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GATE Groupe d'Analyse et de Théorie Économique Lyon-St Étienne

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6, rue Basse des Rives 42023 Saint-Etienne cedex 02 – France
Tel. +33 (0)4 77 42 19 60
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Existence of Nash Networks and Partner Heterogeneity

PASCAL BILLAND[a], CHRISTOPHE BRAVARD[a], SUDIPTA SARANGI[b]

[a] *Université de Lyon, Lyon, F-69003, France ; Université Jean Monnet, Saint-Etienne, F-42000, France ; CNRS, GATE Lyon St Etienne, Saint-Etienne, F-42000, France. email: pascal.billand@univ-st-etienne.fr, christophe.bravard@univ-st-etienne.fr,*

[b] *DIW Berlin and Department of Economics, Louisiana State University, Baton Rouge, LA 70803, USA. email: sarangi@lsu.edu.*

Abstract

In this paper, we pursue the work of H. Haller and al. (2005, [10]) and examine the existence of equilibrium networks, called Nash networks, in the non-cooperative two-way flow model (Bala and Goyal, 2000, [1]) with partner heterogeneous agents. We show through an example that Nash networks do not always exist in such a context. We then restrict the payoff function, in order to find conditions under which Nash networks always exist. We give two properties: increasing differences and convexity in the first argument of the payoff function, that ensure the existence of Nash networks. It is worth noting that linear payoff functions satisfy the previous properties.

Key Words: Nash networks, two-way flow models, partner heterogeneity.

JEL Classification: C72, D85.

1 Introduction

As researchers have become increasingly aware of the importance of networks in determining the outcome of many economic situations, the theoretical analysis of the formation of networks has grown. In particular, Bala and Goyal (2000, [1]) propose a non-cooperative two-way flow model of network formation. In this model agents form links unilaterally with others agents in order to get access to information. A distinctive aspect of this model is that the cost of a link between two agents is incurred only by the agent who forms the link, while both agents can get access to information of each other thanks to this link. Several examples are given that can be interpreted in this spirit, as telephone call in which people exchange information, or investment in personal relationships which creates a social tie yielding values to both agents involved in the tie.

This model has been extended in various directions. Hojman and Szeidl (2006, [12]) develop a general model of decay where the resources obtained by an agent from another agent depend on the distance between the two agents. Bala and Goyal (2000, [2]) and Haller and Sarangi (2005, [11]) propose models with imperfect link reliability. Billand and Bravard (2004, [3]) take into account congestion effect in the values obtained by agents. Goyal, Galeotti and Kamphorst (2006, [9]), Billand Bravard and Sarangi (2011, [5]) propose a model with heterogeneous agents with respect to values as well as the costs of forming links.

In the previous works the central issue concerns the architectures of networks that will emerge in equilibrium. In comparison with this issue, existence of equilibrium networks has been less systematically explored. H. Haller has initiated the study of the existence of equilibrium networks for the non-cooperative network formation models with heterogeneous agents and linear payoff functions (see Haller and

Sarangi 2005, [11], and Haller, Kamphorst and Sarangi 2007, [10]).[1]

In this paper, we pursue the work of H. Haller and examine the existence of equilibrium networks, called Nash networks, in the non-cooperative two-way flow model with partner heterogeneous agents. In the context of partner heterogeneity, the cost for agent i to invest in a link with an agent j, and the benefits to i from accessing j's resources, only depend on the identity of j.[2] Such situations have been examined by Billand, Bravard and Sarangi (2011, [5]). The authors only focus on the characterization of strict Nash networks. Here, we ask under what conditions Nash networks always exist.

We start by considering a general payoff function, where the payoff of an agent i in a network $\boldsymbol{g}$ is increasing in the total value of the resources accessed by i in $\boldsymbol{g}$ and decreasing in the total cost of forming links in $\boldsymbol{g}$. We show through an example that Nash networks do not always exist in such a context. We then restrict the payoff function, in order to find conditions under which Nash networks always exist. We give two properties, increasing differences and convexity in the first argument of the payoff function, that have economic interpretation and ensure the existence of Nash networks.

Our work contributes to the literature on the existence of equilibrium networks for the non-cooperative network formation models with heterogeneous agents in two ways.

[1] Billand, Bravard and Sarangi (2008, [4], [6]) and Derks and Tennekes (2009, [7]) have examined the existence of Nash networks in non-cooperative one-way flow model with heterogeneous agents.

[2] If we consider the phone call example, our model takes into account the fact that the cost incurred by a caller i typically depends on the identity of the receiver j. For instance, if j is a busy person, then it is difficult to access her. It follows that the time that i spends (the cost that i incurs) to obtain an answer from j will depend on j's characteristics. Likewise, the value obtained by i depends on the information owned by j, and this information depends both on the characteristics of j and on her social network.

1. We give conditions which ensure the existence of Nash networks in non-cooperative network formation models with partner heterogeneity. In other words, we delineate situations where the partner heterogeneity framework is consistent with the concept of Nash network. Hence, we complete the work of Billand, Bravard and Sarangi (2011, [5]) who focus on the characterization of strict Nash networks.

2. In the characterization studies of equilibrium networks for the non-cooperative network formation models with heterogeneous agents, the authors generally assume that the payoff function is linear. More precisely, the payoff of an agent i in a network $\boldsymbol{g}$ is equal to the sum of the values of resources that agent i obtains from the other agents in $\boldsymbol{g}$ minus the sum of the costs incurred by i due to the links she forms in $\boldsymbol{g}$. However, it is worth noting that the results qualitatively carry on when relaxing the linearity assumption of the payoff function. In particular, if we introduce a more general payoff function, which is increasing in the total value of the resources accessed by the agent i and decreasing in the total cost that i incurs, then the characterization results given by Galeotti, Goyal and Kamporst (2006, [9]) or Billand, Bravard and Sarangi (2011, [5]) remain qualitatively the same. By contrast, we show that existence of Nash networks results are very different when the payoff function is linear and when the payoff function is more general. More specifically, we show that even if there always exists a Nash network when the payoff function is linear, this existence result is no longer true when we introduce the more general payoff function.

The paper is organized as follows. In section 2 we present the model setup. In section 3 we establish that Nash networks do not always exist for a general payoff function in the partner heterogeneity framework. In section 4 we give two conditions on the payoff function and show that under these conditions, Nash

networks always exist. Section 5 concludes.

2 Model setup

Let $N = \{1, \ldots, n\}$ be the set of agents. Each agent is assumed to possess some information of value to himself and to other players. She can augment her information by communicating with other people; this communication takes resources, time and effort and is made possible via pair-wise links.

Each agent i chooses a strategy $\boldsymbol{g}_i = (g_{i,1}, \ldots, g_{i,i-1}, g_{i,i+1}, \ldots, g_{i,n})$ where $g_{i,j} \in \{0, 1\}$ for all $j \in N \setminus \{i\}$. The interpretation of $g_{i,j} = 1$ is that agent i forms an arc with agent $j \neq i$, and the interpretation of $g_{i,j} = 0$ is that i forms no arc with agent j. By convention, we assume that agent i cannot form an arc with herself. In the following we only use pure strategies. Let $\mathcal{G}_i$ be the set of strategies of agent $i \in N$. The set $\mathcal{G} = \prod_{i=1}^{n} \mathcal{G}_i$ is the set of pure strategies of all the agents. A strategy profile $\boldsymbol{g} = (\boldsymbol{g}_1, \ldots, \boldsymbol{g}_i, \ldots, \boldsymbol{g}_n)$ can be represented as a directed network. We use $\boldsymbol{g} + g_{i,j}$ ($\boldsymbol{g} - g_{i,j}$) to refer to the network obtained when a link $g_{i,j} = 1$ is added in (deleted from) $\boldsymbol{g}$. The *empty network* is a network in which there is no arc. For a directed network, $\boldsymbol{g}$, a *path* from agent k to agent j, $j \neq k$, is a finite sequence $j_0, j_1, \ldots, j_m$ of distinct agents such that $j_0 = j$, $j_m = k$ and $g_{j_\ell, j_{\ell+1}} = 1$ for $\ell = 0, \ldots, m-1$. We define a *chain* between agent k and agent j, $j \neq k$ by replacing $g_{j_\ell, j_{\ell+1}} = 1$, by $\max\{g_{j_\ell, j_{\ell+1}}, g_{j_{\ell+1}, j_\ell}\} = 1$.

Define $N_i(\boldsymbol{g}) = \{i\} \cup \{j \in N \setminus \{i\} \mid \text{there is a chain between } i \text{ and } j \text{ in } \boldsymbol{g}\}$ as the set of agents who are observed by agent i with the convention that agent i always "observes" herself. We assume that values and costs are partner heterogeneous. More precisely, each agent i obtains $V_j > 0$ from each agent $j \in N_i(\boldsymbol{g}) \setminus \{i\}$, and incurs a cost $C_j > 0$ when she forms an arc with agent $j \neq i$. Also since we wish to focus only on the network formed, we assume that agent i obtains no additional resources from herself. Indeed, agent i obtains her own resources even if she forms

no arcs and there is no network. We now introduce the set of agents who have the minimal cost of being linked, $S = \{j \in N \mid j \in \arg\min_{j \in N}\{C_j\}\}$.
Let $\phi : \mathbb{R}^2_+ \to \mathbb{R}$ be a function such that $\phi(x, y)$ is strictly increasing in x and strictly decreasing in y. The payoff of agent i is given by:

$$\pi_i(\boldsymbol{g}) = \phi\left(\sum_{j \in N_i(\boldsymbol{g}) \setminus \{i\}} V_j, \sum_{j \in N} g_{i,j} C_j\right). \tag{1}$$

Given the properties we have assumed for the function ϕ, the first term can be interpreted as the "benefits" that agent i receives in network $\boldsymbol{g}$, while $\sum_{j \in N \setminus \{i\}} g_{i,j} C_j$ measures the "costs" that i incurs from forming her arcs. In the following, we define $\sum_{j \in N_i(\boldsymbol{g}) \setminus \{i\}} V_j = v(i; \boldsymbol{g})$ and $\sum_{j \in N} g_{i,j} C_j = c(i; \boldsymbol{g})$.

Definition 1 *If $v_2 > v_1$ and $c_2 > c_1$ implies $\phi(v_2, c_2) - \phi(v_1, c_2) \geq \phi(v_2, c_1) - \phi(v_1, c_1)$, then the function ϕ admits increasing differences (ID).*

To present situations captured by this property, let us define the following networks. Let $\boldsymbol{g}^{x,y}$, $x \in \{1, 2\}$ and $y \in \{1, 2\}$, be networks where agent i obtains benefits equal to v_x and incurs a cost equal to c_y, with $v_2 > v_1$ and $c_2 > c_1$. Consider two alternative networks changes, from $\boldsymbol{g}^{2,1}$ to $\boldsymbol{g}^{2,2}$ (case 1), and from $\boldsymbol{g}^{1,1}$ to $\boldsymbol{g}^{1,2}$ (case 2). Note that in both cases only the cost incurred by i varies when the network changes. Property ID says that, although in both cases the increase in i's cost is equal, the variation in i's payoffs are different. More precisely, this variation is higher in the first case (when benefits that i incurs are high) than in the second case (when benefits that i incurs are low).
To sum up, if the payoff function of agent i admits ID, then the marginal loss that agent i incurs due to an higher cost decreases with the benefits she obtains. Below, in Example 3, we illustrate how this property can capture some economic effects.

Definition 2 *Suppose $v_1, v_2 \in \left(0, \sum_{j \in N} V_j\right)$, $v_2 > v_1$ and $\kappa \in (0, v_1]$. If $\phi(v_2, c) -$*

$\phi(v_2 - \kappa, c) \geq \phi(v_1, c) - \phi(v_1 - \kappa, c)$, *then the function* ϕ *is convex in its first argument.*

If the payoff function of agent i is convex in its first argument, then the variation in i's payoff increases with the resources she obtains when the cost of forming arcs is given. More precisely, for given cost of forming arcs, the higher are the benefits already obtained by player i, the more this player will take advantage of an increase in the benefits obtained.

Example 1 Suppose ϕ is an additively separable function, that is $\phi(x, y) = \varphi(x) + \eta(y)$. Then ϕ admits ID.

Example 2 Suppose $\phi(x, y) = ax - by$. Then ϕ admits ID since it is an additively separable function. Likewise ϕ is convex in its first argument since if $x > x'$, then $\phi(x, y) - \phi(x - k, y) - (\phi(x', y) - \phi(x' - k, y)) = 0$. It follows that the following linear payoff function $\pi_i(\boldsymbol{g}) = v(i; \boldsymbol{g}) - c(i; \boldsymbol{g})$ is convex in its first argument and admits ID.

In the following example, we present a payoff function which is not additively separable. Till now this kind of function has not been used in the literature on networks with heterogeneous agents (Galeotti, Goyal and Kamphorst, 2006, [9], Galeotti, 2005, [8] use additively separable functions).

Example 3 Suppose $\phi(x, y) = x^3 - (y^2 - xy)$. Then ϕ admits ID. Indeed, if $x > x'$ and $y > y'$, then we have: $\phi(x, y) - \phi(x', y) - (\phi(x, y') - \phi(x', y')) = (y - y')(x - x') > 0$. Likewise ϕ is convex in its first argument. Indeed, if $x > x'$, then we have: $\phi(x, y) - \phi(x - k, y) - (\phi(x', y) - \phi(x' - k, y)) = 3k(x - x')(x + x' - k) > 0$, for $k < x'$.
Let $\pi_i(\boldsymbol{g}) = \phi(v(i; \boldsymbol{g}), c(i; \boldsymbol{g})) = (v(i; \boldsymbol{g}))^3 - \Big[(c(i; \boldsymbol{g}))^2 \ - v(i; \boldsymbol{g})c(i; \boldsymbol{g})\Big]$. This payoff function is increasing in $v(i; \boldsymbol{g})$, decreasing in $c(i; \boldsymbol{g})$, convex in its first argument and admits ID.

In the previous payoff function, there is some synergy between benefits and costs, captured by the interactive term $v(i;\boldsymbol{g})c(i;\boldsymbol{g})$. The assumption built into this function is that as agents handle more resources, their ability to manage the arcs with other agents increases.

Given a network $\boldsymbol{g} \in \mathcal{G}$, let $\boldsymbol{g}_{-i}$ denote the network obtained when all of agent i's arcs are removed. The network $\boldsymbol{g}$ can be written as $\boldsymbol{g} = \boldsymbol{g}_i \oplus \boldsymbol{g}_{-i}$ where '$\oplus$' indicates that $\boldsymbol{g}$ is formed as the union of the arcs of $\boldsymbol{g}_i$ and $\boldsymbol{g}_{-i}$. The strategy $\boldsymbol{g}_i$ is said to be a best response of agent i to the network $\boldsymbol{g}_{-i}$ if:

$$\pi_i(\boldsymbol{g}_i \oplus \boldsymbol{g}_{-i}) \geq \pi_i(\boldsymbol{g}'_i \oplus \boldsymbol{g}_{-i}), \text{ for all } \boldsymbol{g}'_i \in \{0,1\}^{n-1}.$$

The set of all of agent i's best responses to $\boldsymbol{g}_{-i}$ is denoted by $\mathcal{BR}_i(\boldsymbol{g}_{-i})$. A network $\boldsymbol{g}$ is said to be a Nash network if $\boldsymbol{g}_i \in \mathcal{BR}_i(\boldsymbol{g}_{-i})$ for each agent $i \in N$.

3 Existence of Nash Networks in Pure Strategies

First, we show that there does not always exist a Nash network in the partner heterogeneity two-way flow model.

Proposition 1 *Suppose payoff function satisfies (1). Then there does not always exist a Nash network in pure strategies*

Proof To prove this result, we construct an example where there is no Nash network.
Let $N = \{1,2,3\}$ be the set of agents. We assume that $C_3 > C_1 = C_2 > 0$, $V_2 = V_1 > 0$ and $V_3 > V_1 + V_2$. Set $\phi(0,0) = 0$.

1. We assume that $\phi(V_2 + V_1, C_2) < 0$. Therefore agent 3 will never form an arc.

2. We assume that $\phi(V_3, C_3) > 0$. In the empty network agent 1 and agent 2 have an incentive to form an arc with agent 3.

3. We assume that $\phi(V_1, C_1) < 0$. Agent 2 has no incentive to form an arc with agent 1 if 1 is an isolated agent and vice versa.

4. We assume that $\phi(V_2 + V_3, C_2) > 0$. Agent 1 (agent 2) has an incentive to form an arc with agent 2 (agent 1) if the latter allows him to obtain resources from agent 3.

5. We assume that $\phi(V_2, 0) > \phi(V_2 + V_3, C_3)$. Agent 1 (agent 2) has no incentive to form an arc with agent 3 if agent 2 (agent 1) forms an arc with her.

6. Let $\mathcal{Y} = \{Y \in 2^N \mid |Y| = 2\}$ be the set of subsets of N which have a size equal to 2. We suppose that $\phi\left(\sum_{\ell \in Y, Y \in \mathcal{Y}} V_\ell, \sum_{\ell \in Y, Y \in \mathcal{Y}} C_\ell\right) < 0$. Therefore no agent has an incentive to form two arcs.

These assumptions on the payoff function do not contradict the fact that ϕ is strictly increasing in its first argument and strictly decreasing in its second argument.

Clearly a Nash network contains at most 2 arcs. Moreover, the empty network is not Nash by point 2.

Firstly, we show that networks with one arc cannot be Nash. Let $\boldsymbol{g}^1$ be the network where the only link agents have formed is $g^1_{1,3} = 1$ and let $\boldsymbol{g}^2$ be the network where the only link agents have formed is $g^2_{2,3} = 1$. We know by Point 1, Point 3, and Point 4 that $\boldsymbol{g}^1$ and $\boldsymbol{g}^2$ are the only networks with one arc which can be Nash. These networks are not Nash since agent 2 (agent 1) has an incentive to form an arc with agent 1 (agent 2) by point 4 in $\boldsymbol{g}^1$ (in $\boldsymbol{g}^2$).

Secondly, we show that networks with two arcs cannot be Nash. Let $\boldsymbol{g}^3$ be the network where the only links agents have formed are $g^3_{1,3} = g^3_{2,1} = 1$, Let $\boldsymbol{g}^4$ be the network where the only links agents have formed are $g^4_{2,3} = g^4_{1,2} = 1$, and let $\boldsymbol{g}^5$ be the network where the only links agents have formed are $g^5_{2,3} = g^5_{1,3} = 1$. We

know by point 1 and point 6 that $\boldsymbol{g}^3$, $\boldsymbol{g}^4$ and $\boldsymbol{g}^5$ are the only networks with two arcs which can be Nash. By Point 5, $\boldsymbol{g}^3$ ($\boldsymbol{g}^4$) is not Nash since agent 1 (agent 2) has an incentive to remove her arc with agent 3. Finally, $\boldsymbol{g}^5$ is not Nash. Indeed, since $C_1 < C_3$, we have $\phi(V_1 + V_3, C_1) > \phi(V_1 + V_3, C_3)$. It follows that agent 2 has an incentive to replace her arc with agent 3 by an arc with agent 1. $\square$

Now we give conditions that ensure the existence of Nash networks. To establish this existence result we use two lemmas. The first one shows that there always exists a Nash network when the payoff function is convex in its first argument, admits ID and there is an agent $\ell \in S$ who has an incentive to form an arc in the empty network. The second lemma shows that there always exists a Nash network if the payoff function is convex in its first argument, admits ID and there is no agent $\ell \in S$ who has an incentive to form an arc in the empty network.
In the proof of these lemmas given in Appendix, for each agent $j \in N$ we need the set $\mathcal{X}_j^*$ defined as follows:

$$\mathcal{X}_j^* = \left\{ X_j' \in 2^{N\setminus\{j\}} \mid \phi\left(\sum_{i\in X_j'} V_i, \sum_{i\in X_j'} C_i\right) \geq \phi\left(\sum_{i\in X_j} V_i, \sum_{i\in X_j} C_i\right), \forall X_j \in 2^{N\setminus\{j\}} \right\}.$$

We denote by X_j^* a typical member of $\mathcal{X}_j^*$.

Lemma 1 *Suppose payoff function satisfies (1), ϕ is convex in its first argument and satisfies ID. If there is $\ell \in S$ for whom $\phi\left(\sum_{i\in X_\ell^*} V_i, \sum_{i\in X_\ell^*} C_i\right) \geq \phi(0,0)$, $X_\ell^* \in \mathcal{X}_\ell^*$, then there exists a Nash network in pure strategies.*

The intuition is as follows. Suppose that there is a minimal partner cost agent ℓ who has an incentive to form arcs in the empty networks $\boldsymbol{g}^e$. We build a process in two steps. In Step 1, we let agent ℓ play a best response in $\boldsymbol{g}^e$. We denote by $\boldsymbol{g}^0$ the resulting network. If ℓ forms $n-1$ arcs, then $\boldsymbol{g}^0$ is a Nash network and the process stops. Suppose now that ℓ has formed less than $n-1$ arcs in $\boldsymbol{g}^0$. In Step 2, we let all agents who are not linked with ℓ in $\boldsymbol{g}^0$ form an arc with ℓ. Let j' be

one of these agents. We denote by $\boldsymbol{g}^1$ the resulting network. We show that $\boldsymbol{g}^1$ is a Nash network. First, no agent has an incentive to form additional arcs since each agent already receives all resources in $\boldsymbol{g}^1$. Moreover agent j' has no incentive to replace her arc with ℓ since agent ℓ is a minimal partner cost agent. Let us now show that no agent has an incentive to delete arcs in $\boldsymbol{g}^1$.

We begin with agent j'. By construction (a) j' obtains more resources in $\boldsymbol{g}^1$ than ℓ obtained in $\boldsymbol{g}^0$ and (b) the cost incurred by j' in $\boldsymbol{g}^1$ cannot be higher than the cost incurred by ℓ in $\boldsymbol{g}^0$. It follows that the payoff of j' in $\boldsymbol{g}^1$ is higher than the payoff of ℓ in $\boldsymbol{g}^0$. Since agent ℓ obtains a higher payoff in $\boldsymbol{g}^0$ than in $\boldsymbol{g}^e$, agent j' obtains a higher payoff in $\boldsymbol{g}^1$ than in $\boldsymbol{g}^e$ -if j' deletes her arc with ℓ, then she will obtain the same payoff as in $\boldsymbol{g}^e$.

We now give an intuition about the fact that agent ℓ has no incentive to delete some arcs in $\boldsymbol{g}^1$. We restrict our attention on situations where ℓ removes one of her arcs, say the arc with agent j (in the proof we deal with the deletion of several arcs). We define two networks which will be useful in the following. Let $\hat{\boldsymbol{g}}^k$, $k \in \{0,1\}$, be a network similar to $\boldsymbol{g}^k$ except that $\hat{g}^k_{\ell,j} = 0$ and $\hat{g}^k_{j,\ell} = 1$. Firstly, we note that agent ℓ obtains the same resources in $\boldsymbol{g}^k$ and in $\hat{\boldsymbol{g}}^k$, for $k \in \{0,1\}$. Likewise, agent ℓ incurs the same cost in $\boldsymbol{g}^k - g^k_{\ell,j}$ and in $\hat{\boldsymbol{g}}^k$, for $k \in \{0,1\}$. We show that in network $\boldsymbol{g}^1$ agent ℓ has less incentive to delete the arc she has formed with agent j than in network $\boldsymbol{g}^0$. The proof is divided into two parts.

1. By construction for agent ℓ, the variation of cost between $\boldsymbol{g}^1$ and $\hat{\boldsymbol{g}}^1$ is the same as the variation of cost between $\boldsymbol{g}^0$ and $\hat{\boldsymbol{g}}^0$. But, agent ℓ obtains more resources in $\boldsymbol{g}^1$ and $\hat{\boldsymbol{g}}^1$ than in $\boldsymbol{g}^0$ and $\hat{\boldsymbol{g}}^0$. Consequently, by ID, we have $\pi_\ell(\boldsymbol{g}^1) - \pi_\ell(\hat{\boldsymbol{g}}^1) \geq \pi_\ell(\boldsymbol{g}^0) - \pi_\ell(\hat{\boldsymbol{g}}^0)$.

2. By construction for agent ℓ, the variation of value between $\hat{\boldsymbol{g}}^1$ and $\boldsymbol{g}^1 - g^1_{\ell,j}$ is the same as the variation of total value between $\hat{\boldsymbol{g}}^0$ and $\boldsymbol{g}^0 - g^0_{\ell,j}$. But, the total value obtained by agent ℓ is greater in $\hat{\boldsymbol{g}}^1$ than in $\hat{\boldsymbol{g}}^0$. Clearly,

agent ℓ incurs the same cost in networks $\boldsymbol{g}^0 - g^0_{\ell,j}$, $g^0 - g^0_{\ell,j}$, $\hat{\boldsymbol{g}}^0$ and in $\hat{\boldsymbol{g}}^1$. Consequently, by convexity, we have $\pi_\ell(\hat{\boldsymbol{g}}^1) - \pi_\ell(\boldsymbol{g}^1 - g^1_{\ell,j}) \geq \pi_\ell(\hat{\boldsymbol{g}}^0) - \pi_\ell(\boldsymbol{g}^0 - g^0_{\ell,j})$.

Therefore, we have $\pi_\ell(\boldsymbol{g}^1) - \pi_\ell(\boldsymbol{g}^1 - g^1_{\ell,j}) = \pi_\ell(\boldsymbol{g}^1) - \pi_\ell(\hat{\boldsymbol{g}}^1) + \pi_\ell(\hat{\boldsymbol{g}}^1) - \pi_\ell(\boldsymbol{g}^1 - g^1_{\ell,j}) \geq \pi_\ell(\boldsymbol{g}^0) - \pi_\ell(\hat{\boldsymbol{g}}^0) + \pi_\ell(\hat{\boldsymbol{g}}^0) - \pi_\ell(\boldsymbol{g}^0 - g^0_{\ell,j}) = \pi_\ell(\boldsymbol{g}^0) - \pi_\ell(\boldsymbol{g}^0 - g^0_{\ell,j})$. In other words, agent ℓ has less incentive to delete the arc she has formed with agent j in network $\boldsymbol{g}^1$ than in network $\boldsymbol{g}^0$.

Lemma 2 *Suppose payoff function satisfies (1), ϕ is convex in its first argument and satisfies ID. If there is no agent $\ell \in S$ for whom $\phi\left(\sum_{i \in X^*_\ell} V_i, \sum_{i \in X^*_\ell} C_i\right) \geq \phi(0,0)$, $X^*_\ell \in \mathcal{X}^*_\ell$, then there exists a Nash network in pure strategies.*

The logic behind the intuition of Lemma 2 is similar to those of Lemma 1 and therefore is omitted.

Proposition 2 *Suppose payoff function satisfies (1), ϕ is convex in its first argument and satisfies ID. Then, a Nash network always exists in pure strategies.*

Proof The proof follows the two previous lemmas. □

Remark 1 Suppose payoff function satisfies (1) where ϕ is an additively separable function: $\phi(x,y) = \varphi(x) + \eta(y)$ such that (i) φ is an increasing function and is convex in its first argument, and (ii) η is a decreasing function. Then, a Nash network always exists.

Remark 2 Suppose payoff function satisfies (1) where ϕ is linear: $\phi(x,y) = ax - by$, $a, b > 0$. By Example 2, we know that ϕ satisfies ID and convexity. It follows that a Nash network always exists.

Remark 3 By Proposition 2, a Nash network always exists in a network formation game associated with the payoff function given in Example 3.

Conclusion

In this paper, we show two main results. First there does not always exist a Nash network in the partner heterogeneous two-way flow model with a general payoff function. Second, we demonstrate that if the payoff function is convex in its first argument and admits ID, then there always exists a Nash network. In other words, we show the existence of Nash networks for a large class of payoff functions (in particular for linear payoff functions) in two-way flow models with heterogeneous partner.

Appendix

Proof of Lemma 1. To prove this lemma, we construct a two steps process. We begin with the empty network $\boldsymbol{g}^e$.

Step 1. *We let agent $\ell \in S$ play a best response in $\boldsymbol{g}^e$, that is she chooses to form an arc with each agent $j \in X_\ell^*$, $X_\ell^* \in \mathcal{X}_\ell^*$.*

At the end of this step we obtain a network called $\boldsymbol{g}^0$. If in $\boldsymbol{g}^0$ agent ℓ has formed $n-1$ arcs, then $\boldsymbol{g}^0$ is clearly a Nash network. In the following we study the case where agent ℓ has not formed $n-1$ arcs in $\boldsymbol{g}^0$. We have $\sum_{j \in N_\ell(\boldsymbol{g}^0)} V_j = \sum_{j \in X_\ell^*} V_j = v(\ell; \boldsymbol{g}^0)$ and $\sum_{j \in N_\ell(\boldsymbol{g}^0)} C_j = \sum_{j \in X_\ell^*} C_j = c(\ell; \boldsymbol{g}^0)$.

Step 2. *We let agents $i \notin X_\ell^* \cup \{\ell\}$ form an arc with agent ℓ.*

At the end of Step 2, we obtain a network $\boldsymbol{g}^1$. We show that $\boldsymbol{g}^1$ is a Nash network. Let $v^{-i} = \sum_{j \in N \setminus \{i\}} V_j$. Clearly, we have for all $i \in N$, $\sum_{j \in N_i(\boldsymbol{g}^1)} V_j = v^{-i}$. We show that no agent has an incentive to modify her strategy in $\boldsymbol{g}^1$, that is $\boldsymbol{g}^1$ is a Nash network.

1. Each agent j in X_ℓ^* has no incentive to add an arc since j obtains all resources in $\boldsymbol{g}^1$.

2. Agent $\ell \in S$ has no incentive to add an arc since she obtains all resources in

$\boldsymbol{g}^1$. We show that agent ℓ has no incentive to remove one of her arcs. Indeed, we have $\phi\left(v(\ell;\boldsymbol{g}^0), c(\ell;\boldsymbol{g}^0)\right) - \phi\left(\sum_{i\in X_\ell} V_i, \sum_{i\in X_\ell} C_i\right) \geq 0$, for all $X_\ell \in 2^{N\setminus\{\ell\}}$ so in particular, we have for all $M_\ell(\boldsymbol{g}^0) \subset N_\ell(\boldsymbol{g}^0)$

$$\begin{aligned} 0 &\leq \phi\left(v(\ell;\boldsymbol{g}^0), c(\ell;\boldsymbol{g}^0)\right) - \phi\left(\textstyle\sum_{i\in M_\ell(\boldsymbol{g}^0)} V_i, \sum_{i\in M_\ell(\boldsymbol{g}^0)} C_i\right) \\ &= \phi\left(v(\ell;\boldsymbol{g}^0), c(\ell;\boldsymbol{g}^0)\right) - \phi\left(v(\ell;\boldsymbol{g}^0) - \kappa_{M_\ell(\boldsymbol{g}^0)}, c(\ell;\boldsymbol{g}^0) - \kappa'_{M_\ell(\boldsymbol{g}^0)}\right) \end{aligned} \tag{2}$$

with $\kappa_{M_\ell(\boldsymbol{g}^0)} = \sum_{i\in N_\ell(\boldsymbol{g}^0)\setminus M_\ell(\boldsymbol{g}^0)} V_i$, $\kappa'_{M_\ell(\boldsymbol{g}^0)} = \sum_{i\in N_\ell(\boldsymbol{g}^0)\setminus M_\ell(\boldsymbol{g}^0)} C_i$.

The payoff that agent $\ell \in S$ obtains in $\boldsymbol{g}^1$ is $\phi\left(v^{-\ell}, c(\ell;\boldsymbol{g}^0)\right)$ since $v(\ell;\boldsymbol{g}^1) = v^{-\ell}$ and $c(\ell;\boldsymbol{g}^1) = c(\ell;\boldsymbol{g}^0)$. We have for all $M_\ell(\boldsymbol{g}^0) \subset N_\ell(\boldsymbol{g}^0)$:

$$\begin{gathered} \phi\left(v^{-\ell}, c(\ell;\boldsymbol{g}^0)\right) - \phi\left(v(\ell;\boldsymbol{g}^0), c(\ell;\boldsymbol{g}^0)\right) \\ \geq \\ \phi\left(v^{-\ell}, c(\ell;\boldsymbol{g}^0) - \kappa'_{M_\ell(\boldsymbol{g}^0)}\right) - \phi\left(v(\ell;\boldsymbol{g}^0), c(\ell;\boldsymbol{g}^0) - \kappa'_{M_\ell(\boldsymbol{g}^0)}\right) \\ \geq \\ \phi\left(v^{-\ell} - \kappa_{M_\ell(\boldsymbol{g}^0)}, c(\ell;\boldsymbol{g}^0) - \kappa'_{M_\ell(\boldsymbol{g}^0)}\right) - \phi\left(v(\ell;\boldsymbol{g}^0) - \kappa_{M_\ell(\boldsymbol{g}^0)}, c(\ell;\boldsymbol{g}^0) - \kappa'_{M_\ell(\boldsymbol{g}^0)}\right). \end{gathered}$$

The first inequality comes from ID since $v^{-\ell} \geq v(\ell;\boldsymbol{g}^0)$ and $c(\ell;\boldsymbol{g}^0) \geq c(\ell;\boldsymbol{g}^0) - \kappa'_{M_\ell(\boldsymbol{g}^0)}$. The second one comes from convexity of ϕ. Rearranging terms, we have for all $M_\ell(\boldsymbol{g}^0) \subset N_\ell(\boldsymbol{g}^0)$:

$$\begin{gathered} \phi\left(v^{-\ell}, c(\ell;\boldsymbol{g}^0)\right) - \phi\left(v^{-\ell} - \kappa_{M_\ell(\boldsymbol{g}^0)}, c(\ell;\boldsymbol{g}^0) - \kappa'_{M_\ell(\boldsymbol{g}^0)}\right) \\ \geq \\ \phi\left(v(\ell;\boldsymbol{g}^0), c(\ell;\boldsymbol{g}^0)\right) - \phi\left(v(\ell;\boldsymbol{g}^0) - \kappa_{M_\ell(\boldsymbol{g}^0)}, c(\ell;\boldsymbol{g}^0) - \kappa'_{M_\ell(\boldsymbol{g}^0)}\right) \end{gathered}$$

It follows that agent ℓ does not have any incentive to remove arcs since the last difference is positive by (2).

3. We show that agents $i \notin X^*_\ell \cup \{\ell\}$ have no incentive to change their strategy. Clearly, they have no incentive to replace their arc since they are linked with the

minimal partner cost agent and they have no incentive to add arcs since they obtain all the resources in $\boldsymbol{g}^1$. We now show that each agent $i \notin X_\ell^* \cup \{\ell\}$ has an incentive to maintain her arc with agent ℓ. Note that $v(\ell; \boldsymbol{g}^0) \leq v^{-i}$. We have:

$$\phi\left(v^{-i}, C_\ell\right) > \phi\left(v(\ell; \boldsymbol{g}^0), c(\ell; \boldsymbol{g}^0)\right) \geq \phi\left(0, 0\right).$$

The first inequality comes from the fact that ϕ is increasing in its first argument and decreasing in its second argument; the second inequality comes from the assumption made on the payoff obtained by agent $\ell \in S$ in $\boldsymbol{g}^0$.

□

Proof of Lemma 2. Suppose that there is no agent in S for whom $\phi\left(\sum_{i \in X_\ell^*} V_i, \sum_{i \in X_\ell^*} C_i\right) \geq \phi\left(0, 0\right)$, $X_\ell^* \in \mathcal{X}_\ell^*$. If there is no agent $k \in N \setminus S$ such that $\phi\left(\sum_{i \in X_k} V_i, \ \sum_{i \in X_k} C_i\right) \geq \phi\left(0, 0\right)$, for some $X_k \in 2^{N \setminus \{k\}}$, then the empty network is Nash. We suppose now that such an agent k exists, and we construct a process similar to the one proposed in Lemma 1. We begin with the empty network $\boldsymbol{g}^e$.

Step 1. *We let agent k play a best response in $\boldsymbol{g}^e$, that is she chooses to form an arc with each agent $j \in X_k^*$, $X_k^* \in \mathcal{X}_k^*$.*

At the end of this step we obtain a network called $\boldsymbol{g}^2$. If in $\boldsymbol{g}^2$ agent k has formed $n-1$ arcs, then $\boldsymbol{g}^2$ is clearly a Nash network. In the following we study the case where agent k has not formed $n-1$ arcs in $\boldsymbol{g}^2$. We have $\sum_{j \in N_k(\boldsymbol{g}^2)} V_j = \sum_{j \in X_k^*} V_j = v(k; \boldsymbol{g}^2)$ and $\sum_{j \in N_k(\boldsymbol{g}^2)} C_j = \sum_{j \in X_k^*} C_j = c(k; \boldsymbol{g}^2)$. It is worth noting that $S \subset X_k^*$. Indeed, if there is an agent $\ell \in S$, such that $\ell \notin X_k^*$, then for $X_\ell = X_k^*$, we have $\phi\left(\sum_{i \in X_\ell^*} V_i, \sum_{i \in X_\ell^*} C_i\right) \geq \phi\left(\sum_{i \in X_\ell} V_i, \sum_{i \in X_\ell} C_i\right)$ $= \phi\left(\sum_{i \in X_k^*} V_i, \sum_{i \in X_k^*} C_i\right) \geq \phi\left(0, 0\right)$, a contradiction.

Step 2. *Agents $i \notin X_k^* \cup \{k\}$ form an arc with an agent $\ell \in S$.*

At the end of Step 2, we obtain a network $\boldsymbol{g}^3$. Clearly, we have for all $i \in N$, $\sum_{j \in N_i(\boldsymbol{g}^3)} V_j = v^{-i}$.

We show that no agent has an incentive to modify her strategy in $\boldsymbol{g}^3$, that is $\boldsymbol{g}^3$

is a Nash network.

1. Each agent j in X_k^* has no incentive to add an arc since j obtains all resources in $\boldsymbol{g}^3$.

2. Agent k has no incentive to add an arc since she obtains all resources in $\boldsymbol{g}^3$. We show that agent k has no incentive to remove one of her arcs. We have $\phi\left(v(k;\boldsymbol{g}^2), c(k;\boldsymbol{g}^2)\right) - \phi\left(\sum_{i\in X_k} V_i, \sum_{i\in X_k} C_i\right) \geq 0$, for all $X_k \in 2^{N\setminus\{k\}}$ so in particular, we have for all $M_k(\boldsymbol{g}^2) \subset N_k(\boldsymbol{g}^2)$

$$\begin{aligned} 0 &\leq \phi\left(v(k;\boldsymbol{g}^2), c(k;\boldsymbol{g}^2)\right) - \phi\left(\textstyle\sum_{i\in M_k(\boldsymbol{g}^2)} V_i, \sum_{i\in M_k(\boldsymbol{g}^2)} C_i\right) \\ &= \phi\left(v(k;\boldsymbol{g}^2), c(k;\boldsymbol{g}^2)\right) - \phi\left(v(k;\boldsymbol{g}^2) - \kappa_{M_k(\boldsymbol{g}^2)}, c(k;\boldsymbol{g}^2) - \kappa'_{M_k(\boldsymbol{g}^2)}\right) \end{aligned} \tag{3}$$

with $\kappa_{M_k(\boldsymbol{g}^2)} = \sum_{i\in N_k(\boldsymbol{g}^2)\setminus M_k(\boldsymbol{g}^2)} V_i$, $\kappa'_{M_k(\boldsymbol{g}^2)} = \sum_{i\in N_k(\boldsymbol{g}^2)\setminus M_k(\boldsymbol{g}^2)} C_i$.

The payoff that agent k obtains in $\boldsymbol{g}^3$ is $\phi\left(v^{-k}, c(k;\boldsymbol{g}^2)\right)$ since $v(k;\boldsymbol{g}^3) = v^{-k}$ and $c(k;\boldsymbol{g}^3) = c(k;\boldsymbol{g}^2)$. We have for all $M_k(\boldsymbol{g}^2) \subset N_k(\boldsymbol{g}^2)$:

$$\begin{gathered} \phi\left(v^{-k}, c(k;\boldsymbol{g}^2)\right) - \phi\left(v(k;\boldsymbol{g}^2), c(k;\boldsymbol{g}^2)\right) \\ \geq \\ \phi\left(v^{-k}, c(k;\boldsymbol{g}^2) - \kappa'_{M_k(\boldsymbol{g}^2)}\right) - \phi\left(v(k;\boldsymbol{g}^2), c(k;\boldsymbol{g}^2) - \kappa'_{M_k(\boldsymbol{g}^2)}\right) \\ \geq \\ \phi\left(v^{-k} - \kappa_{M_k(\boldsymbol{g}^2)}, c(k;\boldsymbol{g}^2) - \kappa'_{M_k(\boldsymbol{g}^2)}\right) - \phi\left(v(k;\boldsymbol{g}^2) - \kappa_{M_k(\boldsymbol{g}^2)}, c(\ell;\boldsymbol{g}^2) - \kappa'_{M_\ell(\boldsymbol{g}^2)}\right). \end{gathered}$$

The first inequality comes from ID since $v^{-k} \geq v(k;\boldsymbol{g}^2)$ and $c(k;\boldsymbol{g}^2) \geq c(k;\boldsymbol{g}^2) - \kappa'_{M_k(\boldsymbol{g}^2)}$. The second one comes from convexity of ϕ. Rearranging terms, we have for all $M_k(\boldsymbol{g}^2) \subset N_k(\boldsymbol{g}^2)$:

$$\begin{gathered} \phi\left(v^{-k}, c(k;\boldsymbol{g}^2)\right) - \phi\left(v^{-k} - \kappa_{M_k(\boldsymbol{g}^2)}, c(k;\boldsymbol{g}^2) - \kappa'_{M_k(\boldsymbol{g}^2)}\right) \\ \geq \\ \phi\left(v(k;\boldsymbol{g}^2), c(k;\boldsymbol{g}^2)\right) - \phi\left(v(k;\boldsymbol{g}^2) - \kappa_{M_k(\boldsymbol{g}^2)}, c(k;\boldsymbol{g}^2) - \kappa'_{M_k(\boldsymbol{g}^2)}\right) \end{gathered} \tag{4}$$

There are now two cases.

(a)Suppose $\ell \notin M_k(\boldsymbol{g}^2)$. Then the marginal payoff of agent k from deleting arcs with agents in $M_k(\boldsymbol{g}^2)$ is

$$\phi\left(v^{-k} - \kappa_{M_k(\boldsymbol{g}^2)}, c(k;\boldsymbol{g}^2) - \kappa'_{M_k(\boldsymbol{g}^2)}\right) - \phi\left(v^{-k}, c(k;\boldsymbol{g}^2)\right).$$

Agent k does not have any incentive to remove arcs with agents $j \neq \ell$ since the last difference is negative by (3) and (4).

(b)Suppose $\ell \in M_k(\boldsymbol{g}^2)$. We know that agent ℓ allows access to more resources in $\boldsymbol{g}^3$ than in $\boldsymbol{g}^2$ since some players have formed links with ℓ in Step 2. It follows that agent k incurs a loss in resources $x > \kappa_{M_k(\boldsymbol{g}^2)}$ when she removes the set of agents $M_k(\boldsymbol{g}^2)$. We have

$$\phi\left(v^{-k} - \kappa_{M_k(\boldsymbol{g}^2)}, c(k;\boldsymbol{g}^2) - \kappa'_{M_k(\boldsymbol{g}^2)}\right) > \phi\left(v^{-k} - x, c(k;\boldsymbol{g}^2) - \kappa'_{M_k(\boldsymbol{g}^2)}\right),$$

since ϕ is increasing in its first argument. Therefore, we have

$$\begin{gathered}\phi\left(v^{-k}, c(k;\boldsymbol{g}^2)\right) - \phi\left(v^{-k} - x, c(k;\boldsymbol{g}^2) - \kappa'_{M_k(\boldsymbol{g}^2)}\right) \\ \geq \\ \phi\left(v^{-k}, c(k;\boldsymbol{g}^2)\right) - \phi\left(v^{-k} - \kappa_{M_k(\boldsymbol{g}^2)}, c(k;\boldsymbol{g}^2) - \kappa'_{M_k(\boldsymbol{g}^2)}\right)\end{gathered}$$

It follows that agent k does not have any incentive to remove arcs with agents $j \neq \ell$ since the last difference is positive by (3) and (4).

3. Each agent $i \notin X_k^* \cup \{k\}$ has no incentive to add an arc since she obtains all resources in $\boldsymbol{g}^3$. She has no incentive to replace an arc since she is linked with a minimal partner cost agent. We now show that she has no incentive to break her arc. Note that $v(k;\boldsymbol{g}^2) \leq v^{-i}$. We have:

$$\phi\left(v^{-i}, C_\ell\right) > \phi\left(v(k;\boldsymbol{g}^2), c(k;\boldsymbol{g}^2)\right) \geq \phi\left(0,0\right).$$

The first inequality comes from the fact that ϕ is increasing in its first argument and decreasing in its second argument; the second inequality comes from the assumption made on the payoff obtained by agent $k \in N$ in $\boldsymbol{g}^2$. $\square$

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