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**Bundling, tying and collusion**

**David Spector**

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**Keywords : bundling, tying, collusion**

# Bundling, tying and collusion\*

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## Abstract

Tying a good produced monopolistically with a complementary good produced in an oligopolistic market in which there is room for collusion can be profitable if some buyers of the oligopoly good have no demand for the monopoly good. The reason is that a tie makes part of the demand in the oligopolistic market out of the reach of the tying firm's rivals, which decreases the profitability of deviating from a collusive agreement. Tying may thus facilitate collusion. It may also allow the tying firm to alter market share allocation in the collusive oligopolistic market.

## 1 Introduction

This paper shows that a firm enjoying a monopoly in market  $A$  while also being active in an oligopolistic market  $B$  may find it profitable to tie products in both markets in order to facilitate collusion in market  $B$ .<sup>1</sup> We consider a model in which a fraction of buyers wants goods  $A$  and  $B$  in fixed proportions while other buyers want only product  $B$  and do not care about product  $A$ . In such a context, the "Chicago critique" (or "single monopoly critique") of the leveraging theory, which stresses that raising the price of product  $A$  is

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<sup>1</sup>In this introduction, we use both the words "tying" and "bundling", because, as is shown below, the results hold both in the case of an irreversible technical tie and in the case of a reversible (e.g., contractual) bundling strategy.

a sufficient instrument for the product  $A$  monopolist to extract consumers' surplus, breaks down, because raising the price of product  $A$  does not allow the monopolist to extract surplus from the consumers who only want product  $B$ .

One way to extract surplus from these consumers is to collude in market  $B$ , and this paper shows that tying may help the product  $A$  monopolist to extract surplus by colluding in market  $B$ . We prove two results. First, for some parameter values, tying can make collusion in market  $B$  possible, which benefits both the tying firm and its competitors in market  $B$ , and harms consumers. Second, for parameter values such that collusion in market  $B$  is feasible irrespective of whether products  $A$  and  $B$  are tied, tying may allow the market  $A$  monopolist to increase its market share in market  $B$ , and thus its profit.

The mechanism behind this result is very simple. Collusion is feasible to the extent that each firm is deterred from deviating from the collusive agreement by slightly undercutting competitors. This is where tying helps: if the product  $A$  monopolist makes it mandatory for purchasers of product  $A$  to also buy product  $B$  from it, it insulates a fraction of the demand (that emanating from the fraction of consumers who want both products) from competition. As a consequence, tying makes it less attractive for market  $B$  firms (other than the product  $A$  monopolist) to deviate from any given collusive agreement, because the consumers who want both products are out of their reach anyway. This has two consequences. First, since the incentive compatibility constraint facing the firms other than the product  $A$  monopolist is relaxed, the set of parameter values inducing the existence of a collusive equilibrium expands: collusion becomes more likely. Second, for parameter values inducing collusion both with and without tying, tying expands the set of market share allocations which can be sustained in a collusive agreement. More precisely, since all firms' (except the market  $A$  monopolist's) incentive compatibility conditions are relaxed, collusion becomes feasible for market share divisions which are more favorable to the market  $A$  monopolist. Thus, tying a good in a monopolistic market with a complementary good in a collusive market may allow the tying firm to increase its market share in the collusive market. These results are proved first in the case where the tying good monopolist has the option to irreversibly tie its products before the repeated oligopolistic interaction takes place (sections 2 and 3). We then show that they carry over to the case where the tying good monopolist cannot make any once-and-for-all commitment to bundle, but instead can decide in

every period whether or not to offer a bundle (section 4).

### *Relation to the literature*

Two strands of the vast literature on bundling<sup>2</sup> are related to the present paper. One of them, initiated by Whinston (1990) has shown that in the presence of economies of scale, a tying-market monopolist can use tying in order to deter entry in the tied market by preventing potential entrants from reaching the minimum viable scale. This strategy can be profitable if some tied-market customers have no demand for the tying product: in this case, the Chicago critique breaks down, because the extension of monopoly to the tied market increases the tying good monopolist's profit by allowing it to exploit these consumers.<sup>3</sup> The present paper is based on a similar demand structure: some consumers in the tied market also want the tying product, some do not, and tying allows the tying market monopolist to increase the profit it extracts from the latter group. However, in the present paper the profitability of bundling does not result from its entry-detering or exit-inducing virtues, but rather from its impact on the feasibility of collusion. Accordingly, the results of this paper do not rely on any assumption about economies of scale in the tied market.

A second strand of literature has shown that bundling may allow a firm to mitigate competition by increasing differentiation.<sup>4</sup> While the idea in the present paper is also that bundling may mitigate competition, the mechanism is different. The existing literature compares non-cooperative equilibria with and without bundling, while the present paper shows that bundling may allow firms to shift from non-cooperation to cooperation, or may change the division of market shares when the tied market is collusive anyway. Two other differences distinguish the present paper from that literature. First, the present results rely on the assumption that the two goods are complements. Second, in the present paper, the tying firm sells both a bundle and the tied product alone, and this point is crucial, because it is through the sale of the tied product alone that the tying firm can reap the benefits of collusion. On the contrary, the results showing that bundling can mitigate competition

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<sup>2</sup>The literature on bundling is surveyed in Nalebuff (2002), which in particular summarises the main critiques of the single monopoly critique.

<sup>3</sup>Carlton and Waldman (2002) present a modified version of this argument in the case where there are intertemporal economies of scale or network effects.

<sup>4</sup>Carbajo et al. (1990), Seidmann (1991), Chen (1997).

by increasing differentiation usually consider the situation in which the firm offering a bundle does not also offer the tied product alone. This is because the differentiation between different firms' offerings drives the results: some of them offer the bundle alone, while others offer products on a standalone basis.

## 2 The model

### 2.1 Agents, preferences and costs

There are two homogeneous goods, labeled A and B. Firm  $M$  can produce both goods at zero cost, while Firm  $E$  can produce only product  $B$ , at zero cost as well. Product  $B$  is not differentiated: the varieties produced by Firms  $M$  and  $E$  are perfect substitutes for consumers.

A continuum of consumers of total mass one exists and is subdivided into two groups. A mass  $b$  of consumers want to consume goods A and B in fixed amounts (one unit of each). All these consumers have a valuation  $V_1$  for the pair consisting of one unit of each good. A mass  $(1 - b)$  of consumers has no utility for product A. These consumers have a unit demand for product  $B$  and a common valuation  $V_2 < V_1$ . We assume that price discrimination is not possible since the good can be resold.

The rate of time preference of both firms is denoted  $\delta$ .<sup>5</sup>

### 2.2 The game

We consider the following timing.

- In period 0, Firm  $M$  decides whether or not it will tie products A and B. This decision is irreversible and cannot be made at any later stage. If it decides to tie, Firm  $M$  will be able to offer two products at the later stages of the game: (i) a pair containing one unit of product A and one unit of product B, and (ii) product B alone. If it decides not

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<sup>5</sup>The results of this paper would carry over to the case of heterogeneous rates of time preference. The assumption of a uniform rate is made for the sake of simplicity (with heterogeneous rates, the optimal collusive equilibria may not be stationary). For a general analysis of collusion between firms with heterogeneous discount rates, see Harrington (1989).

to tie, it will be able to offer two products: (i) product  $A$  alone, and (ii) product  $B$  alone.

- In periods 1,2,3,... until infinity, both firms simultaneously set prices for the products they sell (i.e. Firm  $M$  sets the prices of its two products, which are either products  $A$  and  $B$ , or an  $A - B$  pair and product  $B$ , and Firm  $E$  sets the price of the product  $B$  it sells). In addition, we assume that each firm announces, in each period, the maximum amount of product  $B$  it is willing to sell. Then, if both firms set the same price for product  $B$ , consumers who want to purchase product  $B$  are allocated to both firms pro rata the announced amounts.

The last assumption (i.e. the assumption that each firm sets the maximum amount of product  $B$  it is willing to sell) is there to take into account the possibility of different divisions of market  $B$ .<sup>6</sup>

### 3 Tying and collusion

#### 3.1 Conditions for the existence of a collusive equilibrium

In order to solve the above game by backward induction, we investigate the feasibility of collusion depending on whether Firm  $E$  chose to tie products  $A$  and  $B$  in period zero. The following lemma, proved in the appendix, characterizes optimal equilibria.

**Lemma** *Assume that the repeated game has collusive equilibria (i.e. equilibria with payoffs different from those of the one-shot game), and consider a Pareto-optimal subgame perfect equilibrium EQ (i.e. one which is optimal from the point of view of the two firms, within the set of all subgame perfect equilibria). Then (i) this equilibrium is such that the price of product  $B$  is equal to  $V_2$  in every period; and there exists a subgame perfect equilibrium yielding the same payoffs as EQ and such that (ii) each player's equilibrium payoff is identical across periods, and (iii) after any deviation off the equilibrium path, each firm's action in the continuation game is the one prescribed by the Nash equilibrium of the one-shot game.*

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<sup>6</sup>This is similar to the modeling of market share proposals in Athey and Bagwell (2001).

This lemma means that the analysis can be carried out by limiting our attention to simple equilibria, namely those in which the equilibrium path involves both players quoting the monopoly price  $V_2$  and a constant market share allocation, while any deviation from these strategies leads each player to earn the profit it would earn in the equilibrium of the one-shot game forever after the period during which the deviation took place. This simplification allows us to analyze the possibility of collusion as follows, depending on whether Firm  $M$  decided to tie both products or not in period 0.

**Case 1: Firm  $M$  chose not to tie products  $A$  and  $B$  in period 0.**

The above lemma means that a hypothetical collusive equilibrium is equivalent to one characterized by the following "tit-for-tat" strategies for some  $\alpha$  between zero and one (denoting the share of the total market for product  $B$  attributed to Firm  $M$ ):

- If no deviation occurred in any past period (i.e. if the actions described hereafter have been played by both firms in all previous periods) then both firms play as follows:
  - Firm  $M$  sets the price of product  $A$  at  $V_1 - V_2$  and offers product  $B$  at price  $V_2$ . It announces that it wants to sell at most  $\alpha$  of product  $B$ .
  - Firm  $E$  offers product  $B$  at price  $V_2$ . It announces that it wants to sell at most  $1 - \alpha$  of product  $B$ .
- If either firm deviated from the above-mentioned actions in any previous period, then
  - Firm  $M$  sets the price of product  $A$  at  $V_1$  and offers product  $B$  at price zero.
  - Firm  $E$  offers product  $B$  at price zero.

These strategies are equilibrium strategies only if, for each firm, the short-term gain from deviating is smaller than the long-term loss from Bertrand competition in market  $B$ . For Firm  $M$ , the long-term benefit from collusion comes from the fact that without collusion, Firm  $M$  would only extract surplus from the group of consumers (of total mass  $b$ ) who want



both products, earning per-period profits equal to  $bV_1$ , while collusion allows it to extract the entire surplus of a population of mass  $\alpha$ , earning  $b(V_1 - V_2) + \alpha V_2 = bV_1 + (\alpha - b)V_2$  per period. If it deviates, Firm  $M$  earns  $bV_1 + (1 - b)V_2$  during one period and  $bV_1$  in all subsequent periods. With a rate of time preference  $\delta$ , the condition for Firm  $M$  not to deviate is thus

$$\alpha - b \geq (1 - b)(1 - \delta). \quad (1)$$

Similarly, Firm  $E$  earns a per-period profit of  $(1 - \alpha)V_2$  if no firm ever deviates, while if it deviates it earns  $V_2$  for one period and zero ever after. The condition for Firm  $E$  not to deviate is thus

$$1 - \alpha \geq 1 - \delta. \quad (2)$$

Collusion with a division of market  $B$  according to the proportions  $(\alpha, 1 - \alpha)$  is feasible only if conditions (1) and (2) hold, i.e.

$$1 - (1 - b)\delta \leq \alpha \leq \delta. \quad (3)$$

**Case 2:** Firm  $M$  chose to tie products  $A$  and  $B$  in period 0.

Just as in Case 1, the lemma means that a hypothetical collusive equilibrium is equivalent to one characterized by the following "tit-for-tat" strategies for some  $\alpha$  between zero and one (denoting the share of the total market for product  $B$  attributed to Firm  $M$ , including the part that is tied to market  $A$ ):

- If no deviation occurred in any past period (i.e. if the actions described hereafter have been played by both firms in all previous periods) then both firms play as follows:
  - Firm  $M$  sets the price of the  $A - B$  pair at  $V_1$  and offers product  $B$  at price  $V_2$ . It announces that it wants to sell at most  $\alpha - b$  of product  $B$  on a standalone basis (amounting to total sales of product  $B$  equal to  $\alpha$  once the sales of the  $A - B$  pair are accounted for).
  - Firm  $E$  offers product  $B$  at price  $V_2$ . It announces that it wants to sell at most  $1 - \alpha$  of product  $B$ .

- If either firm deviated from the above-mentioned actions in any previous period, then
  - Firm  $M$  sets the price of the  $A - B$  pair at  $V_1$  and offers product  $B$  at price zero.
  - Firm  $E$  offers product  $B$  at price zero.

The incentives facing Firm  $M$  are the same as in the previous case: not deviating is optimal if condition (1) holds. But the incentives facing Firm  $E$  are different from those arising in the previous case: if it deviates by slightly lowering its price, Firm  $E$  captures the entire set of consumers who want only product  $B$  (their total mass is  $1 - b$ ), but no more. As a consequence, the one-period profit from deviating is not  $V_2$  anymore, but  $(1 - b)V_2$  instead. The condition for Firm  $E$  not to deviate when Firm  $M$  tied both products is thus

$$1 - \alpha \geq (1 - b)(1 - \delta). \quad (4)$$

Collusion with a division of market  $B$  according to the proportions  $(\alpha, 1 - \alpha)$  is feasible only if conditions (1) and (4) both hold, i.e.

$$1 - (1 - b)\delta \leq \alpha \leq \delta + b(1 - \delta). \quad (5)$$

## 3.2 Main results

### 3.2.1 Tying in order to facilitate collusion

Conditions (3) and (5) yield the following result.

**Proposition 1** *If the rate of time preference  $\delta$  is greater than  $\frac{1}{2}$  but smaller than  $\frac{1}{2-b}$ , then collusion is possible only if Firm  $M$  decides to tie products  $A$  and  $B$  before the repeated pricing game took place.*

This proposition is in accordance with the intuition explained in the introduction: tying decreases the profitability of deviation for Firm  $E$  because it insulates the tied part of market  $B$  from competition. This in turn makes it easier to deter it from deviating from collusive equilibria.

### 3.2.2 Tying in order to increase market share in a collusive market

Even in situations in which collusion would be feasible both with and without tying (i.e. when both conditions (3) and (5) hold for some values of  $\alpha$ ), tying may be profitable because it increases the maximal market share which can be allocated to Firm  $M$  in a collusive equilibrium. In order to see this, we simply assume that after the decision whether to tie is made, firms bargain over the choice of a collusive equilibrium (i.e., they bargain over  $\alpha$ ), without making any side payments - this last assumption is very common in the analysis of collusion, because side payments are often more difficult to conceal than price-fixing or market share allocation.

We assume that firms coordinate on a collusive outcome using Nash bargaining, i.e. they choose an equilibrium maximizing  $\theta\Pi_M + (1 - \theta)\Pi_E$ , where  $\Pi_i$  denotes the expected discounted sum of Firm  $i$ 's profits, and the parameter  $\theta$  denotes Firm  $M$ 's bargaining power relative to Firm  $E$ . The above characterization of feasible market share allocations in collusive equilibria (conditions (3) and (5) above, corresponding respectively to the case without and with tying) yields the following result.

**Proposition 2** *If  $\frac{1}{2-b} < \delta < \theta$ , then collusion is feasible both without and with tying, but tying causes the tying firm's market share to rise from  $\delta$  to  $\text{Min}(\theta, \delta + b(1 - \delta))$ , and its profit to increase by  $\text{Min}(\theta - \delta, b(1 - \delta))V_2$ .*

This result is related to the familiar idea that tying may help a firm to profitably increase its market share in the market in which it does not hold a monopoly. If, absent tying, Firm  $M$ 's market share in the tied market is limited not by its bargaining power, but by the need to leave Firm  $E$  a large enough market share (so as to deter it from deviating), then tying allows Firm  $M$  to increase its market share, and thus its profit, because it relaxes Firm  $E$ 's incentive compatibility constraint.

## 4 Bundling and collusion

We assumed so far that the choice whether to tie products A and B was made once and for all before competitive interaction takes place. This restriction means that the above results, stated as such, can only apply to situations involving an irreversible (or at least costly enough) technical tie, or when a credible commitment to bundle (economically equivalent to a technical tie)

is possible. However, many real-world situations involve mere commercial bundling, which often is easily reversible. We show in this section that the above results extend to commercial bundling without commitment.

We modify the game in order to model a situation in which the choice whether to bundle products  $A$  and  $B$  is made not once and for all before competitive interaction takes place, but rather in every period, simultaneously as decisions over prices. More precisely, we assume that in every period, both firms simultaneously make and announce the following choices:

- Firm  $M$  announces which products it wants to sell (i.e.  $A$  and  $B$ , or the  $A - B$  pair and  $B$ ) and at what price, as well as the maximum amount of product  $B$  it is willing to sell should both firms offer it at the same price.
- Firm  $E$  sets the price at which it sells product  $B$  as well as the maximum amount of product  $B$  it is willing to sell should both firms offer it at the same price.

One can easily check that, if parameters induce the existence of a collusive equilibrium with tying under the assumptions of the previous section, then an equilibrium with bundling in every period and exactly the same prices and market shares exists under the assumptions of the previous section. Consider the following strategies:

- If no deviation occurred in any past period (i.e. if the actions described hereafter have been played by both firms in all previous periods) then both firms play as follows:
  - Firm  $M$  decides to offer only an  $A - B$  bundle priced  $V_1$  as well as product  $B$  priced  $V_2$ , and it announces that it wants to sell at most  $\alpha - b$  of product  $B$  when sold alone (amounting to total sales of product  $B$  equal to  $\alpha$  once the sales of the  $A - B$  pair are accounted for).
  - Firm  $E$  offers product  $B$  at price  $V_2$  and announces that it wants to sell at most  $1 - \alpha$  of product  $B$ .
- If either firm deviated from the above-mentioned actions in any previous period, then
  - Firm  $M$  offers product  $A$  at price  $V_1$ , and product  $B$  at price zero.
  - Firm  $E$  offers product  $B$  at price zero.

The conditions for deviation to be profitable are exactly the same as in the game analyzed in the previous section, assuming that in period 0 Firm  $M$  had decided to tie both products. Indeed, if Firm  $E$  deviates, its profit during the deviation period is  $(1 - b) V_2$  rather than  $V_2$ , because bundling by Firm  $M$  prevents Firm  $E$  from selling anything to a mass  $b$  of consumers.

The fact that bundling may arise in equilibrium is not in itself very interesting, because equilibrium multiplicity in repeated games is a general phenomenon. The interesting result is that the possibility of bundling (i) increases the set of parameters for which collusion is feasible, and (ii) increases the set of market share divisions which are compatible with a collusive equilibrium. Indeed, the above reasoning implies that the results of the previous section carry over to the present one: the word "tying" can be replaced by "bundling" in Propositions 1 and 2.

## 5 Conclusion

This paper shows that bundling or tying may be a profitable strategy because it may facilitate collusion in the tied market or it may allow the tying firm to alter market share allocation in the collusive tied market. Contrary to much of the recent literature, the anticompetitive use of bundling or tying is possible even in the absence of economies of scale or scope in the tied market. The only conditions for the above model to be relevant is that collusion be feasible in the tied market, i.e. in particular that this market be transparent enough, that quick retaliation be feasible, and that entry be difficult enough to make the collective exercise of market power possible. How relevant this simple model is to real markets is an issue left for future empirical research.

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## 7 Appendix

### Proof of the lemma.

In order to simplify notations, we consider the case where  $b = 0$ , which avoids the need to consider tying; the proof easily carries over to the case  $b > 0$  (the important assumption is that each firm can guarantee itself at least its one-shot Bertrand profit whatever its rivals' strategies are). Consider a hypothetical collusive equilibrium (i.e. an equilibrium such that aggregate expected profits are strictly positive) which is Pareto-optimal (from the firms' viewpoint) within the set of subgame perfect equilibria. We introduce the following notations:

- $U_{it}$  denotes the profit earned in equilibrium by agent  $i$  (i.e. Firm  $M$  or  $E$ ) in period  $t$ .

- $V_{it} = (1 - \delta) \times \sum_{t^0 \geq t+1} \delta^{t^0-t-1} U_{it^0}$  denotes agent  $i$ 's average equilibrium per-period profit from period  $t + 1$  on.
- $p_t$  denotes the equilibrium price in the period  $t$ .
- $P_t$  denotes the average equilibrium price in periods  $t + 1$  and after, defined as  $P_t = (1 - \delta) \times \sum_{t^0 \geq t+1} \delta^{t^0-t-1} p_{t^0}$
- $\alpha_{it}$  denotes Firm  $i$ 's equilibrium market share in period  $t$ .

These notations imply the identities  $\alpha_{Mt} + \alpha_{Et} = 1$  and  $U_{it} = \alpha_{it} p_t$ . We can apply the results about optimal penal codes derived in Abreu (1988): all equilibria are equivalent (in terms of payoffs) to equilibria such that out-of-equilibrium strategies involve optimal punishments. In the simple case of price competition with identical marginal cost, the optimal punishment is simply to play the strategies corresponding to the equilibrium of the one-shot game, because each player can guarantee itself the corresponding payoffs (i.e., zero) whatever the other player's strategy. We thus limit our attention to equilibria such that after any deviation, both players forever play the strategies corresponding to the one-shot game (i.e., they set a zero price).

In equilibrium, no firm has an incentive to deviate from its equilibrium strategy by slightly undercutting its rival. Therefore, for  $i = E$  and  $M$ :

$$U_{jt} = (1 - \alpha_{it}) p_t \leq \frac{\delta}{1 - \delta} V_{it}. \quad (\text{IC}_{it})$$

We prove first that for every  $t$ ,  $p_t$  is equal to the monopoly price  $V_2$ . Assume that this is not the case, and that for some  $t_0$ ,  $p_{t_0} < V_2$ .

Two cases must be distinguished, according to whether  $p_{t_0}$  is smaller or greater than  $P_{t_0}$ .

First case:  $p_{t_0} \geq P_{t_0}$ . We define

$$\bar{p} = (1 - \delta) p_{t_0} + \delta P_{t_0}$$

and  $\bar{\alpha}_i$  by the equality

$$\bar{\alpha}_i \bar{p} = (1 - \delta) \times \sum_{t \geq t_0} \delta^{t-t_0} \alpha_{it} p_t.$$

These definitions imply in particular that  $\bar{\alpha}_M + \bar{\alpha}_E = 1$ . We consider now the following change to the original equilibrium strategies: in period  $t$  and in all subsequent periods,

Firm  $i$  announces a price  $\bar{p}$  and a market share  $\bar{\alpha}_i$ ; out-of-equilibrium strategies are the same as in the original equilibrium. Clearly, these new strategies do not affect conditions  $(IC_{it})$  for  $t < t_0$  since they leave each firm's expected discounted payoff between  $t_0$  and infinity unchanged. In order to check whether these strategies can be equilibrium strategies, it is necessary to check whether the condition  $(IC_{it})$  still holds for  $t \geq t_0$ . This condition is equivalent to

$$\bar{\alpha}_i \geq 1 - \delta. \quad (6)$$

But  $(IC_{it_0})$  implies that

$$\bar{\alpha}_i \bar{p} \geq (1 - \delta)p_{t_0},$$

implying that (6) holds, because the inequality  $p_{t_0} \geq P_{t_0}$  implies that  $\bar{p} \leq p_{t_0}$ . Also,  $\bar{p} > 0$  : if the equilibrium under consideration yields strictly positive aggregate profits, then it is not possible that all prices be equal to zero after a certain period (otherwise by backward induction prices could be shown to be zero in all periods), so that  $P_{t_0} > 0$ , and thus  $\bar{p} > 0$ . Consider now the following modified strategies: in every period  $t$ , Firm  $i$  announces a price  $V_2$  and a market share  $\bar{\alpha}_i$ . Clearly, these strategies are compatible with an equilibrium: if  $t \geq t_0$ ,  $(IC_{it})$  still holds since the left-hand and right-hand term are both multiplied by  $V_2/\bar{p}$ ; if  $t < t_0$ ,  $(IC_{it})$  still holds since the left-hand side is unchanged while the right-hand side increases. But the corresponding equilibrium yields each firm a greater profit than the original one, since the average price after period  $t_0$  is multiplied by  $V_2/\bar{p}$ , which is greater than one (since  $p_{t_0} < V_2$ ). Therefore the initial equilibrium could not be optimal from the point of view of firms.

Second case:  $p_{t_0} < P_{t_0}$ . Notice first that  $(IC_{Et_0})$  and  $(IC_{Mt_0})$  are necessarily both binding: if  $(IC_{it_0})$  were not binding, a small price increase in period  $t_0$ , together with a market share reallocation leaving Firm  $i$ 's profit unchanged (and thus slightly increasing Firm  $j$ 's) would be a Pareto improvement and would leave all the  $(IC_{it})$  and  $(IC_{jt})$  constraints satisfied. Therefore,

$$(1 - \alpha_{Et_0})p_{t_0} = \frac{\delta}{1 - \delta}V_{Et_0}$$

and

$$(1 - \alpha_{Mt_0})p_{t_0} = \frac{\delta}{1 - \delta}V_{Mt_0},$$

implying that

$$p_{t_0} = \frac{\delta}{1 - \delta}(V_{Et_0} + V_{Mt_0}) = \frac{\delta}{1 - \delta}P_{t_0}$$

and thus that

$$\frac{\delta}{1 - \delta} = \frac{p_{t_0}}{P_{t_0}} < 1.$$



But conditions  $(IC_{Et})$  and  $(IC_{Mt})$  imply that for every  $t$ ,  $(U_{Et} + U_{Mt}) < \frac{\delta}{1-\delta}(V_{Et} + V_{Mt})$ , or  $(V_{Et} + V_{Mt}) > \frac{1-\delta}{\delta}(U_{Et} + U_{Mt})$ . Since  $(V_{Et} + V_{Mt})$  is a weighted average of all values of  $(U_{Et^0} + U_{Mt^0})$  in periods  $t' \geq t + 1$ , this implies that for any  $t$ , there exists  $t' > t$  such that  $\frac{U_{Et^0} + U_{Mt^0}}{U_{Et} + U_{Mt}} \geq \frac{1-\delta}{\delta} > 1$ . This implies that there exists a sequence  $t_n$  such that  $U_{Et_n} + U_{Mt_n} \geq (U_{Et_0} + U_{Mt_0}) \left(\frac{1-\delta}{\delta}\right)^n = p_{t_0} \left(\frac{1-\delta}{\delta}\right)^n$ , which tends toward infinity as  $n$  does. This is clearly a contradiction, since  $U_{Et} + U_{Mt}$  is bounded above by the one-period monopoly profit  $V_2$ .

This proves that the price in any optimal collusive equilibrium is equal to  $V_2$  in every period. Then, let us define  $\bar{\alpha}_i$  by the equality

$$\bar{\alpha}_i = (1 - \delta) \sum_{t \geq 0} \delta^t \alpha_{it}.$$

Condition  $(IC_{i0})$  is equivalent to  $\bar{\alpha}_i \geq 1 - \delta$ , which implies that there exists an equilibrium such that in every period the price is equal to  $V_2$  and market shares are  $\bar{\alpha}_M, \bar{\alpha}_E$ . This equilibrium yields the same payoffs as the original one. This completes the proof: an optimal collusive equilibrium is such that the price is equal to the short-run monopoly price in every period; there necessarily exists an equilibrium yielding the same payoffs and characterized by constant market shares, and such that any deviation is followed by the repetition of the short-run the Bertrand equilibrium.