no item's score becomes arbitrarily small because its growth rate is superior to 1 for a low enough score. We can say more for generalized handicap-based methods. Lemma 1 extends as follows. Let a method satisfy the conditions (17), which guarantee the dynamics to converge to a unique rest point for a linear influence function. Then the dynamics converges for influence functions $B(x) = x^\alpha$ any $\alpha < 1$. The proof follows the same lines as that of Lemma 1 by considering the function L defined by $L(\rho) = \sum_j \Psi_j(s_j)$ where Ψ_j is a primitive of ψ_j but with the sum s_j given by $s_j = \sum_i \pi_{i,j} \rho_i^\alpha$. For Ψ_j concave (ψ_j decreasing) and $\alpha > 1$, L is strictly concave. The rest point coincides the unique maximizer of L over the rankings, and in addition L is a Lyapounov function (see the details at the end of the proof of Lemma 1). This result implies that the process (24) converges for any generalized handicap-based method adjusted by $g(x) = x^{1-\gamma}$ and influence function $B(r) = r^\alpha$ for which $\gamma \geq 0$ and $\alpha \leq 1$ with at least one strict inequality.

Increasing marginal impact ($\alpha > 1$) For $\alpha > 1$, the limit rankings may be expected to be more discriminative than in the linear case as there are increasing marginal rewards to scores. We show here that this is indeed true for the invariant method since any ranking concentrated on a single point is a stable point.

The proof is as follows. Starting with a score for item i low enough, we show that i 's score will decrease exponentially to 0. Thus the sequence $r^{(t)}$ converges to $\mathbf{1}_{\{\ell\}}$ for initial rankings in a neighborhood of $\mathbf{1}_{\{\ell\}}$, that is rankings for which all scores except that of ℓ are small enough. The proof relies on the following inequality : For some C

$$\forall t > 0, r_i^{(t+1)} \leq C r_i^{(t)\alpha}. \quad (27)$$

Assuming (27), iteration from 0 up to t implies

$$r_i^{(t+1)} \leq C^{1+\alpha+\dots+\alpha^t} r_i^{(0)\alpha^{t+1}}, \text{ or } r_i^{(t+1)} \leq C^{\frac{1}{1-\alpha}} \left[C^{\frac{1}{\alpha-1}} r_i^{(0)} \right]^{\alpha^{t+1}}.$$

Hence if $C^{\frac{1}{\alpha-1}} r_i^{(0)} < 1$ then the sequence $r_i^{(t)}$ converges to 0 as t tends to ∞ . As a consequence, if all $r_i^{(0)}$ for $i \neq \ell$ are small enough, then the rankings converge to $\mathbf{1}_{\{\ell\}}$.

It remains to show (27). For the invariant method, the dynamics is given by

$$\frac{r_i^{(t+1)}}{r_i^{(t)\alpha}} = \sum_j \pi_{i,j} \frac{r_j^{(t+1)}}{\sum_{\ell \in N} \pi_{\ell,j} r_\ell^{(t)\alpha}}.$$

We have to show that the right hand side is bounded above. Let us denote by π_{max} and π_{min} respectively the maximum and minimum of the elements in matrix π . We first provide a lower bound to $\sum_{\ell \in N} \pi_{\ell,j} r_\ell^{(t)\alpha}$. Observe that

$$\pi_{min} \sum_{\ell \in N} r_\ell^{(t)\alpha} \leq \sum_{\ell \in N} \pi_{\ell,j} r_\ell^{(t)\alpha}. \quad (28)$$

To bound $\sum_{\ell \in N} r_\ell^{(t)\alpha}$, we apply Holder inequality¹⁷ to the vectors $r^{(t)}$ and $\mathbf{1}_N$ with the parameters $p = \alpha$ and $q = \alpha/(\alpha - 1)$ (q is positive since $\alpha > 1$). This yields

$$\sum_{\ell \in N} r_\ell^{(t)} \leq \left(\sum_{\ell \in N} r_\ell^{(t)\alpha} \right)^{\frac{1}{\alpha}} n^{\frac{\alpha-1}{\alpha}}.$$

Since $\sum_{\ell \in N} r_\ell^{(t)} = 1$, this writes $1 \leq \left(\sum_{\ell \in N} r_\ell^{(t)\alpha} \right)^{\frac{1}{\alpha}} n^{\frac{\alpha-1}{\alpha}}$ or $n^{(1-\alpha)} \leq \sum_{\ell \in N} r_\ell^{(t)\alpha}$. Using inequality (28) gives

$$\pi_{min} n^{(1-\alpha)} \leq \pi_{min} \sum_{\ell \in N} r_\ell^{(t)\alpha} \leq \sum_{\ell \in N} \pi_{\ell,j} r_\ell^{(t)\alpha}.$$

So together with $\pi_{i,j} \leq \pi_{max}$ we obtain

$$\frac{r_i^{(t+1)}}{r_i^{(t)\alpha}} \leq \sum_j \pi_{max} \frac{r_j^{(t+1)}}{\pi_{min} n^{(1-\alpha)}} = \frac{\pi_{max}}{\pi_{min} n^{(1-\alpha)}}.$$

Inequality (27) holds for C the value on the right hand side. ■

5.2 Concluding remarks

This paper introduces new ranking methods based on the notion of handicaps and analyzes the influence of rankings based on the premise that rankings induce a coordination on attention. It shows that, for an identical influence mechanism, the interplay between preferences and the ranking method results in a variety of different outcomes. While a class of methods (the class based on handicaps) ensures convergence of the system, self-enforcing mechanisms at play in peers' methods are strong enough to sustain multiple equilibria.

Several developments could be contemplated. New methods could be introduced associated with various equilibrium relationships between scores and experts' weights. It would be interesting to

¹⁷Holder inequality is $\sum_{\ell} x_{\ell} y_{\ell} \leq (\sum_{\ell} x_{\ell}^p)^{1/p} (\sum_{\ell} y_{\ell}^q)^{1/q}$, for p and q positive related by $1/p + 1/q = 1$.

analyze more closely the support of the rest points and to understand better their links with the method at hand. Finally the paper concentrates on the influence of a single ranking. Although assuming a single ranking is appropriate for modeling the Web environment given the dominance of the use of Google Search, the assumption should be relaxed. A natural issue is to relate the number of rankings with the preferences, thereby making rankings endogenous.

6 Proofs

Proof of Lemma 1. Let us assume (17) for some functions ψ_j . Let Ψ_j be a primitive of ψ_j and L the function $L(\rho) = \sum_j \Psi_j(\sum_i \pi_{i,j} \rho_i)$, or for short $\sum_j \Psi_j(s_j)$. Consider the program (\mathcal{P})

$$(\mathcal{P}) : \max_{\rho} L(\rho) \text{ over } \rho \geq 0, \sum_i \rho_i \leq 1.$$

Let μ denote the multiplier associated to the constraint $\sum_i \rho_i \leq 1$. The first order conditions satisfied by a solution r of (\mathcal{P}) are

$$\frac{\partial L}{\partial \rho_i}(r) = \sum_j \psi_j(s_j) \pi_{i,j} \leq \mu \text{ each } i \text{ with } = \text{ for } r_i > 0. \quad (29)$$

Multiplying by r_i the inequality for each i and summing over i yields $\sum_i \sum_j \psi_j(s_j) \pi_{i,j} r_i = \mu(\sum_i r_i)$. Exchanging the sums yields $\sum_i \sum_j \psi_j(s_j) \pi_{i,j} r_i = \sum_j \psi_j(s_j) \sum_i \pi_{i,j} r_i = \sum_j \psi_j(s_j) s_j$. Hence using $\sum_i r_i = 1$ gives the value of μ : $\mu = \sum_j \psi_j(s_j) s_j$ and the first order conditions (29) are equivalent to

$$\sum_j \pi_{i,j} \psi_j(s_j) \leq \sum_k s_k \psi_k(s_k) \text{ each } i \text{ with } = \text{ for } r_i > 0. \quad (30)$$

Now given that $(\mathbf{q}_j(\rho))$ satisfy (17), i.e. are proportional to $(\psi_j(s_j))$ we show that the conditions (15) on a rest point

$$\sum_j \frac{\pi_{i,j}}{\sum_{\ell \in N} \pi_{\ell,j} r_{\ell}^*} \mathbf{q}_j(r^*) \leq 1 \text{ for each } i \text{ with an equality if } r_i^* > 0$$

coincide with the first order conditions (30) of program (\mathcal{P}). This is immediate since $\mathbf{q}_j(r) = \frac{s_j \psi_j(s_j)}{\sum_k s_k \psi_k(s_k)}$ and $s_j = \sum_{\ell \in N} \pi_{\ell,j} r_{\ell}$ gives

$$\sum_j \frac{\pi_{i,j}}{\sum_{\ell \in N} \pi_{\ell,j} r_{\ell}} \mathbf{q}_j(r) = \sum_j \pi_{i,j} \frac{\psi_j(s_j)}{\sum_k s_k \psi_k(s_k)}. \quad (31)$$

Let us assume in addition the ψ_j to be decreasing. Then the functions Ψ_j are strictly concave and there is a unique solution to program (\mathcal{P}) characterized by (30). This implies that there is a unique rest point.

To prove convergence, we show that L is a Lyapounov function: the sequence $L(r^{(t)})$ strictly increases with t as long as $r^{(t)}$ differs from $r^{(t-1)}$. We have $\frac{\partial L}{\partial r_i}(r) = \sum_j \psi_j(s_j) \pi_{i,j}$. Multiplying by

r_i , summing over i , and exchanging the sums yields $\sum_{\ell \in N} r_\ell \frac{\partial L}{\partial r_\ell}(r) = \sum_{j \in M} s_j \psi_j(s_j)$. Hence from (31), the dynamics (12) followed by $r^{(t+1)}$: $r_i^{(t+1)} = \sum_j \frac{\pi_{i,j} r_i^{(t)}}{\sum_{\ell \in N} \pi_{\ell,j} r_\ell^{(t)}} \mathbf{q}_j(r^{(t)})$ each i , can be written as

$$r_i^{(t)} = \frac{\frac{\partial L}{\partial r_i}(r^{(t-1)})}{S^{(t-1)}} r_i^{(t-1)} \text{ each } i \text{ where } S^{(t-1)} = \sum_{\ell \in N} r_\ell^{(t-1)} \frac{\partial L}{\partial r_\ell}(r^{(t-1)}). \quad (32)$$

These equations mean that the growth rates of the components of r are proportional to the gradient of L . Consider the difference $L(r^t) - L(r^{t-1})$. The concavity of L implies

$$L(r^t) - L(r^{t-1}) \geq \sum_i \frac{\partial L}{\partial r_i}(r^{(t-1)}) [r_i^{(t)} - r_i^{(t-1)}] \text{ with } > \text{ if } r^{(t)} \neq r^{(t-1)}. \quad (33)$$

We first show that

$$\frac{\partial L}{\partial r_i}(r^{(t-1)}) [r_i^{(t)} - r_i^{(t-1)}] \geq r_i^{(t-1)} \left[\frac{\partial L}{\partial r_i}(r^{(t-1)}) - S^{(t-1)} \right] \quad (34)$$

Rewriting (32) as

$$r_i^{(t)} - r_i^{(t-1)} = \frac{r_i^{(t-1)}}{S^{(t-1)}} \left[\frac{\partial L}{\partial r_i}(r^{(t-1)}) - S^{(t-1)} \right]$$

The term on the right inside the brackets is positive (resp. negative) if the partial derivative $\frac{\partial L}{\partial r_i}$ is larger (resp. smaller) than $S^{(t-1)}$: inequality (34) follows by multiplying by $\frac{\partial L}{\partial r_i}$.

Summing the inequalities (34) over i yields

$$\sum_i \frac{\partial L}{\partial r_i}(r^{(t-1)}) [r_i^{(t)} - r_i^{(t-1)}] \geq \sum_i r_i^{(t-1)} \frac{\partial L}{\partial r_i}(r^{(t-1)}) - S^{(t-1)} = 0$$

Hence from (33), the value of L strictly increases as long as $r^{(t)}$ differs from $r^{(t-1)}$, that is as long as the rest point is not reached: L is a Lyapounov function and the sequence converges to the rest point.

Extension to influence functions $B(x) = x^\alpha$, $\alpha < 1$. Let L be defined by $L(\rho) = \sum_j \Psi_j(\sum_i \pi_{i,j} \rho_i^\alpha)$ where Ψ_j is a primitive of ψ_j . Consider as above the program \mathcal{P} of maximization of L over Δ_N . The program is strictly concave and the unique rest point is interior. By similar computation as for the linear influence function, one checks that the first order conditions of \mathcal{P} are

$$\sum_j \pi_{i,j} r_i^{\alpha-1} \frac{\psi_j(s_j)}{\sum_k s_k \psi_k(s_k)} = 1 \text{ each } i \text{ with } s_j = \sum_i \pi_{i,j} r_i^\alpha. \quad (35)$$

Under the assumption on \mathbf{q}_j these conditions write $\sum_j \frac{\pi_{i,j} r_i^{*1-\alpha}}{\sum_\ell \pi_{\ell,j} r_\ell^{*\alpha}} \mathbf{q}_j(r^{*\alpha}) = 1$ for each i , which coincide with the conditions (26) on a rest point. The proof that L is a Lyapounov function for the process with $\alpha < 1$ is identical to that for $\alpha = 1$. ■

Proof of Proposition 4. We proved in the text that conditions (22) are necessary for the existence of a rest point with support I . Let us show the converse. Recall the notation $\mathbf{q}(r) = Q^F(dg(r)\pi)$.

Under Assumption (C), function \mathbf{q} is continuous over the whole set Δ_N and furthermore, r_i null implies $\mathbf{q}_i(r)$ null by the peers' property. Given x that satisfies (22) we need to prove the existence of r^* positive in Δ_I that satisfies (23):

$$x_j = \frac{\mathbf{q}_j(r^*)}{\sum_i \pi_{i,j} r_i^*}, \text{ each } j \in I.$$

Consider the correspondence from Δ_I to itself defined by

$$\Phi(r) = \{\rho \in \Delta_I \text{ s.t. } \rho_k = 0 \text{ for each } k \text{ that does not minimize } \frac{\mathbf{q}_j(r)}{x_j(\sum_i \pi_{i,j} r_i)} \text{ over } j \in I\}. \quad (36)$$

It is easy to check that the continuity of the function \mathbf{q} implies that the correspondence Φ is upper hemi-continuous. Since Φ is convex-valued, it has a fixed point by Kakutani theorem, $r^* \in \Phi(r^*)$. We show that r^* is positive and satisfies (23).

By contradiction, assume $r_i^* = 0$ for some i . The peers' property implies that $\mathbf{q}_i(r^*)$ is null. Since \mathbf{q} takes values in Δ_I , there is k in I with $\mathbf{q}_k(r^*) > 0$. Thus the minimum of the $\frac{\mathbf{q}_j(r^*)}{x_j(\sum_i \pi_{i,j} r_i^*)}$ over j in I is zero (achieved at $j = i$) and is not achieved at $j = k$. Hence any ρ in $\Phi(r^*)$ has $\rho_k = 0$. However, applying the peers' property again, r_k^* must be positive because $\mathbf{q}_k(r^*) > 0$. Hence $r^* \in \Phi(r^*)$ cannot hold, which gives the contradiction.

Since a fixed point r^* of Φ has all its components positive and $r^* \in \Phi(r^*)$, $\Phi(r^*)$ contains a strictly positive vector. By the definition (36), this implies that the ratios $\mathbf{q}_j(r^*)/(x_j(\sum_i \pi_{i,j} r_i^*))$ are equalized across j : there is some λ such that $\mathbf{q}_j(r^*) = \lambda x_j(\sum_i \pi_{i,j} r_i^*)$ for each j . Summing these equations over j yields that λ is equal to 1: the sum on the left hand side is $\sum_j \mathbf{q}_j(r^*)$ which is equal to 1, and the sum on the right hand side is $\lambda \sum_j x_j(\sum_i \pi_{i,j} r_i^*)$ which is equal to $\lambda \sum_i r_i^*(\sum_j \pi_{i,j} x_j) = \lambda$ since x satisfies (22). Hence $\lambda = 1$, which proves (23). ■

Proof of Proposition 5. Let a peers' method. The proof is divided in three steps.

In the first step we find conditions on a matrix π under which the scores on $N - I$ converge to zero if their initial values are low enough, whatever the behavior of the scores on I . In the second step we strengthen these conditions so that they bear only on the columns indexed by $N - I$, and the third step concludes.

Step 1. The growth rate of i 's score satisfies

$$\frac{r_i^{(t+1)}}{r_i^{(t)}} = \sum_j \pi_{i,j} \frac{\mathbf{q}_j(r^{(t)})}{\sum_\ell \pi_{\ell,j} r_\ell^{(t)}}. \quad (37)$$

Let a matrix π be such that

$$\text{for some } k < 1, \sum_j \pi_{i,j} \frac{\mathbf{q}_j(r)}{\sum_\ell \pi_{\ell,j} r_\ell} \leq k \text{ for each } i \notin I \text{ and each } r \text{ null on } N - I. \quad (38)$$

We show that, starting from rankings with small enough values outside I , the growth rate of the scores on $N - I$ is strictly smaller than 1, hence the scores converge to zero.

By continuity of the \mathbf{q} , similar inequalities to (38) hold for r with small enough components on $N - I$. Formally, given $\epsilon > 0$ let $\mathcal{V}(\epsilon)$ denote the subset of Δ_N composed with the rankings whose components on $N - I$ are smaller than ϵ . By the continuity of the \mathbf{q} , for k' with $k < k' < 1$, there is $\epsilon > 0$ such that

$$\sum_j \pi_{i,j} \frac{\mathbf{q}_j(r)}{\sum_\ell \pi_{\ell,j} r_\ell} \leq k', i \notin I, \text{ for each } r \in \mathcal{V}(\epsilon). \quad (39)$$

Using (37), this inequality says that the growth rates of all components on $N - I$ are less than k' . Assume that the ranking belongs to $\mathcal{V}(\epsilon)$ at some date t . Since k' is strictly smaller than 1, the ranking at date $t + 1$ also belongs to $\mathcal{V}(\epsilon)$. By induction, the sequence stays in $\mathcal{V}(\epsilon)$ at any further date and furthermore the components on $N - I$ converge to zero because their growth rates is smaller than k' .

Step 2. The following conditions imply Conditions (38):

$$\frac{\beta}{\alpha} \leq k \text{ where } \beta = \max_{(i,j) \in N-I \times I} \pi_{i,j} \text{ and } m = \min_{(i,j) \in I \times I} \pi_{i,j}. \quad (40)$$

Conditions (40) bear on the columns indexed by I , and ask the values on the rows $N - I$ to be small enough relative to those in I . Many matrices satisfy (40). The simplest example is

$$\begin{pmatrix} \alpha & \alpha & \times, \times \\ \dots & \dots & \dots \\ \alpha & \dots & \alpha & \times, \times \\ \beta & \dots & \beta & \times, \times \\ \beta & \dots & \beta & \times, \times \end{pmatrix}$$

where $\alpha|I| + \beta|N - I| = 1$, and β small enough so that $\beta/\alpha < 1$.

Let us prove that (40) imply (38). For r with null components on $N - I$, we have $\mathbf{q}_j(r) = 0$ for any j not in I by definition of a peers' method, and $\sum_\ell \pi_{\ell,j} r_\ell = \sum_{\ell \in I} \pi_{\ell,j} r_\ell$. Since for j in I we have $\sum_{\ell \in I} \pi_{\ell,j} r_\ell \geq \alpha$ the ratio $\frac{\mathbf{q}_j(r)}{\sum_\ell \pi_{\ell,j} r_\ell}$ is either 0 (for j not in I) or smaller than $\frac{\mathbf{q}_j(r)}{\alpha}$ (for j in I). This yields

$$\sum_j \pi_{i,j} \frac{\mathbf{q}_j(r)}{\sum_\ell \pi_{\ell,j} r_\ell} \leq \sum_{j \in I} \pi_{i,j} \frac{\mathbf{q}_j(r)}{\alpha}$$

Now consider i not in I : $\pi_{i,j} \leq \beta$ for j in I yields

$$\sum_{j \in I} \pi_{i,j} \frac{\mathbf{q}_j(r)}{\alpha} \leq \sum_{j \in I} \beta \frac{\mathbf{q}_j(r)}{\alpha} \leq \frac{\beta}{\alpha}$$

This proves

$$\sum_j \pi_{i,j} \frac{\mathbf{q}_j(r)}{\sum_\ell \pi_{\ell,j} r_\ell} = \frac{\beta}{\alpha} \text{ } i \notin I$$

Thus if (40) holds, $\beta/\alpha < 1$, (38) is met with $k = \beta/\alpha$.

Step 3. There exist matrices with several locally stable points. To show this, note that the columns indexed by I in matrix π can be chosen so that there is a locally stable point with support I , independently of the columns indexed by $N - I$. To see this, observe that conditions (38), which ensure local convergence to zero of the components on $N - I$, only bear on the columns indexed by I . Furthermore the values of these columns on rows I can be chosen freely given the lower bound m so they can be chosen so that there is convergence within I (that is starting with a ranking null on $N - I$). Combining these two conditions, which bear only on the columns indexed by I , gives a locally stable point with support I . The values of the matrix on $N - I$ can also be chosen so that the same result hold on $N - I$: this ensures the existence of another stable point with support included in $N - I$. Clearly, the argument can be extended so as to show that several (more than two) stable points may exist. ■

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