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Savings behavior with imperfect capital markets: when hyperbolic discounting leads to discontinuous strategies

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Abstract

This paper provides a detailed study of a simple life-cycle consumption model with quasi-hyperbolic discounting and an imperfect financial market. It gives a complete characterization of savings behaviors. The joint assumptions of quasi-hyperbolic discount factors and no-borrowing constraints may lead to non-convexities in selves' objective functions that may imply discontinuous equilibrium strategies. Savings function may undergo jumps and non-monotonicities when the income or the interest rate reach a threshold value. These "anomalies" may exist even for reasonable parameters values.

JEL classification: D03, D91

Keywords: quasi-hyperbolic preferences, no-borrowing constraint, discontinuous strategies.

1 Introduction

Laibson (1997) analyzes the decisions of a hyperbolic consumer who has access to imperfect capital markets. He points out that some restrictions on parameters (interest rate and labor income) are necessary to use marginal conditions to characterize the equilibrium strategies. Otherwise, it is possible that selves face a "nonconvex reduced-form choice set" that may generate discontinuous equilibrium strategies. Therefore, he restricts the set of parameters in order to eliminate this problem.

Some other studies have considered similar frameworks in which the same properties of nonconvexity may occur (e. g. Laibson (1998), Harris and Laibson (2001)). But they do not focus on the occurrence of these discontinuous strategies. Harris and Laibson (2002) provide the most detailed study of this question. They give an intuition of such strategies. They present the results of numerical simulations, using for labor income a random variable generated by a shift symmetric Beta density. The parameter β being the bias for the present of the hyperbolic consumer (with $\beta \leq 1$), and ρ the relative risk aversion parameter, they show that irregularities in the consumption function tend to disappear when β is close to 1 and ρ is high enough. They conclude that discontinuous strategies do not arise when the model is calibrated with empirically sensible parameter values.

The aim of this paper is to challenge Harris and Laibson's conclusions, in showing that discontinuous equilibrium strategies should not be neglected too soon. Firstly, their conclusions were obtained by numerical simulations with a particular law on labor-income. One aim of this paper is to go beyond these simulation results, by a detailed study of consumption and savings behavior in a simple framework that allows a complete characterization of such strategies.

Secondly, Harris and Laibson's results are not very surprising. If β is close to 1 or if ρ is high, the behavior with hyperbolic discounting is very close to the behavior with exponential discounting. When hyperbolic discounting does not matter, it is not surprising to find that discontinuities vanish. On the other hand, if hyperbolic discounting actually modifies agents' behaviors, it is also associated with discontinuous equilibrium strategies. The value of β may also depend on the time duration of the period, and therefore on the frequency of agents' decisions. If the time separating two decisions is long, it can be pertinent to assume that β is low. In this case, hyperbolic discounting should have a great effect on behaviors. If the length of the period is short, β should be close to 1 and hyperbolic discounting should have a low impact.

Thirdly, discontinuous equilibrium strategies can be interesting, as they

provide an example of a behavior that qualitatively differs from the standard model. In the standard model with exponential discount factors, consumption behaviors are continuous functions of the different parameters: interest rates, labor incomes, etc. A consequence of hyperbolic discounting is that a continuous change of one parameter can induce a jump in consumption and savings. Therefore, the two models of consumption and savings lead to results with qualitative differences. They are not observationally equivalent.

This paper considers a simple framework to obtain a complete characterization of solutions: a three-period quasi-hyperbolic discounting model of life-cycle consumption. At each period the consumer earns some revenue. She/he has access to a financial market with a given interest rate on savings. The financial market is imperfect in the sense that borrowing is not possible. In periods 1 and 2, selves 1 and 2 strategically choose their amount of savings. The paper studies the Nash equilibrium of this game in which self 1 plays first and knows the best response function of self 2. In the literature, this equilibrium is often named the "sophisticated solution". From the assumption of quasi-hyperbolic discounting, the discount factor between period 2 and period 3 is higher for self 1 than for self 2. This discrepancy in preferences may cause a non-concavity of the objective function of self 1, when it is associated with the non-borrowing constraint. For some values of incomes and interest rate, it is possible that self $2'$s decision to save or not depends on the amount of savings chosen by self 1. At the limit value of self $1'$s savings for which self $2'$s savings cancel out, the derivative of self $1'$s objective undergoes a positive jump. This jump results from the discrepancy between the preferences of the two selves. In this case, self $1'$s objective function may have two local maxima that must be compared to find the Nash equilibrium. When some parameter of income or interest rate changes, it is possible that the best strategy of self 1 undergoes a jump from one local maximum to the other one.

These results have important consequences on the properties of the resulting savings functions. Some examples are provided for which savings functions can be non-monotonic and may undergo jumps. Initially, the benchmark calibration of Harris and Laibson (2002) is used. Considering first period savings with respect to first and second period incomes, it is shown that savings function may undergo a jump upward for some threshold value of incomes. After the jump, there is a significative change in the propensity to save with respect to incomes. Harris and Laibson's calibration considers an elasticity of substitution equal to 0.5 (a relative risk aversion parameter equal to 2). Increasing the elasticity of substitution allows us to obtain higher jumps in savings function, as the consumer has less taste for consumption

smoothing.

Section 2 presents the model and gives the basic results in the case without non-convexity problems. Section 3 studies the model when the non-borrowing constraint and the quasi-hyperbolic discounting assumption create non-convexity problems. Section 4 gives an overall picture of the results and point out some consequences on savings functions. Section 5 concludes, and Section 6 presents the technical Appendixes.

2 The model

2.1 Basic assumptions

The model is simple, and is restricted to a three-period quasi-hyperbolic discounting model. In period 1, self 1 preferences are given by the utility function:

$$U(c,d,e) = u(c) + \beta\delta u(d) + \beta\delta^2 u(e)$$

with

$$u(x) = \frac{x^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}} \tag{1}$$

and $\sigma > 0$. c, d and e are respectively the consumption levels in period 1, 2 and 3.

In period 2, self 2's preferences are given by the utility function:

$$u(d) + \beta\delta u(e)$$

The budget constraints are:

$$\begin{aligned} c + x &= A \\ d + y &= B + Rx \\ e &= C + Ry \end{aligned}$$

A, B and C are respectively the income amounts earned in period 1, 2 and 3. x and y are first and second period savings. R is the factor of interest. β and δ are two positive coefficients not greater than 1. δ is the usual discount parameter and β is the bias for the present.

Capital markets are imperfect: $x \geq 0$ and $y \geq 0$. These constraints mean that the agent cannot be indebted[1].

[1]It could also represent a more general setting. Assume that the constraints follow

The discount factor between period 2 and period 3 is δ if it is computed by self 1, and $\beta\delta$ if it is computed by self 2. The parameter β indicates whether there is a self-control problem ($\beta < 1$) or not ($\beta = 1$). x is the decision variable of self 1 and y the decision variable of self 2. The sequence of decisions of the agent results from the equilibrium of the game between selves 1 and 2, in which self 1 plays first. Therefore, self 2 chooses y, x being taken as given. Self 1 chooses x, taking into account the best response function of self 2.

2.2 Self 2's behavior

The program of self 2 is:

$$\begin{cases} \max\limits_{(y)} u\left(B + Rx - y\right) + \beta\delta u\left(C + Ry\right) \\ \text{s. t. } y \geq 0 \end{cases}$$

The solution to this program can be interior ($y > 0$) or not. Defining the threshold

$$\tilde{x} \equiv \frac{1}{R}\left[\frac{C}{(R\beta\delta)^{\sigma}} - B\right] \tag{2}$$

the best response function of self 2 is:

$$Y(x) = \begin{cases} \frac{(R\beta\delta)^{\sigma}(B+Rx)-C}{R+(R\beta\delta)^{\sigma}} & \text{if } x \geq \tilde{x} \\ 0 & \text{if } x \leq \tilde{x} \end{cases} \tag{3}$$

Y is a non-decreasing function of x.

In the case $C \leq B\left(R\beta\delta\right)^{\sigma}$, y is always non-negative, even if $x = 0$. The optimal choice of y is not constrained by the constraint $y \geq 0$. In the opposite case $C > B\left(R\beta\delta\right)^{\sigma}$, it is possible that the optimal behavior without

the form: $x \geq -v$ and $y \geq -w$. In this case, with the change of variables: $x' \equiv x + v$, $y' \equiv y + w$, $A' \equiv A + v$, $B' \equiv B + w - Rv$, $C \equiv C' - Rw$, the budget constraints of the consumer can be written again:

$$\begin{aligned} c + x' &= A' \\ d + y' &= B' + Rx' \\ e &= C' + Ry' \end{aligned}$$

with the constraints: $x' \geq 0$ and $y' \geq 0$.

the constraint $y \geq 0$ leads to a negative optimal value for y. In this case the constraint is binding and $y = 0$. As a consequence, the choice of x by self 1 can determine if self 2 is constrained or not.

2.3 The case $C \leq B\left(R\beta\delta\right)^{\sigma}$

This case is simple to study as y is non-negative whatever the choice of x. This is the situation that is considered in general in the literature on savings with hyperbolic discounting.

The optimal behavior of self 1 is obtained in maximizing his objective function, taking into account the best response function of self 2 (3):

$$\begin{cases} \max\limits_{(x)} u\left(A-x\right)+\beta\delta u\left(B+Rx-Y(x)\right)+\beta\delta u\left(C+RY(x)\right) \\ \text{s. t. } x \geq 0 \end{cases}$$

Replacing $Y(x)$ by the equation (3), this program can be written after some calculations:

$$\begin{cases} \max\limits_{(x)} \frac{1}{1-\frac{1}{\sigma}}\left(A-x\right)^{1-1/\sigma}+\frac{1}{1-\frac{1}{\sigma}}Z\left(B+Rx+\frac{C}{R}\right)^{1-1/\sigma} \\ \text{s. t. } x \geq 0 \end{cases} \tag{4}$$

with

$$Z=\beta\delta\frac{\left[1+\delta^{\sigma}\left(R\beta\right)^{\sigma-1}\right]}{\left[1+R^{\sigma-1}\left(\beta\delta\right)^{\sigma}\right]^{1-1/\sigma}} \tag{5}$$

The solution is:

- If $R^{\sigma}Z^{\sigma}A > B+C/R$

$$x=\frac{R^{\sigma}Z^{\sigma}A-B-\frac{C}{R}}{R+R^{\sigma}Z^{\sigma}}\equiv\hat{x} \tag{6}$$

- If $R^{\sigma}Z^{\sigma}A \leq B+C/R$, $x=0$.

Using the optimal value of x (either $\hat{x}$ given by (6) or 0), it is possible to find the equilibrium value of y given by $Y(x)$ (cf. equation (3)).

3 Resolution in the case $C > B\left(R\beta\delta\right)^{\sigma}$

3.1 The non-concavity of self 1's objective function

The case $C > B\left(R\beta\delta\right)^{\sigma}$ is more interesting as it is possible that the optimal choice of y is positive or 0, depending on the choice of x. The problem of the non-concavity of the objective function of self 1 may occur. To see this, let us consider the objective function of self 1, taking into account the optimal response of self 2.

$$\begin{cases} \max\limits_{(x)} u\left(A-x\right)+\beta\delta u\left(B+Rx-Y(x)\right)+\beta\delta^{2}u\left(C+RY(x)\right) \\ \text{s. t. } x\geq 0 \end{cases}$$

Taking the derivative with respect to x of this objective function (with $x \neq \tilde{x}$), we obtain:

$$u'(c)+R\beta\delta u'(d)+Y'(x)\beta\delta\left[-u'(d)+R\delta u'(e)\right] \tag{7}$$

If the consumer had no problem of self control ($\beta = 1$), the expression $[-u'(d)+R\delta u'(e)]$ would correspond to the derivative of self 2's objective function. This term cancels out ($-u'(d)+R\delta u'(e) = 0$) along the best response function of self 2. This is a simple consequence of the envelope theorem. Therefore, the discontinuity of the derivative Y' in $\tilde{x}$ is not a problem and the derivative of self 1's objective function (7) is continuous.

In the case of a quasi-hyperbolic consumer ($\beta < 1$), there is a discrepancy between the objective functions of self 1 and self 2. From the first order condition of self 2, it is obtained that $-u'(d)+\beta R\delta u'(e) = 0$, which implies $-u'(d)+R\delta u'(e) > 0$. Moreover, for $x = \tilde{x}$, the derivative $Y'(x)$ is discontinuous: the derivative is 0 to the left of $\tilde{x}$, and is positive to the right.

The consequence of this analysis is that the derivative of self 1's objective function is discontinuous at the point $\tilde{x}$, with a higher value to the right of $\tilde{x}$. It is then possible that self 1's objective function admits two local maxima. The function is concave on each interval $(0,\tilde{x})$ and $(\tilde{x},+\infty)$ and continuous, but the derivative is discontinuous in $\tilde{x}$.

When the objective function of self 1 has two local maxima, one on the interval $(0,\tilde{x})$ and one on $(\tilde{x},+\infty)$, it is necessary to compare the values of the function at these two extrema to find its optimal behavior.

Figure 1 presents a simple numerical simulation with two local maxima for the following values of parameters: $\beta = 0.7$, $\delta = 0.9571$, $R = 1.0375$, $\sigma = 0.5$, $A = 12$ and $C = 6$. Depending on the value of B, the optimal value of first period savings x can imply either $y = 0$ or $y > 0$. $B = 3.28565$ is the threshold value such that the two local maxima are equal.

3.2 Self 1's objective function

In this part the objective function of self 1 is studied with respect to x, self 1's decision variable. Two cases are considered: on the interval $(\tilde{x}, +\infty)$, $Y(x)$ is positive; on the interval $(0, \tilde{x})$, $Y(x)$ cancels out. Finally, the solution of the game between selves 1 and 2 is found.

3.2.1 Self 1's objective function on $(\tilde{x}, +\infty)$

First the solution is assumed to be such that $y > 0$, or $(B + Rx)(R\beta\delta)^{\sigma} > C$. In this case, the value of x chosen by self 1 is again the solution of program (4). It has been shown that program (4) can lead either to a positive value of x given by (6), or to $x = 0$. As $C > B(R\beta\delta)^{\sigma}$, the solution $x = 0$ is not compatible with the assumption $y > 0$, because $x = 0$ implies $y = 0$. Therefore, to obtain the case $y > 0$ as a solution, it is necessary that $R^{\sigma} Z^{\sigma} A > B + C/R$, and x is given by (6). Moreover, x being given by (6), to obtain a positive value for y, the study of self 2's behavior has shown that it is necessary that $\hat{x} > \tilde{x}$. Using (2) and (6), the inequality $\hat{x} > \tilde{x}$ leads to:

$$A + \frac{B}{R} > \frac{C}{R^{\sigma} Z^{\sigma} (R\beta\delta)^{\sigma}} \left[1 + R^{\sigma-1} Z^{\sigma} + R^{\sigma-1} (\beta\delta)^{\sigma}\right] \equiv K \tag{8}$$

When $C > B(R\beta\delta)^{\sigma}$, as $\tilde{x} > 0$ the inequality $\hat{x} > \tilde{x}$ is stronger than $\hat{x} > 0$. Therefore, if (8) is true, the condition $R^{\sigma} Z^{\sigma} A > B + C/R$ holds.

To sum up, if (8) is satisfied, self 1's objective function has a local maximum $\hat{x} \in (\tilde{x}, +\infty)$ given by (6), with $Y(\hat{x}) > 0$. If (8) is not satisfied, self 1's objective function is decreasing on $(\tilde{x}, +\infty)$, as the maximum of the objective function is obtained for a value of x smaller than $\tilde{x}$.

3.2.2 Self 1's objective function on $(0, \tilde{x})$

The second type of solution is such that $y = 0$. In this case, self 1's program is given by:

$$\begin{cases} \max\limits_{(x)} \frac{1}{1-1/\sigma} (A - x)^{1-1/\sigma} + \frac{\beta\delta}{1-1/\sigma} (B + Rx)^{1-1/\sigma} + \frac{\beta\delta^2}{1-1/\sigma} (C)^{1-1/\sigma} \\ \text{s. t. } x \geq 0 \end{cases}$$

The solution is:

- If $(R\beta\delta)^{\sigma} A \geq B$

$$x = \frac{(R\beta\delta)^{\sigma} A - B}{R + (R\beta\delta)^{\sigma}} \equiv \check{x} \tag{9}$$

- If $(R\beta\delta)^{\sigma} A < B$, $x = 0$. The objective function is decreasing on the interval $(0, \tilde{x})$, as the maximum is reached for a negative value of x.

In the case $(R\beta\delta)^{\sigma} A \geq B$, the value of x given by (9) is admissible only if it belongs to $(0, \tilde{x})$. In the converse case, y should be positive.

Using (9) and (2), the inequality $\check{x} < \tilde{x}$ leads to

$$A + \frac{B}{R} < \frac{C}{(R\beta\delta)^{2\sigma}} \left[1 + R^{\sigma-1} (\beta\delta)^{\sigma}\right] \equiv L \tag{10}$$

To sum up, if $(R\beta\delta)^{\sigma} A \geq B$ and if (10) is satisfied, self 1's objective function has a local maximum $\check{x} \in (0, \tilde{x})$ given by (9), with $Y(\check{x}) = 0$. If (10) is not satisfied, it means that self 1's objective function is increasing on the interval $(0, \tilde{x})$, as the maximum is obtained for a value of x greater than $\tilde{x}$.

If $(R\beta\delta)^{\sigma} A < B$, self 1's objective function is decreasing on the interval $(0, \tilde{x})$.

3.3 The solution of the game between selves 1 and 2

From (8) and (10), it is straightforward to check that $K < L$ as $\beta < 1$. Therefore, the comparison of the two preceding types of solution leads to the following results.

3.3.1 Solution in the case $A + B/R \leq K$

If $A + B/R \leq K$, condition (8) is not satisfied but (10) holds. From the preceding analysis, it is known that self 1's objective function is decreasing on the interval $(\tilde{x}, +\infty)$ ((8) does not hold). On the interval $(0, \tilde{x})$, as (10) holds, two cases may happen: If $(R\beta\delta)^{\sigma} A \geq B$, self 1's objective function reaches a local maximum in $x = \check{x} > 0$ given by (9). If $(R\beta\delta)^{\sigma} A < B$, self 1's objective function is decreasing on the interval $(0, \tilde{x})$. From these properties, the result follows:

Proposition 1 *If $C > B(R\beta\delta)^{\sigma}$ and $A + B/R \leq K$, the solution of the game between selves 1 and 2 is:*

- *if $(R\beta\delta)^{\sigma} A \geq B$, $x = \check{x}$ given by (9) and $y = 0$;*
- *if $(R\beta\delta)^{\sigma} A < B$, $x = 0$ and $y = 0$.*

These first results show that the intertemporal wealth of the consumer during the two first periods $(A + B/R)$ plays a crucial role in savings. If this wealth is low and if C is high enough, self 1 has no interest in saving an amount x allowing self 2 to save a positive amount y. Expecting $y = 0$, the savings choice of self 1 is positive if A is relatively high with respect to B and zero otherwise.

3.3.2 Solution in the case $A + B/R \geq L$

If $A + B/R \geq L$, condition (8) holds and condition (10) is not satisfied:

$$A + \frac{B}{R} \geq \frac{C}{(R\beta\delta)^{2\sigma}} \left[1 + R^{\sigma-1} (\beta\delta)^{\sigma}\right]$$

By assumption, $C > B (R\beta\delta)^{\sigma}$. These two inequalities imply that $(R\beta\delta)^{\sigma} A > B$. Therefore it is possible to deduce in this case that:

- on the interval $(0, \tilde{x})$, self 1's objective function is increasing on the interval $(0, \tilde{x})$.
- on the interval $(\tilde{x}, +\infty)$, self 1's objective function has a maximum $\hat{x}$.

Finally, the result is obtained:

Proposition 2 *If $C > B (R\beta\delta)^{\sigma}$ and $A + B/R \geq L$, the solution of the game between selves 1 and 2 is: $x = \hat{x}$ and $y = Y(\hat{x})$.*

In this case, the intertemporal wealth of the consumer during the two first periods $(A + B/R)$ is high. The optimal behavior of self 1 is to save an amount x high enough to allow self 2 to save a positive amount y.

3.3.3 Solution in the case $K < A + B/R < L$

When $A + B/R < L$, the preceding study (section 3.2.2) has shown that two cases may happen, depending on the sign of $(R\beta\delta)^{\sigma} A - B$. Therefore, these two cases are studied separately.

First case: $(R\beta\delta)^{\sigma} A \geq B$. In this case, previous analysis have shown that self 1's objective function has two local maxima: on $(0, \tilde{x})$, $\check{x}$ given by (9) is a local maximum, and $Y(\check{x}) = 0$; on $(\tilde{x}, +\infty)$, $\hat{x}$ given by (6) is another one, with $Y(\hat{x}) > 0$. To know what is the final choice of self 1, it is necessary to compare the value of the objective function at these two points $\check{x}$ and $\hat{x}$. When $x = \check{x}$ and $y = 0$, the corresponding values for consumption are denoted by $\check{c}$, $\check{d}$ and $\check{e}$ and the utility level of self 1 is $U(\check{c}, \check{d}, \check{e})$. When $x = \hat{x}$ and $y = Y(\hat{x})$, the corresponding values for consumption are denoted by $\hat{c}$, $\hat{d}$ and $\hat{e}$ and the utility level of self 1 is $U(\hat{c}, \hat{d}, \hat{e})$.

Lemma 1 • *The values of $U(\check{c}, \check{d}, \check{e})$ and $U(\hat{c}, \hat{d}, \hat{e})$ are given by*

$$\begin{aligned} U(\check{c}, \check{d}, \check{e}) &= \frac{\sigma}{\sigma - 1}\left(A + \frac{B}{R}\right)^{1-\frac{1}{\sigma}} \left[1 + R^{\sigma-1}(\beta\delta)^{\sigma}\right]^{\frac{1}{\sigma}} + \frac{\sigma}{\sigma - 1}\beta\delta^2 (C)^{1-\frac{1}{\sigma}} \\ U(\hat{c}, \hat{d}, \hat{e}) &= \frac{\sigma}{\sigma - 1}\left(A + \frac{B}{R} + \frac{C}{R^2}\right)^{1-\frac{1}{\sigma}} \left[1 + R^{\sigma-1} Z^{\sigma}\right]^{\frac{1}{\sigma}} \end{aligned}$$

- *Let us define the function F as:*

$$\begin{aligned} F(\xi) &= \frac{\sigma}{\sigma - 1}\left(\xi + \frac{C}{R^2}\right)^{1-\frac{1}{\sigma}} \left[1 + R^{\sigma-1} Z^{\sigma}\right]^{\frac{1}{\sigma}} \\ &\quad - \frac{\sigma}{\sigma - 1}(\xi)^{1-\frac{1}{\sigma}} \left[1 + R^{\sigma-1}(\beta\delta)^{\sigma}\right]^{\frac{1}{\sigma}} - \frac{\sigma}{\sigma - 1}\beta\delta^2 (C)^{1-\frac{1}{\sigma}} \end{aligned}$$

  F is defined in such a way that $F(A + B/R) = U(\hat{c}, \hat{d}, \hat{e}) - U(\check{c}, \check{d}, \check{e})$.

  F is strictly increasing on $[K, L]$, with $F(K) < 0$ and $F(L) > 0$.

Proof. See Appendix 6.1. ■

This technical lemma allows us to conclude on the optimal behavior of self 1.

Proposition 3 *Assume that $C > B(R\beta\delta)^{\sigma}$, $K < A + B/R < L$ and $(R\beta\delta)^{\sigma} A \geq B$. There exists $M \in (K, L)$ such that,*

- *if $K < A + B/R < M$, the game between selves 1 and 2 leads to the decisions $x = \check{x}$ and $y = 0$.*
- *if $M < A + B/R < L$ the game between selves 1 and 2 leads to the decisions $x = \hat{x}$ and $y = Y(\hat{x})$*
- *if $A + B/R = M$, self 1 is indifferent between the two solutions $x = \check{x}$ and $y = 0$, or $x = \hat{x}$ and $y = Y(\hat{x})$.*

Following propositions (1) and (2), this proposition shows that for $(R\beta\delta)^{\sigma} A \geq B$, the line $A + B/R = M$ is the pertinent frontier that separates the two types of savings behaviors: $x = \hat{x}$ and $y = Y(\hat{x})$ or $x = \check{x}$ and $y = 0$.

A consequence of this result is that, if $A + B/R$ is close to M, a small change in one parameter (A, or B or R) can have a dramatic effect on savings behavior: y can jump from 0 to $Y(\hat{x})$, and x can jump from $\check{x}$ to $\hat{x}$. This point will be developed later in Section 4.

Second case: $(R\beta\delta)^{\sigma} A < B$. In this case, it is not possible to reach the solution $x = \check{x}$, $y = 0$ associated with $(\check{c}, \check{d}, \check{e})$. Therefore, the equilibrium of the game between selves 1 and 2 is obtained by the comparison between $U(\hat{c}, \hat{d}, \hat{e})$ and $U(A, B, C)$. The last value of the utility corresponds to the solution $x = 0$ and $y = 0$.

Before doing this comparison, it is useful to note that the indirect utility

$$U(\check{c}, \check{d}, \check{e}) = \frac{\sigma}{\sigma - 1}\left(A + \frac{B}{R}\right)^{1-\frac{1}{\sigma}}\left[1 + R^{\sigma-1}(\beta\delta)^{\sigma}\right]^{\frac{1}{\sigma}} + \frac{\sigma}{\sigma - 1}\beta\delta^{2}(C)^{1-\frac{1}{\sigma}}$$

can also be interpreted as the utility that self 1 could get in the absence of the constraint $x \geq 0$ (but with the constraint $y \geq 0$ which is binding). Consequently, a first property is obtained: $U(\check{c}, \check{d}, \check{e}) > U(A, B, C)$. With no constraint on x, it would be optimal to have a negative amount of savings in period 1 as $(R\beta\delta)^{\sigma} A < B$.

In the case $M \leq A + B/R < L$, it is easy to conclude. Indeed, it is known that $U(\hat{c}, \hat{d}, \hat{e}) \geq U(\check{c}, \check{d}, \check{e})$ and $U(\check{c}, \check{d}, \check{e}) > U(A, B, C)$. Consequently $U(\hat{c}, \hat{d}, \hat{e}) > U(A, B, C)$. The game between selves 1 and 2 leads to the decisions $x = \hat{x}$ and $y = Y(\hat{x})$. This result is summarized in the following proposition:

Proposition 4 *If $C > B(R\beta\delta)^{\sigma}$, $(R\beta\delta)^{\sigma} A < B$ and $M \leq A + B/R < L$, the solution of the game between selves 1 and 2 is: $x = \hat{x}$ and $y = Y(\hat{x})$.*

The case $K < A + B/R < M$ needs a particular study as $U(\hat{c}, \hat{d}, \hat{e}) < U(\check{c}, \check{d}, \check{e})$ and $U(\check{c}, \check{d}, \check{e}) > U(A, B, C)$. Thus, it remains to compare $U(\hat{c}, \hat{d}, \hat{e})$ with $U(A, B, C)$.

The following proposition gives the results:

Proposition 5 *Assume that $C > B(R\beta\delta)^{\sigma}$, $(R\beta\delta)^{\sigma} A < B$ and $K < A + B/R < M$. There exists a decreasing function $\Delta(A)$ defined on $[A_H, A_I]$,*

with

$$\begin{aligned} A_H &= \frac{C}{(RZ)^{\sigma}(R\beta\delta)^{\sigma}}\left[1+R^{\sigma-1}(\beta\delta)^{\sigma}\right],\ \Delta(A_H)=\frac{C}{(R\beta\delta)^{\sigma}} \\ A_I &= \frac{M}{1+R^{\sigma-1}(\beta\delta)^{\sigma}},\ \Delta(A_I)=\frac{M(R\beta\delta)^{\sigma}}{1+R^{\sigma-1}(\beta\delta)^{\sigma}} \end{aligned}$$

such that,

- *if* $B<\Delta(A)$, $U(\hat{c},\hat{d},\hat{e})<U(A,B,C)$: *the game between selves 1 and 2 leads to the decisions* $x=0$ *and* $y=0$.
- *if* $B>\Delta(A)$, $U(\hat{c},\hat{d},\hat{e})>U(A,B,C)$: *the game between selves 1 and 2 leads to the decisions* $x=\hat{x}$ *and* $y=Y(\hat{x})$.
- *if* $B=\Delta(A)$, $U(\hat{c},\hat{d},\hat{e})=U(A,B,C)$: *self 1 is indifferent between these two solutions.*

Proof. See Appendix 6.2. ■

Propositions (4) and (5) allow us to conclude when $K<A+B/R<L$, $C>B(R\beta\delta)^{\sigma}$ and $(R\beta\delta)^{\sigma}A<B$. When the intertemporal wealth of the consumer during the two first periods is high enough $(A+B/R\geq M)$, the optimal choice for self 1 is $x=\hat{x}$ (and $y=Y(\hat{x})$ for self 2). When $A+B/R<M$, the choice of self 1 remains $x=\hat{x}$ only if B is high enough with respect to A ($B>\Delta(A)$). In the converse case, self 1 chooses $x=0$ and self 2 $y=0$.

4 Final results and consequences

4.1 An overall picture of the consumer's choice

For a given value of the third period income C, it is possible to obtain an overall characterization of the consumer's savings choices, using the preceding results. Four cases may arise, depending on A and B that allow us to define four sets:

$$Z_1=\{(A,B)\text{ s. t. }(R\beta\delta)^{\sigma}A>B\text{ and }A+B/R<M\}$$

$$\begin{aligned} Z_2 &= \{(A,B)\text{ s. t. }(R\beta\delta)^{\sigma}A<B,\ B<C(R\beta\delta)^{-\sigma}, \\ &\quad A+B/R<M\text{ and }B<\Delta(A)\text{ for }A\in(A_H,A_I)\} \end{aligned}$$

$$Z_3 = \{(A,B) \text{ s. t. } A > A_H,\ (RZ)^\sigma A > B + C/R,\ B > \Delta(A) \text{ for } A \in (A_H, A_I),\ A + B/R > M \text{ for } A > A_I\}$$

$$Z_4 = \left\{(A,B) \text{ s. t. } B > C\,(R\beta\delta)^{-\sigma} \text{ and } (RZ)^\sigma A > B + C/R\right\}$$

The preceding results give the consumer's choice in each set[2]:

- Set 1: $x = \check{x}$ given by (9) and $y = 0$.
- Set 2: $x = 0$ and $y = 0$.
- Set 3: $x = \hat{x}$ given by (6) and $y = Y(\hat{x})$ given by (3).
- Set 4: $x = 0$ and $y = Y(0)$ given by (3).

Figure (2) represents the characterization of the different zones of the plane (A, B) corresponding to the four sets. The main frontiers are in bold lines. The figure is obtained for the following values of the parameters: $\beta = 0.7$, $\delta = 0.9571$, $R = 1.0375$, $C = 6$ and $\sigma = 2$. The three first parameters are chosen according to the benchmark calibration of Harris and Laibson (2002). The value of σ is set equal to 2 in order to make more apparent the curve $\Delta(A)$. The numerical values have no incidence on the general form of the figure. In the next part, figures associated with $\sigma = 0.5$ and $\sigma = 0.99$ will be provided.

4.2 Some consequences of quasi-hyperbolic discounting

The assumption of quasi-hyperbolic discounting can have strong consequences on savings behaviors, when self 1's objective function is not concave. Different illustrations are provided, with different parameters values.

Initially, the benchmark calibration of Harris and Laibson (2002) is used: $\beta = 0.7$, $\delta = 0.9571$, $R = 1.0375$, $\sigma = 0.5$. The value of C is fixed to $C = 6$. Figure (3) presents the different zones for these parameter values. The impact of A and B on the amount of savings x is studied. In Figure

[2] To simplify the exposition, the frontiers have not been included in the 4 sets. Indeed, for some frontiers corresponding to a discontinuity in the optimal strategy, two different choices are equivalent for self 1, as it was established in the preceding results (see propositions (3) and (5)).

(4), x is represented as a function of B for two different values of $A : A = 12$ and $A = 8.16$[3]. With $A = 12$, starting from $B = 0$ and increasing B, the incomes of the consumer first belong to zone 1, and successively reach zone 3 and 4 (cf. Figure (3)). When B reaches the value such that $M = A + B/R$, x jumps from $x = \check{x}$ to $x = \hat{x} > \check{x}$. The slope of the savings function goes from -0.53 to -0.37.

With $A = 8.16$, as B increases, the incomes of the consumer belong to zone 1 at the beginning, and successively reach zone 2, 3 and 4 (cf. Figure (3)). In zone 1, $x = \check{x}$ and decreases with B till 0. In zone 2, x remains equal to 0. When crossing the frontier $\Delta(A)$, x jumps from 0 to a positive value $\hat{x}$. In zone 3, $x = \hat{x}$ and decreases with B till 0. In zone 4, x remains equal to 0.

The effect of A on savings for a given value of B is simulated for $B = 4$ (cf. Figure (5)). Starting from $A = 0$ and increasing A, consumer incomes successively cross zones 2, 1 and 3 (cf. Figure (3)). In zone 2, $x = 0$. In zone 1, $x = \check{x}$ and increases with A. When A reaches the value such that $M = A + B/R$, x jumps from $x = \check{x}$ to $x = \hat{x} > \check{x}$. The slope of the savings function goes from 0.44 to 0.61.

The same numerical simulations are done for $\sigma = 0.99$. The overall picture of frontiers is given in Figure (6). The impact of A and B on the amount of savings x is studied. In Figure (7), x is represented as a function of B for two different values of $A : A = 16$ and $A = 11.13$. The values of A are changed with respect to the preceding simulations, in order to obtain the same qualitative features, as the frontiers have moved. Figure (8) shows the evolution of x with respect to A when B is fixed to $B = 4$. The comparison with the case $\sigma = 0.5$ shows that increasing the elasticity of substitution allows us to obtain more dramatic jumps in savings functions. The same increase in the jumps could be obtained through a decrease of β.

In these simple examples, the evolution of savings x with respect to A and B has been considered. It is possible to make the same study with the other parameters. For instance, a jump of savings can also be obtained through a change in the interest factor R.

Our numerical simulations show that savings functions under quasi hyperbolic discounting present significative differences with respect to the standard case, when discontinuous strategies are taken into account, even for the benchmark parametrization of Harris and Laibson. Firstly, savings functions may undergo upward jumps, resulting from an increase in incomes. Secondly, after these jumps, the slope of savings with respect to income may go up or down.

[3] The value 8.16 corresponds to $(A_H + A_I)/2$.

These theoretical results may have important consequences for the empirical analysis of savings. If agents behave as predicted by the theory, savings functions should be estimated through functional forms able to deal with these jumps and changes in slopes.

5 Conclusion

In this paper, a simple life-cycle consumption model with quasi-hyperbolic discounting and imperfect capital markets has been studied. Using three periods of life for the consumer, it has been possible to reach a complete resolution and a characterization of savings behavior. It has been shown that the savings function may experience discontinuities when consumer incomes reach some threshold values. These discontinuities may arise even for "reasonable" values of the parameters. They show that quasi-hyperbolic discounting can lead to important differences in agents' behaviors, compared with the standard assumption of exponential discounting.

This paper could be extended in different directions. Firstly, a technical improvement could be made by introducing more than three periods and more than two decisions. Secondly, discontinuous strategies could be studied in more complex financial environment. For example, in an economy with one liquid and one illiquid asset, as in Laibson (1997), discontinuous strategies could exist as a consequence of the non-borrowing constraint on the two assets. Thirdly, discontinuous strategies could exist in other frameworks. For example, Wigniolle (2011) shows that discontinuous strategies can exist in the standard quantity-quality trade-off model of fertility with quasi-hyperbolic discounting. In the problem of the decision of retirement that is analyzed by Diamond and Koszegi (2003), discontinuous strategies could also play a role. In their article, they introduce at some period the retirement decision as a discrete variable. If the time spent working in this period and the retirement time were continuous variables, the existence of discontinuous strategies would become apparent.

6 Appendix

6.1 Proof of lemma (1).

The first part of lemma (1) is a straightforward calculation.

The second part is concerned with the study of the function $F(\xi) = U(\hat{c}, \hat{d}, \hat{e}) - U(\check{c}, \check{d}, \check{e})$ with $\xi = A + B/R$.

Firstly it is proved that $F(K) < 0$. The case $\xi = K$ corresponds to the limit case $\hat{x} = \tilde{x}$ when self 1's objective function is studied on $(\tilde{x}, +\infty)$. From Section 3.1, it is known that the derivative of self 1's objective function is discontinuous at the point $\tilde{x}$, with a higher value to the right of $\tilde{x}$. When $\hat{x} = \tilde{x}$ the value of the derivative on the right is 0. Therefore, the value of the derivative on the left is negative. This proves that $U(\hat{c}, \hat{d}, \hat{e}) < U(\check{c}, \check{d}, \check{e})$ or $F(K) < 0$.

Secondly it is proved that $F(L) > 0$. The case $\xi = L$ corresponds to the limit case $\check{x} = \tilde{x}$ when self 1's objective function is studied on $(0, \tilde{x})$. From Section 3.1, it is known that the derivative of self 1's objective function is discontinuous at the point $\tilde{x}$, with a higher value to the right of $\tilde{x}$. When $\check{x} = \tilde{x}$ the value of the derivative on the left is 0. Therefore, the value of the derivative on the right is positive. This proves that $U(\hat{c}, \hat{d}, \hat{e}) > U(\check{c}, \check{d}, \check{e})$ or $F(L) > 0$.

Finally, it remains to show that $F'(\xi) > 0$. The derivative is:

$$F'(\xi) = \left(\xi + \frac{C}{R^2}\right)^{-\frac{1}{\sigma}} \left[1 + R^{\sigma-1} Z^{\sigma}\right]^{\frac{1}{\sigma}} - (\xi)^{-\frac{1}{\sigma}} \left[1 + R^{\sigma-1} (\beta\delta)^{\sigma}\right]^{\frac{1}{\sigma}}$$

The condition $F'(\xi) > 0$, after some calculations, can be expressed:

$$\xi > C \frac{1 + R^{\sigma-1} (\beta\delta)^{\sigma}}{R^{\sigma-1} \left[Z^{\sigma} - (\beta\delta)^{\sigma}\right]} \tag{11}$$

where it is important to note that $Z^{\sigma} - (\beta\delta)^{\sigma} > 0$. Indeed, from (5), the condition $Z^{\sigma} > (\beta\delta)^{\sigma}$ is equivalent to:

$$\vartheta(\beta) \equiv \frac{\left[1 + \delta^{\sigma} (R\beta)^{\sigma-1}\right]^{\sigma}}{\left[1 + R^{\sigma-1} (\beta\delta)^{\sigma}\right]^{\sigma-1}} > 1$$

For $\beta = 1$, $\vartheta(1) = 1 + \delta^{\sigma} R^{\sigma-1} > 1$. The derivative is

$$\frac{d \ln \left[\vartheta(\beta)\right]}{d\beta} = \frac{\sigma(\sigma-1)\delta^{\sigma} R^{\sigma-1} \beta^{\sigma-2} (1-\beta)}{\left(1 + \delta^{\sigma} \beta^{\sigma-1} R^{\sigma-1}\right)\left(1 + \delta^{\sigma} \beta^{\sigma} R^{\sigma-1}\right)}$$

For $\sigma < 1$, $\vartheta(\beta)$ is a decreasing function of β when $\beta < 1$ with $\vartheta(1) > 1$. Therefore, $\vartheta(\beta) > 1$.

For $\sigma > 1$, $\vartheta(\beta)$ is an increasing function of β when $\beta < 1$ with $\vartheta(0) = 1$. Therefore, $\vartheta(\beta) > 1$. Finally, in any case, $\vartheta(\beta) > 1$ and $Z^{\sigma} - (\beta\delta)^{\sigma} > 0$.

Coming back to condition (11), it remains to prove that this condition holds when $K < \xi < M$. A sufficient condition for that is:

$$K = \frac{C}{R^{\sigma} Z^{\sigma} (R\beta\delta)^{\sigma}} \left[1 + R^{\sigma-1} Z^{\sigma} + R^{\sigma-1} (\beta\delta)^{\sigma}\right] > C \frac{1 + R^{\sigma-1} (\beta\delta)^{\sigma}}{R^{\sigma-1} \left[Z^{\sigma} - (\beta\delta)^{\sigma}\right]}$$

After some calculations, taking into account the expression of Z given by (5), this inequality is equivalent to $\beta < 1$ which is true by assumption.

6.2 Proof of proposition (5).

The proof is based on the following lemma:

Lemma 2 *Assume that* $C > B(R\beta\delta)^{\sigma}$, $(R\beta\delta)^{\sigma} A < B$ *and* $K < A + B/R < M$. *Let us consider a given value of* C *and a given value of* $\xi = A + B/R$ *with* $K < \xi < M$. A *and* B *may vary in such a way that* $\xi = A + B/R$, *with a higher bound* $A^h(\xi)$ *for* A *such that* $A^h(\xi) = \xi / [1 + R^{\sigma-1}(\beta\delta)^{\sigma}]$ *and a lower bound* $A^l(\xi)$ *for* A *such that* $A^l(\xi) = \xi - C/[R(R\beta\delta)^{\sigma}]$. *There exists an increasing function* $\alpha(\xi)$ *that satisfies* $A^l(\xi) < \alpha(\xi) < A^h(\xi)$ *and such that*

- *for* $A^l(\xi) < A < \alpha(\xi)$, $U(\hat{c}, \hat{d}, \hat{e}) > U(A, B, C)$: *the game between selves 1 and 2 leads to the decisions* $x = \hat{x}$ *and* $y = Y(\hat{x})$.
- *for* $\alpha(\xi) < A < A^h(\xi)$, $U(\hat{c}, \hat{d}, \hat{e}) < U(A, B, C)$: *the game between selves 1 and 2 leads to the decisions* $x = 0$ *and* $y = 0$.
- *for* $A = \alpha(\xi)$, *self 1 is indifferent between these two solutions.*

The function $\alpha(\xi)$ *is implicitly defined by:*

$$u(\alpha) + \beta\delta u[R(\xi - \alpha)] + \beta\delta^2 u(C) = U(\hat{c}, \hat{d}, \hat{e})$$

Proof:

Let us consider a given value of C and a given value of $\xi = A + B/R$ with $K \leq \xi < M$. $U(\hat{c}, \hat{d}, \hat{e})$ and $U(\check{c}, \check{d}, \check{e})$ only depend on ξ and C and they are fixed. For this given value of ξ, it is possible to consider different values for A and B such that $\xi = A + B/R$. The highest possible value of A is such that $(R\beta\delta)^{\sigma} A = B$, and it corresponds to the lowest possible value for B. This gives the values

$$\begin{aligned} A^h(\xi) &= \frac{\xi}{1 + R^{\sigma-1}(\beta\delta)^{\sigma}} \\ B^l(\xi) &= \frac{(R\beta\delta)^{\sigma}\xi}{1 + R^{\sigma-1}(\beta\delta)^{\sigma}} \end{aligned}$$

For $A = A^h(\xi)$ and $B = B^l(\xi)$, $U(\check{c}, \check{d}, \check{e}) = U(A, B, C)$. Indeed the optimal choice of x without any constraint for x (and $y = 0$) is such that $x = 0$. As

$U(\hat{c},\hat{d},\hat{e}) < U(\check{c},\check{d},\check{e})$ (and $\xi < M$), it is clear that $U(\hat{c},\hat{d},\hat{e}) < U(A,B,C)$. The game between selves 1 and 2 leads to the decisions $x = 0$ and $y = 0$.

The highest possible value of B is such that $C = B\,(R\beta\delta)^{\sigma}$, and it corresponds to the lowest value of A. This gives the values

$$\begin{aligned} A^{l}(\xi) &= \xi - \frac{C}{R\,(R\beta\delta)^{\sigma}} \\ B^{h}(\xi) &= \frac{C}{(R\beta\delta)^{\sigma}} \end{aligned}$$

For B such that $C = B\,(R\beta\delta)^{\sigma}$, $\tilde{x} = 0$. The objective function of self one is concave on $(0,+\infty)$. The optimal choice of x again is given by (6). It is positive as

$$R^{\sigma}Z^{\sigma}A - B - \frac{C}{R} > 0$$

Indeed, for $A = A^{l}(\xi)$ and $B = B^{h}(\xi)$, this inequality becomes:

$$\xi > \frac{C}{R^{\sigma}Z^{\sigma}\,(R\beta\delta)^{\sigma}}\left[1 + R^{\sigma-1}Z^{\sigma} + R^{\sigma-1}\,(\beta\delta)^{\sigma}\right] = K$$

which is true as $\xi > K$ by assumption. Therefore, in this case, $U(\hat{c},\hat{d},\hat{e})$ corresponds to the optimal choice of self 1 and $U(\hat{c},\hat{d},\hat{e}) > U(A,B,C)$.

Finally, the expression of $U(A,B,C)$ is studied as a function of A and ξ :

$$G(A,\xi) \equiv u(A) + \beta\delta u\left[R\,(\xi - A)\right] + \beta\delta^{2}u(C)$$

The derivative $\partial G(A,\xi)/\partial x$ is positive as it leads to

$$A^{-\frac{1}{\sigma}} - R\beta\delta B^{-\frac{1}{\sigma}} > 0 \Leftrightarrow (R\beta\delta)^{\sigma}\,A < B$$

$G(A,\xi)$ is an increasing function of A. G is such that $G(A^{l}(\xi),\xi)) < U(\hat{c},\hat{d},\hat{e})$ and $G(A^{h}(\xi),\xi) > U(\hat{c},\hat{d},\hat{e})$. Therefore, the existence and uniqueness of $\alpha(\xi)$ is proved and

- for $A^{l}(\xi) < A < \alpha(\xi)$, $U(\hat{c},\hat{d},\hat{e}) > U(A,B,C)$: the game between selves 1 and 2 leads to the decisions $x = \hat{x}$ and $y = Y(\hat{x})$.
- for $\alpha(\xi) < A < A^{h}(\xi)$, $U(\hat{c},\hat{d},\hat{e}) < U(A,B,C)$: the game between selves 1 and 2 leads to the decisions $x = 0$ and $y = 0$.
- for $A = \alpha(\xi)$, $U(\hat{c},\hat{d},\hat{e}) = U(A,B,C)$.

It remains to prove that $\alpha(\xi)$ is an increasing function. The function $\alpha(\xi)$ is implicitly defined by: $U(A, R(\xi-\alpha), C) = U(\hat{c}, \hat{d}, \hat{e})$, or

$$\frac{\sigma}{\sigma-1}\left[(\alpha)^{1-\frac{1}{\sigma}} + (R(\xi-\alpha))^{1-\frac{1}{\sigma}} + (C)^{1-\frac{1}{\sigma}}\right] = \frac{\sigma}{\sigma-1}\left(\xi + \frac{C}{R^2}\right)^{1-\frac{1}{\sigma}}\left[1 + R^{\sigma-1}Z^{\sigma}\right]^{\frac{1}{\sigma}}$$

The derivative $\alpha'(\xi)$ is such that:

$$\left[A^{-\frac{1}{\sigma}} - R\beta\delta B^{-\frac{1}{\sigma}}\right]\alpha'(\xi) = \left(\xi + \frac{C}{R^2}\right)^{-\frac{1}{\sigma}}\left[1 + R^{\sigma-1}Z^{\sigma}\right]^{\frac{1}{\sigma}} - R\beta\delta B^{-\frac{1}{\sigma}}$$

The term $A^{-\frac{1}{\sigma}} - R\beta\delta B^{-\frac{1}{\sigma}}$ is positive as by assumption $(R\beta\delta)^{\sigma} A < B$. It remains to prove that the right-hand side is positive. To prove that, it is first proved that

$$\left(\xi + \frac{C}{R^2}\right)^{-\frac{1}{\sigma}}\left[1 + R^{\sigma-1}Z^{\sigma}\right]^{\frac{1}{\sigma}} > A^{-\frac{1}{\sigma}}$$

this inequality is equivalent to

$$A\left[1 + R^{\sigma-1}Z^{\sigma}\right] > A + \frac{B}{R} + \frac{C}{R^2} \tag{12}$$

which gives $AR^{\sigma-1}Z^{\sigma} > B + C/R$. This last inequality is true. Indeed, the inequality $K < A + B/R$ leads to

$$A + \frac{B}{R} > \frac{C}{R^{\sigma}Z^{\sigma}(R\beta\delta)^{\sigma}}\left[1 + R^{\sigma-1}Z^{\sigma} + R^{\sigma-1}(\beta\delta)^{\sigma}\right]$$

or

$$AR^{\sigma}Z^{\sigma} > -\frac{BR^{\sigma}Z^{\sigma}}{R} + \frac{C}{R} + \frac{C}{(R\beta\delta)^{\sigma}}\left[1 + R^{\sigma-1}Z^{\sigma}\right]$$

By assumption, $C > B(R\beta\delta)^{\sigma}$, which allows us to write:

$$AR^{\sigma}Z^{\sigma} > -\frac{BR^{\sigma}Z^{\sigma}}{R} + \frac{C}{R} + B\left[1 + R^{\sigma-1}Z^{\sigma}\right]$$

that gives $AR^{\sigma-1}Z^{\sigma} > B + C/R$. Finally, it has been proved that

$$\left(\xi + \frac{C}{R^2}\right)^{-\frac{1}{\sigma}}\left[1 + R^{\sigma-1}Z^{\sigma}\right]^{\frac{1}{\sigma}} > A^{-\frac{1}{\sigma}}$$

As it is assumed that $A^{-\frac{1}{\sigma}} > R\beta\delta B^{-\frac{1}{\sigma}}$, it implies that

$$\left(\xi + \frac{C}{R^2}\right)^{-\frac{1}{\sigma}}\left[1 + R^{\sigma-1}Z^{\sigma}\right]^{\frac{1}{\sigma}} > R\beta\delta B^{-\frac{1}{\sigma}} \tag{13}$$

Finally, α is an increasing function of ξ .■

Proof of proposition (5).

From the relation $A = \alpha(\xi)$, it is possible to find a relation between A and B given by: $B = R\left[\alpha^{-1}(A) - A\right] \equiv \Delta(A)$. It is easy to show that $\Delta(A)$ is a decreasing function. Indeed, $\Delta(A)$ is implicitly defined by $U(A, B, C) = U(\hat{c}, \hat{d}, \hat{e})$, or

$$\frac{\sigma}{\sigma-1}\left[(A)^{1-\frac{1}{\sigma}} + (B)^{1-\frac{1}{\sigma}} + (C)^{1-\frac{1}{\sigma}}\right] = \frac{\sigma}{\sigma-1}\left(A + \frac{B}{R} + \frac{C}{R^2}\right)^{1-\frac{1}{\sigma}}\left[1 + R^{\sigma-1}Z^{\sigma}\right]^{\frac{1}{\sigma}} \tag{14}$$

The derivative $\Delta'(A)$ is such that:

$$\begin{aligned} & \left[R\beta\delta B^{-\frac{1}{\sigma}} - \left(A + \frac{B}{R} + \frac{C}{R^2}\right)^{-\frac{1}{\sigma}}\left(1 + R^{\sigma-1}Z^{\sigma}\right)^{\frac{1}{\sigma}}\right]\frac{\Delta'(A)}{R} \\ = & \left(A + \frac{B}{R} + \frac{C}{R^2}\right)^{-\frac{1}{\sigma}}\left(1 + R^{\sigma-1}Z^{\sigma}\right)^{\frac{1}{\sigma}} - A^{-\frac{1}{\sigma}} \end{aligned}$$

From the preceding results (cf. (12) and (13)), it is known that

$$\begin{aligned} \left(A + \frac{B}{R} + \frac{C}{R^2}\right)^{-\frac{1}{\sigma}}\left(1 + R^{\sigma-1}Z^{\sigma}\right)^{\frac{1}{\sigma}} - A^{-\frac{1}{\sigma}} &> 0 \\ R\beta\delta B^{-\frac{1}{\sigma}} - \left(A + \frac{B}{R} + \frac{C}{R^2}\right)^{-\frac{1}{\sigma}}\left(1 + R^{\sigma-1}Z^{\sigma}\right)^{\frac{1}{\sigma}} &< 0 \end{aligned}$$

Therefore $\Delta'(A) < 0$.

From (14), it is easy to check that

$$\begin{aligned} \Delta(A_H) &= \Delta\left(\frac{C}{(RZ)^{\sigma}(R\beta\delta)^{\sigma}}\left[1 + R^{\sigma-1}(\beta\delta)^{\sigma}\right]\right) = \frac{C}{(R\beta\delta)^{\sigma}} \\ \Delta(A_I) &= \Delta\left(\frac{M}{1 + R^{\sigma-1}(\beta\delta)^{\sigma}}\right) = \frac{M(R\beta\delta)^{\sigma}}{1 + R^{\sigma-1}(\beta\delta)^{\sigma}} \end{aligned}$$

■

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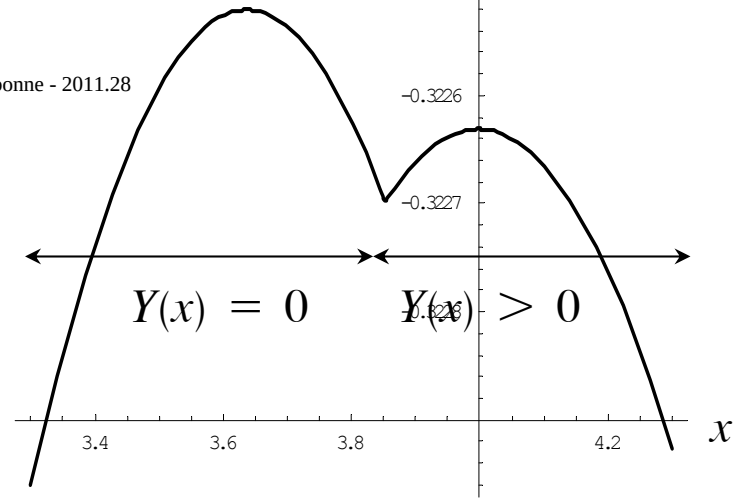
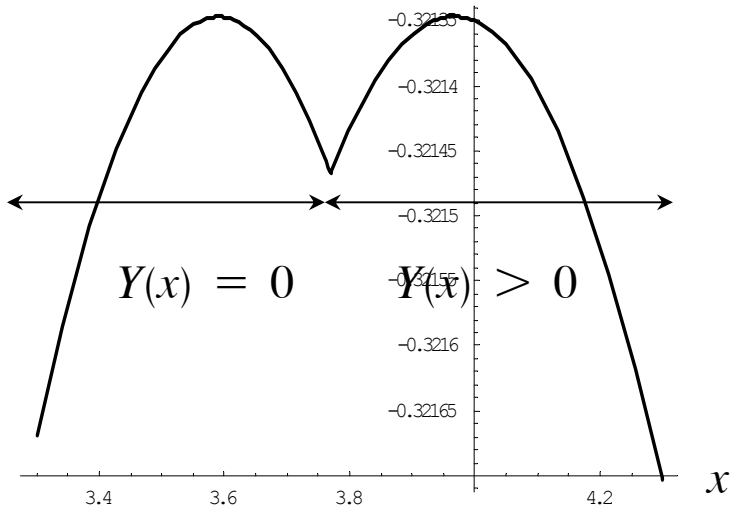
$B = 3.2$



$B = 3.28565$



$B = 3.4$

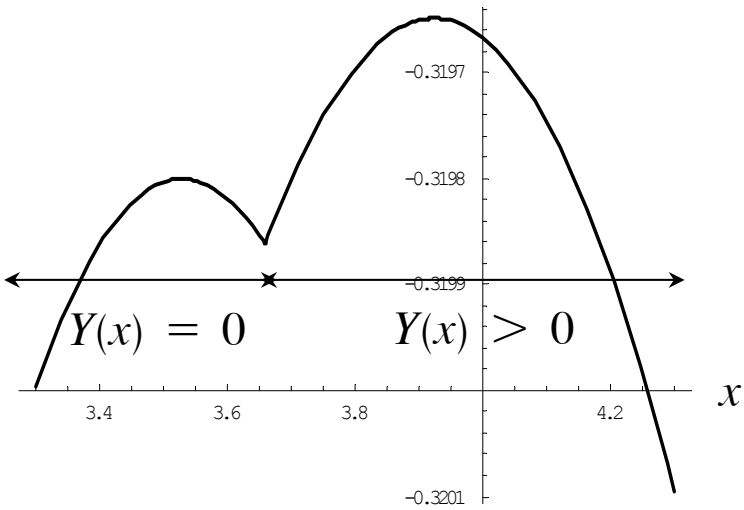


Figure 1: self 1's objective function for $A = 6$, $\beta = 0.7$, $\delta = 0.9571$, $R = 1.0375$, $C = 6$, $\sigma = 0.5$.

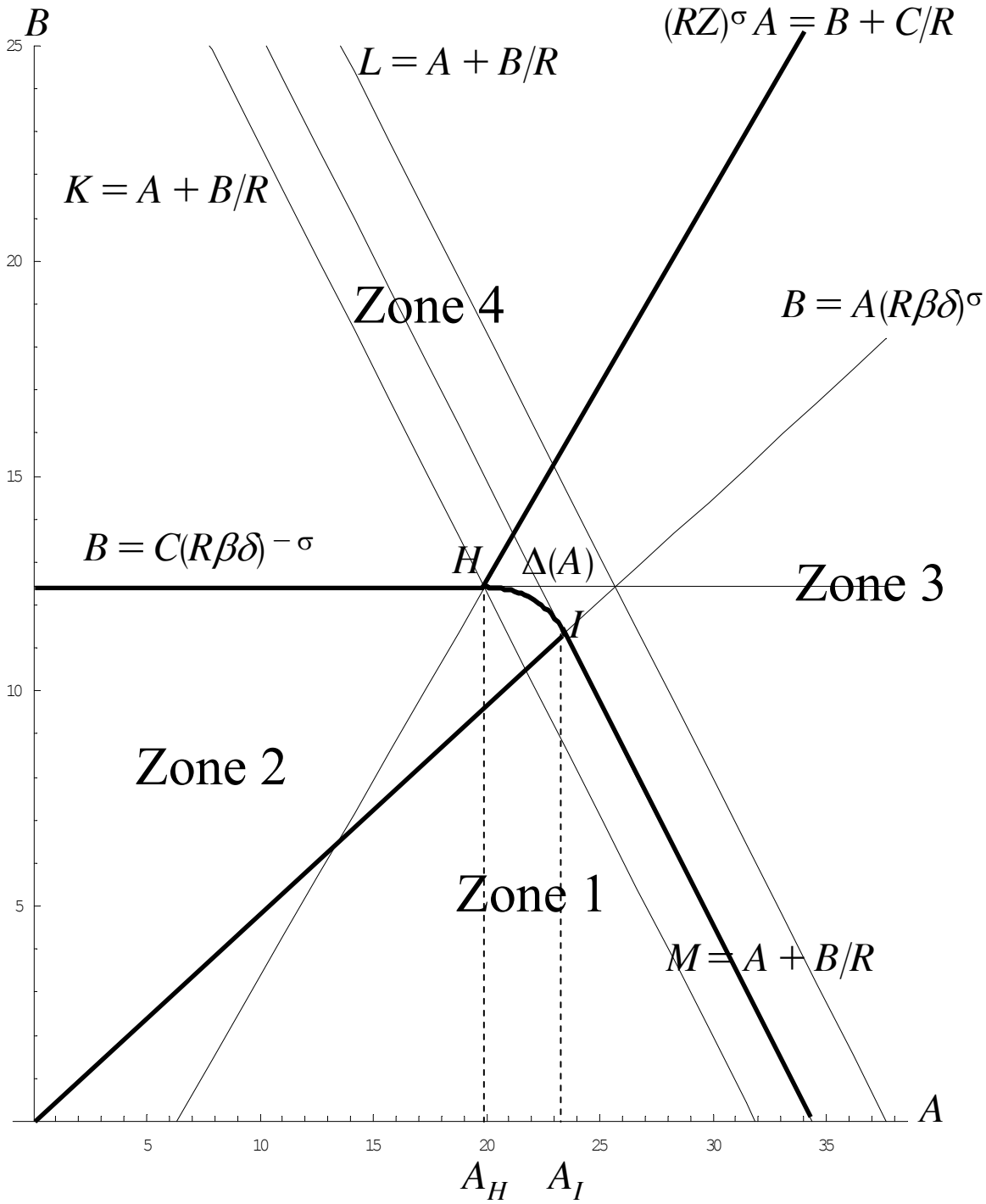


Figure 2: characterization of results with respect to A and B

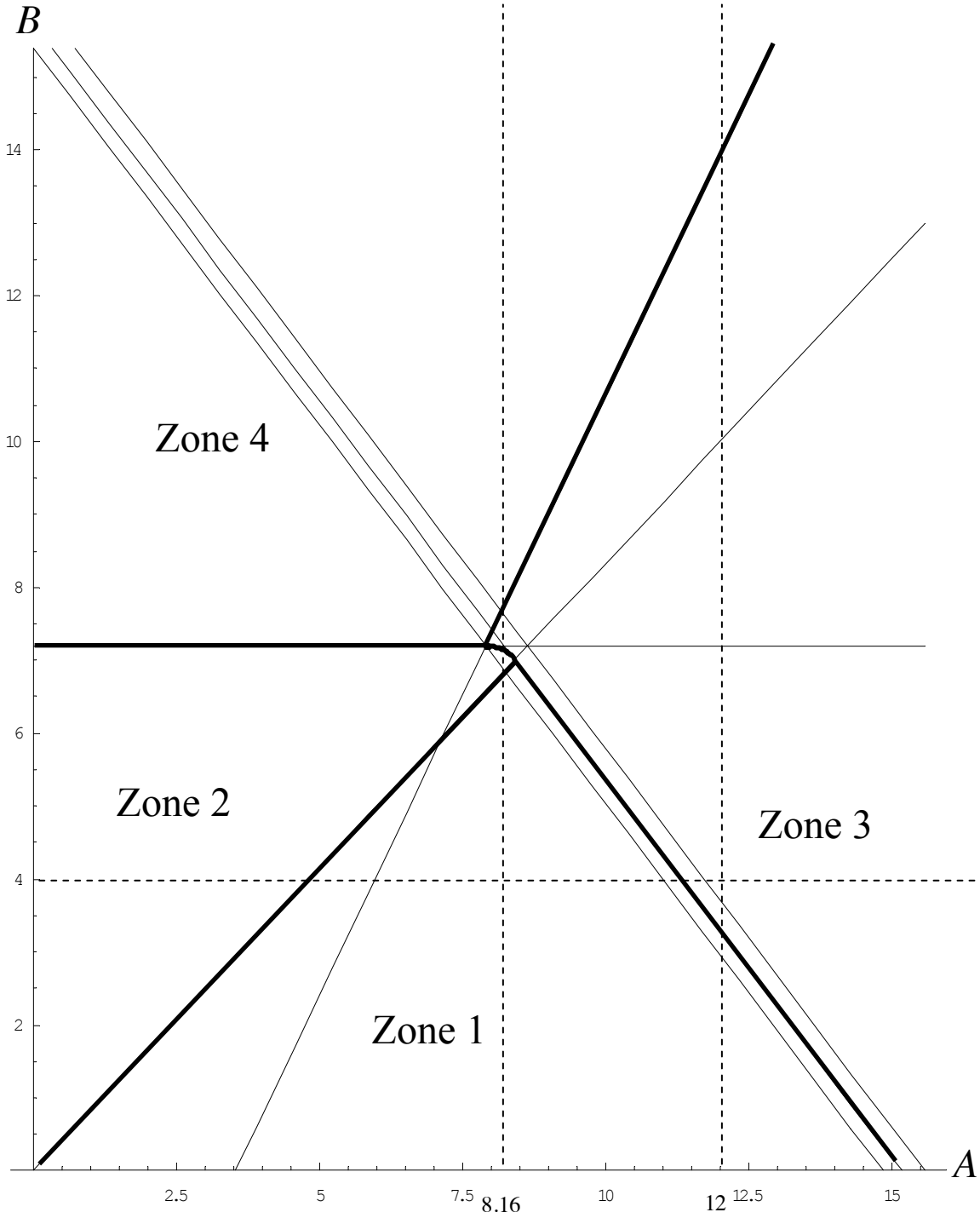


Figure 3: the different zones with $\sigma = 0.5$

$\beta = 0.7, \delta = 0.9571, R = 1.0375, C = 6, \sigma = 0.5$

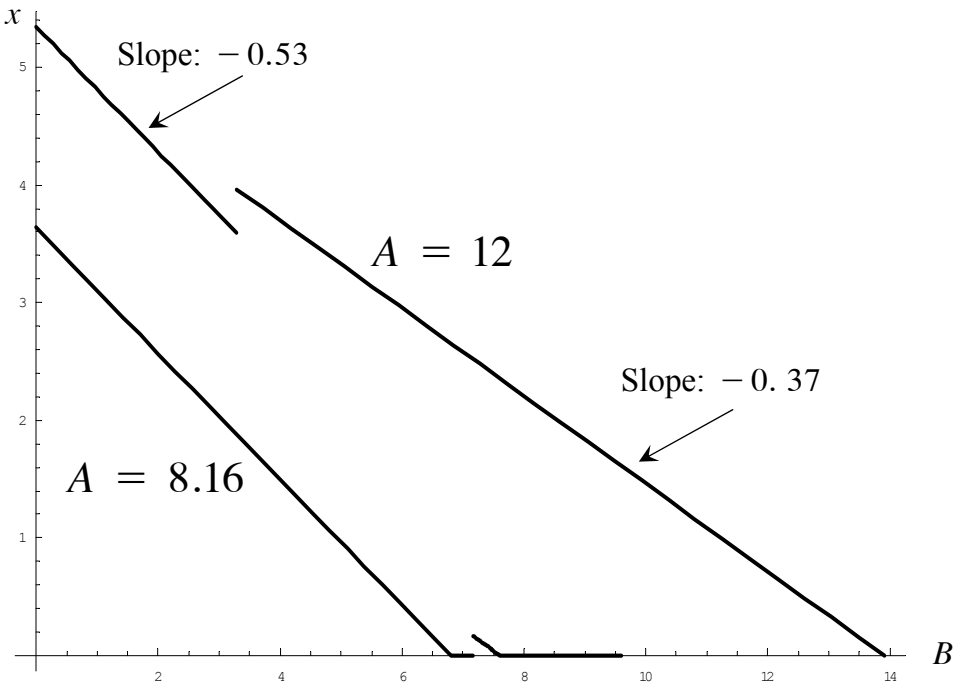


Figure 4: savings x with respect to B

$B = 4, \beta = 0.7, \delta = 0.9571, R = 1.0375, C = 6, \sigma = 0.5$

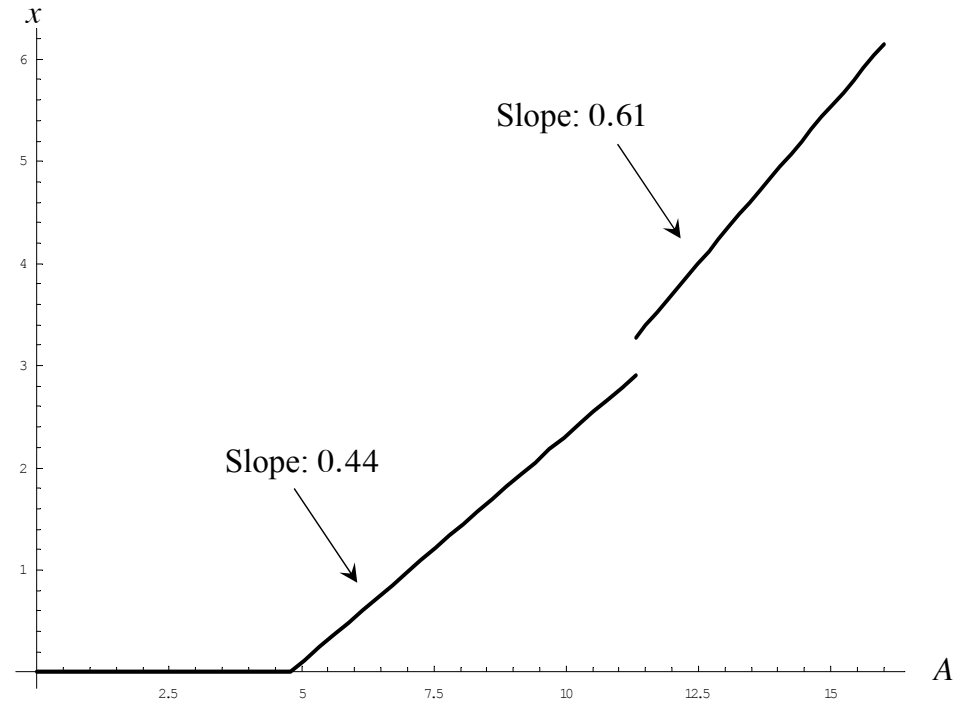


Figure 5: savings x with respect to A

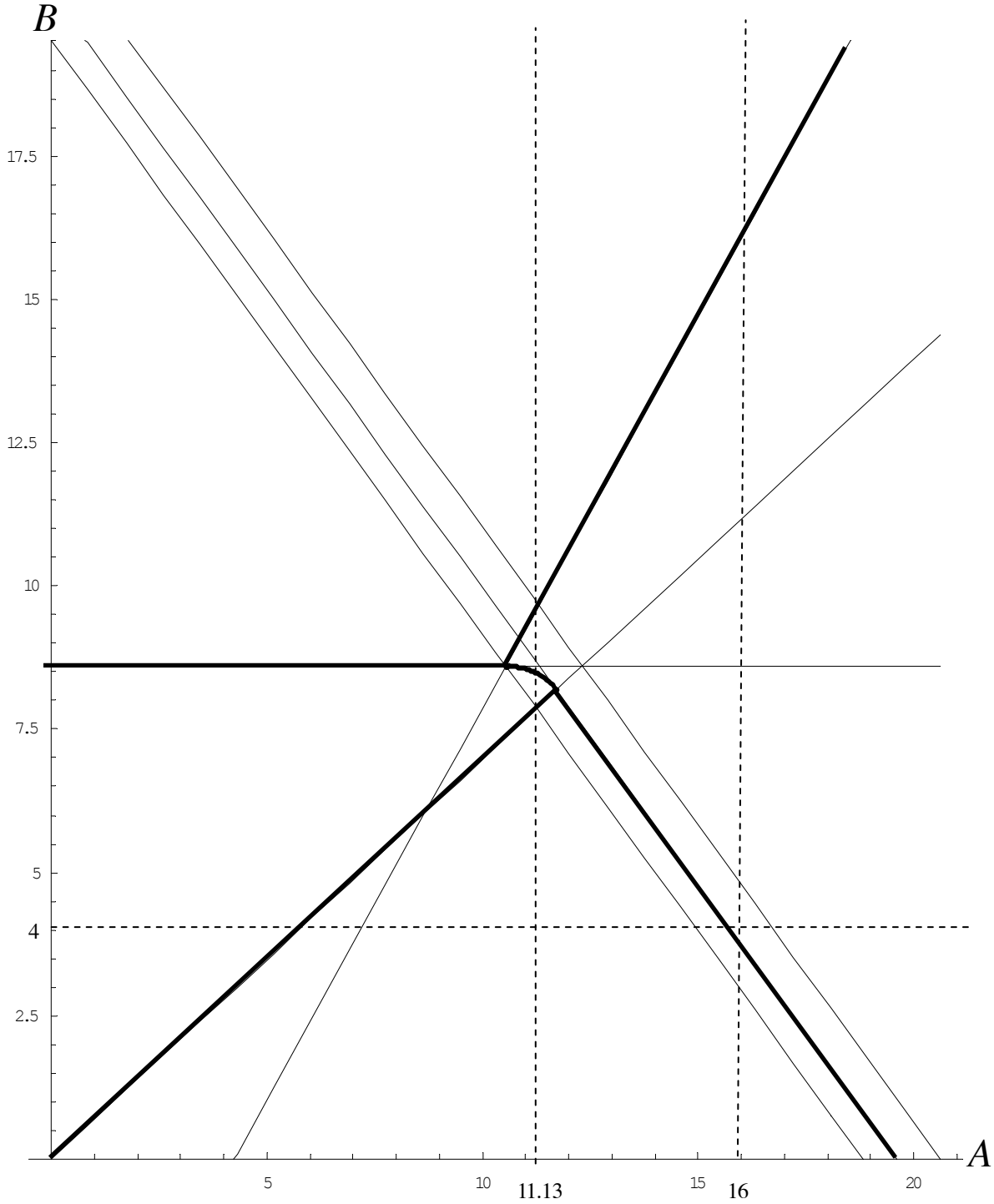


Figure 6: the different zones with $\sigma = 0.99$

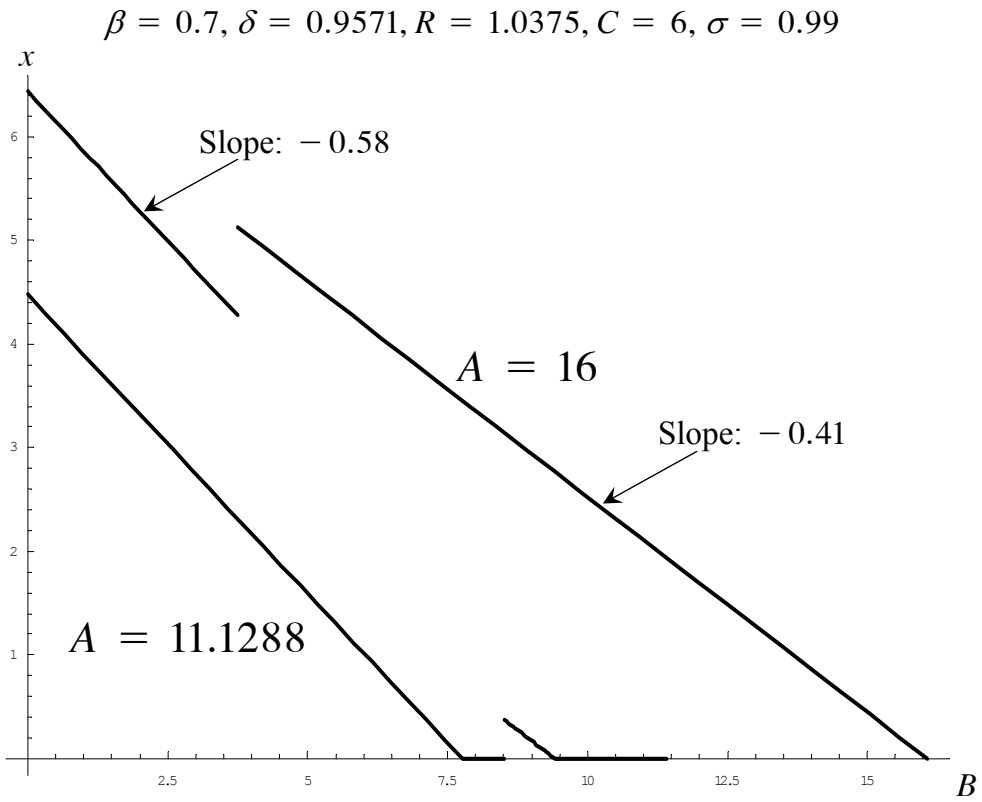


Figure 7: savings x with respect to B

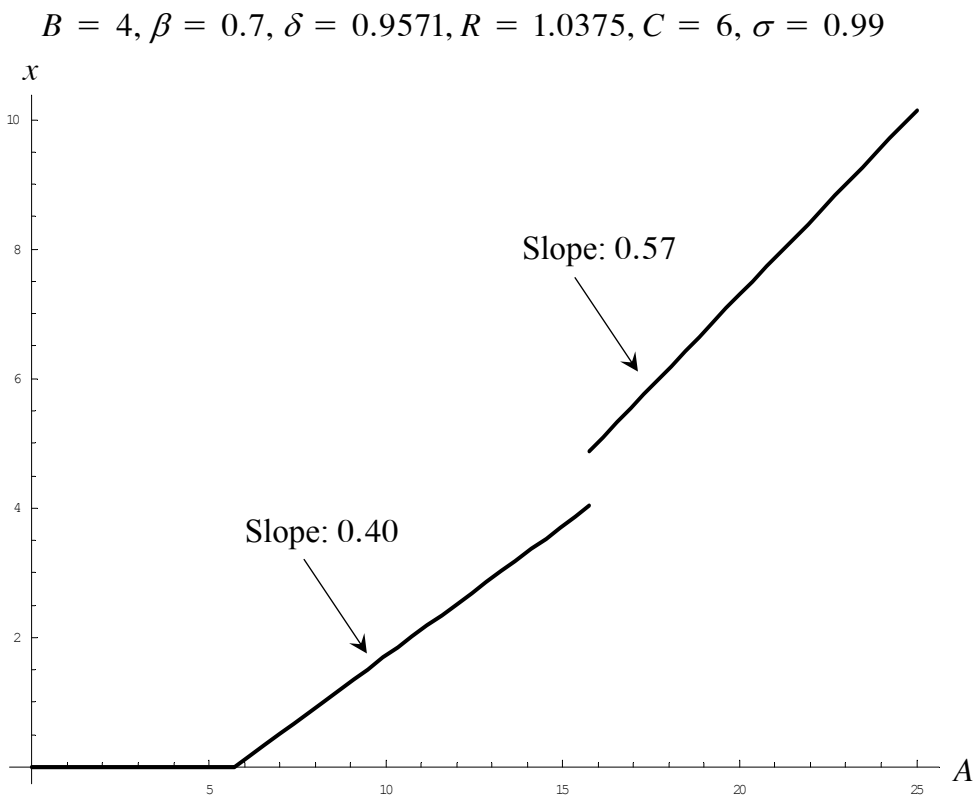


Figure 8: savings x with respect to A