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On Group Stability in Hierarchies and Networks

Gabrielle Demange

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A hierarchical structure is a widespread organizational form in many areas. My aim in this paper is to provide a rationale for this fact based on two premises. First, a group organizes itself so as to achieve efficient coordination. Second, efficient coordination is achieved only if subgroups as well as individuals agree to cooperate. Even in situations in which there are gains to coordination, the agreement of each possible subgroup may be impossible to reach, resulting in instabilities. I argue that a hierarchical organization avoids such instabilities by distributing in an optimal way autonomy and blocking power to a restricted set of subgroups. Comparisons with nondirected networks are drawn.

I. Introduction

A hierarchical structure is a widespread form of large, long-lived organizations, in a wide range of activities, including economic, political, and military activities. My aim here is to propose an explanation for this prevalence.

The basic premise of this explanation is that gains to coordinated actions primarily drive the formation of organized groups. If indeed a group organizes itself to exploit gains from coordination, a goal of a good organization should be to promote coordination. A major difficulty is that to achieve efficient coordination, subgroups (coalitions) as well as individuals must agree to cooperate. To illustrate the difficulty, consider the provision of a service, say a given amount of computing capability, to the different units within a firm. If providing the service is

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less costly through a unique system than through several separate systems, efficiency requires building a unique system. How does one allocate the overall cost among the different units? A natural requirement is that a subgroup of units should not be charged with a greater cost than it would pay for the same service with a separate system; otherwise it may credibly threaten to acquire a separate system and “block” the decision. In other words, to achieve efficiency, the allocation of costs should be immune to blocking. However, if *any* subgroup is allowed to block, such an allocation may not exist. Indeed, for a large range of cost functions, the core is empty.¹

There is nothing special in cost-sharing problems.² Any organization that faces many various decision problems requiring coordination is bound to encounter similar difficulties: the possibility for subgroups to block if they find it in their interest to do so may result in instabilities and presumably prevent efficient coordination. This paper argues that a hierarchical structure solves these difficulties in an optimal way. More precisely, a hierarchical structure specifies a rigid and quite complex collection of subgroups. These subgroups, called hereafter teams and defined later on, enjoy some autonomy within a certain range of decisions. For instance, a division in a multidivisional firm can refuse to participate in a project, ask for some transfers, or negotiate with outside parties. As a result, decisions are not fully centralized nor fully controlled by the top of the hierarchy because, given their autonomy, even limited, the teams can credibly challenge decisions. Emphasizing this aspect, (1) I propose to view a hierarchy as allocating the blocking power to the teams, and only to the teams, and (2) I argue that this allocation of power guarantees stability and is maximal in doing so.³

A hierarchical structure is represented by a pyramidal network. A single individual, called the principal, is at the top, and each other individual is assigned a unique direct superior. The hierarchy structure limits communication between the different individuals or units and precisely defines the authority. Two units that want to perform a task jointly typically have to inform a common superior and all the intermediary units between them and this common superior. A team is de-

¹ Shubik (1962) was the first to model the allocation of overhead costs within a firm as a cooperative game. He initiated a large literature, surveyed by Young (1994).

² Developing a new line of products and defining compatibility between the products are subject to blocking within a firm. In public economics, deciding on levels of public goods and their financing are also under the threat of secession and blocking. For an explicit analysis, see Bewley (1981), Guesnerie and Oddou (1988), and Ray and Vohra (2001), among others.

³ My explanation is in line with the analysis of Chandler (1962), who argues that the emergence of large multidivisional firms in the middle of the nineteenth century is largely attributed to the development of new technologies that generated large increasing returns to scale and called for coordination.

defined as a subgroup of individuals who can communicate through the channels defined by the hierarchical structure and take actions on their own, within a specific range, without the agreement of outsiders. The set of teams depends on the structure of the hierarchy. In a two-tier hierarchy in which each "agent" is directly subordinate to the principal, any nontrivial team includes the principal. In general structures, the set of teams is quite rich and complex.

I first show that, given a hierarchy, whatever the problem faced by the group, there is a very natural decision process leading to an outcome that no team can credibly block. The outcome, called hereafter the *hierarchical outcome*, is essentially unique. So, by giving the blocking power to the teams only, a hierarchical organization achieves efficiency and avoids instability. One could argue that the blocking power is very restricted. It turns out, however, that the blocking power is *maximal*. More precisely, allowing any coalition in addition to the teams of a hierarchy may generate instability.

An organized group has to solve many kinds of decision problems. On important issues, the group is likely to face problems that are *super-additive*, in which efficiency is achieved only through a joint coordination of all members within the group. On a day-to-day basis, on some operational options or minor issues, problems are likely not to be super-additive, in which case the group optimally splits into smaller noninteracting groups. Interestingly, the hierarchical process works whether the problem is super-additive or not. In the former case, the hierarchical process selects an efficient joint action for all members in the group; in the latter, it selects a partition of the group into self-sufficient teams, each one taking an action for its own members. In both cases the outcome is not blocked by any team.

Whereas a hierarchy defines a set of directed bilateral relationships, nondirected networks also play an important role in organizations. Transportation and telecommunication networks connect units through bilateral links, which are not directed. Also, the sociological literature stresses the importance of networks of relationships. I compare the relative performance of the two alternative organizations, hierarchies and nondirected networks, still from the point of view of group stability. Quite naturally the group of individuals who are connected through the links are assumed to be the only coalitions that can block. Despite the added flexibility, the networks that guarantee stability with respect to connected coalitions are very similar to the hierarchical structures. Furthermore, a hierarchical decision process is shown to be, in some sense, much more efficient in reaching stable outcomes than other processes.

This paper is related to several strands of literature. First, hierarchical organizations differ from other organizational forms in a number of

important ways. So, not surprisingly, the properties that drive their choice have been investigated from many perspectives that are complementary to mine: To name a few, Sah and Stiglitz (1986) compare the screening properties of nonhierarchical versus hierarchical structures, and Radner (1993) and Bolton and Dewatripont (1994) study the optimal network structure for processing information from a planner's viewpoint. Second, collusion in principal-agent models has been the subject of some recent studies, following Kofman and Lawarrée (1993). Their main concern is collusion among agents in reporting information to the principal. In contrast, the type of collusion I focus on always involves individuals at different levels of the hierarchy. Finally, the recent literature on network games mostly investigates the formation of links between pairs of individuals from a noncooperative viewpoint (see, e.g., Jackson and Wolinsky 1996; Bala and Goyal 2000).

The paper is organized as follows. Section II describes the model, illustrates the approach with examples within a firm and within a political party, and defines stability concepts. Section III constructs hierarchical outcomes, studies their stability, and considers information revelation. Section IV analyzes nondirected networks, and Section V presents conclusions. Proofs are gathered in the Appendix.

II. Stability

Throughout the paper I consider a set of individuals, denoted by N , who organize themselves as a group. A *coalition* is a nonempty subset S of N . "Individuals" may be regarded as the units within a firm, the subsidiaries within a conglomerate, or the citizens within a party. Once formed, the group faces various decision problems. I start by describing a given decision problem and the no-blocking condition in an abstract setup. The examples that follow may be read first without inconvenience.

A. *Stability within a Problem*

The main point of this analysis is the impact that coalitions exercise on final decisions through their blocking power. The *effective* blocking power depends on two factors. First, what decisions can a coalition make if its members decide to act together? Second, what are the benefits derived by each member in exercising this possibility? A problem describes these two factors for each possible coalition.

Description of a problem.—All the individuals in N may collaborate together and make a collective decision, or take an action, which possibly includes some personalized transfers. The notion of a collective decision or action can accommodate various interpretations, as shown below in some examples. Also, a coalition may form if its members are willing

to act together. Coalition S can choose an action a in a nonempty set $A(S)$. An action in $A(S)$ is said to be *feasible for S* . The welfare of an individual depends on the action taken by the coalition to which he belongs.⁴ If a is chosen, agent i obtains a utility level $u_i(a)$. So a *problem* is given by (A, u) , which specifies the feasibility sets $A = (A(S))_{\emptyset \neq S \subset N}$ and the utility profile⁵ $u = (u_i)_{i=1, \dots, n}$.

Gains to coordination.—Within a problem, there are gains to coordination if two disjoint coalitions can only improve their prospects by acting together, a property called super-additivity. Formally, a problem (A, u) is *super-additive* if for every two disjoint coalitions S and T , every two actions a in $A(S)$ and b in $A(T)$, there exists c in $A(S \cup T)$ such that

$$u_i(c) \geq \begin{cases} u_i(a) & \text{for any } i \in S \\ u_i(b) & \text{for any } i \in T. \end{cases}$$

Faced with a super-additive problem, the entire group gains by acting together: a splitting of the whole group into two distinct groups, or more generally into a partition of several groups, each one taking a feasible action for its own members, is always Pareto-dominated by a joint action. An organized group presumably gains by coordinated action. Thus, on important issues, it is likely to face problems that are super-additive. So until Section III C, we shall focus on super-additive problems.

Blocking condition.—In a super-additive problem, efficiency is reached only through an action coordinated within the whole group N . Such a collective action may, however, be subject to some objections stemming from coalitions. The incentives for a coalition to form depend on the payoffs the members of S can attain on their own. In what follows, we shall assume that a coalition can credibly object to a proposed outcome only if it can make each of its members better off by seceding: this is the blocking condition used in cooperative game theory. Let us start with the situation in which an action includes personalized transfers and preferences are quasi-linear in money. Since monetary transfers within a coalition amount to transfer utility among the members, the payoffs that members of S can achieve on their own are described by a

⁴ This assumption excludes externalities (or spillovers) across coalitions in the following sense: the feasibility set of a coalition and the members' welfare if it forms are not affected by the organization of outside agents (see Yi [1997] for an analysis of group formation under some form of externalities). Note, however, that if the whole group splits into coalitions as considered in Sec. IV, the opportunity for a group to deviate crucially depends on this organization.

⁵ Throughout the paper, feasibility sets are assumed to be compact and individual utility functions to be continuous.

single number, the “value” $v(S)$.⁶ The utility payoffs $x = (x_i)_{i \in S}$ that can be achieved by S are those that satisfy $\sum_{i \in S} x_i \leq v(S)$. So a decision is blocked by S if

$$\sum_{i \in S} x_i < v(S). \tag{1}$$

More generally, given a problem (A, u) , the set of utility levels that members of S can attain on their own is given by

$$V(S) = \{x \in R^S \mid x_i \leq u_i(a) \ \forall i \in S, \text{ for some } a \in A(S)\}.$$

The definitions of blocking and stability with respect to a collection of coalitions follow.

DEFINITION 1. Let (A, u) be a super-additive problem. Action a is blocked by coalition S if there is an action b in $A(S)$ such that $u_i(b) > u_i(a)$ for each i in S . Given \mathcal{C} , a collection of coalitions of N , action a is said to be \mathcal{C} -stable if a is feasible for N and is not blocked by any coalition in \mathcal{C} . The core is the set of \mathcal{C} -stable actions in which \mathcal{C} is the collection of all coalitions.

I illustrate the setup first with problems within a firm under quasi-linear preferences. Transferable utility is, however, a very restrictive assumption, as illustrated by example 3 below, which describes problems within a party.⁷

Example 1: Coordination within a firm.—Consider a firm that undertakes a new project, say develops a new line of products. Each product is developed and produced by a specialized unit. The return to be expected from the line of products depends not only on the characteristics of each product but also on whether the products can be sold separately and on their degree of compatibility, in short, on the complementarity among the products. Choosing the characteristics of the products and splitting the rewards between the different units define a problem as follows.

Each unit’s utility function is given by the monetary amount the unit gets. A coalition of units S that jointly determine the characteristics of their products expects a maximal joint return denoted by $v(S)$. The complementarity across products is described by the super-additivity of the function v :

$$v(S \cup T) \geq v(S) + v(T) \quad \text{for every } S, T \text{ s.t. } S \cap T = \emptyset. \tag{2}$$

Under super-additivity, efficiency is achieved only through a *joint* de-

⁶ A decision within S may be represented by $(d, (t_i)_{i \in S})$, where $(t_i)_{i \in S}$ describes balanced transfers within the coalition. So the value $v(S)$ of coalition S is defined as the maximum of the sum of utilities $\sum_{i \in S} v_i(d)$ over the set of decisions d available to S .

⁷ Furthermore, as will be made clear later on, restricting attention to problems with transferable utility does not facilitate the analysis and does not affect the results.

termination of all product characteristics. How does one allocate the overall profit, $v(N)$, among the different units? An allocation of the profit is described by the profit x_i to be allocated to unit i under the constraint $\sum_{i \in N} x_i = v(N)$. The no-blocking condition states the natural requirement that a coalition should not get a lower profit than it would get with a separate determination of the coalition products. Indeed, if $\sum_{i \in S} x_i < v(S)$, coalition S has simply no incentives to comply with the proposed product characteristics: by coordinating together, each unit in S can achieve a larger payoff. Coalition S blocks the profit allocation if it has the power to do so.

A simple example motivates the concept of guarantee of stability introduced in the next section. There are only three units, $N = \{1, 2, 3\}$, which face the super-additive value function v :⁸ $v(i) = 0$, $0 < v(i, j) < 1$, $v(N) = 1$. An allocation of the profit belongs to the core if $x_1 + x_2 + x_3 = 1$ (feasibility), $x_i \geq 0$, and $x_i + x_j \geq v(i, j)$ (no blocking).

Whenever $v(1, 2) + v(2, 3) + v(3, 1) > 2$, no allocation satisfies all conditions: the core is empty. Instability results; hence it is dubious whether efficiency can be achieved. Notice that the core is empty for a large class of nonpathological games. For example, in symmetric games in which each doubleton gets an identical value, $v(i, j) = b$, the core is empty for b strictly larger than two-thirds, whereas super-additivity holds for b not greater than one.

Example 2: Cost sharing.—A firm, or any organization, has to allocate various common costs among its units. Simple cost-sharing and coordination problems have a very similar structure. Consider a service, say a given amount of computing capability, to be provided to units. Let $c(S)$ be the cheapest way that provides all the units of coalition S with the service. Assume i 's utility to be linear, given by $b_i - m_i$, where b_i is i 's utility for the service, and m_i is i 's contribution. The problem is described by the value function v : $v(S) = \max(0, \sum_{i \in S} b_i - c(S))$. The no-blocking condition states two natural requirements: no coalition should be charged with a greater cost than it would pay for the same service with a separate system, and no coalition should be charged with a cost that outweighs the overall benefits derived from the service by its members.

Example 3: Agreeing on a position within a political party.—A political party, whether in power or not, must take a position or make a proposal on various political and economic issues. Whereas presumably the members of the party agree on some general principles, they do not agree on each particular issue. Furthermore, unexpected issues arise. Hence, before making public statements on a new issue, the party members

⁸ To simplify notation, $v(i)$ denotes $v(\{i\})$, and similarly for $v(i, j)$. Notice that any super-additive v can be normalized so as to be null on singletons and equal to one on N .

need to coordinate among themselves. This naturally leads one to associate to each issue a “problem” among N , the set of party members, as I describe now.

Given the issue to be discussed, party members contemplate various alternatives or proposals. For instance, if reforming social security is under discussion, each proposal describes a new architecture, determines the level of contribution rates, specifies how the pension benefits are computed, and so on. The set of possible alternatives is denoted by X . A coalition that forms announces a proposal. The impact that the announcement may have on political life depends on the composition of the coalition through the members’ notoriety, their number, and the range of their political sensibilities. Accordingly, a party member who joins a group S is concerned not only by the proposal chosen by S , say x , but also by the composition of the group itself. To account for this, a feasible action a for coalition S is described by a pair formed with the proposal and the group itself, $A(S) = \{a = (x, S) \text{ where } x \text{ in } X\}$, and preferences over actions are represented by $u_i(x, S)$ for each i member of S . It is quite natural to assume that (1) for a fixed S to which i belongs, $u_i(\cdot, S)$ coincides with i ’s “intrinsic” preferences over proposals; and (2) new supporters for a proposal are always welcome: for each x , $u_i(x, S)$ increases with the coalition S to which i belongs. Two basic forces are at work.⁹ On one hand, the (possible) diversity in intrinsic preferences encourages the members of the party to split into distinct groups, each one stating a distinct opinion. On the other hand, the fact that the impact of a message increases with its support promotes coordination.

What does super-additivity mean in this problem? Super-additivity holds if two disjoint coalitions, each one stating an opinion, can always find a joint announcement that is mutually advantageous: for every disjoint S and T , every proposal x and y , there is a proposal z for which

$$u_i(z, S \cup T) \geq \begin{cases} u_i(x, S) & \text{for any } i \in S \\ u_i(y, T) & \text{for any } i \in T. \end{cases}$$

Notice that this condition may hold even if each member of S prefers x to z and each member of T prefers y to z . Basically, the problem is super-additive if the diversity in preferences is sufficiently small relative to the benefit to be expected from a unique message (the increasing returns to size effect). In such a case, a splitting of the party into distinct groups, each one stating its position on this particular issue, can always be improved on by an adequate common position. The question is whether the party will indeed agree on a common position. Indeed to reach an agreement, the various interests of the subgroups to secede must be taken into account. This is precisely the concern of the no-

⁹ These assumptions are not necessary but allow me to describe a usual trade-off.

blocking conditions. To illustrate this concern, consider a simple example with three individuals who can choose among three alternatives $X = \{x, y, z\}$. Individual 1 prefers x to y to z . Also, he cares only about the number of individuals in a coalition he joins. His preferences are given as follows:

	x	y	z
$u_1(\cdot, N)$	1	1/2	1/4
$u_1(\cdot, \{1, i\})$	2/3	1/3	0
$u_1(\cdot, \{1\})$	1/5	0	0

The preferences of individuals 2 and 3 are similar except that individual 2 prefers y to z to x , and 3 prefers z to x to y . Super-additivity holds.¹⁰ So, from the Pareto point of view, $\{1, 2, 3\}$ should agree on a common position. Assume that they attempt to agree on announcing x . Utility levels of 1, 2, and 3 are, respectively, 1, 1/4, and 1/2. So individuals 2 and 3 can credibly object and threaten to announce together z (which gives them utility levels of 1/3 and 2/3). Any proposal can be blocked in the same way: it is not at all clear whether an efficient agreement will be reached.

B. *The Guarantee of Stability*

A group should organize itself so as to benefit from gains to coordination. Ideally, an action in the core should be chosen in super-additive problems. However, as illustrated in the previous examples, the core is empty for a large class of super-additive problems. A long-lived organized group is likely to face many problems with an empty core. A natural idea is that the group organizes itself so as to avoid instability. This leads to the following definitions.¹¹

DEFINITION 2. A collection of coalitions \mathcal{C} guarantees stability if every super-additive problem admits a \mathcal{C} -stable action. If, in addition, whatever coalition S not in \mathcal{C} , $\mathcal{C} \cup S$, does not guarantee stability, \mathcal{C} is said to be maximal.

If \mathcal{C} is composed of N and all the singletons, \mathcal{C} -stability coincides with Pareto optimality and individual rationality. Stability is guaranteed, but the stability requirements are rather weak. In contrast, if \mathcal{C} is the whole set of coalitions, stability, which coincides with the core concept, is not guaranteed. A natural question is whether and how some blocking power

¹⁰ For example, $\{1\}$ and $\{2\}$ are surely better off by joining and choosing either x or y . Also consider $\{1\}$, who chooses x , and $\{2, 3\}$, who choose y . Then 1, 2, and 3 are all better off if they coordinate on y . All other cases are derived by symmetry.

¹¹ Kaneko and Wooders (1982) study a similar concept of stability in the context of partitioning games.

can be distributed to intermediate coalitions so as to guarantee stability. Since the larger the collection \mathcal{C} the stronger the stability concept is, the maximal collections are of particular interest.

Requiring the existence of a stable action in *every* super-additive problem may be thought of as strong. This would be the case if a set of coalitions failed to guarantee stability only because of some “pathological” or extreme problems. It turns out that this is not the case: if a set does not guarantee stability, then no \mathcal{C} -stable action exists in a large class of super-additive problems (see n. 12).

The guarantee of stability imposes quite severe restrictions. In a group N of three individuals, a collection \mathcal{C} that contains all doubletons does not guarantee stability (see example 1 in subsection A). The condition extends to any set N , whatever its cardinality, as follows. Let us say that three coalitions S_i , $i = 1, 2, 3$, form a *Condorcet triple* if they intersect each other but their overall intersection is empty: $S_i \cap S_j \neq \emptyset$ and $S_1 \cap S_2 \cap S_3 = \emptyset$. Using an argument similar to that for three individuals, one can easily show that *a collection of coalitions \mathcal{C} that contains a Condorcet triple does not guarantee stability*. Furthermore, not only does stability fail to be guaranteed, but also no stable action exists in a large class of games. While necessary, the absence of a Condorcet triple is unfortunately not sufficient to guarantee stability when N contains more than three individuals. A mathematical characterization for a set to guarantee stability is quite easy to obtain but difficult to interpret.¹² We are now ready to discuss the stability properties that hold in hierarchical organizations.

III. On the Stability of Hierarchical Outcomes

I first define a hierarchy and the teams that are associated with it. Then I construct a hierarchical outcome and study its stability.

A. Hierarchy and Teams

A hierarchy singles out an individual, called the principal, and each other individual is assigned a unique direct superior. Furthermore, starting from any individual and taking iteratively direct superiors, one reaches the principal at some step. This can be formally stated as follows.

¹² From Scarf's theorem, a set \mathcal{C} guarantees stability if and only if any balanced family composed of coalitions in \mathcal{C} contains a partition (see Kaneko and Wooders 1982). Note also that, thanks to Shapley, exactly the same characterization holds for \mathcal{C} to guarantee stability in transferable utility problems. Thus restricting attention to a transferable utility setup has no effect on the analysis. Finally, arguing as in example 1 of subsection A, one can show that if a balanced family in \mathcal{C} does not contain a partition, then a large class of super-additive games have no \mathcal{C} -stable actions.

A hierarchical structure on the set of individuals N is given by a principal, say 1, and a superior function s from $N - \{1\}$ to N that satisfies the following property: for each i other than 1, there is an integer r for which $s^r(i) = 1$;¹³ r is called the rank of i . Each individual $j = s(i)$, ..., $s^r(i)$ is said to be *superior to i* and i *subordinate to j* . Finally, if j is superior to i , that is, $j = s^\rho(i)$ for some integer ρ , the coalition $\{i, s(i), \dots, j = s^\rho(i)\}$ is called an interval and is denoted $[i, j]$.

Two simple hierarchies are the standard principal-agent model, in which every agent is directly subordinate to the principal, and a “chain,” a hierarchy that totally orders individuals. In general, the relation “superior to” (and its reverse “subordinate to”) partially orders the individuals. The principal is, of course, superior to each other individual. Also, individuals in an interval are all comparable among each other. Two individuals may not be compared, however, as, for instance, two distinct agents in a principal-agent model. Given a hierarchy, the teams respect the hierarchy structure: they may be considered as “sub-hierarchies.”

DEFINITION 3. Given hierarchy s , a coalition T is a team if there is a member i of T who is superior to each other member j of T , and the interval $[j, i]$ is included in T .

To understand better under which types of constraints only the teams can form, the following characterization is useful.

CHARACTERIZATION. A coalition T is a team if and only if, for every i and j in T , there is a member of T who is superior or equal to both i and j , and, furthermore, whenever j is superior to i , $s(i)$ belongs to T .

Hence, the teams form only if the two following constraints bear on a coalition: first, if two members of the coalition are not comparable, a common superior belongs to the coalition; and second, an individual cannot be “hired” by a superior without the agreement of his direct superior.

Singletons and intervals are teams. Two important teams directed by i are the *full team* of i , denoted by T^i , which is the set composed of i and all his subordinates, and the *direct team* of i , which is composed of i and the (possibly empty) set D^i of i 's direct subordinates.

B. Hierarchical Outcomes

In the principal-agent model, a natural outcome is an action that maximizes the principal's utility under the constraints that each agent gets at least his reservation level. In other hierarchies, such an outcome,

¹³ If the direct superior of i , $s(i)$, is the principal, take $r = 1$. If not, consider the direct superior of $s(i)$, $s(s(i)) = s^2(i)$. If $s^2(i)$ is the principal, $r = 2$; if not, consider his direct superior $s(s^2(i)) = s^3(i)$, and so on.

which treats all agents on an equal footing, does not reflect the implicit power of intermediate agents. A natural idea is to take into account the power of an individual on his full team. For that purpose, the individual reservation levels need to be modified, as is performed in the following algorithm.

Given a hierarchy s and a problem (A, u) , compute *guarantee levels* $g = (g_i)$ by backward induction starting from the individuals whose rank is maximal R (hence who have no subordinates) as follows:

Step 0: For each i with maximal rank R , g_i is set at i 's reservation level: $g_i = \max\{u_i(a), a \in A(i)\}$.

Step r , $r = 1, \dots, R$: This step defines the guarantee level g_i for each i with rank $R - r$. Note that the guarantee levels of i 's subordinates have been set at previous steps since their ranks are larger than i 's. Level g_i is defined as the maximum that i can get by choosing an action feasible for his full team T^i that gives each of i 's subordinates his guarantee level at least

$$g_i = \sup\{u_i(a) \text{ over } a \in A(T^i), \text{ and } u_j(a) \geq g_j, j \in T^i - \{i\}\}, \quad (3)$$

where $g_i = -\infty$ if no feasible action satisfies the constraints.

At step R , the guarantee level g_1 for the principal is defined, and an action that solves (3), if any, is called a *hierarchical outcome*.

At a hierarchical outcome, each individual gets at least his guarantee level, and the principal gets g_1 . To be interesting, guarantee levels should not be $-\infty$.

THEOREM 1. Let a hierarchy be given and \mathcal{T} denote the set of its teams.

1. Given a super-additive problem, guarantee levels are finite, and a hierarchical outcome exists. A hierarchical outcome is not blocked by any team.
2. The set \mathcal{T} , which guarantees stability from assertion 1, is maximal.

That guarantee levels are finite means that an individual, in particular the principal, can always find an action feasible for his full team that gives each of his subordinates his guarantee level. This easily follows from super-additivity (and the compactness and continuity assumptions). The stability of a hierarchical outcome with respect to *every* team is less obvious. Whereas, by construction, a hierarchical outcome is not blocked by any full team, an individual may contemplate forming a team with some, but not all, of his subordinates. According to assertion 1, he never has any incentive to do so if, in order to attract these subordinates, he must make them better off than under the hierarchical outcome. Property 2 states that, allowing any other coalition to block in addition to the set of teams leads to instability for some problems. This applies

in particular to coalitions composed of individuals with identical ranks, say agents in a two-tier hierarchy.

A hierarchical outcome is unique in most problems. Actually, under a smoothness assumption, even if multiple, all hierarchical outcomes yield identical payoffs, which are exactly equal to the guarantee levels. This is the case in transferable utility problems. Given a value function v , guarantee levels are easily computed as

$$g_i = v(T^i) - \sum_{j \in D^i} v(T^j).$$

If units are completely ordered, say from one to n , the unit with maximal rank, n , gets his “standing-alone” value $v(n)$, and each other i gets the incremental value over the set of all his subordinates: $v([n, i]) - v([n, i + 1])$. If individuals are not completely ordered, a unit that directly supervises several subordinates gets more than this incremental value: By super-additivity of v , $\sum_{j \in D^i} v(T^j) \leq v(T^i - \{i\})$. More generally, an individual’s guarantee payoff is always maximal when all his subordinates are directly subordinate to him. In particular, without management costs, a principal prefers the principal-agent structure.

Consider now the problem within a party in which the core is empty (example 3). In a hierarchy in which 2 and 3 are directly subordinate to 1, guarantee levels are $(1, 1/5, 1/5)$, and the hierarchical outcome is x . In the chain hierarchy with 3 subordinate to 2, guarantee levels are $(1/2, 2/3, 1/5)$, and the outcome is y .

So far, only the full teams determine the hierarchical outcome. Thanks to super-additivity, the teams that are not full do not have to exercise their power. This is no longer true in problems that are not super-additive.

C. On Stable Coalition Structures

Whereas super-additivity seems natural—why could not the union replicate what each coalition can do?—in some environments the union may generate some inefficiencies due, for instance, to congestion, increasing marginal cost of dissemination of information (with respect to the size of a group), or increasing marginal cost of control. In problems that are not super-additive, there is no reason, on efficiency grounds, to focus on an action that is taken by the whole group. Coalition structures take into account the possibility for the whole group to split into noninteracting self-sufficient coalitions.

DEFINITION 4. A coalition structure is a family $(a_p, S_l)_{l=1, \dots, L}$, where $(S_l)_{l=1, \dots, L}$ is a partition of N , and a_l is feasible for S_p , $l = 1, \dots, L$. It is blocked by coalition S if an action b feasible for S makes every member of S better off: for any i in S , $u_i(b) > u_i(a_i)$ if i belongs to S_p .

A coalition structure specifies simultaneously the coalitions that form and the action each one is taking. An individual is required to belong to one coalition only, a coalition that may be restricted to himself. A coalition blocks if it can make each of its members better off than under the standing coalition structure (the stability of coalition structures has been introduced in Aumann and Dreze [1974]).

In order to extend hierarchical outcomes to a setup in which a partition may form, guarantee levels are defined so as to allow an individual to form any team of which he is the principal. Given a hierarchy, guarantee levels $\hat{g} = (\hat{g}_i)$ are set by backward induction as follows:

- Step 0: Set $\hat{g}_i = \max\{u_i(a), a \in A(i)\}$ for all agents i with rank R .
- Step $r, r = 1, \dots, R$: If i 's rank is equal to $R - r$, \hat{g}_i is the maximum that i can get by forming a team with some of his subordinates while giving them at least their guarantee level:

$$\hat{g}_i = \max\{u_i(a), \text{ where } a \text{ is feasible for a team } T$$

whose principal is i and $u_j(a) \geq \hat{g}_j$, for all $j \in T - \{i\}$.

An individual can stay alone, so that \hat{g}_i is at least equal to i 's reservation level. It can be checked that g and \hat{g} coincide in super-additive problems. Otherwise, an individual possibly gets a higher utility level by forming a team that is smaller than his full team. This may be true for the principal. If so, he chooses a team T_1 that differs from N . Whereas the members of T_1 get their "guarantee" levels \hat{g}_i , outsiders must organize themselves. So it is unclear whether the payoff \hat{g} is feasible. Theorem 1 is extended as follows.

THEOREM 2. Given a hierarchy and a problem, guarantee levels $\hat{g} = (\hat{g}_i)$ are feasible: there exists a coalition structure $(a_b, T_l)_{l=1, \dots, L}$ under which

$$u_i(a_l) \geq \hat{g}_i \quad \text{for all } i \in T_l, \text{ all } l = 1, \dots, L.$$

The coalition structure $(a_b, T_l)_{l=1, \dots, L}$ is not blocked by any team.

According to this result, a hierarchical outcome leads to a stable partition of the group into teams. The partition is obtained from the guarantee levels by starting from the top of the hierarchy as follows. Consider the team and action chosen by the principal; if the team T_1 is not the whole set, choose an individual who is not a member of T_1 but whose direct superior is; pick the team and action chosen by this individual and so on¹⁴ (see details in the Appendix).

¹⁴ Greenberg and Weber (1986) provide an algorithm reaching a stable coalition structure in consecutive games in which individuals are ordered on a line, starting from one of the two individuals located at an extreme point. The hierarchical process just defined may be viewed as an extension of their algorithm to hierarchies.

Let me illustrate the algorithm in the cost-sharing problem (example 2, Sec. IIA). The utility derived by i for the service is b_i , and the cost to provide a system to S is $c(S)$. Without further assumption on c , the problem may not be super-additive. Guarantee levels (\hat{g}_i) are characterized by two functions, interpreted as costs and prices, \hat{c}_i and p_i , $i = 1, \dots, n$, as follows (where by convention $\sum_{\emptyset} p_j = 0$):

Step 0: For any i whose rank is maximal, $\hat{c}_i = c(i)$, $\hat{g}_i = \max(b_i - \hat{c}_i, 0)$, and $p_i = \min(b_i, \hat{c}_i)$.

Step r , $r = 1, \dots, R$: For each agent i with rank $R - r$, set

$$\hat{c}_i = \min \left\{ c(T) - \sum_{j \in T - \{i\}} p_j \text{ over the teams } T \text{ directed by } i \right\},$$

$$\hat{g}_i = \max(b_i - \hat{c}_i, 0), \text{ and } p_i = \min(b_i, \hat{c}_i).$$

The value \hat{c}_i is the minimal residual cost for i if the service is provided to one of his teams, each subordinate j in the team contributing p_j . Comparing i 's utility level if he gets the service at this minimal cost, $b_i - \hat{c}_i$, with the null payoff if he gets no service gives i 's guarantee level: $\hat{g}_i = \max(b_i - \hat{c}_i, 0)$. Accordingly, i agrees to get the service provided through a superior only if he is asked a contribution no larger than $p_i = \min(b_i, \hat{c}_i)$ (which gives him the guarantee level \hat{g}_i). The hierarchical outcome is composed of a partition into teams, together with the contribution of each unit to the service within each team. The overall contribution of each team exactly covers the cost of providing the service to its members. Units that get no service pay nothing.

D. Information Revelation

If units are asked to reveal a piece of information in order to implement a hierarchical outcome, some of them have incentives to lie. In a hierarchy, it seems fair to assume that individuals are able to observe their direct subordinates' characteristics such as the current workload or the ability to accomplish a given task. As shown by the implementation literature, more powerful mechanisms may be used to extract a piece of information that is shared by several individuals.

This insight can be developed in the cost-sharing problem when the individuals' utility parameters b_i are unknown (but the cost function c is still known). Recall that no strategy-proof mechanism makes efficient decisions and has a balanced budget. In a carefully designed revelation game, truthful behavior is not only a Nash equilibrium, as is standard, but also a strong equilibrium, meaning that it is immune to coordinated

deviations.¹⁵ Each individual i announces a characteristic for himself, β_i , and for each of his direct subordinates, α_p if any: a strategy for i is given by $\sigma_i = (\beta_i, (\alpha_j)_{j \in D^i})$. Thus, except for the principal, individual i 's characteristic is announced twice, once by himself, β_i , and once by his direct superior, α_i . If $\beta_i \neq \alpha_i$, i and $s(i)$ are said to disagree. The basic idea is to compute the hierarchical outcome for the utility characteristics given by the minimum of the announced ones, $\min(\beta_i, \alpha_i)$ for each i , but to modify the contribution asked to each subordinate i by assuming his characteristic to be $\max(\beta_i, \alpha_i)$. More precisely, given the strategy profile (σ_i) , compute recursively costs and prices, \hat{c}_i , p_i and q_i , $i = 1, \dots, n$, as follows:

Step 0: For each i whose rank is maximal, $\hat{c}_i = c(i)$, $q_i = \min(\max(\beta_i, \alpha_i), \hat{c}_i)$, and $p_i = \min(\min(\beta_i, \alpha_i), \hat{c}_i)$.

Step r , $r = 1, \dots, R$: For each i with rank $R - r$, set

$$\hat{c}_i = \min \left\{ c(T) - \sum_{j \in T - \{i\}} p_j \text{ over the teams } T \text{ directed by } i \right\},$$

$$q_i = \min(\max(\beta_i, \alpha_i), \hat{c}_i), \text{ and } p_i = \min(\min(\beta_i, \alpha_i), \hat{c}_i).$$

The final outcome is defined by the partition of N into teams given by the hierarchical outcome associated with profile $(\min(\beta_i, \alpha_i))_{i \in N}$; in each team that is provided with the service, the principal of the team receives p_j from each subordinate j within the team, whereas the subordinate contributes q_j . As a consequence, if a subordinate disagrees with his direct superior, there is a gap between the contribution asked to the subordinate and the price received by his direct superior, the latter being smaller than the former.

If every individual announces the true valuation for himself and for all his direct subordinates, the hierarchical outcome associated with the true profile is obtained. It can be shown that these truth-telling strategies form a *strong* equilibrium, that is, are robust to coordinated deviations. It should be noted that the deviation of any coalition, whether a team or not, is contemplated. The basic idea of the proof (available on request) is as follows. At the truth-telling profile, each individual gets a nonnegative payoff (since the hierarchical outcome with respect to the true profile is reached). Therefore, to be strictly better off by lying, an individual must get the service and contribute less than under truth

¹⁵ In line with this model, the principal's preferences are announced by him only. Therefore, the truth-telling profile may not be the unique Nash equilibrium. Indeed, mechanisms can be designed with full implementation if there is no truly private information; i.e., each characteristic is known by at least two distinct individuals (see Postlewaite and Schmeidler 1986; Saijo 1988). These mechanisms are, however, quite abstract and may not be immune to coordinated deviations.

telling. This is used to show that his direct superior lies as well. By backward induction starting from individuals with rank R , a coalition that makes all its members better off by lying is a team directed by the principal. This gives a contradiction: the principal cannot get a higher monetary payoff than under the “true” hierarchical outcome while paying his subordinates more.

IV. Networks

A hierarchy specifies bilateral relationships that are ordered through the superior function. In various contexts, individuals are related among each other but without any ordering. These relationships are represented by a nondirected network (from now on, network means nondirected network). For instance, in transportation and telecommunication, communication between units is processed through a network of physical links, as given by roads, pipes, cables, or wires. Various problems, such as cost sharing among customers, must be solved taking into account the network structure (see Sharkey 1982). A network can also represent social and economic relationships, such as family links, friendships, personal contacts, alliances, and so on. Following Myerson (1977), the literature is currently developing fast (see, e.g., the recent survey by Jackson [2004]). Finally, a network can represent ideological proximity, as, for example, in Black’s (1948) voting model, where individuals are ranked from left to right. The ranking does not reflect any superiority, but specifies whether an individual is “intermediate” or “between” two others, as far as preferences are concerned (for a development of these ideas, see Demange [1994]).

My purpose here is to analyze networks (also called graphs) from the group stability viewpoint.¹⁶ Let us first recall some definitions. A *network* G on N is a set of unordered pairs of distinct elements of N . A *path* of G is a sequence i_0, \dots, i_m in which the pairs (i_k, i_{k+1}) for $k = 0, \dots, m - 1$ are in G and are all distinct. A *cycle* is a path from a point to itself. A network is a *tree* if two distinct elements are linked by a unique path. As in a hierarchy, some coalitions respect the existing links. Formally, given a network G on N , a coalition S is *G-connected* if for every two distinct agents in S there is a path between them that is contained in S .

Hierarchies and trees define similar structures of bilateral relationships. More precisely, if in a hierarchy the relationship between an agent

¹⁶ In an exchange economy, interpreting connected coalitions as the ones that can trade among each other, Kalai, Postlewaite, and Roberts (1978) study the impact of the network on the core, in particular whether being a middleman, i.e., an individual who connects two disjoint sets, is advantageous. Since the core is always nonempty in this setup, the guarantee of stability is not an issue.

and his direct superior is “undirected” and becomes a link between the two individuals, the obtained network is a tree: two individuals are connected through a unique path. Conversely, given a tree, a hierarchy is obtained by picking an individual as the principal and “directing” the links on the paths starting from him.¹⁷ The close relationships between hierarchies and trees extend to stability properties.

PROPOSITION 1.

1. Given a network G , the set of G -connected subsets guarantees stability only if G is contained in a tree.
2. Given a tree G , all the hierarchies associated with G admit the same set of teams, which coincides with the set of G -connected coalitions.

Assertion 1 is easy to understand. A network G that is not included in a tree contains a cycle. Then a Condorcet triple of G -connected coalitions is easily found, and stability is not guaranteed. In view of this result, nondirected networks do not enrich the set of graph structures that guarantee stability. How does one explain the current development of networks that are clearly not tree-hierarchical structures? Two distinct arguments can be put forward. First, in industries, networks such as alliances typically bear on a limited range of decision problems. For example, in airline industries, companies agree on sharing some codes. If they contemplated a more complete integration, our analysis suggests that they would need a tighter organization. Second, most of the sociological networks one can think of are primarily used to share pieces of information rather than to take actions. The concept of stability considered in this paper addresses the problem of taking actions.

Two consequences can be drawn from assertion 2 of proposition 1. First, given a set of undirected relationships, tree-network outcomes that are stable with respect to G -connected coalitions are easy to obtain. It suffices to pick an individual as principal and to take the hierarchical outcome in the obtained hierarchy.¹⁸ Second, given a hierarchy, whereas the hierarchical outcome is not blocked by any team, it is typically not the unique one: the hierarchical outcome associated with the same tree but with a different principal is stable as well. A natural question is whether hierarchical outcomes enjoy some special properties among all stable outcomes. In particular, are they in some sense easier to obtain and more likely to emerge than other stable outcomes? The answer is

¹⁷ Formally, given a hierarchy s , the network G_s is defined as the set of pairs (i, j) , where either $i = s(j)$ or $j = s(i)$. Given a tree G , the hierarchy with individual i as principal is defined by s_i as follows: for any j different from i , let i_0, \dots, i_{r-1}, i_r be the unique path in G from $i_0 = i$ to $i_r = j$. Set $s_i(j) = i_{r-1}$.

¹⁸ Demange (1994) gives a nonconstructive proof of this result, which relies on the Scarf theorem on balanced games. The basic argument is that any balanced family of G -connected coalitions contains a partition if G is a tree.

positive in a sense that I now make precise. For that purpose, attention may be restricted to super-additive problems with transferable utility.

Given a tree G that links individuals in N and a value function v , the set of stable payoffs, for a short stable set, is described by the following system of linear inequalities:

$$\begin{aligned} \sum_{i \in N} x_i &= v(N), \\ \sum_{i \in S} x_i &\geq v(S) \quad \text{for each } G\text{-connected } S. \end{aligned} \quad (4)$$

The extreme points, which characterize the stable set, play an important role.¹⁹ Intuitively at each extreme point, many coalitions get their minimum value since many constraints in (4) are binding. To formalize this intuition, let us say that a collection \mathcal{E} of n teams of G is *complete* if (a) N belongs to \mathcal{E} and (b) for each v there is a unique solution $x(v, \mathcal{E})$ to the system of equations

$$\sum_{i \in S} x_i = v(S) \quad \text{for each } S \text{ in } \mathcal{E}.$$

It can be shown that *each extreme point of the stable set is equal to $x(v, \mathcal{E})$ for some complete collection \mathcal{E}* . Hence hierarchical payoffs are extreme points. Indeed denote by \mathcal{F} the set of full teams in the hierarchy with i as principal. The hierarchical payoff is unique, obtained by solving in a recursive way the n equations $\sum_{i \in S} x_i = v(S)$ for each full team: \mathcal{F} is complete, and the hierarchical payoff $x(v, \mathcal{F})$ is an extreme point. More generally, the following process finds any other extreme point: first, pick a complete family \mathcal{E} , and compute the payoff $x(v, \mathcal{E})$ that gives to each coalition in \mathcal{E} its value; second, check whether the payoff is stable, that is, whether it satisfies all inequalities (4). Note that the second step is not necessary if \mathcal{E} is a set of full teams.

Example.—Let three players form the links (1, 2) and (2, 3). Consider a problem with transferable utility described by the value v : $v(1, 2, 3) = 1$, $v(1, 2) = a$, $v(2, 3) = b$, $v(1, 3) = 0$, and $v(i) = 0$. Super-additivity holds for a and b nonnegative and smaller than one. Two shapes for the stable set obtain depending on the value of $a + b$ with respect to one, as shown in figure 1 (each point (x_1, x_2, x_3) that satisfies $x_1 + x_2 + x_3 = 1$ and $x_i \geq 0$ is represented by a point in the triangle). In addition to the three hierarchical payoffs $H^1 = (1 - b, b, 0)$, $H^2 = (0, 1, 0)$, and $H^3 = (0, a, 1 - a)$ obtained with 1, 2, and 3, respectively, as principal, the other extreme points are as follows: if $1 \leq a + b$, $A = (1 - a, a + b - 1, 1 - b)$ with coalitions $\{1, 2\}$, $\{2, 3\}$, and $\{1,$

¹⁹ Recall that an extreme point is a stable payoff that is not a convex combination of other stable payoffs. The characterization below is a kind of “folk” result, probably due to Shapley.

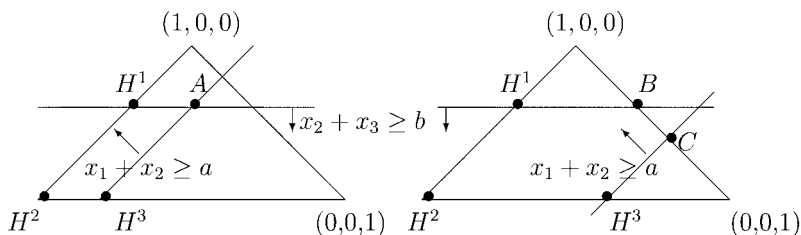


FIG. 1.—Stable sets for $a + b > 1$ ($a = 4/5, b = 2/5$) and $a + b < 1$ ($a = 2/5, b = 2/5$)

2, 3} binding; and if $1 \geq a + b$, $C = (a, 0, 1 - a)$ with {1, 2}, {2}, and {1, 2, 3} binding and $B = (1 - b, 0, b)$ with {2, 3}, {2}, and {1, 2, 3} binding. Except for the hierarchical payoffs, the stability of a “tentative” extreme point depends on the problem at hand and has to be checked. This extends as follows.

THEOREM 3. Given a tree G , let \mathcal{C} be a complete family of G -connected sets. If \mathcal{C} is not a set of full teams, there are super-additive games v for which the payoff vector $x(v, \mathcal{C})$ is not stable.

In other words, for any family other than a full-teams family, the “tentative” extreme point may be blocked by a connected set. Accordingly, a process that would reach stable outcomes other than the hierarchical ones is likely to be much more complex than a hierarchical process.

V. Conclusion

I have shown that hierarchical outcomes enjoy nice properties. They are easy to implement and satisfy strong-group rationality requirements. These properties are valid for *any* hierarchical structure. This calls for two remarks. First, hierarchical organizations are obviously not all alike, some appearing more “centralized” than others, independently of the tree structure itself. This does not contradict the analysis, which allows one to separate two sources of power: the possibility for a coalition to make independent decisions, which is determined by the tree structure only, and the scope of these decisions, described here by the feasible sets. Hierarchies that give a very limited scope of autonomy to their teams are indeed centralized. Incorporating the choice of the feasible sets into the framework of this paper would be most interesting. Second, according to our results, the choice of a particular hierarchical structure should be explained by factors other than group stability. Comparing structures among each other, determining which ones are more appropriate to a given situation, would be worth investigating. For example, without the cost of managing subordinates, a principal has nothing to

lose to have them all directly subordinate to him; that is, he prefers the principal-agent structure. In the presence of significant management costs, however, a trade-off between the cost of directly managing subordinates and the gains of extracting surplus over them would justify more complicated structures. Another interesting issue would be to determine who is more efficient or, more likely, to be a principal (on this point, see the paper by Barbera and Ehlers [2002], who study the choice of a principal under majority rule).

Appendix

Proof of Theorem 1

Proof of part 1.—By induction, assume that at step r of the algorithm, for each i with rank $R - r$, i 's guarantee level is finite, and a hierarchical outcome exists in the problem restricted to the full team of i ; that is, there is an action, say a , that maximizes $u_i(a)$ under the constraints

$$a \in A(T^i), \quad u_j(a) \geq g_j, \quad j \in T^i - \{i\}. \quad (\text{A1})$$

The induction assumption is true at step 0 since an individual of rank R has no subordinate. Consider at step r individual i whose rank is $R - r$. By the continuity and compactness assumptions, it suffices to show that there is an action that satisfies constraints (A1). By the induction assumption, for each direct subordinate j in D^i , there is an action a_j in $A(T^j)$ that gives to j and to each of his subordinates his guarantee level. Note that the family of teams $\{i\}$ and T^j , $j \in D^i$, forms a partition of i 's full team of i (because a subordinate of i is either subordinate or equal to one, and only one, direct subordinate of i). Hence, by super-additivity, given any action b in $A(i)$ and the feasible actions a_j in $A(T^j)$, $j \in D^i$, there is an action c in $A(T^i)$ that gives at least the utility level of a_j to each member of T^j , for every j , and, hence, their guarantee levels. Therefore, action c satisfies constraints (A1) for i , which proves the induction assumption.

Let us prove by contradiction that a hierarchical outcome a_1 is not blocked by any team. If a team T blocks a_1 , there is a b in $A(T)$ for which

$$u_j(b) > u_j(a_1) \geq g_j \quad \text{for every } j \in T. \quad (\text{A2})$$

Let i be the principal of team T . Since by construction no full team blocks a_1 , T is a strict subset of T^i , the full team of i . So there is at least a subordinate of i who does not belong to T but whose direct superior belongs to it. Let K be the set of such subordinates: $K = \{k \in T^i, k \notin T, s(k) \in T\}$. I claim that the family of teams $(T, T^k, k \in K)$ forms a partition of T^i . Suppose it to be true. Since b and a_k are feasible for T and T^k , respectively, by super-additivity, there is an action feasible for T^i that gives to each member j of T at least $u_j(b)$ and to each other member of T^i his guarantee level: by (A2), i gets strictly more than g_i , a contradiction.

It remains to show that the family of teams $(T, T^k, k \in K)$ forms a partition of T^i . Consider j in T^i . First, he belongs to at most one of the sets: if j belongs to T^k , the interval $[j, i]$ is the disjoint union of $[j, k]$ and $[s(k), i]$, where $[j, k]$ is disjoint from T (because otherwise k would belong to T), and $[s(k), i]$ is included in T . This implies that j does not belong to any other set of the family. Second, j belongs to one of the sets: if j does not belong to T , since i does,

there is surely some k in the interval $]j, i]$ who does not belong to T and whose direct superior does. Therefore, k belongs to K , and since j is either equal or subordinate to k , j belongs to T^k . Q.E.D.

Proof of part 2.—Let S_0 be a coalition that is not a team. To prove that adding S_0 to the set of teams may lead to instability, it suffices to find two teams, S_1 and S_2 , such that $S_i, i = 0, 1, 2$, form a Condorcet triple. Since S_0 is not a team, there are two distinct i and j in S_0 such that at least one of the following cases is true. In case 1, no member of S_0 is superior or equal to both i and j . In case 2, j is superior to i but $s(i)$ does not belong to T .

In case 1, take $S_1 = [i, 1]$ as the interval from i to 1, which is composed of i and all his superiors, and similarly $S_2 = [j, 1]$. Clearly the sets $S_i, i = 0, 1, 2$, intersect each other. Moreover, $S_1 \cap S_2$ is formed with the individuals who are superior or equal to both i and j , so by assumption $S_1 \cap S_2$ does not intersect S_0 . In case 2, note that $s(i)$ is surely subordinate to j . It suffices to take $S_1 = [i, s(i)]$ and $S_2 = [s(i), j]$. Q.E.D.

Proof of Theorem 2

A coalition structure that gives utility levels at least equal to \hat{g} is clearly not blocked by any team. So we have to show only that $\hat{g} = (\hat{g}_i)$ is a feasible payoff vector.

Let (T_1, a_1) solve \hat{g}_1 . By construction, each member of T_1 gets at least his guarantee level at a_1 . Hence, if T_1 is the whole set N , (N, a_1) ensures the feasibility of \hat{g} . If not, there is a member of T_1 who has a direct subordinate, say 2, who does not belong to T_1 .

Let (T_2, a_2) solve \hat{g}_2 . We now show that T_1 and T_2 are disjoint and $T_1 \cup T_2$ is a team. A member i in T_2 is subordinate to 2, so he does not belong to T_1 because otherwise 2 would belong to T_1 as well: $T_1 \cap T_2 = \emptyset$. Moreover i 's superiors belong to the interval $]i, 1]$, which is the union of $]i, 2]$ and $[s(2), 1]$. Since $]i, 2]$ is included in T_2 and $[s(2), 1]$ in T_1 , this proves that $T_1 \cup T_2$ is a team. If $T_1 \cup T_2$ is the whole set N , we are done: (T_1, a_1) and (T_2, a_2) ensure the feasibility of \hat{g} . The argument can be repeated so as to obtain, at the end of step $l-1$, disjoint teams $T_k, k = 1, \dots, l-1$, whose union is a team. If this union is not the whole set N , one may find (T_l, a_l) , where a_l gives to each member of T_l his guarantee level, T_l is disjoint from the previous teams, and the overall union is a team. Surely the union is N at some step. Q.E.D.

Proof of Proposition 1

Given a tree G on N and individual i , let s^i be the hierarchy associated with G .

We first show that each team in s^i is G -connected. An interval is clearly G -connected. Now a team T with principal, say, j is the union of all intervals $[k, j]$ for k running in T . So T is G -connected, as the union of G -connected sets that have an overall nonempty intersection.

To show that conversely a G -connected coalition T is a team of s^i , pick j in T whose rank r is minimal among the members of T (rank defined in the hierarchy s^i). We claim that T is a team of j . First let us consider the path from j to principal i . By definition of s^i , any agent other than j in this path has a rank strictly lower than r and, hence, does not belong to T . Now let k be in T , distinct from j . The path from j to k is included in the G -connected set T , so that, from the remark above, it intersects the path from j to i only at j . It follows that the path from k to i is composed of the path from k to j and j to i : this implies that in the

hierarchy s^i , k is subordinate to j , and the interval $[k, j]$ coincides with the path from k to j , which is included in T . Q.E.D.

Proof of Theorem 3

The proof is divided into several steps. In what follows, \mathcal{E} is assumed to be a complete family for which the payoff vector $x(v, \mathcal{E})$ is stable whatever the super-additive game v .

Proof of step 1.—If S and T are elements of \mathcal{E} , either $S \cap T = \emptyset$, $S \subset T$, or $T \subset S$. If not, $S \cap T$ is nonempty, so both $S \cup T$ and $S \cap T$ are teams. Moreover, the game v defined by

$$v(C) = \begin{cases} \alpha & \text{for any } C \text{ that contains either } S \text{ or } T \text{ but not both} \\ 1 & \text{for any } C \text{ that contains } S \cup T \\ 0 & \text{otherwise} \end{cases}$$

is super-additive for any α between zero and one. At the payoff $x = x(v, \mathcal{E})$, the equations $x(S) = x(T) = \alpha$ and $x(N) = 1$ hold. Stability applied to each individual and to the team $S \cup T$ requires $x \geq 0$ and $x(S \cup T) \geq 1$, which implies $x(S \cup T) = 1$. Now

$$x(S \cup T) = x(S) + x(T) - x(S \cap T)$$

yields $x(S \cap T) = 2\alpha - 1$. Choosing $\alpha < 1/2$ gives a negative payoff to the team $S \cap T$, which contradicts stability. Q.E.D.

Proof of step 2.—For each i , there is a smallest set C_i in \mathcal{E} to which i belongs. Let us consider the sets in \mathcal{E} to which i belongs (the whole set N , which belongs to \mathcal{E} , is such a set). All these sets intersect each other (at i). So by step 1, they are nested by inclusion: there is a smallest set. Q.E.D.

Proof of step 3.—If C_i and C_j intersect for $i \neq j$ distinct, then C_i is a strict subset of C_j or the converse. The sets C_i and C_j are in \mathcal{E} , so if they intersect, one is a subset of the other one (from step 1). Now, since the family \mathcal{E} is complete, coordinates x_i and x_j , and not just the sum $x_i + x_j$, are determined by equations (4). So there must be a set T in \mathcal{E} that contains one, and only one, element of the pair i, j . This implies that C_i and C_j are distinct. Q.E.D.

Proof of step 4.—There is i such that C_i is the singleton $\{i\}$. If C_i is not a singleton, pick j in C_i , j different from i . Since j belongs to C_i , surely C_j is a strict subset of C_i from step 3. If C_j is not a singleton, repeat the argument. Q.E.D.

Proof of step 5.— If $C_i = \{i\}$ and (i, j) is a link, then i belongs to C_j . Note that $C_j \cup \{i\}$ is a team if (i, j) is a link. By contradiction, if i does not belong to C_j , one may define a super-additive v that satisfies $v(i) = 0$, $v(C_j) = 0$, and $v(C_j \cup \{i\}) > 0$. Since $\{i\}$ and C_j belong to \mathcal{E} , at $x(v, \mathcal{E})$, $x_i = v(i) = 0$ and $x(C_j) = v(C_j) = 0$, which gives $x(C_j \cup \{i\}) = 0$, so $C_j \cup \{i\}$ blocks. Q.E.D.

Proof of step 6.— If $C_i = \{i\}$, then i is linked to a unique individual; that is, i is an extreme point of the tree G . Assume by contradiction that (i, j) and (i, k) are links for j, k distinct. By step 5, C_j and C_k intersect at i , so by step 3, we may assume $C_j \subset C_k$ and surely $k \notin C_j$. Given α between zero and one, let v be defined by $v(N) = 1$, $v(C) = \alpha$ for any C that contains C_j or $\{i, k\}$, and zero otherwise. The game v is super-additive: if two coalitions do not intersect, one at least does not contain i , so its value is zero. At $x = x(v, \mathcal{F})$, since $\{i\}$, C_j , and N belong to \mathcal{E} , we have $x_i = v(i) = 0$, $x(C_j) = v(C_j) = \alpha$, and $x(N) = 1$. Stability requires $x \geq 0$ and $x_i + x_k \geq v(\{i, k\}) = \alpha$ since $\{i, k\}$ is a team. Since $x_i = 0$, this implies

that $x(N) \geq x(C_i) + x_k \geq 2\alpha$, which is impossible whenever α is higher than one-half. This gives the desired contradiction. Q.E.D.

We end the proof by induction on the number of individuals. Assume theorem 3 to be true for any set with at most $n - 1$ individuals (it is trivially true for a unique agent), and consider n individuals. By step 4, there exists an individual, say n , with $C_n = \{n\}$, who, moreover, by step 6, is linked to a unique individual, say $n - 1$. Consider the network G_{-n} obtained from G by dropping the link $(n - 1, n)$: it is a tree on $N - \{n\}$. For each C element of \mathcal{E} distinct from $\{n\}$, take its intersection with $N - \{n\}$, and let \mathcal{E}_{-n} be the obtained family. One easily checks that \mathcal{E}_{-n} is a complete family of teams in G_{-n} . By the induction assumption, there is $i < n$ for which the family \mathcal{E}_{-n} is the set of full teams in G_{-n} with i as principal. Since any set in \mathcal{E} that contains $n - 1$ contains n (by step 5), this yields that \mathcal{E} is the set of full teams \mathcal{F}_i in G , which ends the proof. Q.E.D.

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