



HAL
open science

Fertility in the absence of self-control

Bertrand Wigniolle

► **To cite this version:**

| Bertrand Wigniolle. Fertility in the absence of self-control. 2011. halshs-00565321

HAL Id: halshs-00565321

<https://shs.hal.science/halshs-00565321>

Submitted on 11 Feb 2011

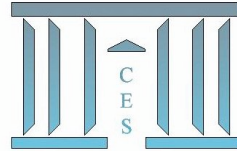
HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Documents de Travail du Centre d'Économie de la Sorbonne

C
E
S
W
o
r
k
i
n
g
P
a
p
e
r
s



Fertility in the absence of self-control

Bertrand WIGNIOLLE

2011.07



Fertility in the absence of self-control*

Bertrand Wigniolle[†]

January 27, 2011

Abstract

This paper studies the quantity-quality trade-off model of fertility, under the assumption of hyperbolic discounting. It shows that the lack of self-control may play a different role in a developed economy and in a developing one. In the first case, characterized by a positive investment in quality, the lack of self control may tend to reduce fertility. In the second case, it is possible that the lack of self-control leads to both no investment in quality and a higher fertility rate. It is also proved that if parents cannot commit on their investment in quality, a small change of parameters may lead to a jump in fertility and education.

JEL classification: D91, J13, O12

Keywords: endogenous fertility, quasi-hyperbolic discounting.

*I thank participants to Public Economic Theory conference in Seoul (2008), where a preliminary version of this paper was presented. I have also benefited from helpful comments at the EQUIPPE seminar in the University of Lille 1.

[†]Paris School of Economics and University of Paris 1. Address: C.E.S., Maison des Sciences Economiques, 106-112, boulevard de l'hôpital, 75647 Paris Cedex 13, France. Tel: +33 (0)1 44 07 81 98. Email : wignioll@univ-paris1.fr.

1 Introduction

From the seminal articles of Becker and Lewis (1973), and Becker and Tomes (1976), the benchmark theory of fertility decisions within the family is the quantity-quality trade-off model. According to this model, the quality and quantity of children are both endogenous variables. Fertility behaviors and investments in children's human capital are consciously and jointly determined by parents. This theory explains fertility and education behaviors as an optimal choice of the household, depending on its income and on the costs of quality and quantity.

In this paper, I argue that this theory is founded on the implicit assumption of perfect self-control of the household. Indeed, as education decisions are taken after the fertility decision, it is not obvious that the education decision ex-post is consistent with the education decision planned at the time of the fertility choice. This problem of self-control exists if agents are endowed with a non recursive utility function, for instance if the flows of instantaneous utilities are discounted with (quasi)-hyperbolic discounting.

Recently, a growing literature has stressed the assumption of (quasi)-hyperbolic discount rates. It seems more consistent with laboratory experiments that find a negative relationship between discount rates and time delay (see e. g. Loewenstein, and Thaler (1989)). The consequences of quasi-hyperbolic discounting have been studied in various frameworks. Many articles have been concerned with savings behavior, mainly Harris and Laibson (2001) and Laibson (1997). Diamond and Köszegi (2003) applied hyperbolic discounting to the early retirement pattern of workers. Barro (1999) introduced this assumption in a standard growth model. Wrede (2009) applies quasi-hyperbolic discounting to the timing and number of births, pointing out a possible postponement of births.

A recent article by Salanié and Treich (2006) has made a breakthrough in this literature. In discrete time, quasi-hyperbolic discounting is introduced in the intertemporal utility function of the consumer by adding an extra parameter $\beta \leq 1$ that represents the bias for the present. The instantaneous flows of utility are weighted by the discount factors: $1, \beta\delta, \beta\delta^2, \beta\delta^3$, etc. The standard assumption of exponential consumers is obtained for $\beta = 1$. Hyperbolic consumers have a bias for the present $\beta < 1$. In order to evaluate the impact of self-control on behaviors, most articles have compared the results obtained for $\beta < 1$ with that obtained for $\beta = 1$. The point made by Salanié and Treich is that this comparison is not appropriate to isolate the effect of a lack of self-control, as β also modifies the preferences of the consumer. The only pertinent comparison is between the behavior of a consumer with

commitment power, and that of a consumer without this power.

In this paper I consider a simple model in which parents arbitrate between the quantity and the quality of their children. The household's utility depends on the flows of instantaneous utility obtained during three periods. These flows are discounted with a quasi-hyperbolic discount factor. In the first period, self 1 chooses the quantity of children. Each Child entails a cost in time for the household (mainly for the wife) and implies a reduction of income. This cost comes from child rearing and the primary education given inside the family. In the second period, self 2 chooses the quality level (the education level) given to each child. The education cost is proportional to the number of children and to the level of quality. Finally, in the third period the flow of utility depends positively on both quantity and quality levels. This last assumption can be interpreted as the altruistic feeling of parents that value both the number and the quality of their children. It could also be viewed as the total gain received from children, if they are altruistic towards their parents and make them a gift.

Following Salanié and Treich, the commitment solution (C in abbreviated form) for fertility and education is compared to the solution without commitment, obtained as the Nash equilibrium reached by selves 1 and 2 in their game. I call this last solution the temporary consistent solution (TC in abbreviated form)¹. Two cases are studied. In the first one, interpreted as the case of a developed economy, both C and TC solutions lead to a positive investment in quality. The impact of the absence of self control depends on the elasticity of substitution of preferences. In the case of an elasticity of substitution greater than one, the absence of self control implies a smaller fertility. The investment in quality is also lower for β close to 1, but higher for a small value of β .

The second case corresponds to a situation for which the investment in quality cancels out along the TC solution, whereas it is positive for the C-solution. This case is thinkable for a developing economy². It leads to a higher fertility rate for the TC solution than for the C solution. It means that if the household could commit on its future education investment, it would choose a lower fertility level. For instance, a policy that imposes compulsory attendance at school for children can be viewed as a commitment technology, which is expected to reduce fertility.

¹In the literature, the temporary consistent behavior is often named the "sophisticated solution".

²A third case with no investment in quality for both solutions is not studied as it is not interesting. Indeed, for no investment in quality in both solutions, TC and C solutions give the same value of fertility.

The influence of different parameters is considered. The wife's income w_0 in the model can be viewed as the opportunity cost of fertility. Considering the TC solution and starting from a low value of w_0 , the fertility level is high and no education investment occurs. An increase of w_0 reduces fertility. At some threshold value, the household begins to invest in quality. At this value, fertility undergoes a jump downward and continues to decrease as w_0 increases. A second parameter of interest is the cost of education τ . Starting from a high value, the economy features a high fertility level with no education investment. As long as no investment in quality occurs, a decrease in τ has of course no impact on fertility. At some threshold level, the household starts to invest in quality. At this value, fertility undergoes a jump downward and continues to decrease as τ decreases³.

This model offers two novel features with respect to the existing literature. In other words, two characteristics make it difficult to infer directly the impact of self-control on fertility and education from preceding studies of savings, retirement behaviors, etc. The first characteristic is the non-linearity of the budget constraint deriving from the quantity-quality trade-off. The cost of education is the product of quality time quantity. The second characteristic comes from the property that no investment in quality is a possible solution. This solution represents the case of a developing economy, for which no investment in education is provided to children, except primary education.

Concerning the non-linearity of the budget constraint, one consequence is that the lack of self-control may imply lower investment in quality for β close to 1, but higher investment for a small value of β . This result comes from the property that the cost of quality depends on quantity, and quantity increases with β . In a model with a linear budget constraint, the lack of self-control would have a monotonic impact.

The second novel feature comes from the case for which no investment in quality is reached along the TC solution. When the quality level chosen by self 2 cancels out, the optimal response for self 1 corresponds to a jump in fertility. In other words, fertility is not continuous at the point for which quality cancels out. This property is interesting, as it means that in the neighborhood of this point, a small change in some parameters can lead to a big change in fertility. For example, a small increase in the opportunity cost of quantity can lead to a big reduction in fertility. This result can be explained considering the objective function of self 1, along the TC solution. When quality cancels out, the response function of self 2 undergoes a discontinuity of its derivative. Whereas this derivative is negative for a positive investment

³For a high elasticity of substitution in the household preferences, it is possible that the evolution of fertility becomes non-monotonic with τ .

in quality (quality is a decreasing function of quantity), the derivative is equal to zero when quality cancels out. As there is a discrepancy between the objective functions of selves 1 and 2, the derivative of the self 1 objective function undergoes a jump when quality cancels out. For this reason, two levels of fertility may exist that are local maxima of the objective function of self 1. If the change of a parameter leads to a jump from one local maximum to the other one, there is a high variation in fertility at this point.

Few studies have been devoted to this property, that a continuous change in some parameter can induce a jump of an endogenous variable, under quasi-hyperbolic discounting. It can be true in all models in which the decision of a self is subject to a constraint. Laibson (1997) was the first to point out the existence of discontinuous optimal strategies with quasi-hyperbolic discounting, in a model of savings with imperfect capital markets. To avoid the difficulties related to the non-convexity of the problem, he introduced a restriction on the labor income process that ruled out the possibility of corner solutions and discontinuous equilibrium strategies. Harris and Laibson (2002) have provided the most detailed study of this question. They give an intuition of such pathologies. They present the results of numerical simulations, and conclude that such pathologies do not arise when the model is calibrated with empirically sensible parameter values. Wigniolle (2010) remarks that the calibrations in Harris and Laibson that can eliminate the discontinuous strategies (a value of β close to 1, a small value for the elasticity of substitution) are also those that make negligible the impact of hyperbolic discounting. In other words, when hyperbolic discounting matters, it is necessary to deal with such pathologies. He provides a detailed study of such discontinuous strategies in a simple framework that allows a complete characterization.

These different studies point out the role of β : if β can depart significantly from 1, the existence of discontinuous strategies may occur. The value of β may depend on the time horizon of decisions. If the frequency of decisions is high, a value close to 1 is expected. If the interval of time between two decisions is high, a low value of β may be relevant. For decisions concerning fertility and education, it is reasonable to assume a low frequency and a small value of β . Therefore, it seems relevant to expect a strong impact of quasi-hyperbolic discounting on decisions and the occurrence of discontinuous strategies cannot be ignored.

Section 2 presents the model. Section 3 gives the fertility decisions for developed and developing economies. Section 4 studies how fertility and education decisions respond to changes in their costs. Section 5 concludes. A final appendix gives the proofs.

2 The model

2.1 Basic assumptions

A simple model is presented, for a household living during three periods and endowed with a quasi-hyperbolic discounting factor. In period 1, self 1 preferences are given by the utility function:

$$u[w_1 + w_0(1 - \phi m)] + \beta\delta u[w_2 - \tau m q] + \beta\delta^2 u[m(q_0 + q)]$$

with

$$u(x) = \frac{x^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}} \quad (1)$$

and $\sigma > 0$. m is the number of children and q the quality of each child. w_1 , w_0 , ϕ , β , δ , w_2 , τ and q_0 are positive parameters. Child quantity m is chosen in period 1 by self 1, whereas child quality q is a decision of self 2. As usual in this literature, m is considered as a continuous variable. Moreover, it is assumed that parents choose the same level of quality for each child.

In period 1, the family income consists of two parts: a constant part w_1 , and a variable part $w_0(1 - \phi m)$ that depends on child quantity m . w_1 can be viewed as husband's income, whereas w_0 is wife's income. Giving birth and raising one child takes a fraction ϕ of wife's time. Therefore, ϕw_0 is the opportunity cost for each child. The resulting consumption level of the household is $w_1 + w_0(1 - \phi m)$.

In period 2, the family income is w_2 . τ is the unit cost for one unit of quality for one child. Therefore $\tau m q$ is the cost of providing a quality q to each of the m children, and $w_2 - \tau m q$ the resulting second period consumption level of the household.

In period 3, the total revenue earned by children is assumed to be equal to $m(q_0 + q)$. q_0 is the human capital level of an uneducated agent. Parents care about the total revenue of their children. This assumption can represent either intergenerational altruism or implicit concern about potential support by children in old age.

Finally, β and δ are two positive coefficients not greater than 1.

In period 2, self 2 preferences are given by:

$$u[w_2 - \tau m q] + \beta\delta u[m(q_0 + q)]$$

The discount factor between period 3 and period 2 is δ if it is computed by self 1, and $\beta\delta$ if it is computed by self 2. The parameter β indicates whether there is a self-control problem ($\beta < 1$) or not ($\beta = 1$).

Following Salanié and Treich (2006), the time-consistent solution is compared to the commitment solution. The time-consistent solution (TC) is the non cooperative equilibrium obtained from the game played by selves 1 and 2. More precisely, self 2 chooses q , m being given. Self 1 chooses m , taking into account the best response function of self 2. The commitment solution (C) is obtained by assuming that self 1 can choose both m and q .

2.2 Investment in quality

The best response function of self 2 for the TC solution

Self 2 takes m as given and chooses q following his best response function:

$$q^{TC}(m) = \arg \max_{(q)} \begin{cases} u[w_2 - \tau m q] + \beta \delta u[m(q_0 + q)] \\ \text{s. t. } q \geq 0 \end{cases}$$

The solution to this program can be interior ($q > 0$) or not. Defining the threshold

$$\bar{m}^{TC} \equiv \frac{(\beta \delta / \tau)^\sigma w_2}{q_0}$$

the best response function of self 2 is:

$$q^{TC}(m) = \begin{cases} \frac{(\beta \delta / \tau)^\sigma \frac{w_2}{m} - q_0}{1 + (\beta \delta)^\sigma \tau^{1-\sigma}} & \text{if } m \leq \bar{m}^{TC} \\ 0 & \text{if } m \geq \bar{m}^{TC} \end{cases} \quad (2)$$

$q^{TC}(m)$ is a non-increasing function of m .

The commitment solution

Assume that self 1 can commit in period 1 on a choice of q in period 2. To compare this solution with the preceding one, it is useful to split the resolution in two steps: firstly the optimal choice of q for m given, secondly the optimal value of m , in taking into account the effect of m on the optimal choice of q . For m given, defining a new threshold

$$\bar{m}^C \equiv \frac{(\delta / \tau)^\sigma w_2}{q_0}$$

the optimal value of q if self 1 can commit on it in period 1 is:

$$q^C(m) = \begin{cases} \frac{(\delta / \tau)^\sigma \frac{w_2}{m} - q_0}{1 + \delta^\sigma \tau^{1-\sigma}} & \text{if } m \leq \bar{m}^C \\ 0 & \text{if } m \geq \bar{m}^C \end{cases} \quad (3)$$

$q^C(m)$ is a non increasing function of m . It is clear that, for m given, $q^C(m) \geq q^{TC}(m)$ with a strict inequality when $q^C(m) > 0$. For a given value of fertility, self 1's optimal investment in quality is higher than that chosen by self 2.

Remark 1 *As usual, the fertility rate is assumed to be a continuous variable. This simplifying assumption leads to meaningless results for m tending toward 0. Indeed, $q^C(m)$ and $q^{TC}(m)$ tend to be infinite when m tends toward 0, with a discontinuity in $m = 0$. Thus, it will be appropriate to eliminate parameter values leading to fertility rates close to 0.*

3 Fertility decisions under quasi-hyperbolic discounting

This section studies the impact of quasi-hyperbolic discounting on fertility and education decisions. The time-consistent solution is compared to the commitment solution. Two cases are analyzed. In the case of a developed economy, both solutions are associated with a positive investment in education. In the case of a developing economy, investment in education may cancel out. It is shown that the lack of self-control may have opposite results on fertility in these two cases: it decreases fertility in the developed economy, while it increases fertility in the developing one. Finally, a complete characterization of these two cases is provided related to parameter values.

3.1 The developed economy

This part compares the time-consistent solution with the commitment solution, when both are interior solutions: $q > 0$.

The time-consistent solution

Along the time-consistent solution, self 1 chooses m , taking into account the best response function of self 2 given by equation (2). By assumption, m is such that $q^{TC}(m) > 0$ for a developed economy. Self 1's program is:

$$\max_{m \geq 0} u [w_1 + w_0(1 - \phi m)] + \beta \delta u [w_2 - \tau m q^{TC}(m)] + \beta \delta^2 u [m(q_0 + q^{TC}(m))]$$

Defining

$$A(\beta) \equiv \frac{(1 + \delta^\sigma \beta^{\sigma-1} \tau^{1-\sigma})^\sigma}{(1 + \delta^\sigma \beta^\sigma \tau^{1-\sigma})^{\sigma-1}} \quad (4)$$

$$B \equiv \left(\frac{\tau q_0}{\phi w_0} \right)^\sigma \quad (5)$$

the time-consistent solution is:

$$m^{TC} = \frac{(\beta\delta)^\sigma A(\beta)B(w_1 + w_0) - w_2}{\tau q_0 + \phi w_0 (\beta\delta)^\sigma A(\beta)B} \quad (6)$$

This solution is valid only if $m^{TC} > 0$, which is satisfied if

$$H(\beta) \equiv (\beta\delta)^\sigma A(\beta)B > \frac{w_2}{w_1 + w_0} \quad (7)$$

Following the preceding remark, the parameter values will be restricted in such a way that (7) will hold in what follows.

The commitment solution

Along the commitment solution, self 1 chooses both m and q . This solution can be obtained using equation (3) with $q^C(m) > 0$ by assumption. The program is:

$$\max_{m \geq 0} [w_1 + w_0(1 - \phi m)] + \beta\delta u [w_2 - \tau m q^C(m)] + \beta\delta^2 u [m(q_0 + q^C(m))]$$

The commitment solution is

$$m^C = \frac{(\beta\delta)^\sigma A(1)B(w_1 + w_0) - w_2}{\tau q_0 + \phi w_0 (\beta\delta)^\sigma A(1)B} \quad (8)$$

This solution is valid only if $m^C > 0$, which gives the condition

$$(\beta\delta)^\sigma A(1)B > \frac{w_2}{w_1 + w_0} \quad (9)$$

Comparison between TC and C solutions

The only difference between the two expressions (6) and (8) is the term $A(\beta)$ in place of $A(1)$. As m is increasing with respect to A , $m^{TC} < m^C$ if and only if $A(\beta) < A(1)$. It is easy to find:

$$\frac{d \ln [A(\beta)]}{d\beta} = \frac{\sigma(\sigma - 1)\delta^\sigma \tau^{1-\sigma} \beta^{\sigma-2}(1 - \beta)}{(1 + \delta^\sigma \beta^{\sigma-1} \tau^{1-\sigma})(1 + \delta^\sigma \beta^\sigma \tau^{1-\sigma})}$$

As $\beta < 1$, $A(\beta) < A(1) \iff \sigma > 1$.

If $\sigma > 1$, $m^{TC} < m^C$: the time-consistent solution leads to a lower fertility. As self 2 does not invest enough in education from the point of view of self 1, and as education choice decreases with fertility, self 1's best response is a reduction in fertility. If self 2 could commit on a higher level of quality (for instance, if he could commit on the behavior $q^C(m)$), self 1 would invest more in the quantity of children.

In the opposite case $\sigma < 1$, the result is reversed. As self 2 under-invests in quality, self 1 increases quantity with respect to the commitment solution.

This result is close to the one obtained by Salanié and Treich (2006), in a model in which the decision variable of agents is savings. Applying their results to a CES utility function (1), they find that the time-consistent solution leads to undersavings iff $\sigma > 1$.

In the case $\sigma < 1$, the lack of self control leads to higher fertility $m^{TC} > m^C$. Therefore, it also leads to a lower quality investment : as $m^{TC} > m^C$, $q^C(m^C) > q^C(m^{TC}) > q^{TC}(m^{TC})$. The absence of commitment implies more quantity and less quality.

In the case $\sigma > 1$, it is not so easy to conclude on quality. Indeed, $q^C(m)$ and $q^{TC}(m)$ are decreasing functions, with $q^C(m) > q^{TC}(m)$ for a given level of fertility m . But, as $m^{TC} < m^C$, it is not possible yet to conclude if $q^C(m^C) \gtrless q^{TC}(m^{TC})$. Proposition 1 proves that parents under-invest in quality when β is close to 1, but they over invest for a low value of β .

The different results are summarized in the following proposition:

Proposition 2 *Assuming an interior solution for m and q (m and $q > 0$),*

- *In the case $\sigma < 1$, the lack of self control leads to higher fertility $m^{TC} > m^C$ and lower investment in education $q^C > q^{TC}$.*
- *In the case $\sigma > 1$, the lack of self-control leads to lower investments in quantity $m^{TC} < m^C$. The investment in quality is also lower for β close to one, but higher for a low β .*

Proof. See Appendix 1 ■

Assumption: $\sigma > 1$.

The assumption $\sigma > 1$ is retained in what follows. It corresponds to the case favored by Salanié and Treich (2006), in which the lack of self-control leads to under-savings. As a consequence of Proposition 1, for $\sigma > 1$ and β close to 1, every commitment mechanism on a higher investment in quality increases fertility. For instance, a public policy in favor of commitment such as compulsory schooling will lead to a higher fertility level. But for a low value of β , there is over investment in quality. The intuition behind this result is that, for a low value of β , as m^{TC} becomes weak, the cost of quality is very low. This result is due to the non linearity of the cost of education which depends also on quantity. This non linearity is a particular feature of the quantity-quality trade-off model of fertility.

Another consequence of the case $\sigma > 1$ is that the constraint (9) is weaker than (7). Therefore, only (7) will be retained.

Existence of an interior solution

The time-consistent and commitment solutions must satisfy the following inequalities: $0 < m^{TC} < \bar{m}^{TC}$, $0 < m^C < \bar{m}^C$. As $m^{TC} < m^C$, only three inequalities must be considered: $0 < m^{TC}$, $m^{TC} < \bar{m}^{TC}$ and $m^C < \bar{m}^C$.

Condition $m^{TC} > 0$ is given by (7).

The inequality $m^{TC} < \bar{m}^{TC}$ gives:

$$Z(\beta) < \frac{w_2}{w_1 + w_0} \quad (10)$$

with Z defined as:

$$Z(\beta) \equiv \frac{1}{\frac{\phi w_0 \beta^\sigma \delta^\sigma \tau^{-\sigma}}{q_0} + \frac{(\phi w_0)^\sigma}{(\tau q_0)^\sigma (\beta \delta)^\sigma} \left(\frac{1 + \beta^\sigma \delta^\sigma \tau^{1-\sigma}}{1 + \beta^{\sigma-1} \delta^\sigma \tau^{1-\sigma}} \right)^\sigma}$$

Finally, the inequality $m^C < \bar{m}^C$ gives:

$$G(\beta) < \frac{w_2}{w_1 + w_0} \quad (11)$$

with

$$G(\beta) \equiv \frac{1}{\frac{\phi w_0 \delta^\sigma \tau^{-\sigma}}{q_0} + \frac{(\phi w_0)^\sigma}{(\tau q_0)^\sigma (\beta \delta)^\sigma}} \quad (12)$$

It is straightforward to see that $G(\beta) < Z(\beta)$. Therefore there remain two necessary conditions for the existence of an interior solution of the household program: (7) and (10).

3.2 The developing economy

This part focuses on the case in which the time-consistent solution is a corner solution with no investment in quality ($q^{TC} = 0$). If the commitment solution is also associated with no education investment ($q^C = 0$), it is straightforward to see that the fertility level will be the same for the two solutions C and TC. Therefore, this case is not interesting as the lack of self-control has no impact on decisions.

More interesting is the case in which the commitment solution is associated with some positive education investment ($q^C > 0$). In this case, the lack of commitment influences education, and thus fertility behaviors.

The time-consistent solution without investment in quality

Considering the TC behavior in the corner solution with $q^{TC} = 0$, the fertility level m^{TC} is given by the first order condition:

$$-\phi w_0 [w_1 + w_0(1 - \phi m)]^{-1/\sigma} + \beta \delta^2 m^{-1/\sigma} q_0^{1-1/\sigma} = 0$$

The solution is denoted by \tilde{m}^{TC} and is equal to:

$$\tilde{m}^{TC} = \frac{(\beta\delta^2)^\sigma (\phi w_0)^{-\sigma} q_0^{\sigma-1} (w_1 + w_0)}{1 + (\beta\delta^2)^\sigma (\phi w_0)^{1-\sigma} q_0^{\sigma-1}} \quad (13)$$

Using (2), condition $\tilde{m}^{TC} > \bar{m}^{TC}$ ensuring that $q^{TC} = 0$ gives the following inequality:

$$\frac{w_2}{w_1 + w_0} < D(\beta) \quad (14)$$

with

$$D(\beta) \equiv \frac{1}{\frac{\phi w_0 (\beta\delta)^\sigma \tau^{-\sigma}}{q_0} + \frac{(\phi w_0)^\sigma}{(\tau q_0)^\sigma (\delta)^\sigma}} \quad (15)$$

Comparison with the commitment solutions

By assumption, the commitment solution is associated with some positive education investment ($q^C > 0$). Therefore, m^C is still given by (8), and condition (11) must be fulfilled. The comparison between \tilde{m}^{TC} given by (13), and m^C given by (8) gives the following result:

Proposition 3 *When the lack of self-control leads to no investment in quality for the time-consistent solution and to a positive investment for the commitment solution, the fertility level is higher for the first time-consistent solution: $\tilde{m}^{TC} > m^C$.*

Proof. From (13) and (8), the inequality $\tilde{m}^{TC} > m^C$ is equivalent to

$$G(\beta) < \frac{w_2}{w_1 + w_0}$$

This condition holds by assumption, as it corresponds to (11), which was obtained in writing the inequality $m^C < \bar{m}^C$. ■

This proposition shows that the lack of self-control has a different impact in the developing economy, as it tends to increase fertility. If self 2 could commit on some positive investment in quality, self 1 would invest less in quantity. In a developed country, a policy measure that favors commitment increases fertility. In a developing economy, such a measure will reduce fertility.

How to understand this result? For the TC solution, while q^{TC} remains positive, self 1 gives birth to fewer children in order to obtain more investment in quality by self 2. But, when q^{TC} cancels out, decreasing fertility has no more impact on quality. The optimal response of self 1 is now to increase his fertility level.

Conditions for a positive investment in quality

Considering the TC behavior, two solutions have been found: one interior solution associated with a positive investment in quality and one constrained solution with no investment in quality. The first one must satisfy the condition $Z(\beta) < \frac{w_2}{w_1+w_0}$ and the second one $\frac{w_2}{w_1+w_0} < D(\beta)$. It is easy to check that $Z(\beta) < D(\beta)$. Therefore, three cases may exist. If $\frac{w_2}{w_1+w_0} > D(\beta)$, only the interior solution exists. If $\frac{w_2}{w_1+w_0} < Z(\beta)$ only the constrained solution exists. If $Z(\beta) < \frac{w_2}{w_1+w_0} < D(\beta)$, the problem is to choose between the two solutions.

To understand this point, it is useful to consider the first order condition of the program of self 1. Self 1 chooses the optimal value of m , taking into account the best response function of self 2 $q^{TC}(m)$:

$$\begin{aligned} 0 = & -\phi w_0 u' [w_1 + w_0(1 - \phi m)] - \tau q^{TC}(m) \beta \delta u' [w_2 - \tau m q^{TC}(m)] \\ & + \beta \delta^2 (q_0 + q^{TC}(m)) u' [m(q_0 + q^{TC}(m))] \\ & + \beta \delta m \frac{dq^{TC}(m)}{dm} \{ -\tau u' [w_2 - \tau m q^{TC}(m)] + \delta u' [m(q_0 + q^{TC}(m))] \} \end{aligned}$$

For the commitment solution, the first order condition is the same, except that $q^{TC}(m)$ is replaced by $q^C(m)$. But, for the commitment solution, the expression $-\tau u' [w_2 - \tau m q^C(m)] + \delta u' [m(q_0 + q^C(m))]$ cancels out by definition of $q^C(m)$. For the time-consistent solution, the expression

$-\tau u' [w_2 - \tau m q^{TC}(m)] + \delta u' [m(q_0 + q^{TC}(m))]$ is positive, as $q^{TC}(m)$ is implicitly defined by $-\tau u' [w_2 - \tau m q^{TC}(m)] + \beta \delta u' [m(q_0 + q^{TC}(m))] = 0$. This is the consequence of the discrepancy between the objective functions of self 1 and self 2. For the derivative $dq^{TC}(m)/dm$, there is a discontinuity in \bar{m}^{TC} : this derivative is negative to the left of \bar{m}^{TC} , and is zero to the right.

The consequence of this analysis is that the derivative of self 1's objective function is always continuous for the commitment solution. But, for the time-consistent solution, the derivative of self 1's objective function is discontinuous at the point \bar{m}^{TC} , with a higher value to the right of \bar{m}^{TC} . It is then possible that self 1 objective function admits two local maxima. The function is concave on each interval $(0, \bar{m}^{TC})$ and $(\bar{m}^{TC}, +\infty)$ and continuous, but the derivative is discontinuous in \bar{m}^{TC} .

Figure 1 presents a numerical simulation with the following values of parameters: $\sigma = 2$, $\tau = 0.5$, $\phi = 0.17$, $\beta = 0.5$, $\delta = 1$, $w_2 = 2$, $q_0 = 0.5$ and $w_1 = 1$. The different curves are obtained for different values of the parameter w_0 ⁴. The objective function of self 1 with respect to m is drawn. The value

⁴The same type of analysis could be carried out with respect to another parameter than w_0 .

$w_0 = 0.951856$ is such that the two local maxima give the same value to the utility. For w_0 smaller than this value, the optimal behavior is to give birth to many children (\tilde{m}^{TC}) and to not invest in their education. For w_0 higher than this value, the optimal behavior is to have a small number of children (m^{TC}) and to invest in their quality.

Considering self 1's objective function, under the condition $Z(\beta) < \frac{w_2}{w_1+w_0} < D(\beta)$, this function of m has two local maxima: one associated with a positive investment in education (m^{TC}) and one for which education cancels out (\tilde{m}^{TC}). Therefore it is necessary to compare the utility levels obtained for each local maximum. U^{TC} denotes the indirect utility level when q^{TC} is positive and \tilde{U}^{TC} the utility level when q^{TC} is zero. The following lemma shows that the equality $U^{TC} = \tilde{U}^{TC}$ implicitly defines a function $V(\beta)$ such that $\frac{w_2}{w_1+w_0} = V(\beta) \Leftrightarrow U^{TC} = \tilde{U}^{TC}$. Moreover, $U^{TC} > \tilde{U}^{TC} \Leftrightarrow \frac{w_2}{w_1+w_0} > V(\beta)$.

Lemma 1 *Assume that $Z(\beta) < \frac{w_2}{w_1+w_0} < D(\beta)$. The condition $U^{TC} > \tilde{U}^{TC}$ holds iff:*

$$\begin{aligned} & \left(\frac{w_2}{w_1+w_0} \frac{\phi w_0}{\tau q_0} + 1 \right)^{1-1/\sigma} \left[1 + \left(\frac{\phi w_0}{\tau q_0} \right)^{1-\sigma} (\beta \delta)^\sigma A(\beta) \right]^{1/\sigma} \\ & > \left[1 + \left(\frac{\phi w_0}{q_0} \right)^{1-\sigma} (\beta \delta^2)^\sigma \right]^{1/\sigma} + \beta \delta \left(\frac{w_2}{w_1+w_0} \right)^{1-1/\sigma} \end{aligned} \quad (16)$$

This inequality implicitly defines a function $V(\beta)$ such that

$$U^{TC} > \tilde{U}^{TC} \Leftrightarrow \frac{w_2}{w_1+w_0} > V(\beta), \quad (17)$$

and this function satisfies:

$$Z(\beta) < V(\beta) < D(\beta).$$

Proof. See Appendix 2. ■

This lemma allows characterizing the optimal solution for $\frac{w_2}{w_1+w_0} \in [Z(\beta), D(\beta)]$. If $\frac{w_2}{w_1+w_0} > V(\beta)$, the optimal TC-solution is such that $q^{TC} > 0$. If $\frac{w_2}{w_1+w_0} < V(\beta)$, the optimal TC-solution is such that $q^{TC} = 0$.

3.3 Existence of the different regimes

This part provides a characterization of the existence of the different regimes with respect to the parameter β . It is based on a technical lemma:

Lemma 2 • $H(\beta)$ is an increasing function of β , and when β goes from 0 to 1, $H(\beta)$ goes from 0 to $(\delta\tau q_0)^\sigma (\phi w_0)^{-\sigma} (1 + \delta^\sigma \tau^{1-\sigma}) > D(1)$. Moreover, for every β , $H(\beta) > Z(\beta)$.

- G, Z, V and D are such that: $\forall \beta \in (0, 1)$,

$$G(\beta) < Z(\beta) < V(\beta) < D(\beta)$$

and

$$G(1) = Z(1) = V(1) = D(1) = \frac{(\delta\tau q_0)^\sigma (\phi w_0)^{-\sigma}}{1 + q_0^{\sigma-1} \delta^{2\sigma} (\phi w_0)^{1-\sigma}}$$

- G increases with β and D decreases with β .

Proof. The proof results from straightforward calculations. ■

This lemma allows a complete characterization of the different cases. Parameters are restricted to be such that (7) holds, or $w_2/(w_1 + w_0) < H(\beta)$. In this zone, the preceding analysis has shown that the functions $G(\beta)$ and $V(\beta)$ are the pertinent frontiers. The set of parameters can be divided into 3 sub-zones A, B and C . The following proposition gives for each zone the corresponding expressions of fertility and education. A numerical illustration (see Figure 2) is provided for the following values of parameters : $\sigma = 2$, $\tau = 0.5$, $\phi = 0.17$, $\delta = 1$, $q_0 = 0.5$.

Proposition 4 • The plan $(\beta, w_2/(w_1 + w_0))$ can be separated into three zones:

$$\text{Zone } A = \{(\beta, w_2/(w_1 + w_0)), w_2/(w_1 + w_0) < G(\beta)\},$$

$$\text{Zone } B = \{(\beta, w_2/(w_1 + w_0)), G(\beta) < w_2/(w_1 + w_0) < V(\beta)\},$$

$$\text{Zone } C = \{(\beta, w_2/(w_1 + w_0)), V(\beta) < w_2/(w_1 + w_0)\}.$$

- Assuming that parameters are such that $w_2/(w_1 + w_0) < H(\beta)$,

$$\text{in zone } A, q^C = q^{TC} = 0 \text{ and } \tilde{m}^C = \tilde{m}^{TC},$$

$$\text{in zone } B, q^{TC} = 0, q^C > 0 \text{ and } \tilde{m}^{TC} > m^C,$$

$$\text{in zone } C, q^{TC} > 0, q^C > 0 \text{ and } m^{TC} < m^C.$$

Proof. The proof is a direct consequence of lemmas (1) and (2). ■

In zone A , the optimal behavior in both cases leads to no investment in quality. When investment in quality cancels out, both solutions are associated with the same level of fertility.

In zone B , the time consistent solution leads to a higher level of fertility than the commitment solution, and to no investment in children's quality. If self 2 could commit on a higher investment in education, self 1 would invest less in quantity.

Zone C corresponds to the developed economy with a positive investment in quality. The temporary consistent solution leads to lower investment in quantity. If self 2 could commit on a higher investment in education, self 1 would invest more in quantity.

For a given value of $w_2/(w_1 + w_0)$, it is possible that all three zones A , B and C are successively reached depending on the value of β .

Two cases may happen. In the case $w_2/(w_1 + w_0) < G(1)$, zone A appears for β close to 1; zone B for β such that $G(\beta) < w_2/(w_1 + w_0) < V(\beta)$; zone C appears only if there exist values of β such that $V(\beta) < w_2/(w_1 + w_0) < H(\beta)$. In the case $w_2/(w_1 + w_0) > G(1)$, only zones B and C may exist because $G(\beta)$ is always smaller than $G(1) < w_2/(w_1 + w_0)$. These results show that the impact of β on the investments in quality can be ambiguous for the TC solution. Indeed, an increase of β has a twofold effect. For a given level of fertility, increasing β rises the investment in quality. But an increase in β also rises the investment in quantity, which has a negative impact on the investment in quality.

4 Impact of the costs of fertility and education

This section studies how fertility and education behaviors respond to changes of w_0 and τ .

4.1 Effect of w_0

w_0 play a crucial role in education and fertility. An increase of w_0 has a twofold impact: first it increases the opportunity cost of the quantity of children; second, for a given level of fertility, it increases the first period income of the family. The first effect (effect on the price) is expected to dominate the second one (effect on the revenue), as in the standard trade-off model between consumption and leisure. Therefore, the increase of w_0 is expected to imply a fall in fertility.

In writing equation (6) under the form:

$$m^{TC} = \frac{(\beta\delta)^\sigma A(\beta)(\tau q_0)^\sigma \frac{(w_1+w_0)}{\phi w_0} - w_2(\phi w_0)^{\sigma-1}}{\tau q_0(\phi w_0)^{\sigma-1} + (\beta\delta)^\sigma A(\beta)(\tau q_0)^\sigma}$$

it is straightforward that m^{TC} is a decreasing function of w_0 as the numerator is decreasing and the denominator is increasing. Using the same argument, m^C given by (8) is also decreasing with respect to w_0 . Finally, when education cancels out, equation (13) can be written again as

$$\tilde{m}^{TC} = \frac{(\beta\delta^2)^\sigma q_0^{\sigma-1} \frac{(w_1+w_0)}{\phi w_0}}{(\phi w_0)^{\sigma-1} + (\beta\delta^2)^\sigma q_0^{\sigma-1}}$$

which is decreasing with w_0 .

In all cases, fertility decreases with respect to w_0 . A change of w_0 can also result in a change of regime, and a drop in fertility. Starting from the fertility level without education \tilde{m}^{TC} , an increase of w_0 implies a decrease in fertility. This change of w_0 may induce such a decrease in fertility that it becomes optimal to invest in quality. At this point, there is a discontinuity in fertility that experiences a fall between \tilde{m}^{TC} and m^{TC} . In the neighborhood of the frontier value of w_0 , a small increase of w_0 induces a great drop in fertility. This jump is the consequence of the discrepancy between the objective functions of self 1 and self 2. Figure 3 gives a numerical illustration for the following values of parameters : $\sigma = 2$, $\tau = 0.5$, $\phi = 0.17$, $\beta = 0.5$, $\delta = 1$, $w_2 = 2$, $q_0 = 0.5$, $w_1 = 1$. Parameters are such that for $w_0 = 0.951856$ there is the discontinuity in fertility.

The frontiers between the different regimes can be characterized with respect to w_0 . They cannot be deduced from Figure 2, as the different functions H , V and G depend on w_0 . The characterization is made in the plan (w_0, w_1) . As before, parameters are constrained in such a way that $m^{TC} > 0$, which corresponds to condition (7). This constraint defines in the plan (w_0, w_1) a zone such that $w_1 > W^H(w_0)$, with W^H a function defined in Appendix (3). The same method is used for condition (11): a function $W^G(w_0)$ is introduced, such that the condition holds iff $w_1 < W^G(w_0)$. Finally, the function $W^V(w_0)$ is introduced, such that condition (17) holds iff $w_1 < W^V(w_0)$. The three functions $W^H(w_0)$, $W^G(w_0)$ and $W^V(w_0)$ allow to obtain a characterization of the different regimes in the plan (w_0, w_1) . This characterization is equivalent to the one given in section 3.3 in the plan $\left(\beta, \frac{w_2}{w_1+w_0}\right)$, but it shows the role of w_0 in the existence of the different regime.

The following proposition gives the complete characterizations of the different regimes in the plan (w_0, w_1) , using the frontiers defined by the three functions $W^H(w_0)$, $W^G(w_0)$ and $W^V(w_0)$.

Proposition 5 • *It is possible to define three functions $W^H(w_0)$, $W^G(w_0)$ and $W^V(w_0)$ that are non-decreasing functions of w_0 , and for all w_0 , $W^G(w_0) > W^V(w_0)$.*

- *The plan (w_0, w_1) can be separated in three zones:*

$$\text{Zone A} = \{(w_0, w_1), w_1 > W^G(w_0)\},$$

$$\text{Zone B} = \{(w_0, w_1), W^V(w_0) < w_1 < W^G(w_0)\},$$

$$\text{Zone C} = \{(w_0, w_1), w_1 < W^V(w_0)\}.$$

- *Assuming that parameters are such that $w_1 > W^H(w_0)$,*

$$\text{in zone A, } q^C = q^{TC} = 0 \text{ and } \tilde{m}^C = \tilde{m}^{TC},$$

$$\text{in zone B, } q^{TC} = 0, q^C > 0 \text{ and } \tilde{m}^{TC} > m^C,$$

$$\text{in zone C, } q^{TC} > 0, q^C > 0 \text{ and } m^{TC} < m^C.$$

Proof. See Appendix 3. ■

Zone A is obtained for a low value of w_0 , $q^C = q^{TC} = 0$ and $\tilde{m}^C = \tilde{m}^{TC}$. As the opportunity cost of children is small, fertility is high and parents do not invest in quality.

In zone B, $q^{TC} = 0$ but $q^C > 0$ and $\tilde{m}^{TC} > m^C$. For an intermediate value of w_0 , the TC behavior leads to no investment in quality, whereas parents invest in quality along the commitment solution. Fertility is lower for the commitment solution.

In zone C, q^{TC} and q^C are both positive and $m^{TC} < m^C$. For a high value of w_0 , the TC and C solutions are associated with a positive investment in quality, and fertility is higher for the commitment solution.

A consequence of these results for the TC behavior is that fertility experiences a strong discontinuity for $w_0 = (W^V)^{-1}(w_1) \equiv w_0^l$. In the neighborhood of this value w_0^l , a small increase of w_0 leads to a large drop in fertility.

Figure 4 shows a numerical simulation of the different zones in the plan (w_0, w_1) , for the same values of parameters as Figure 3. For $w_1 = 1$, fertility experiences a discontinuity at the value $w_0 = 0.951856$.

The discontinuity in the optimal strategy of self 1 is a particular feature of the model with quasi-hyperbolic discounting. In the model with exponential discounting, a change in the value of some parameter results in a continuous effect on the choices of the agent. In the model with quasi-hyperbolic discounting, it is possible to observe jumps that are related to the non-concavity

of self one's objective. This property introduces a qualitative difference in the two models that may have important empirical consequences. If the model with quasi-hyperbolic discounting is relevant, fertility behaviors may undergo large changes for some critical values of the parameters. This may have consequences for the empirical analysis of fertility and for the dynamics of demographic transitions.

4.2 Effect of τ

The parameter τ is the cost of education. An increase of τ changes the optimal trade-off between quality and quantity. The following proposition summarizes the effect of τ on fertility and education.

Proposition 6 • m^C increases when the cost of education τ increases, and q^C decreases.

- If σ is small enough, i.e. $\sigma < 1/(1 - \beta)$, m^{TC} increases when the cost of education τ increases, and $q^{TC}(m^{TC})$ decreases. If $\sigma > 1/(1 - \beta)$, m^{TC} can be a non monotonic function of τ .
- There exists a threshold $\bar{\sigma}$, with $\bar{\sigma} > 1/(1 - \beta)$, such that if $\sigma < \bar{\sigma}$, $q^{TC}(m^{TC})$ decreases with τ .

Proof. See Appendix 4. ■

For the commitment solution, an increase of τ reduces the investment in education, and increases fertility. This result is standard in the basic model of quantity-quality trade-off. For a given level of fertility, an increase of τ reduces the investment in education q . As q is lower, the cost of fertility decreases and fertility increases. For the TC-solution, the same effect is obtained for a small value of σ . But if σ is very high, τ can have a non monotonic effect.

As for w_0 , there exists a threshold level τ^l such that education cancels out for $\tau > \tau^l$. At this point τ^l , fertility is not continuous and jumps to a higher value, as education falls to zero. Figures 5 and 6 show how fertility may evolve with respect to τ . Figure 5 uses the preceding values for the different parameters: $\sigma = 2$, $\phi = 0.17$, $\beta = 0.5$, $\delta = 1$, $q_0 = 0.5$, $w_0 = 1$, $w_1 = 1$, $w_2 = 2$. The threshold level from which education cancels out is $\tau^l = 0.516066$.

Figure 6 is an example of parameters leading to a non-monotonic evolution of fertility for a high value of σ . $\sigma = 4$, $\phi = 0.28$, $\beta = 0.5$, $\delta = 1$, $q_0 = 1.5$, $w_0 = 1$, $w_1 = 1$, $w_2 = 2$. The threshold level is $\tau^l = 0.341284$.

An increase of τ also influences the existence of the different regimes. This question is studied in the plan $(\beta, w_2/(w_1 + w_0))$, considering how the different frontiers $G(\beta)$ and $V(\beta)$ are modified. For β given, it is straightforward from (12) that $G(\beta)$ is an increasing function of τ . As could be expected, the region A in which no education occurs increases with τ . Consequently there is less space for regions B and C . The frontier between regions B and C is defined with the function $V(\beta)$. Appendix 5 shows that $V(\beta)$ increases with τ . A numerical experiment is provided in Figure 7, with the following parameters $\sigma = 2$, $\phi = 0.17$, $\delta = 1$, $q_0 = 0.5$, $\tau = 0.5$ and $\tau = 0.7$. The case $\tau = 0.7$ is presented with bold lines.

5 Conclusion

This paper has studied the quantity-quality fertility model under the assumption of quasi-hyperbolic discounting. The impact of the absence of self control is isolated through the comparison between the TC solution (sophisticated behavior) and the C solution (commitment solution). The lack of self control may have different impact on fertility in a developed economy and in a developing one. In a developed economy characterized by a positive investment in quality, the lack of self control tends to reduce fertility. In a developing economy, the lack of self-control may lead to both no investment in quality and a higher fertility rate. It is also proved that if parents cannot commit on their investment in quality, a small change of parameters may lead to a jump in fertility and education.

This paper could be extended in different directions. First, the robustness of the results could be studied if the model was enriched by additional assumptions: access to capital markets for the households, imperfect capital markets through borrowing constraints, collective choice within the household, etc. Secondly, a technical improvement could be made by introducing more than three periods and more than two decisions.

References

- [1] Robert J. Barro, (1999), "Ramsey Meets Laibson In The Neoclassical Growth Model," *The Quarterly Journal of Economics*, vol. 114(4), pages 1125-1152, November.
- [2] Becker, G. S. and Lewis, H. G. (1973) 'Interaction between Quantity and Quality in Children'. *Journal of Political Economy*, **81**, S279-S288.

- [3] Becker, G. S. and Tomes, N. (1976) 'Child Endowments and the Quantity and Quality of Children'. *Journal of Political Economy*, **84**, S143-S162.
- [4] Diamond, Peter & Kozzegi, Botond, (2003), "Quasi-hyperbolic discounting and retirement," *Journal of Public Economics*, vol. 87(9-10), pages 1839-1872, September.
- [5] Harris, Christopher & Laibson, David, (2001), "Dynamic Choices of Hyperbolic Consumers," *Econometrica*, vol. 69(4), pages 935-57, July.
- [6] Harris, Christopher and David Laibson. (2002), "Hyperbolic discounting and consumption". Mathias Dewatripont, Lars Peter Hansen, and Stephen Turnovsky, editors. *Advances in Economics and Econometrics: Theory and Applications*, Eighth World Congress (1)258-298.
- [7] Laibson, David, (1997), "Golden Eggs and Hyperbolic Discounting," *The Quarterly Journal of Economics*, MIT Press, vol. 112(2), pages 443-77, May.
- [8] Loewenstein, George & Thaler, Richard H, (1989). "Intertemporal Choice," *Journal of Economic Perspectives*, vol. 3(4), pages 181-93, Fall.
- [9] Salanié, Francois & Treich, Nicolas, 2006. "Over-savings and hyperbolic discounting," *European Economic Review*, Elsevier, vol. 50(6), pages 1557-1570, August.
- [10] Wrede Matthias (2009) Hyperbolic discounting and fertility. *Journal of Population Economics*, Online publication date: 9-Oct-2009.
- [11] Wigniolle Bertrand (2010), "Hyperbolic discounting and discontinuous savings strategies", mimeo, University of Paris 1.

6 Appendixes

6.1 Appendix 1

The comparison between m^{TC} and m^C is made in the text. In the case $\sigma < 1$, the comparison between $q^C(m^C)$ and $q^{TC}(m^{TC})$ is simple and is made in the text. It remains to compare $q^C(m^C)$ and $q^{TC}(m^{TC})$ when $\sigma > 1$.

First, it appears that:

$$q^{TC}(m^{TC}) + q_o = \frac{(\beta\delta/\tau)^\sigma \left(\frac{w_2}{m^{TC}} + \tau q_0\right)}{1 + (\beta\delta)^\sigma \tau^{1-\sigma}}$$

and

$$q^C(m^C) + q_o = \frac{(\delta/\tau)^\sigma \left(\frac{w_2}{m^C} + \tau q_0\right)}{1 + (\delta)^\sigma \tau^{1-\sigma}}$$

From (6), is obtained:

$$\frac{w_2}{m^{TC}} + \tau q_0 = \frac{(\beta\delta)^\sigma A(\beta)B [\tau q_0 (w_1 + w_0) + w_2 \phi w_0]}{(\beta\delta)^\sigma A(\beta)B (w_1 + w_0) - w_2} \quad (18)$$

From (8), is obtained:

$$\frac{w_2}{m^C} + \tau q_0 = \frac{(\beta\delta)^\sigma A(1)B [\tau q_0 (w_1 + w_0) + w_2 \phi w_0]}{(\beta\delta)^\sigma A(1)B (w_1 + w_0) - w_2} \quad (19)$$

As $A(1) = 1 + \delta^\sigma \tau^{1-\sigma}$, it follows:

$$q^{TC}(m^{TC}) + q_o < q^C(m^C) + q_o \Leftrightarrow \frac{(\beta)^\sigma}{1 + (\beta\delta)^\sigma \tau^{1-\sigma}} \frac{A(\beta)}{(\beta\delta)^\sigma A(\beta)B (w_1 + w_0) - w_2} < \frac{1}{(\beta\delta)^\sigma A(1)B (w_1 + w_0) - w_2} \quad (20)$$

After rearranging and using the expression (4) of $A(\beta)$, it is possible to write this inequality:

$$0 < B(w_1 + w_0) \delta^\sigma - w_2 f(\beta) \quad (21)$$

with

$$f(\beta) \equiv \frac{\left(\frac{1 + \delta^\sigma \beta^\sigma \tau^{1-\sigma}}{\beta + \delta^\sigma \beta^\sigma \tau^{1-\sigma}}\right)^\sigma - 1}{1 - \beta^\sigma}$$

Firstly the inequality (21) is studied in a neighborhood of $\beta = 1$. In setting $x = \beta^\sigma$, a function g is introduced such that:

$$g(x) \equiv \frac{\left(\frac{1 + x \delta^\sigma \tau^{1-\sigma}}{x^{1/\sigma} + x \delta^\sigma \tau^{1-\sigma}}\right)^\sigma - 1}{1 - x} = f(\beta)$$

The limit of g when x tends toward 1 is equal to the limit of f in $\beta = 1$. Defining a function $h(x)$ such that:

$$h(x) \equiv \left(\frac{1 + x \delta^\sigma \tau^{1-\sigma}}{x^{1/\sigma} + x \delta^\sigma \tau^{1-\sigma}}\right)^\sigma$$

this limit is equal to $-h'(1)$. Taking the derivative of the logarithm of h in $x = 1$, it is obtained:

$$h'(1) = h'(1)/h(1) = \frac{\sigma\delta^\sigma\tau^{1-\sigma}}{1 + \delta^\sigma\tau^{1-\sigma}} - \frac{1 + \sigma\delta^\sigma\tau^{1-\sigma}}{1 + \delta^\sigma\tau^{1-\sigma}} = -\frac{1}{1 + \delta^\sigma\tau^{1-\sigma}}$$

Thus, with $\beta = 1$, (21) becomes:

$$0 < B(w_1 + w_0)\delta^\sigma - \frac{w_2}{1 + \delta^\sigma\tau^{1-\sigma}}$$

which is satisfied as it corresponds to (9) with $\beta = 1$. It is then proved that $q^{TC} < q^C$ in a neighborhood of $\beta = 1$.

Secondly, the inequality (21) is studied for a low value of β . When β tends toward 0, $f(\beta)$ tends to be infinite, the inequality (21) cannot be satisfied, and $q^{TC} > q^C$. β close to 0 is not possible as it implies negative values for m^{TC} and m^C . The smallest possible value of β corresponds to the constraint (7) ensuring $m^{TC} > 0$. When β tends to this value, the left-hand side of (20) tends to be infinite. Thus, when β is low enough, (21) cannot be satisfied, and $q^{TC} > q^C$.

6.2 Appendix 2

In this appendix, a new notation x is introduced for the expression $\frac{w_2}{w_1+w_0}$. Assume that $Z(\beta) < x < D(\beta)$. The equality $U^{TC} = \tilde{U}^{TC}$ implicitly defines x as a function of β : $f(x, \beta) = 0$ with

$$\begin{aligned} f(x, \beta) \equiv & \frac{\sigma - 1}{\sigma} \left(U^{TC} - \tilde{U}^{TC} \right) \frac{1}{(w_1 + w_0)^{1-1/\sigma}} = \\ & \left(x \frac{\phi w_0}{\tau q_0} + 1 \right)^{1-1/\sigma} \left[1 + \left(\frac{\phi w_0}{\tau q_0} \right)^{1-\sigma} (\beta\delta)^\sigma A(\beta) \right]^{1/\sigma} \\ & - \left[1 + \left(\frac{\phi w_0}{q_0} \right)^{1-\sigma} (\beta\delta^2)^\sigma \right]^{1/\sigma} - \beta\delta x^{1-1/\sigma} \end{aligned}$$

First, it is proved that $\partial f/\partial x > 0$. The condition $\partial f/\partial x > 0$ is equivalent to:

$$\frac{\phi w_0}{\tau q_0} \left(x \frac{\phi w_0}{\tau q_0} + 1 \right)^{-1/\sigma} \left[1 + \left(\frac{\phi w_0}{\tau q_0} \right)^{1-\sigma} (\beta\delta)^\sigma A(\beta) \right]^{1/\sigma} > \beta\delta x^{-1/\sigma}$$

which is equivalent to

$$x > \frac{1}{\left(\frac{\phi w_0}{\tau q_0}\right)^\sigma (\beta \delta)^{-\sigma} + \frac{\phi w_0}{\tau q_0} [A(\beta) - 1]} \equiv \Omega(\beta)$$

From this inequality, as by assumption $x > Z(\beta)$, if $Z(\beta) > \Omega(\beta)$, the property $x > \Omega(\beta)$ will be satisfied and $\partial f / \partial x > 0$.

$Z(\beta) > \Omega(\beta)$ is equivalent to:

$$\frac{\phi w_0}{\tau q_0} [A(\beta) - 1 - (\beta \delta)^\sigma \tau^{1-\sigma}] + \left(\frac{\phi w_0}{\tau q_0}\right)^\sigma (\beta \delta)^{-\sigma} \left[1 - \frac{1 + (\beta \delta)^\sigma \tau^{1-\sigma}}{A(\beta)}\right] > 0$$

From the definition of $A(\beta)$, $A(\beta) > 1 + (\beta \delta)^\sigma \tau^{1-\sigma} \Leftrightarrow 1 + \beta^{\sigma-1} \delta^\sigma \tau^{1-\sigma} > 1 + (\beta \delta)^\sigma \tau^{1-\sigma}$ which is true for $\beta < 1$.

Finally the property $\partial f / \partial x > 0$ is proved.

The next step is to prove that $f(Z(\beta), \beta) < 0$ and $f(D(\beta), \beta) > 0$. These two inequalities with the property $\partial f / \partial x > 0$ will ensure the existence and uniqueness of x as a function $V(\beta)$ of β .

After tedious calculations, it is possible to write $f(Z(\beta), \beta) < 0$ under the form

$$\begin{aligned} & 1 + \left(\frac{\phi w_0}{q_0}\right)^{1-\sigma} (\beta \delta^2)^\sigma \left(\frac{\beta + \delta^\sigma \beta^\sigma \tau^{1-\sigma}}{1 + \delta^\sigma \beta^\sigma \tau^{1-\sigma}}\right)^{\sigma-1} \\ < & \left[1 + \left(\frac{\phi w_0}{q_0}\right)^{1-\sigma} (\beta \delta^2)^\sigma\right]^{1/\sigma} \left[1 + \left(\frac{\phi w_0}{q_0}\right)^{1-\sigma} (\beta \delta^2)^\sigma \left(\frac{\beta + \delta^\sigma \beta^\sigma \tau^{1-\sigma}}{1 + \delta^\sigma \beta^\sigma \tau^{1-\sigma}}\right)^\sigma\right]^{1-1/\sigma} \end{aligned}$$

The following notations are introduced:

$$\begin{aligned} a &= \left(\frac{\phi w_0}{q_0}\right)^{1-\sigma} (\beta \delta^2)^\sigma \\ y(\beta) &= \frac{\beta + \delta^\sigma \beta^\sigma \tau^{1-\sigma}}{1 + \delta^\sigma \beta^\sigma \tau^{1-\sigma}} \end{aligned}$$

It is possible to write the preceding inequality:

$$\frac{[1 + ay(\beta)^{\sigma-1}]^\sigma}{[1 + ay(\beta)^\sigma]^{\sigma-1}} < 1 + a \quad (22)$$

The function $y(\beta)$ increases from 0 toward 1 when β goes from 0 to 1.

As a function of y , the expression

$$\frac{[1 + ay^{\sigma-1}]^\sigma}{[1 + ay^\sigma]^{\sigma-1}}$$

is strictly increasing when y goes from 0 to 1, and is equal to $1 + a$ for $y = 1$.

By these two properties, it is proved that (22) is satisfied.

The condition $f(D(\beta), \beta) > 0$ can be written after some calculations

$$\begin{aligned} & \left[1 + \left(\frac{\phi w_0}{\tau q_0} \right)^{1-\sigma} \delta^\sigma (1 + \delta^\sigma \beta^\sigma \tau^{1-\sigma}) \right]^{1-1/\sigma} \left[1 + \left(\frac{\phi w_0}{\tau q_0} \right)^{1-\sigma} \delta^\sigma \frac{(\beta + \delta^\sigma \beta^\sigma \tau^{1-\sigma})^\sigma}{(1 + \delta^\sigma \beta^\sigma \tau^{1-\sigma})^{\sigma-1}} \right]^{1/\sigma} \\ > & 1 + \left(\frac{\phi w_0}{\tau q_0} \right)^{1-\sigma} \delta^\sigma (\beta + \delta^\sigma \beta^\sigma \tau^{1-\sigma}) \end{aligned}$$

The following notations are introduced:

$$\begin{aligned} b &= \left(\frac{\phi w_0}{\tau q_0} \right)^{1-\sigma} \delta^\sigma \\ \xi &= 1 + \delta^\sigma \beta^\sigma \tau^{1-\sigma} \\ \theta &= \beta + \delta^\sigma \beta^\sigma \tau^{1-\sigma} \end{aligned}$$

By definition, $\theta \leq \xi$ with a strict inequality for $\beta < 1$. It is possible to write the preceding inequality:

$$1 + b \frac{\theta^\sigma}{\xi^{\sigma-1}} > \frac{(1 + b\theta)^\sigma}{(1 + b\xi)^{\sigma-1}} \quad (23)$$

Defining the function g :

$$g(\theta) = 1 + b \frac{\theta^\sigma}{\xi^{\sigma-1}} - \frac{(1 + b\theta)^\sigma}{(1 + b\xi)^{\sigma-1}}$$

it is easy to check that it is strictly decreasing for $\theta \in [0, \xi]$, with $g(\xi) = 0$. Therefore, it is proved that (23) is satisfied, and $f(D(\beta), \beta) > 0$.

6.3 Appendix 3

Condition (11) can be written under a condition on w_0 and w_1 . The inequality $\frac{w_2}{w_1 + w_0} > G(\beta)$ is equivalent to:

$$w_1 < \left(\frac{\phi \delta^\sigma \tau^{-\sigma} w_2}{q_0} - 1 \right) w_0 + \frac{w_2 (\phi w_0)^\sigma}{(\tau q_0)^\sigma (\beta \delta)^\sigma} \equiv \Gamma(w_0)$$

The right-hand side member of this inequality is a function Γ of w_0 such that: if $\frac{\phi\delta^\sigma\tau^{-\sigma}w_2}{q_0} > 1$, Γ is strictly increasing; if $\frac{\phi\delta^\sigma\tau^{-\sigma}w_2}{q_0} < 1$, Γ is U-shaped, first decreasing and then increasing. As w_1 cannot be negative, the negative part of Γ does not play any role. The function W^G is defined a

$$W^G(w_0) = \max \{\Gamma(w_0), 0\}$$

By definition, either $W^G(w_0)$ is strictly increasing, or it is first equal to 0, and then strictly increasing.

For condition (10), the inequality $\frac{w_2}{w_1+w_0} > Z(\beta)$ is equivalent to:

$$w_1 < \left(\frac{\phi\beta^\sigma\delta^\sigma\tau^{-\sigma}w_2}{q_0} - 1 \right) w_0 + \frac{w_2(\phi w_0)^\sigma}{(\tau q_0)^\sigma\delta^\sigma} \left(\frac{1 + \beta^\sigma\delta^\sigma\tau^{1-\sigma}}{\beta + \beta^\sigma\delta^\sigma\tau^{1-\sigma}} \right)^\sigma \equiv \vartheta(w_0)$$

As for the preceding example, a function W^Z is defined as

$$W^Z(w_0) = \max \{\vartheta(w_0), 0\}$$

By definition, $W^Z(0) = 0$, either $W^Z(w_0)$ is strictly increasing, or it is first equal to 0, and then strictly increasing.

For condition (14), the inequality $\frac{w_2}{w_1+w_0} < D(\beta)$ is equivalent to:

$$w_1 > \left(\frac{\phi\beta^\sigma\delta^\sigma\tau^{-\sigma}w_2}{q_0} - 1 \right) w_0 + \frac{w_2(\phi w_0)^\sigma}{(\tau q_0)^\sigma\delta^\sigma} \equiv \Delta(w_0)$$

As for the preceding examples, a function W^D is defined as

$$W^D(w_0) = \max \{\Delta(w_0), 0\}$$

By definition, $W^D(0) = 0$, either $W^D(w_0)$ is strictly increasing, or it is first equal to 0, and then strictly increasing.

Finally condition (7) can be written:

$$w_1 > \frac{w_2(\phi w_0)^\sigma}{(\tau q_0)^\sigma (\beta\delta)^\sigma} \frac{(1 + \delta^\sigma\beta^\sigma\tau^{1-\sigma})^{\sigma-1}}{(1 + \delta^\sigma\beta^{\sigma-1}\tau^{1-\sigma})^\sigma} - w_0 \equiv \Xi(w_0)$$

As for the preceding examples, a function W^H is defined as

$$W^H(w_0) = \max \{\Xi(w_0), 0\}$$

By definition, $W^H(w_0)$ is first equal to 0, and then strictly increasing.

Lemma 4 with Appendix 2 allow to define a function $V(\beta)$ such that $\frac{w_2}{w_1+w_0} = V(\beta) \Leftrightarrow U^{TC} = \tilde{U}^{TC}$. This function $V(\beta)$ depends on different parameters of the model including w_0 , but does not depend on w_1 . Therefore, it is clear that it can be expressed under the form:

$$w_1 = \frac{w_2}{V(\beta)} - w_0$$

A function W^V is defined as:

$$W^V(w_0) = \max \left\{ \frac{w_2}{V(\beta)} - w_0, 0 \right\}$$

As $Z(\beta) < V(\beta) < D(\beta)$, it implies that: $W^D(w_0) < W^V(w_0) < W^Z(w_0)$.

To find how $W^V(w_0)$ evolves with w_0 , it is useful to come back to the definition. When $w_1 = W^V(w_0)$ is positive, the function is implicitly defined by the relation $U^{TC} - \tilde{U}^{TC} = 0$. The derivative is implicitly given by:

$$\frac{dW^V(w_0)}{dw_0} = - \frac{\frac{\partial(U^{TC} - \tilde{U}^{TC})}{\partial w_0}}{\frac{\partial(U^{TC} - \tilde{U}^{TC})}{\partial w_1}}$$

If $\frac{\partial(U^{TC} - \tilde{U}^{TC})}{\partial w_0} \neq 0$, it will prove that $W^V(w_0)$ is monotonic. As $W^D(w_0) < W^V(w_0) < W^Z(w_0)$, with $W^D(w_0)$ and $W^Z(w_0)$ two increasing functions tending to $+\infty$ when $w_0 \rightarrow +\infty$, the only possibility will be that $W^V(w_0)$ is monotonically increasing.

It is possible to prove that $\frac{\partial(U^{TC} - \tilde{U}^{TC})}{\partial w_0} > 0$. U^{TC} is the maximum value of self 1's objective function when $q^{TC}(m) > 0$. \tilde{U}^{TC} is the maximum value of self 1's objective function when $q^{TC}(m) = 0$. The derivatives can be obtained using the envelope theorem:

$$\begin{aligned} \frac{\partial U^{TC}}{\partial w_0} &= (1 - \phi m^{TC}) [w_1 + w_0(1 - \phi m^{TC})]^{-\frac{1}{\sigma}} \\ \frac{\partial \tilde{U}^{TC}}{\partial w_0} &= (1 - \phi \tilde{m}^{TC}) [w_1 + w_0(1 - \phi \tilde{m}^{TC})]^{-\frac{1}{\sigma}} \end{aligned}$$

As $\sigma > 1$, the function $x [w_1 + w_0x]^{-\frac{1}{\sigma}}$ is an increasing function of x , as its logarithmic derivative is

$$\frac{\sigma w_1 + (\sigma - 1)w_0x}{x(w_1 + w_0x)\sigma}$$

Consequently, the function $(1 - \phi m) [w_1 + w_0(1 - \phi m)]^{-\frac{1}{\sigma}}$ is a decreasing function of m . As $\tilde{m}^{TC} > m^{TC}$, it is obtained that,

$$\frac{\partial U^{TC}}{\partial w_0} > \frac{\partial \tilde{U}^{TC}}{\partial w_0}$$

which implies that the function $W^V(w_0)$ is increasing.

6.4 Appendix 4

From (8), m^C can be written:

$$m^C = \frac{x(\tau)(w_1 + w_0) - \frac{w_2}{\tau q_0}}{1 + \phi w_0 x(\tau)}$$

with $x(\tau) \equiv (\beta\delta)^\sigma q_0^{\sigma-1} (\phi w_0)^{-\sigma} (\tau^{\sigma-1} + \delta^\sigma)$. Taking the derivative of m^C with respect to τ , it is obtained that the sign of this derivative is the sign of the expression:

$$\begin{aligned} & \left[x'(\tau)(w_1 + w_0) + \frac{w_2}{\tau^2 q_0} \right] [1 + \phi w_0 x(\tau)] - \left[x(\tau)(w_1 + w_0) - \frac{w_2}{\tau q_0} \right] [\phi w_0 x'(\tau)] \\ &= x'(\tau) \left[(w_1 + w_0) + \phi w_0 \frac{w_2}{\tau q_0} \right] + \frac{w_2}{\tau^2 q_0} [1 + \phi w_0 x(\tau)] > 0 \end{aligned}$$

Thus, m^C is an increasing function of τ .

From (3), the quality level q is such that:

$$q_0 \tau^\sigma + q(\tau^\sigma + \delta^\sigma \tau) = \delta^\sigma \frac{w_2}{m^C}$$

If τ increases, as m^C increases, q must decrease.

From (6), the time-consistent solution can be written:

$$m^{TC} = \frac{y(\tau)(w_1 + w_0) - \frac{w_2}{\tau q_0}}{1 + \phi w_0 y(\tau)}$$

with $y(\tau) \equiv (\beta\delta)^\sigma q_0^{\sigma-1} (\phi w_0)^{-\sigma} \tau^{\sigma-1} A(\beta)$. If $y'(\tau) > 0$, it is known from the preceding calculation that m^{TC} increases with τ . Therefore, it remains to check if $\tau^{\sigma-1} A(\beta)$ increases with τ . After some calculations, it is obtained that

$$\frac{d \ln [\tau^{\sigma-1} A(\beta)]}{d\tau} = (\sigma - 1) \frac{1 - [\sigma(1 - \beta) - 1] \delta^\sigma \beta^{\sigma-1} \tau^{1-\sigma}}{\tau (1 + \delta^\sigma \beta^{\sigma-1} \tau^{1-\sigma}) (1 + \delta^\sigma \beta^\sigma \tau^{1-\sigma})}$$

If $\sigma(1 - \beta) < 1$, $y'(\tau) > 0$ and m^{TC} increases with τ . If $\sigma(1 - \beta) > 1$, it is not possible to achieve a general conclusion.

Assuming that m^{TC} increases with τ , from (2), the quality level q is such that:

$$q_0\tau^\sigma + q(\tau^\sigma + (\beta\delta)^\sigma \tau) = (\beta\delta)^\sigma \frac{w_2}{m^{TC}}$$

If τ increases, as m^{TC} increases, q must decrease.

Considering now $q^{TC}(m^{TC})$, Appendix 1 has shown that

$$q^{TC}(m^{TC}) + q_o = \frac{(\beta\delta/\tau)^\sigma \left(\frac{w_2}{m^{TC}} + \tau q_0\right)}{1 + (\beta\delta)^\sigma \tau^{1-\sigma}}$$

or

$$q^{TC}(m^{TC}) + q_o = \frac{(\beta\delta)^\sigma [\tau q_0 (w_1 + w_0) + w_2 \phi w_0]}{\tau^\sigma + (\beta\delta)^\sigma \tau} \frac{(\beta\delta)^\sigma A(\beta)B}{(\beta\delta)^\sigma A(\beta)B (w_1 + w_0) - w_2}$$

It is easy to check that the first term

$$\frac{(\beta\delta)^\sigma [\tau q_0 (w_1 + w_0) + w_2 \phi w_0]}{\tau^\sigma + (\beta\delta)^\sigma \tau} = \frac{(\beta\delta)^\sigma \left[q_0 (w_1 + w_0) + \frac{w_2 \phi w_0}{\tau} \right]}{\tau^{\sigma-1} + (\beta\delta)^\sigma}$$

is a decreasing function of τ as $\sigma > 1$.

Defining $z(\tau) = (\beta\delta)^\sigma A(\beta)B$, the second term can be written

$$\frac{z(\tau)}{z(\tau) (w_1 + w_0) - w_2}$$

This is a decreasing function with respect to $z(\tau)$. Therefore, if $z(\tau)$ increases with τ , it will be obtained that $q^{TC}(m^{TC})$ is a decreasing function of τ .

$z(\tau)$ can be written:

$$z(\tau) = (\beta\delta)^\sigma \left(\frac{q_0}{\phi w_0} \right)^\sigma \frac{(\tau^\sigma + \delta^\sigma \beta^{\sigma-1} \tau)^\sigma}{(\tau^\sigma + \delta^\sigma \beta^\sigma \tau)^{\sigma-1}}$$

The sign of $z'(\tau)/z(\tau)$ is given by the sign of:

$$\tau^{2\sigma-1} + \tau^\sigma \delta^\sigma \beta^{\sigma-1} [-\sigma^2 (1 - \beta) + (2 - \beta) \sigma + \beta] + \delta^{2\sigma} \beta^{2\sigma-1}$$

A sufficient condition to have $z'(\tau) > 0$ is that $-\sigma^2 (1 - \beta) + (2 - \beta) \sigma + \beta > 0$. Considering this second degree equation in σ , the property $z'(\tau) > 0$ will hold if

$$\sigma < \bar{\sigma} \equiv \frac{2 - \beta + \sqrt{4 - 3\beta^2}}{2(1 - \beta)}$$

It is easy to check that $\bar{\sigma} > 1/(1 - \beta)$ as it is equivalent to $\beta < 1$.

6.5 Appendix 5

This appendix proves that $V(\beta)$ increases with τ . This function has been defined in Appendix 2 as the solution x implicitly defined by the equation: $f(x, \beta) = 0$ with $x = \frac{w_2}{w_1 + w_0}$ and

$$f(x, \beta) = \frac{\sigma - 1}{\sigma} \left(U^{TC} - \tilde{U}^{TC} \right) \frac{1}{(w_1 + w_0)^{1-1/\sigma}}$$

From this definition, it appears that

$$\frac{\partial V(\beta)}{\partial \tau} = - \frac{\frac{\partial f}{\partial \tau}}{\frac{\partial f}{\partial x}}$$

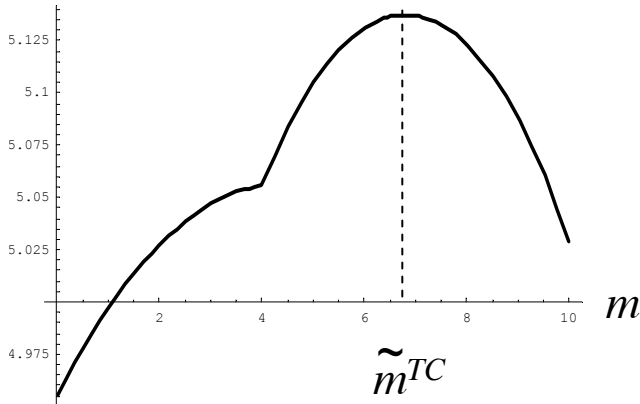
In Appendix 2, it was shown that $\frac{\partial f}{\partial x} > 0$. It remains to prove that $\frac{\partial f}{\partial \tau} < 0$, which is equivalent to prove that: $\frac{\partial(U^{TC} - \tilde{U}^{TC})}{\partial \tau} < 0$. U^{TC} is the maximum value of self 1's objective function when $q^{TC}(m) > 0$. \tilde{U}^{TC} is the maximum value of self 1's objective function when $q^{TC}(m) = 0$. The derivatives can be obtained using the envelope theorem:

$$\begin{aligned} \frac{\partial U^{TC}}{\partial w_0} &= -m^{TC} q^{TC} \beta \delta [w_2 - \tau m^{TC} q^{TC}]^{-\frac{1}{\sigma}} \\ \frac{\partial \tilde{U}^{TC}}{\partial w_0} &= 0 \end{aligned}$$

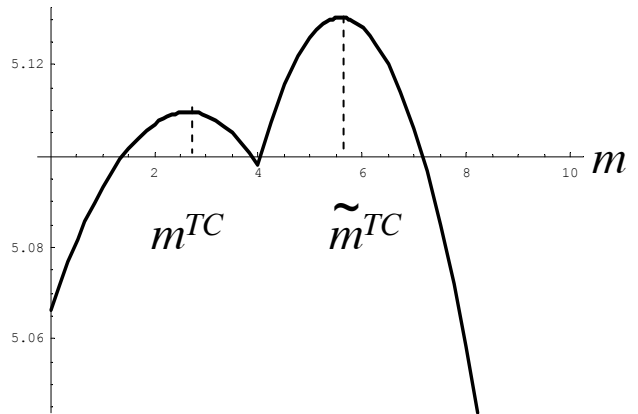
Therefore, $\frac{\partial(U^{TC} - \tilde{U}^{TC})}{\partial \tau} < 0$ and $V(\beta)$ is an increasing function of τ .

Figure 1: self 1's objective function with respect to m

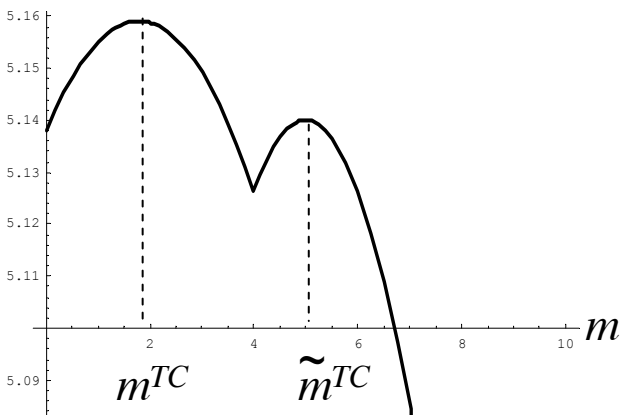
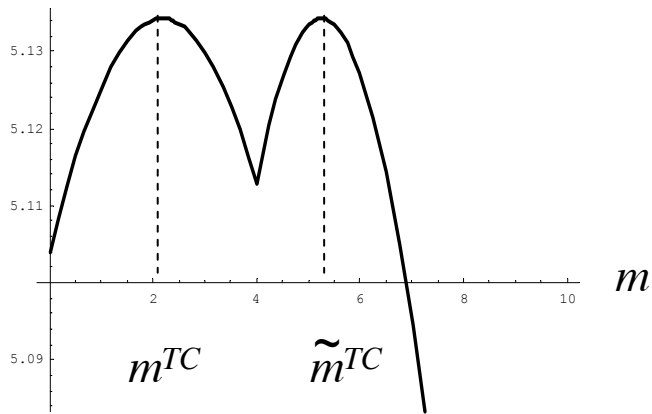
$w_0 = 0.75$



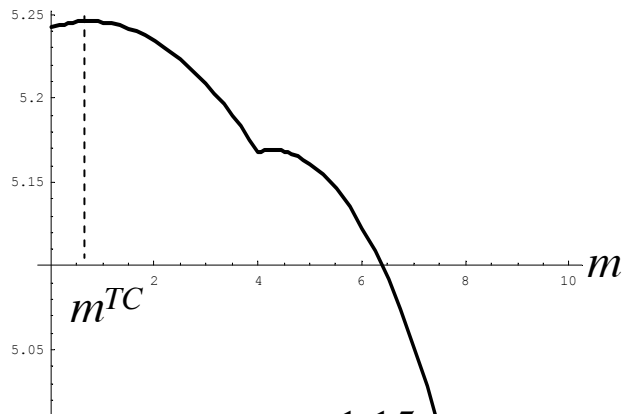
$w_0 = 0.9$



$w_0 = 0.951856$



$w_0 = 1$



$w_0 = 1.15$

$$\sigma = 2, \tau = 0.5, \varphi = 0.17, \delta = 1, q_0 = 0.5$$

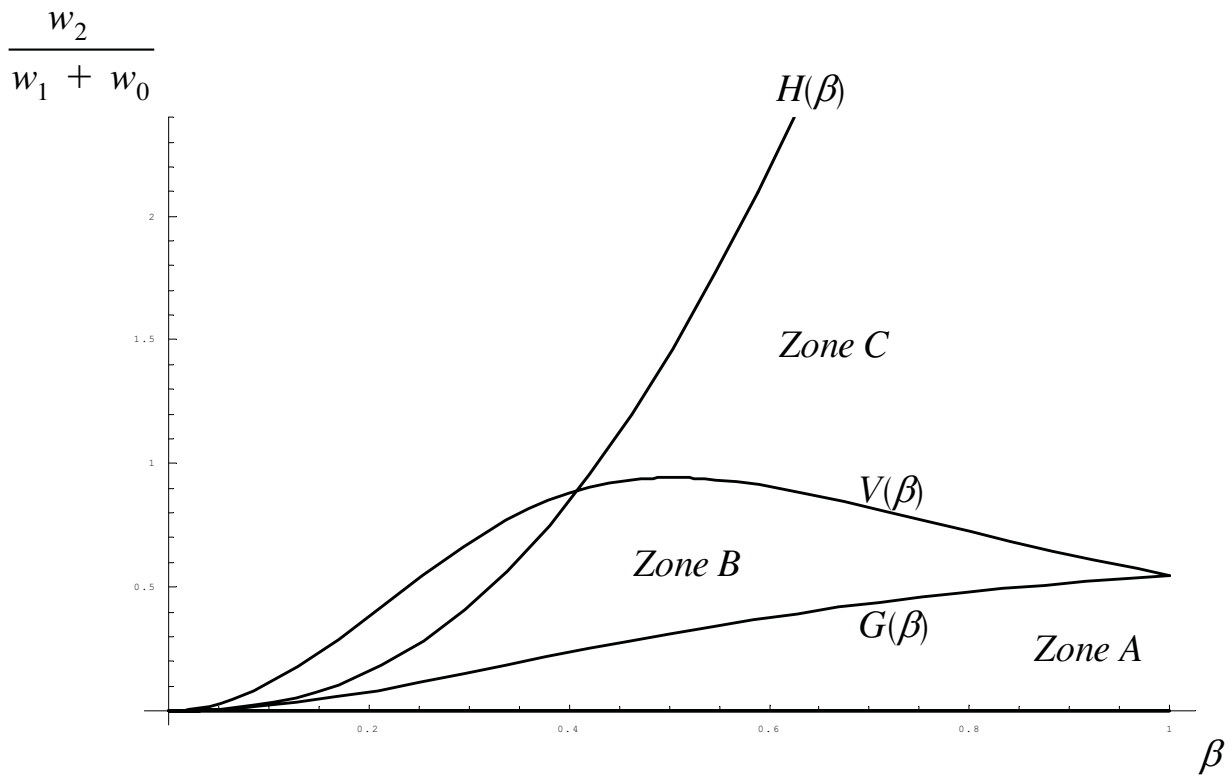


Figure 2

$$\sigma = 2, \tau = 0.5, \varphi = 0.17, \delta = 1, q_0 = 0.5, w_2 = 2, w_1 = 1$$

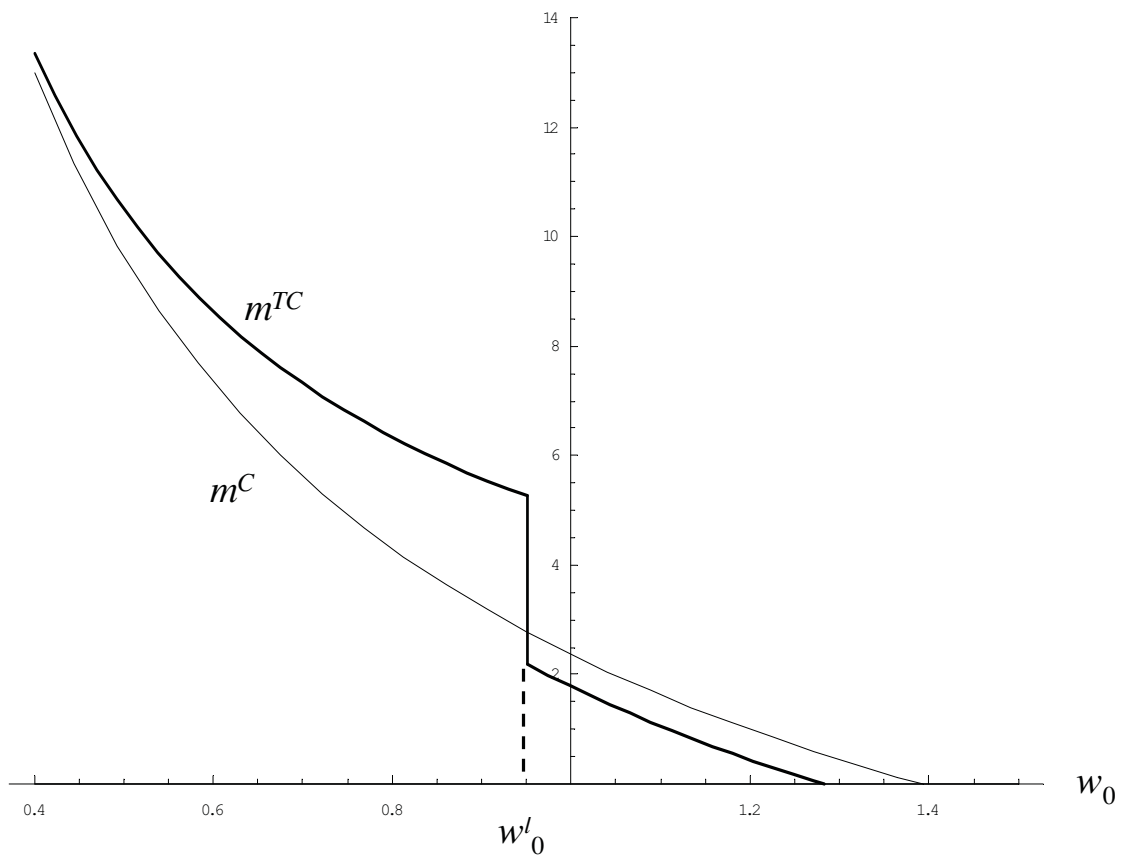


Figure 3

$$\sigma = 2, \tau = 0.5, \varphi = 0.17, \delta = 1, q_0 = 0.5, w_2 = 2$$

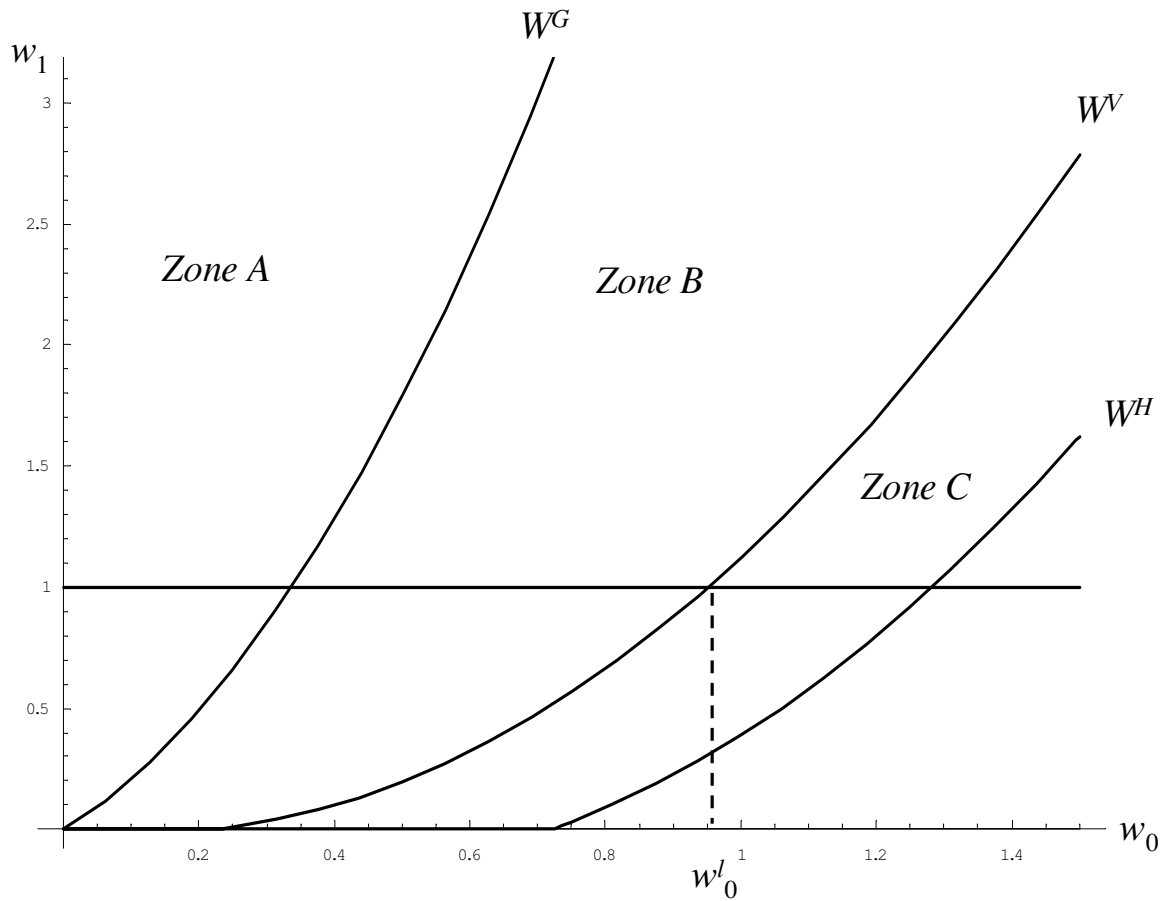


Figure 4

$$\sigma = 2, \beta = 0.5, \varphi = 0.17, \delta = 1, q_0 = 0.5, w_0 = 1, w_2 = 2, w_1 = 1$$

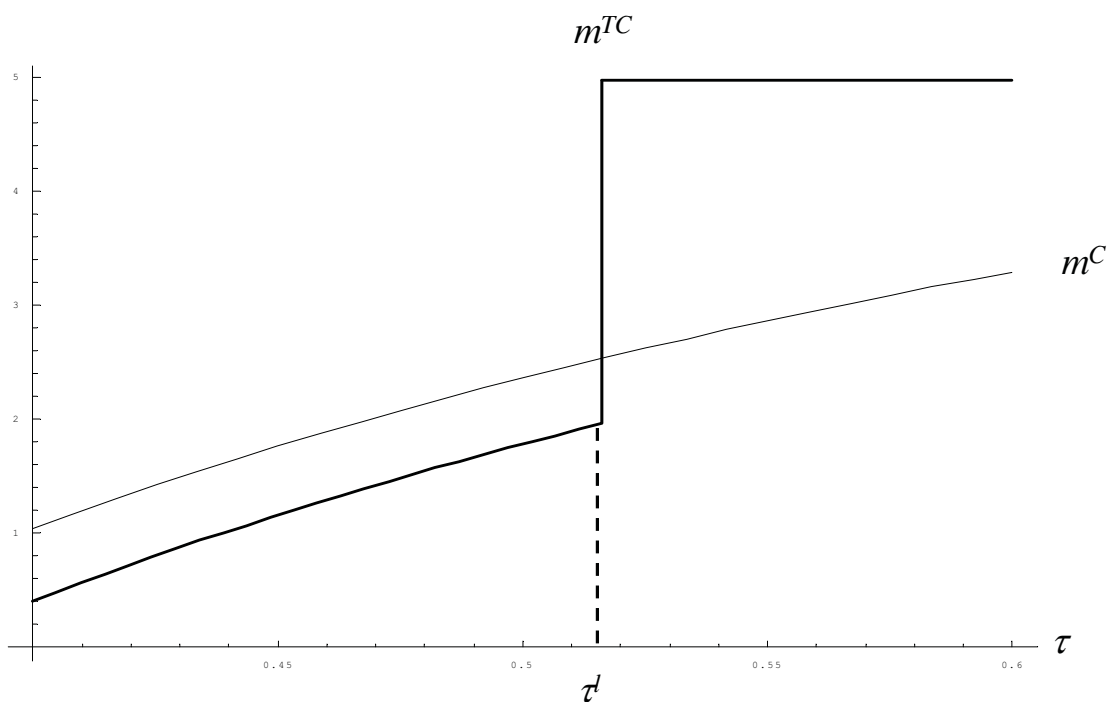


Figure 5

$$\sigma = 4, \beta = 0.5, \varphi = 0.28, \delta = 1, q_0 = 1.5, w_2 = 2, w_1 = 1$$

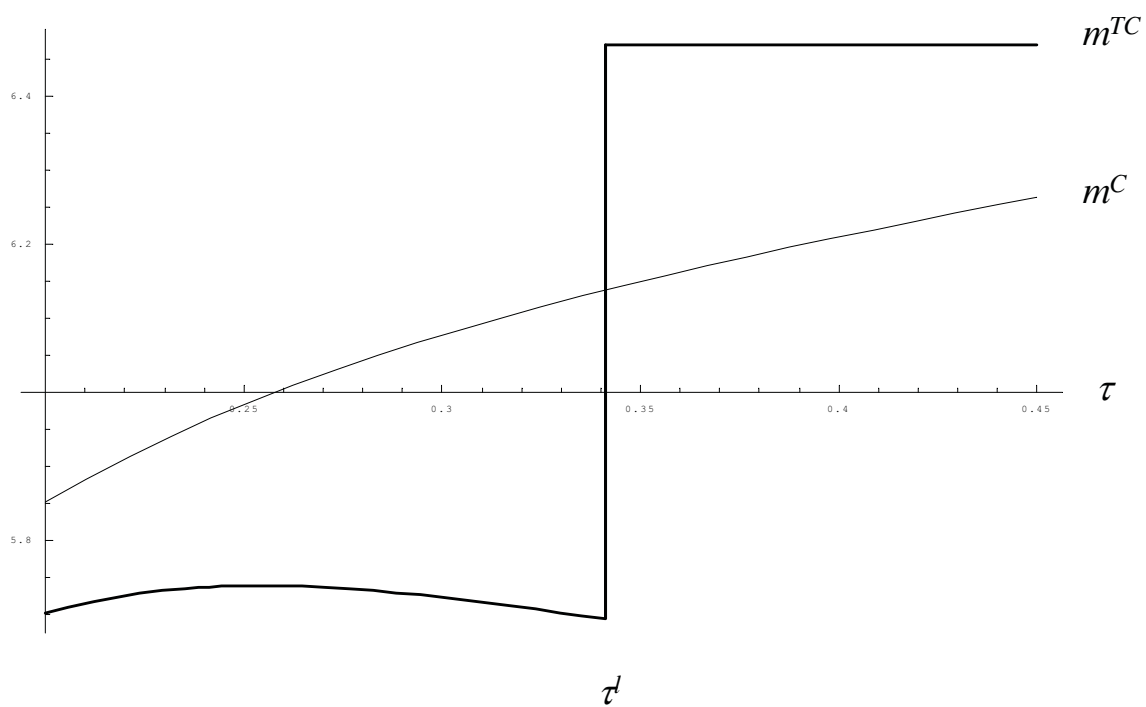


Figure 6

$$\tau = 0.5 \text{ and } \tau = 0.7, \sigma = 2, \varphi = 0.17, \delta = 1, q_0 = 0.5$$

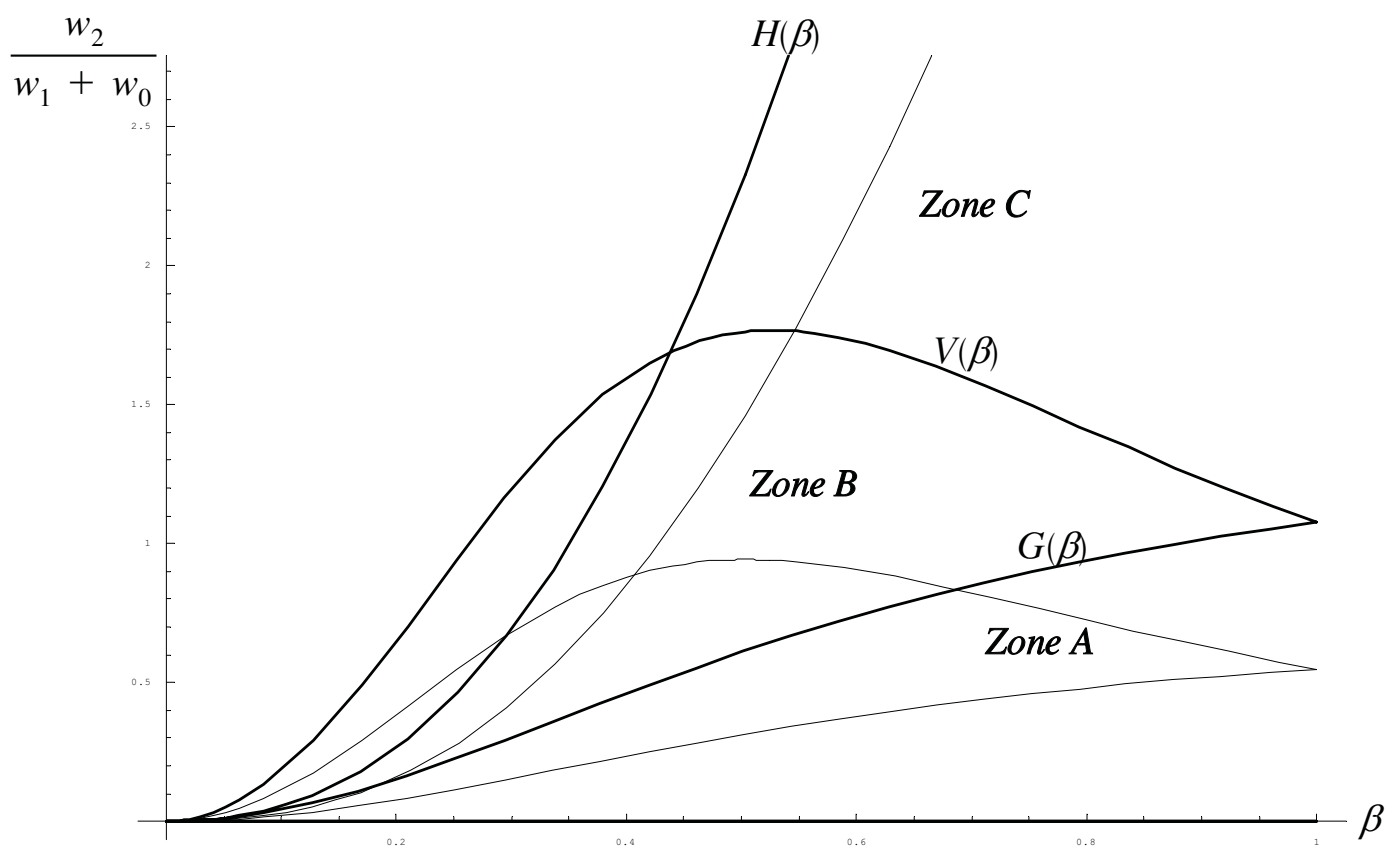


Figure 7