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► **To cite this version:**

| Stéphane Gauthier, Guy Laroque. Second-best Random Redistribution. 2011. halshs-00639834v1

**HAL Id: halshs-00639834**

**<https://shs.hal.science/halshs-00639834v1>**

Submitted on 10 Nov 2011 (v1), last revised 4 Oct 2012 (v2)

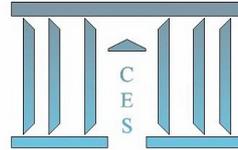
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## Second-best Random Redistribution

Stéphane GAUTHIER, Guy LAROQUE

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# Second-best Random Redistribution

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November 27, 2011

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## Résumé

Une fiscalité aléatoire peut être optimale lorsque les assujettis diffèrent selon leur attitude envers le risque, de sorte qu'un aléa permet de relâcher les contraintes d'incitation auxquelles l'autorité fiscale fait face. Cet article donne une condition nécessaire et suffisante pour que l'introduction de taxes aléatoires soit socialement bénéfique dans le voisinage d'un optimum où les impôts ont été supposés déterministes. Il s'intéresse également à la caractérisation d'un optimum global aléatoire. L'impôt optimal est caractérisé complètement dans une économie à deux biens et deux agents dont les utilités sont isoélastiques.

## abstract

Random taxation may be optimal when the taxpayers differ in their attitudes towards risk, so that tax randomization enables the government to relax the incentive constraints. The paper provides a necessary and sufficient condition for local random deviations to be welfare improving in a neighborhood of a nonrandom optimum. It also derives conditions satisfied by a global random optimum. A full analytical derivation is given for a two goods two agents economy with isoelastic utilities.

**JEL classification numbers:** H21, H23, H26.

**Mots-clés :** taxation aléatoire, contrats aléatoires, second rang, fraude fiscale.

**Keywords:** random taxation, stochastic contract, second best, tax evasion.

# 1 Introduction

In general the actual tax base is not fully known by the tax authority. When administrating and auditing all fiscal reports without error are extremely costly, some taxpayers will pay random taxes depending on administration errors or whether tax evasion is successful, as in Cremer and Gahvari (1993), Cremer and Gahvari (1996), Slemrod (2007) and Slemrod and Traxler (2010). However random taxation is not necessarily driven by the cost of monitoring. A random tax system may be optimal even if exact auditing can be implemented at no cost. This may occur when taxpayers differ in their attitudes towards risk, so that tax randomization enables the government to separate agents who otherwise would have been treated equally. In Pestieau, Possen, and Slutsky (2008) agents differ both in their abilities and in their attitudes towards risks. A nonrandom income tax pools agents with the same taxable income. In this context, randomization within income classes may allow to treat differently the taxpayers depending on their self-revealed risk aversions (see the examples of 1991 and 1994 Italian tax amnesties for an application of this kind of argument in Marchese and Privileggi (2004)).

More generally randomization may be useful whenever risk attitudes are correlated with other individual characteristics that are private information to the agents, but relevant to the government. For instance a random conscription draft with exemptions may select young men with the lowest opportunity costs (Sabin (2008)); a fairy tale king may try to evaluate how much a young prince loves his daughter by putting forward a risky fight against a dragoon, and Solomon may gauge entrepreneurship in the light of a choice involving some exposure to risk (Miller, Wagner, and Zeckhauser (2010)).

The present paper is concerned with a closely related line of argument developed early on in the context of the principal-agent model by Weiss (1976) and Stiglitz (1982), where randomization alleviates the incentive constraints faced by the regulator. Consider an economy with two agents: the first one is skilled and displays a high risk aversion, whereas the second one is unskilled and has a lower risk aversion. Assume that the government would like to redistribute welfare from the skilled (rich) agent towards the unskilled (poor) agent. If neither skills nor risk aversions are publicly observed, the available information puts strong limits on the scope of redistribution: the skilled agent would pretend to be unskilled if she faced too high a tax. A random tax on the unskilled worker may then be part of the optimal second-best tax policy: the risk averse skilled type is discouraged to pretend that she is unskilled,

which yields a social gain possibly overcoming the efficiency loss due to the government created risk bearing on the unskilled. In this example, random taxation can be optimal when skill and risk aversion are positively correlated. Indeed randomized redistribution can only be useful when the agent that the government wants to favor has the lowest risk aversion (Hellwig (2007)).

The most general results on the usefulness of random taxation appear in Brito, Hamilton, Slutsky, and Stiglitz (1995). These authors examine whether a small random redistribution can locally improve social welfare at a nonrandom second best allocation in a standard two good (consumption and labor) Mirrlees economy. They obtain a necessary and sufficient condition (their Theorem III) for the existence of a local random improvement. However, their constructive tax reform methodology does not provide a description of the mechanism at work, so that it is unclear how it could be extended to more general economies. The condition, which bears on a weighted difference of the second derivatives of the agents' utility functions, appears quite intricate (Hellwig (2007)).

Second derivatives of utilities are suggestive of second order conditions, and second order conditions indeed are important since incentive considerations typically generate a nonconvex set of constraints. The present paper provides a general characterization of the conditions under which local random redistribution at a non-random optimum is welfare improving. It shows how the condition obtained by Brito, Hamilton, Slutsky, and Stiglitz (1995) relates to the second order conditions of the nonrandom government program. Local randomization is useful if and only if the Hessian of the Lagrangian is not negative definite, when evaluated at the optimum of the nonrandom program. Then a nonrandom optimum, which of course cannot be improved upon through nonrandom deviations, is not a maximum anymore when small random deviations are allowed.

Our paper provides a careful analysis of the second order conditions. In a standard optimization problem, the second order conditions bear on deviations that belong to the tangent space to the active constraints (see e.g. Simon and Blume (1994)). The tangent space associated with the nonrandom problem is included in the one that allows random deviations, since a nonrandom deviation is also a degenerate random deviation. The inclusion is strict: when random deviations are allowed, the expected deviation must belong to tangent space associated with the nonrandom problem but the dispersion of the transfers around their mean remains free. This freedom can be exploited by the government along specific directions when the Hessian of the

Lagrangian of the nonrandom problem is not negative definite. Furthermore, the paper also presents conditions satisfied by a global random optimum.

The theoretical analysis is complemented with a parametric example with two agents, two goods, and isoelastic utilities. The set of parameters under which the nonrandom optimum (among nonrandom allocations) is dominated by random deviations in its immediate vicinity is derived explicitly. This example highlights the main forces at work: agents' risk aversions and social weights, shape of the distribution of types in the population. A full characterization of the global random optima, including the number of possible outcomes and the probabilities of their occurrences, is also offered.

The paper is organized as follows. Section 2 presents the general framework. The optima among the set of nonrandom allocations are characterized in section 3, with special attention to second order conditions. Necessary and sufficient conditions for these nonrandom optima to be locally dominated by stochastic allocations are provided in section 4. Properties of the global random optima appear in section 5. Finally section 6 gives a full characterization of the global random optimum in a simple parametric example.

## 2 The model

We consider economies with a continuum of agents with a finite number of types  $i$ ,  $i = 1, \dots, I$ . Population size is normalized to 1, and there are  $n_i$  agents of type  $i$ . The government assigns a vector of transfers  $z_i$  to agents of type  $i$ . Transfers apply to  $k$  goods,  $k = 1 \dots, K$ . The government has full control of the vectors of transfers ( $z_i$ ) received by the agents: the components of ( $z_i$ ) are defined in units observable to the government. A vector of transfers  $z_i$  is feasible for type  $i$  agents if it belongs to a subset  $Z_i$  of  $\mathbb{R}^K$ .

Type  $i$  agents who receive a nonrandom transfer  $z_i$  have utility  $v_i(z_i)$ , where  $v_i : Z_i \rightarrow \mathbb{R}$  is an increasing concave von Neumann Morgenstern utility index. Our focus will be on random transfers  $\tilde{z}_i$  in the set  $\mathcal{M}(Z_i)$  of integrable random variables with support in  $Z_i$ . When an agent  $i$  receives such a transfer, her *ex ante* utility is  $\mathbb{E}v_i(\tilde{z}_i)$ . By construction all agents of the same type are *ex ante* equally treated.

The weighted utilitarian government chooses an allocation  $(\tilde{z}_i)$ ,  $i = 1, \dots, I$ ,

solution of the following program

$$\begin{aligned} \max \sum_{i=1}^I a_i n_i \mathbb{E} v_i(\tilde{z}_i) \\ \sum_{i=1}^I n_i \mathbb{E} \tilde{z}_i \leq 0 \end{aligned} \quad (p) \quad (1a)$$

$$\mathbb{E} v_i(\tilde{z}_i) \geq \mathbb{E} v_i(\tilde{z}_j) \quad \text{for all } i \text{ and } j \quad (\lambda_{ij}) \quad (1b)$$

where the nonnegative weights  $a_i$  in the  $I$ -simplex parameterize tastes for redistribution. The solution procedure can accommodate feasibility constraints such as  $\tilde{z}_i$  belongs to  $\mathcal{M}(Z_i)$ , which for simplicity we assume not to be binding. Similarly, we could add participation constraints, as often found in contract theory.

The feasibility constraints (1a) are written in expectations. This is consistent with microeconomic idiosyncratic independent shocks, using the law of large numbers. The government knows the structure of the economy, but does not know the types of the agents. The incentive constraints (1b) make sure that when it announces a profile of possibly random transfers  $(\tilde{z}_i)$  type  $i$  agents voluntarily choose the transfer  $\tilde{z}_i$  designed for them.

Let  $p$  be the  $K$ -dimensional vector of multipliers associated with the  $K$  feasibility constraints (1a), and  $\lambda_{ij}$  the multiplier associated with the constraint (1b) where agent  $i$  compares her bundle to that assigned to type  $j$ . The associated Lagrangian is

$$\tilde{\mathcal{L}} = \sum_{i=1}^I a_i n_i \mathbb{E} v_i(\tilde{z}_i) - \sum_{i=1}^I n_i p' \mathbb{E} \tilde{z}_i + \sum_{i \neq j} \lambda_{ij} [\mathbb{E} v_i(\tilde{z}_i) - \mathbb{E} v_i(\tilde{z}_j)]. \quad (2)$$

### 3 Nonrandom optimum

A local (interior) nonrandom second-best optimum is a profile of nonrandom transfers  $(z_i)$  solutions of a program (1') obtained from (1) by replacing  $\tilde{z}_i$  with  $z_i$  everywhere. It can be characterized using the Lagrangian function associated with program (1'),

$$\mathcal{L} = \sum_{i=1}^I a_i n_i v_i(z_i) - \sum_{i=1}^I n_i p' z_i + \sum_{i=1}^I \sum_{j \neq i} \lambda_{ij} (v_i(z_i) - v_i(z_j)). \quad (3')$$

The profile  $(z_i)$  at such an optimum satisfies the necessary first order conditions

$$\left( a_i n_i + \sum_{j \neq i} \lambda_{ij} \right) \nabla v_i(z_i) - p - \sum_{j \neq i} \lambda_{ji} \nabla v_j(z_i) = 0 \quad \text{for all } i, \quad (3)$$

with the complementary slackness conditions  $\lambda_{ij} (v_i(z_i) - v_i(z_j)) = 0$  for all  $i$  and  $j \neq i$ .

The second order conditions for  $((z_i), p, \lambda)$  to be a local maximum make use of the Hessian  $H$  of the Lagrangian. This Hessian is the  $IK \times IK$  symmetric matrix whose  $i$ th diagonal block is the  $K \times K$  matrix

$$H_i = \left( a_i n_i + \sum_{j \neq i} \lambda_{ij} \right) \nabla^2 v_i(z_i) - \sum_{j \neq i} \lambda_{ji} \nabla^2 v_j(z_i), \quad (4)$$

while all off diagonal blocks are zero.

A direct application of Theorem 3 of Appendix A yields:

**Proposition 1.** *Consider a nonrandom allocation  $(z_i)$  where  $z_i$  lies in the interior of  $Z_i$  for all  $i$ . Suppose that the constraints are qualified at  $(z_i)$  and that the matrix  $H$  is of full rank. Then  $(z_i)$  is a local optimum in the set of nonrandom allocations if and only if*

1. *the first order conditions (3) hold;*
2. *the Hessian  $H$  of the Lagrangian evaluated at the nonrandom allocation  $(z_i)$  is negative definite on the tangent plane to the active constraints, i.e.,*

$$\sum_{i=1}^I x_i' H_i x_i < 0 \quad (5)$$

*for all non zero  $(x_i)$ 's such that*

$$\sum_{i=1}^I n_i x_i = 0, \quad (6)$$

*and*

$$\nabla v_i(z_i)' x_i - \nabla v_i(z_j)' x_j = 0 \quad (7)$$

*for all  $i$  and all  $j \neq i$  such that  $v_i(z_i) = v_i(z_j)$ .*

When a type  $i$  agent is not envied, the multipliers  $\lambda_{ji}$  are equal to 0 for all  $j \neq i$ . The matrix  $H_i$  defined in (4) then is negative definite by concavity of the utility function. As a result, when none of the incentive constraints is binding ( $\lambda_{ji} = 0$  for all  $i$  and  $j$ ), the Hessian  $H$  is negative definite. From Proposition 1, any allocation which satisfies the first order conditions and generates no envy is a local optimum: indeed it is a first best global optimum.

When some incentive constraints bind, the concavity of the utility functions  $v_i(\cdot)$  no longer warrants the negative definiteness of  $H$ . As seen from the last term in (4), the transfers  $z_i$  change the desire of agents  $j$  to pretend that they are of type  $i$ . A large enough value of the multiplier  $\lambda_{ji}$  could make  $H_i$  positive definite. A non concave Lagrangian does not necessarily prevent local optimality: from Proposition 1.2,  $H$  must be negative definite on the tangent space to the active constraints. When there are  $\Lambda$  binding incentive constraints, this space has dimension  $IK - (K + \Lambda)$ . Hence the  $IK$  dimensional square matrix  $H$  has at least  $IK - (K + \Lambda)$  negative eigenvalues at a local optimum. It may consequently have  $K + \Lambda$  positive eigenvalues. Of course, for Proposition 1.2 to hold, the  $K + \Lambda$  dimensional positive eigenspace of  $H$  and the  $IK - (K + \Lambda)$  dimensional tangent space to the active constraints must have no intersection.

## 4 Local random transfers

Consider a local maximum among the nonrandom allocations. The second order conditions leave room for positive eigenvalues in the  $K + \Lambda$  directions that do not belong to the tangent plane to the active constraints at this optimum. This property can be exploited to yield a welfare improvement through random transfers.

The intuition of the argument is as follows. Consider a local maximum  $(z_i)$  among the nonrandom allocations, and small deviations  $(d\tilde{z}_i)$  such that all the active constraints at the nonrandom optimum remain binding at the new point  $(z_i + d\tilde{z}_i)$ . Since the reference allocation satisfies the first order conditions in Proposition 1, the reform only yields a second order change to social welfare,

$$\sum_{i=1}^I a_i n_i (\mathbb{E} v_i(z_i + d\tilde{z}_i) - v_i(z_i)) = \tilde{\mathcal{L}} - \mathcal{L} \approx \frac{1}{2} \mathbb{E} \sum_{i=1}^I (d\tilde{z}_i)' H_i (d\tilde{z}_i), \quad (8)$$

where  $\tilde{\mathcal{L}}$  is the Lagrangian defined by (2) and evaluated at the final random allocation. Whether social welfare rises thus depends on the properties of the matrices  $(H_i)$  in the Hessian  $H$ .

When  $H$  has a positive eigenvalue, the  $IK$ -eigenvector  $x^+$  associated with this positive eigenvalue is such that  $x^{+'}Hx^+ > 0$ . Assume that the positive eigenvalue of  $H$  is an eigenvalue of the matrix  $H_j$ . Let  $x_j^+$  be the  $K$ -eigenvector of  $H_j$  associated with this eigenvalue. Since  $H$  is block diagonal, all the components of  $x^+$  are zero, except those of  $x_j^+$ . Thus we have

$$x^{+'}Hx^+ = \sum_{i=1}^I x_i^{+'}H_i x_i^+ = x_j^{+'}H_j x_j^+ > 0.$$

From (8), to increase the social objective, it seems natural to choose the support of the lottery  $d\tilde{z}$  collinear to  $x^+$ . For simplicity, consider a lottery with two possible values

$$dz_i^1 = \beta_i(t) = dz_i^2 \text{ for all } i \neq j$$

$$dz_j^1 = tx_j^+ + \beta_j(t) \quad dz_j^2 = -tx_j^+ + \beta_j(t),$$

drawn independently with equal probability. The small positive scalar  $t$  measures the scale of the change along the direction  $x^+$ . The deterministic part  $(\beta_i(t))$  ensures that all the active constraints at the nonrandom optimum remain satisfied. Note that this deviation actually involves noise on the allocation received by agents of type  $j$ , while it is deterministic for all the other agents. It turns out that  $(\beta_i(t))$  is of the order of  $t^2$ . For  $t$  close enough to 0, it therefore is negligible in (8), which reduces to

$$\sum_{i=1}^I a_i n_i (\mathbb{E}v_i(z_i + d\tilde{z}_i) - v_i(z_i)) \approx \frac{1}{2} t^2 x_j^{+'} H_j x_j^+ > 0.$$

Economically, the zero expectation  $(tx_j^+)$  terms serve to relax the binding constraints, but they lower the concave objective (type  $j$  agents are risk averse), while the deterministic  $(\beta_i(t))$  terms use the slack in the constraints to redistribute from the least to the most favored agents in the social welfare criterion.

**Proposition 2.** *Consider a nonrandom optimum among the nonrandom allocations, which satisfies the conditions laid down in Proposition 1. Then social welfare can be improved upon through local random transfers at this allocation if and only if the Hessian  $H$  of the nonrandom Lagrangian has at least one positive eigenvalue.*

The result follows from Theorem 5 of Appendix A, which does not depend on the specific form taken by the constraints. Proposition 2 actually applies more generally, e.g., when feasibility constraints appearing in taxation problems are replaced by individual rationality or participation constraints, as is the case in contract theory.

**Remark 1.** Proposition 1 and 2 assume qualified constraints. This allows to compute  $(\beta_i(t))$  so that all the active constraints are exactly satisfied. The constraints are not qualified in the Lerner world examined by Pestieau, Posen, and Slutsky (2002). Suppose that there are two agents ( $I = 2$ ) among whom the government has to distribute a given quantity of a single good ( $K = 1$ ). The government faces two incentive constraints ensuring that both agents self-select. In the absence of randomness, if one agent were receiving more good than the other, incentive compatibility would fail: at the nonrandom optimum, both agents get one half of the aggregate endowment ( $\Lambda = 2$ ). All three, feasibility and the two incentive compatibility, constraints bind at the optimum: the constraints are not qualified ( $IK - (K + \Lambda) < 0$ ).

The argument used to derive Theorem 5 of Appendix A and Proposition 2 can be adapted to the case where the constraints are not qualified (see Appendix B). When the number of active constraints is greater than the number  $IK$  of components of  $\beta(t)$ , one can sometimes consider a subset of  $IK$  constraints and proceed as in the qualified case, making sure *ex post* that these  $IK$  constraints bind (since one only considers small departures from the initial allocation, the initially slack constraints remain slack). In the Lerner case, intuition suggests that the most favored agents will not envy the others once randomness on the former type of agents is introduced: the corresponding incentive constraint will be slack.

**Remark 2.** When some variables linearly enter the Lagrangian function, the corresponding entries of the Hessian of the Lagrangian are identically zero and the structure of the problem may simplify drastically. This is the

case when the utility functions and the constraints are quasi-linear in some variables. For instance, in the model studied by Strausz (2006), there are  $I$  known active constraints with  $I$  associated transfers entering linearly both the constraints and the objective. After substitution and elimination of the transfers, one gets an unconstrained optimization problem with respect to the other variables. At any deterministic solution of this unconstrained problem, the Hessian of the Lagrangian function is negative definite: there is no scope for improvement through local random deviations. However, with a different set of active constraints which would not disappear by elimination, Proposition 1.2 would leave room for a non negative definite Hessian of the Lagrangian and the ensuing optimal stochastic mechanisms.

## 5 Global random optimum

We now turn our attention to the global random optimum. Suppose that the social planner chooses lotteries where agent  $i$  gets  $z_i^s$  with probability  $\pi_i^s$ ,  $s = 1, \dots, S_i$

$$\sum_{s=1}^{S_i} \pi_i^s = 1, \text{ for all } i. \quad (9)$$

Given  $(z_i^s)$ , the program (1) is linear in probabilities. Let  $\pi$  be the non negative vector obtained from stacking up the individual probabilities  $(\pi_i^s)$ . From (1) the optimal probabilities  $\pi$  are solutions of

$$\max A' \pi \text{ subject to } B \pi = b. \quad (10)$$

Here  $A$  is the vector of values of the von Neumann Morgenstern indices  $a_i n_i v_i(z_i^s)$ , and  $B$  is the matrix of the coefficients of the probabilities in the various constraints: feasibility (1a), incentive (1b) and the constraints (9) of definition of the probabilities. All the potential  $I \times (I - 1)$  incentive compatibility constraints are included: the corresponding components of  $b$  on the right hand side are zero for the binding constraints, and by convention they are positive for the ones that are slack. The row dimension of  $B$  therefore is  $K + I(I - 1) + I = K + I^2$ . Using this property of the problem, we prove in Appendix C the following result:

**Lemma 1.** *1. Without loss of generality the number of points in the support of the optimum allocation can be taken to be inferior than or equal to  $K + I^2$ .*

2. *Without loss of generality the number of points in the support of the allocation of an individual agent at a second best optimum can be taken to be inferior than or equal to  $K + 2I - 1$ .*

The bounds could be improved by using specific properties of the program. For instance, a lower row dimension of  $B$  can be used in Lemma 1 when some incentive constraints do not matter at the optimum. In Brito, Hamilton, Slutsky, and Stiglitz (1995), there are two goods ( $K = 2$ ) and two agents ( $I = 2$ ) but one agent is not envied by the other. The nonenvied agent receives a nonrandom allocation: the number of points in the support can be taken to be three.

By Lemma 1 there is no loss of generality in limiting the attention to finite state spaces. The Lagrangian technique then can be applied to the choice of  $(z_i^s)$  and of the new variables  $(\pi_i^s)$  for all  $i$  and all  $s$ . Let  $\mu_i$  denote the nonnegative Lagrange multipliers associated with the  $I$  constraints (9). By Theorem 3 of Appendix A, a necessary first order condition for  $\pi_i^s$  to be optimal is

$$a_i n_i v_i(z_i^s) - n_i p' z_i^s + \sum_{j \neq i} \lambda_{ij} v_i(z_i^s) - \sum_{j \neq i} \lambda_{ji} v_j(z_i^s) \leq \mu_i, \quad (11)$$

with equality if the probability  $\pi_i^s$  is strictly positive. Hence, we have:

**Lemma 2.** *The support of the lottery  $(z_i^s, \pi_i^s)$  offered to agent  $i$  is included into the set of  $(z^*)$  such that*

$$z^* = \arg \max_z \left\{ a_i n_i v_i(z) - n_i p' z + \sum_{j \neq i} \lambda_{ij} v_i(z) - \sum_{j \neq i} \lambda_{ji} v_j(z) \right\}.$$

All the elements in the support of an optimal random allocation maximize a function which has the shape of the Lagrangian associated with the nonrandom problem (3'), with of course different values for the multipliers. As a consequence, the first order condition for an interior  $z_i^s$ , with positive probability of occurrence, is similar to (3):

$$\left( a_i n_i + \sum_{j \neq i} \lambda_{ij} \right) \nabla v_i(z_i^s) - p - \sum_{j \neq i} \lambda_{ji} \nabla v_j(z_i^s) = 0 \quad \text{for all } i. \quad (12)$$

Finally, the feasibility constraints (1a) and the incentive compatibility constraints (1b) of the general program (1) given in Section 2 must be met, as well as (9).

A procedure to solve the problem is combinatorial. Choose a possible subset of active incentive constraints, with corresponding multipliers. The results in Appendix D help to narrow down the set of constraints that may bind at the optimum. Solve for the associated allocation and multipliers that satisfy the optimality conditions. The full optimum is the one that maximizes the objective among all possible cases. In a typical case with  $\Lambda$  binding incentive constraints, a possible procedure is as follows:

1. Express  $(z_i^s)$  as a function of the multipliers  $(\lambda_{ij})$  and  $p$  by finding the maxima of the Lagrangian (3'). In case of interior maxima, they solve the  $K \sum S_i$  first order conditions (12);
2. Reintroduce the expression for  $(z_i^s)$  obtained in Step 1 into the  $K + \Lambda$  (binding) constraints (1a) and (1b). This yields the multipliers as a function of  $(\pi_i^s)$ , and therefore  $(z_i^s)$  as a function of  $(\pi_i^s)$ ;
3. The equality between the values of the nonrandom Lagrangian at all points of support of the lottery required by the constraints (11), together with the  $I$  constraints (9), yield  $(\pi_i^s)$  and  $(\mu_i)$ .

## 6 A simple example

We illustrate the procedure described above on a simple economic example where the government wants to redistribute among two types of agents, 1 and 2, who differ according to their preferences over two goods, consumption and labor. Individual preferences are represented with isoelastic separable utility functions

$$v_1(c, \ell) = \frac{c^{1-\alpha_1}}{1-\alpha_1} - \frac{1}{2}\ell^2, \quad v_2(c) = \frac{c^{1-\alpha_2}}{1-\alpha_2}. \quad (13)$$

Agents of type 2 have an inelastic supply of labor  $\bar{\ell}$ . The subutilities obtained from consumption are concave, with positive coefficients  $\alpha_1$  and  $\alpha_2$  of absolute risk aversion,  $0 \leq \alpha_1 < 1$ ,  $0 \leq \alpha_2 < 1$ .

Let us first suppose that the government chooses a nonrandom allocation  $(c_1, \ell_1)$  for type 1 agents, and  $c_2$  for type 2 agents. Suppose also that the

government only cares about type 1 agents, those who suffer from working ( $a_2 = 0$ ). The government's objective is to maximize  $n_1 v_1(c_1, \ell_1)$ . The technology allows to transform one unit of labor into one unit of consumption good, with constant returns to scale. The government faces the feasibility constraint

$$n_1 c_1 + n_2 c_2 = n_1 \ell_1 + n_2 \bar{\ell}. \quad (14)$$

When the government observes agents' preferences, it maximizes its objective subject to the linear constraint (14). Randomization over consumption would be detrimental to social welfare. If the consumption of agents of type 1 were random, replacing the lottery by its expected value would be feasible (assuming that feasibility (14) is written in expected terms by the law of large numbers) and raise the welfare of type 1 agents.

This argument no longer applies if preferences are private information. When the government does not observe the agents types, it faces both the constraint (14) and two incentive constraints ensuring that the agents do not care for the bundle assigned to the others,

$$\frac{c_1^{1-\alpha_1}}{1-\alpha_1} - \frac{1}{2}\ell_1^2 \geq \frac{c_2^{1-\alpha_1}}{1-\alpha_1} - \frac{1}{2}\bar{\ell}^2, \quad (15)$$

and, whenever type 1 labor supply  $\ell_1$  is smaller than  $\bar{\ell}$ ,

$$\frac{c_2^{1-\alpha_2}}{1-\alpha_2} \geq \frac{c_1^{1-\alpha_2}}{1-\alpha_2}. \quad (16)$$

By Proposition 1.1 an interior nonrandom optimum is a local extremum of the Lagrangian

$$\begin{aligned} \mathcal{L}^{\text{iso}} = & n_1 \left( \frac{c_1^{1-\alpha_1}}{1-\alpha_1} - \frac{1}{2}\ell_1^2 \right) + p (n_1 (\ell_1 - c_1) + n_2 (\bar{\ell} - c_2)) \\ & + \lambda_{12} \left( \frac{c_1^{1-\alpha_1}}{1-\alpha_1} - \frac{1}{2}\ell_1^2 - \frac{c_2^{1-\alpha_1}}{1-\alpha_1} + \frac{1}{2}\bar{\ell}^2 \right) + \lambda_{21} \left( \frac{c_2^{1-\alpha_2}}{1-\alpha_2} - \frac{c_1^{1-\alpha_2}}{1-\alpha_2} \right), \quad (17) \end{aligned}$$

where  $p$  is the Lagrange multiplier associated with the feasibility constraint. In the sequel, we focus on economies where  $\bar{\ell}$  is high enough (a fuller treatment is in Appendix E.1). Then  $\ell_1 < \bar{\ell}$  at the nonrandom optimum, so that the incentive constraint (16) matters. In addition this constraint (16) always binds at the optimum: otherwise a small increase in  $c_1$  accompanied with a decrease in  $c_2$  so as to preserve feasibility would keep all constraints satisfied while improving the welfare of type 1 agents. It follows that

$c_1 = c_2 \equiv c$  at any nonrandom optimum. Finally for  $\bar{\ell}$  large enough, (15) is slack ( $\lambda_{12} = 0$ ). Then the Hessian of the Lagrangian with respect to the three variables  $(c_1, \ell_1, c_2)$  is

$$H = \begin{pmatrix} \lambda_{21}\alpha_2c^{-\alpha_2-1} - \alpha_1n_1c^{-\alpha_1-1} & 0 & 0 \\ 0 & -n_1 & 0 \\ 0 & 0 & -\alpha_2\lambda_{21}c^{-\alpha_2-1} \end{pmatrix}.$$

Using the expression for  $\lambda_{21}$  obtained from the first order conditions, it simplifies into

$$H = \begin{pmatrix} (\alpha_2n_2 - \alpha_1)n_1c^{-\alpha_1-1} & 0 & 0 \\ 0 & -n_1 & 0 \\ 0 & 0 & -\alpha_2n_1n_2c^{-\alpha_1-1} \end{pmatrix}.$$

The Hessian is not necessarily negative definite: the top eigenvalue is positive whenever  $n_2\alpha_2 > \alpha_1$ . Proposition 1.2 shows that this does not prevent the solution to the first order conditions to be a local maximum, since only deviations in the tangent space to the active constraints at the extremum need to be considered to assess optimality. This tangent space is defined by  $dc_1 = dc_2 \equiv dc$  and  $dc = n_1d\ell_1$ . Hence, an extremum  $(c_1, \ell_1, c_2) = (c, \ell_1, c)$  is a local maximum if and only if, for any  $dc$ ,

$$\begin{pmatrix} dc \\ dc/n_1 \\ dc \end{pmatrix}' H \begin{pmatrix} dc \\ dc/n_1 \\ dc \end{pmatrix} \leq 0.$$

This condition is always satisfied (the sum of the first and third eigenvalues of the Hessian is always negative). Therefore, by Proposition 1, any nonrandom deviation from a nonrandom extremum decreases welfare: every interior extremum of the Lagrangian is a local optimum.

Allowing for randomness enlarges the set of admissible deviations. By Proposition 2 social welfare can be improved upon by a local random deviation at the nonrandom optimum if and only if the Hessian  $H$  has at least one positive eigenvalue, i.e.,

$$n_2\alpha_2 > \alpha_1. \tag{18}$$

Economically, this happens when the least favored type 2 agents are more risk averse than type 1 and, in addition, the number  $n_1$  of type 1 agents is not too large. Then the difference in risk aversion can be exploited by the

government to increase the welfare of the most favored type 1 agents while satisfying all the constraints. Indeed randomness in the consumption of type 1 agents discourages type 2 agents from mimicking type 1. The incentive constraint (16) is relaxed: type 2 agents are willing to pay a risk premium to avoid the type 1 lottery, and this premium is larger than that of type 1. The difference can be transferred to type 1. Theorem 5 shows that one can build such a random deviation by leaving labor supply unchanged, decreasing consumption  $c_2$  for sure by a small  $dc_2$ , and changing consumption  $c_1$  by either  $dc_2 + dc$  or  $dc_2 - dc$  both with probability  $1/2$ , where  $dc$  is computed to satisfy at equality the incentive constraint (16).

In this example the Lagrangian is concave in  $\ell_1$ . It is also concave in  $c_2$  in the case where type 2 wants to imitate type 1 ( $\lambda_{12} = 0, \lambda_{21} > 0$ ). Randomness at the global optimum, if any, can only bear on the consumption of type 1 agents. That is,  $z_1^s = (c_1^s, \ell_1)$  and  $z_2^s = c_2$ . By Lemma 2 a global random optimum may only be obtained if the Lagrangian (17) has several maxima in the type 1 agents consumption, i.e., the function of  $c_1$

$$n_1 \left( \frac{c_1^{1-\alpha_1}}{1-\alpha_1} \right) - pn_1 c_1 - \lambda_{21} \frac{c_1^{1-\alpha_2}}{1-\alpha_2}.$$

has several maxima. As shown in Figure 1, for  $0 \leq \alpha_1 < 1$  and  $0 \leq \alpha_2 < 1$ , this function of  $c_1$  drawn for various values of  $\lambda_{21}$  has always a corner local maximum at  $c_1 = 0$ , is first decreasing and convex, and then concave. It has possibly at most one additional interior local maximum at some  $c_1^* > 0$ .

Thus the lottery offered to type 1 agents has at most two possible outcomes for consumption,  $c_1^1 = 0$  and the consumption  $c_1^2 = c_1^*$  which is the only interior maximum of the nonrandom Lagrangian. The global random optimum obtains when there are probabilities consistent with the equality of the function at these two points. Working from the first order conditions, we can characterize the subset of parameters for which a government who wants to maximize type 1 utility would implement a random allocation, with the after tax income of type 1 being set to zero with positive probability:

**Lemma 3.** *Suppose that  $\alpha_2 > \alpha_1$ . Let  $\bar{\ell}$  be large enough. The global second best optimum of the example economy where the government puts all the social weight on agent 1 is random if and only if*

$$n_2 \frac{\alpha_2}{1-\alpha_2} > \frac{\alpha_1}{1-\alpha_1}.$$

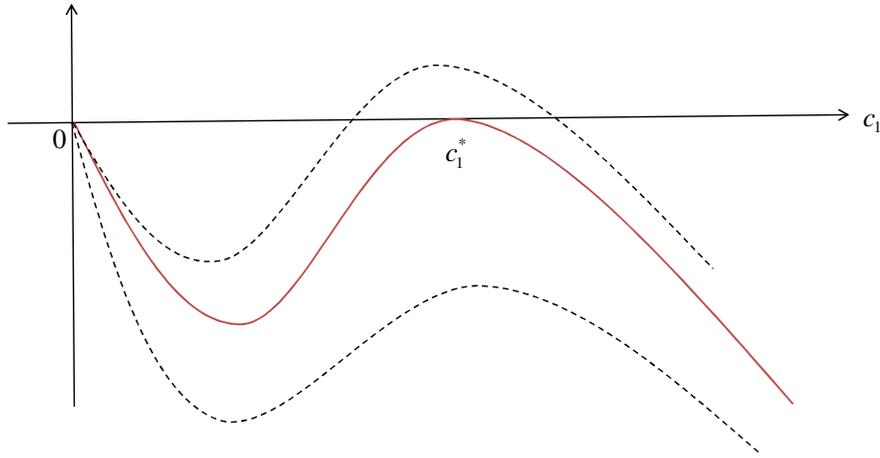


Figure 1: Lagrangian of the nonrandom problem in the isoelastic case

The consumption of type 1 agents will be random when type 2 agents are more risk averse than type 1, and their number  $n_2 = 1 - n_1$  is large enough compared to that of the type 1 agents who bear risk. The condition (18) for random improvements in the neighborhood of the deterministic optimum is of course more restrictive: whenever there is a local improvement, the global optimum cannot be deterministic.

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# Appendices

## A Non convex optimization

Consider the following constrained optimization problem:

$$\begin{cases} \max f(x) \\ g_i(x) \geq 0, \quad i = 1, \dots, I \end{cases}$$

where  $x$  is in  $\mathbb{R}^k$ . The functions  $f$  and  $g_i$ ,  $i = 1, \dots, I$ , are twice continuously differentiable. We do not impose convexity restrictions on the objective  $f(\cdot)$ , nor on the functions  $g_i(\cdot)$ . In the second best problems the objective is typically concave in  $x$ , while the incentive constraints may define a set which is not convex. We shall refer to the above program as the *deterministic problem*. The associated Lagrangian  $\mathcal{L}(x, \lambda)$  is the function  $f(x) + \lambda'g(x)$ , where  $\lambda$  is a vector of  $\mathbb{R}_+^I$ .

Some of the results that follow require that the constraints be qualified at the point we consider. Let the active constraints at a point  $x$  be the set of indices  $i$  such that  $g_i(x) = 0$ . The constraints are qualified at the point  $x$  of the domain when the family of gradient vectors  $(\nabla g_i(x))$  of the active constraints is linearly independent.

The following property is drawn from Simon and Blume (1994), Th. 18.4 and 19.8.

**Theorem 3.** *Let  $x^*$  be an interior local maximum of the deterministic problem where the constraints are qualified.*

1. *There exists  $\lambda^* \geq 0$  such that  $\nabla_x \mathcal{L}(x^*, \lambda^*) = 0$ , and the complementary slackness conditions  $\lambda_i^* g_i(x^*) = 0$  hold for all  $i$ .*
2. *The Hessian  $\nabla_x^2 \mathcal{L}(x^*, \lambda^*)$  is semi negative definite on the tangent space to the active constraints at  $x^*$ , i.e.,  $x' \nabla_x^2 \mathcal{L}(x^*, \lambda^*) x \leq 0$  for all  $x$  such that  $\nabla g_i(x^*)' x = 0$  for all constraints  $i$  with  $g_i(x^*) = 0$ .*

Let  $x^*$  be a point that satisfies the first order conditions (1) of Theorem 3. Suppose that (2) is not satisfied in the following sense: there is a direction  $x^+$  in the tangent space to the active constraints such that  $x^{+'} H x^+ > 0$ . We are interested in the feasible deviations which improve the objective in these

circumstances. To this purpose we define a *deterministic* deviation from a point  $x^*$  in  $\mathbb{R}^k$  as a continuous function  $h(t)$  from  $[0, 1]$  into  $\mathbb{R}^k$  such that  $h(0) = 0$  and

$$g_i(x^* + h(t)) = g_i(x^*) = 0$$

for all  $i$  such that  $g_i(x^*) = 0$ . A *random* deviation  $\tilde{h}(t)$  is defined similarly with the requirement

$$\mathbb{E}g_i(x^* + \tilde{h}(t)) = g_i(x^*) = 0,$$

for all active constraints.

We prove a converse to Theorem 3:

**Theorem 4.** *Let  $x^*$  be a point where the constraints are qualified that satisfies the first order conditions 1. of Theorem 3. Assume that there is a direction  $x^+$  in the tangent space to the active constraints such that  $x^{+'}Hx^+ > 0$ .*

*Then, there exist deterministic deviations  $h(t) = tx^+ + \beta(t)$  satisfying*

$$\nabla g_i(x^*)'\beta(t) + \frac{1}{2}t^2x^{+'}\nabla^2 g_i(x^*)'x^+ \approx 0,$$

*for all the active constraints  $i$  with  $g_i(x^*) = 0$ , such that  $f(x^* + h(t)) > f(x^*)$  for small enough  $t$ .*

Consider now the maximization of  $\mathbb{E}f(\tilde{x})$  with respect to  $\tilde{x}$  where  $\tilde{x}$  is a random variable with values in  $\mathbb{R}^k$ , subject to  $\mathbb{E}g_i(\tilde{x}) \geq 0, i = 1, \dots, I$ . We shall refer to this program as the *random problem*. Suppose that the Hessian  $\nabla_x^2 \mathcal{L}(x^*, \lambda^*)$  evaluated at  $x^*$  of the corresponding deterministic problem is of full rank. Then we have:

**Theorem 5.** *Let  $x^*$  be a local extremum of the deterministic problem where the constraints are qualified.*

1. *If  $\nabla_x^2 \mathcal{L}(x^*, \lambda^*)$  is negative definite,  $x^*$  is a local maximum of the random problem: there exist an open neighborhood  $V(x^*)$  of  $x^*$  in  $\mathbb{R}^k$  such that*

$$f(x^*) \geq \mathbb{E}f(\tilde{x})$$

*for all random variables  $\tilde{x}$  with support contained in  $V(x^*)$  such that  $\mathbb{E}g_i(\tilde{x}) = g_i(x^*) = 0$  for the active constraints.*

2. Suppose that  $\nabla_x^2 \mathcal{L}(x^*, \lambda^*)$  has a positive eigenvalue associated with an eigenvector  $x^+$ . There exist deviations  $\tilde{h}(t)$  equal to  $tx^+ + \beta(t)$  and to  $-tx^+ + \beta(t)$  with equal probabilities satisfying

$$\nabla g_i(x^*)' \beta(t) + \frac{1}{2} t^2 x^{+'} \nabla^2 g_i(x^*)' x^+ \approx 0,$$

for all the active constraints  $i$  with  $g_i(x^*) = 0$ , such that  $\mathbb{E}f(x^* + \tilde{h}(t)) > f(x^*)$  for small enough  $t$ .

**Proof of Theorem 5.** We start with 1. The mathematical expectation of a Taylor expansion of  $\mathcal{L}$  in a suitable neighborhood of  $x^*$  is

$$\mathbb{E}\mathcal{L}(\tilde{x}, \lambda) \approx \mathcal{L}(x^*, \lambda) + \nabla_x \mathcal{L}(x^*, \lambda)(\mathbb{E}\tilde{x} - x^*) + \frac{1}{2} \mathbb{E}(\tilde{x} - x^*)' \nabla_x^2 \mathcal{L}(x^*, \lambda)(\tilde{x} - x^*).$$

From Theorem 3.1, the second term on the right hand side is equal to zero, while the third one is negative. Furthermore, since the active constraints are satisfied at equality, while the inactive constraints hold in a suitable neighborhood,  $\mathbb{E}\mathcal{L}(\tilde{x}, \lambda) = \mathbb{E}f(\tilde{x})$  and  $\mathcal{L}(x^*, \lambda) = f(x^*)$ . This proves the desired result:

$$\mathbb{E}f(\tilde{x}) < f(x^*).$$

We now prove 2. By assumption  $\nabla_x^2 \mathcal{L}(x^*, \lambda)$  has one positive eigenvalue. Let  $x^+ \in \mathbb{R}^K$  be the associated eigenvector. From 2. of Theorem 1, note that  $x^+$  does not belong to the tangent space  $\nabla g_i(x^*)' x = 0$ .

By definition the deviations  $\tilde{h}(t)$  are such that  $\mathbb{E}g_i(x^* + \tilde{h}(t)) = g_i(x^*)$  for the active constraints, i.e.

$$\frac{1}{2} g_i(x^* + tx^+ + \beta(t)) + \frac{1}{2} g_i(x^* - tx^+ + \beta(t)) = g_i(x^*).$$

This is an implicit equation in the unknown  $\beta(t)$ . Since the constraints are qualified, we can fix the number of non zero components of  $\beta$  equal to the number of active constraints, so that the implicit function theorem applies. In a neighborhood of the origin, the function  $\beta(t)$ , with  $\beta(0) = 0$ , is well defined and  $\mathcal{C}^1$ . A Taylor expansion, observing that  $\beta(t) \pm tx^+$  is at most  $O(t)$  yields

$$\begin{aligned} \nabla g_i(x^*)' \beta(t) + \frac{1}{4} (tx^+ + \beta(t))' \nabla^2 g_i(x^*) (tx^+ + \beta(t)) \\ + \frac{1}{4} (-tx^+ + \beta(t))' \nabla^2 g_i(x^*) (-tx^+ + \beta(t)) = o(t^2), \end{aligned}$$

which can be rewritten as

$$\nabla g_i(x^*)'\beta(t) + \frac{1}{2}t^2x^{+\prime}\nabla^2g_i(x^*)x^+ + \frac{1}{2}\beta(t)'\nabla^2g_i(x^*)\beta(t) = o(t^2). \quad (19)$$

It follows that  $\beta(t) = O(t^2)$ . Thus, for  $t$  close enough to 0, the last term in the left-hand side can be neglected ( $\beta(t)'\nabla^2g_i(x^*)\beta(t) = O(t^4)$ ), and one gets

$$\nabla g_i(x^*)'\beta(t) + \frac{1}{2}t^2x^{+\prime}\nabla^2g_i(x^*)x^+ \approx 0.$$

This is the expression given in Theorem 3 that must be satisfied by the deterministic component  $\beta(t)$  of the deviation. Now, by definition of a deviation,

$$\begin{aligned} & \frac{1}{2}f(x^* + tx^+ + \beta(t)) + \frac{1}{2}f(x^* - tx^+ + \beta(t)) - f(x^*) \\ &= \frac{1}{2}\mathcal{L}(x^* + tx^+ + \beta(t)) + \frac{1}{2}\mathcal{L}(x^* - tx^+ + \beta(t)) - \mathcal{L}(x^*) \\ &\approx \nabla\mathcal{L}(x^*)'\beta(t) + \frac{1}{2}t^2x^{+\prime}\nabla^2\mathcal{L}(x^*)x^+ + \frac{1}{2}\beta(t)'\nabla^2\mathcal{L}(x^*)\beta(t). \end{aligned}$$

Since  $x^*$  is a deterministic maximum,  $\nabla\mathcal{L}(x^*) = 0$ . Moreover, by (19),  $\beta(t) = O(t^2)$ , so that  $\beta(t)'\nabla^2\mathcal{L}(x^*)\beta(t) = O(t^4)$ . As a result,

$$\mathbb{E}f(x^* + \tilde{h}(t)) - f(x^*) \approx \frac{1}{2}t^2x^{+\prime}\nabla^2\mathcal{L}(x^*)x^+ > 0, \quad (20)$$

since  $x^+$  is an eigenvector associated with a positive eigenvalue of  $\nabla^2\mathcal{L}(x^*)$ .  $\square$

**Proof of Theorem 4.** The deviation is deterministic,  $h(t) = tx^+ + \beta(t)$ , and the implicit function theorem, with well chosen non zero components of  $\beta(t)$ , gives

$$\nabla g_i(x^*)'\beta(t) + \frac{1}{2}t^2x^{+\prime}\nabla^2g_i(x^*)x^+ + \frac{1}{2}x^{+\prime}\nabla^2g_i(x^*)\beta(t) \approx 0.$$

As a consequence  $\beta(t)$  is  $O(t^2)$ . The expansion of  $\mathcal{L}(x^* + tx^+ + \beta(t)) - \mathcal{L}(x^*) = f(x^* + h(t)) - f(x^*)$  shows that it is of the order of

$$\frac{1}{2}t^2x^{+\prime}\nabla^2\mathcal{L}(x^*)x^+ > 0.$$

$\square$

## B The Lerner case

When the number  $I$  of active constraints at the deterministic optimum is greater than the number  $k$  of nonzero components of the vector  $\beta(t)$ , the implicit function theorem does not apply and the argument in the proof of Theorem 5 does not work. A possible way out however is to restrict attention to a subset of  $k$  qualified constraints, and hope that the  $I - k$  remaining constraints are satisfied when one implements the random deviation exhibited in Theorem 5.

In order to illustrate this point, let us consider the example of the redistribution problem in a Lerner world studied by Pestieau, Possen and Slutsky (2002). In this example, there are two agents,  $i = 1, 2$ , and one consumption good. Utility of agent  $i$  when she consumes  $c$  units of the good is  $u_i(c)$ . Each agent is initially endowed with  $y$  units of the good. The type  $i$  is not publicly observed. Therefore, if the tax authority wants to allocate (nonrandomly)  $c_i$  units of the good to agent  $i$ , the two incentive constraints

$$u_1(c_1) \geq u_1(c_2),$$

$$u_2(c_2) \geq u_2(c_1),$$

must be met. In addition, feasibility requires

$$c_1 + c_2 \leq 2y.$$

In this example, the vector  $(c_1, c_2)$  is two-dimensional, i.e.,  $k = 2$ . Still, at the nonrandom optimum, the three constraints hold at equality: incentive compatibility actually requires  $c_1 = c_2$  and feasibility implies  $c_1 = c_2 = y$ .

We can study the usefulness of randomization by adapting the argument in the proof of Theorem 5. Assume that the tax authority only cares about the welfare of agent 1. Random redistribution toward this agent, if possible, is likely to be limited by the fact that agent 2 will mimic agent 1, when she receives too much. Let us therefore restrict our attention to the following subset of the active constraints at the nonrandom optimum:

$$u_2(c_2) = \mathbb{E}u_2(c_1)$$

and

$$\mathbb{E}c_1 + c_2 = y.$$

We shall verify *ex post* that the remaining constraint,  $u_1(c_2) \leq \mathbb{E}u_1(c_1)$ , is indeed satisfied, with a Lagrange multiplier equal to 0. The corresponding Lagrangian is

$$\mathcal{L} = u_1(c_1) + \lambda [u_2(c_2) - u_2(c_1)] + \rho (y - c_1 - c_2).$$

Its Hessian at the deterministic allocation  $c_1 = c_2 = y$  is

$$\nabla^2 \mathcal{L} = \begin{pmatrix} u_1''(y) - \lambda u_2''(y) & 0 \\ 0 & \lambda u_2''(y) \end{pmatrix}$$

with from the first order condition

$$\lambda = \frac{1}{2} \frac{u_1'(y)}{u_2'(y)}.$$

The eigenvalue  $\lambda u_2''(y)$  is always negative. The other one is positive if and only if

$$\frac{u_1''(y)}{u_1'(y)} > \frac{1}{2} \frac{u_2''(y)}{u_2'(y)}.$$

Let  $x^+ = (1, 0)'$  stand for the eigenvector associated with this eigenvalue. Consider the lottery with the two outcomes  $tx^+ + \beta(t)$  and  $-tx^+ + \beta(t)$ , each one occurring with the same probability, and  $\beta(t) = (\beta_1(t), \beta_2(t))'$ . The first component of the two-dimensional vectors  $\pm tx^+ + \beta(t)$  gives the consumption of agent 1, and the second the consumption of agent 2. For  $t$  close enough to 0, the vector  $\beta(t)$  is characterized by the active constraints as in Theorem 5:

$$\mathbb{E}u_2(c_1) = u_2(c_2),$$

$$\mathbb{E}c_1 + c_2 = 2y,$$

which become

$$\frac{1}{2} [u_2(y + t + \beta_1(t)) + u_2(y - t + \beta_1(t))] = u_2(y + \beta_2(t)),$$

$$\beta_1(t) + \beta_2(t) = 0.$$

This yields

$$\beta_1(t) = -\beta_2(t) \approx -\frac{1}{4} t^2 \frac{u_2''(y)}{u_2'(y)}.$$

When the Hessian has a positive eigenvalue, this randomization increases the welfare of agent 1:

$$Eu_1(c_1) - u_1(y) \approx u_1'(y)\beta_1(t) + \frac{1}{2}u_1''(y)t^2 \approx \frac{1}{2}u_1'(y) \left[ \frac{u_1''(y)}{u_1'(y)} - \frac{1}{2} \frac{u_2''(y)}{u_2'(y)} \right] t^2 > 0.$$

There remains to check that type 1 incentive constraint is satisfied. But we have just seen that  $Eu_1(c_1) > u_1(y)$  and since  $\beta_2(t)$  is negative,  $c_2$  is smaller than  $y$ . This completes the analysis of this variant of the Lerner model.

## C Proof of Lemma 1

We start with 1. The number we are looking for is the minimal number of strictly positive components in the solution to (10). By contradiction suppose that  $\pi$  has  $\ell$  positive components, so that  $A$  and  $B$  are respectively of dimension  $\ell$  and  $(K + I^2) \times \ell$  in (10). Then if  $\ell > (K + I^2)$ , the kernel of  $B$  is at least of dimension  $(K + I^2) - \ell$ . Take an element of the kernel, say  $q$ . By definition  $Bq = 0$ . Also note that because of the constraints  $\sum_{i=1}^{S_i} \pi_i^s = 1$ ,  $q$  has at least one negative and one positive components. Consider now the vector  $\pi + \lambda q$  for  $\lambda$  in  $\mathbb{R}$ . It satisfies all the constraints: in particular the slack incentive constraints are satisfied for all  $\lambda$ , with the same value of  $b$ . Also  $A'q$  is equal to zero: otherwise  $\pi + \lambda q$  for small positive or negative  $\lambda$  would lead to a higher objective than  $\pi$ , a contradiction with the fact that  $\pi$  is the optimum. While  $\pi$  has all its components positive, there is a  $\lambda_+ > 0$  such that  $\pi + \lambda_+ q$  has all its components non negative and at least one zero component, and yield the same value of the objective. Iterating on  $\ell$  yields the desired result.

A similar argument shows 2. The subsystem analogous to (10) but reduced to some  $\pi_i$  has row dimension equal to  $K + 2(I - 1) + 1$ :  $K$  is the number of feasibility constraints; agent  $i$  can envy the other  $I - 1$  agents, who in turn can envy her, so that there are  $2(I - 1)$  incentive constraints; finally the probabilities sum up to 1. The result follows.  $\square$

## D Patterns of envy

The suggested procedure to find the optimal allocations involves considering in turn all the possible subsets of binding incentive constraints among the possible  $I(I - 1)$ , and choosing the one that yields the largest value of the objective. It turns out that one can reduce the burden of these computations by looking more closely at the problem and finding patterns of the binding incentive constraints at the optimum.

Under qualification of constraints, we know that any solution of the non-convex program (1) satisfies the necessary first-order condition (see e.g. Simon and Blume (1994) Theorem 18.1) that for all small (possibly random) changes ( $\tilde{z}$ ) in the allocation, the change in the Lagrangian (2) is small, i.e.  $o(\tilde{z})$ .

**Lemma 4.** *At a qualified solution of (1), an agent does not envy someone whose expected trade has a lower value than her:*

$$p' \mathbb{E} \tilde{z}_j > p' \mathbb{E} \tilde{z}_i \text{ implies } V_j(\tilde{z}_j) > V_j(\tilde{z}_i).$$

**Proof:** Otherwise,  $V_j(\tilde{z}_j) = V_j(\tilde{z}_i)$ . For  $q$  in  $[0, 1]$ , consider the allocation  $\tilde{z}^q$  that is identical to  $\tilde{z}$  for all agents different from  $j$ , and gives  $\tilde{z}_i$  with probability  $q$  and  $\tilde{z}_j$  with probability  $1 - q$  to type  $j$ . All agents have identical utility levels at  $\tilde{z}$  and  $\tilde{z}^q$ . The incentive constraints are satisfied since, for all  $k$ :

$$\left. \begin{array}{l} V_k(\tilde{z}_k) \geq V_k(\tilde{z}_i) \\ V_k(\tilde{z}_k) \geq V_k(\tilde{z}_j) \end{array} \right\} \implies V_k(\tilde{z}_k) \geq qV_k(\tilde{z}_i) + (1 - q)V_k(\tilde{z}_j) = V_k(\tilde{z}_j^q).$$

Now, for any (small) positive  $q$ , the change in the Lagrangian is

$$d\mathcal{L} = p' \sum_k \mathbb{E}(\tilde{z}_k^q - \tilde{z}_k) = qp' \mathbb{E}(\tilde{z}_j - \tilde{z}_i) = O(q),$$

which yields the desired contradiction.  $\square$

**Remark 3.** The proof of Lemma 4 relies on the possibility of random deviations in  $\mathcal{M}(Z_i)$ . To prove a similar result for solutions of the non random program (1'), we seem to need a (mild) additional condition, akin to a non-satiation assumption. This is in line with Proposition 1 of Brito, Hamilton, Slutsky, and Stiglitz (1990) which relies on a transformation of the second best program (see their footnote 6, page 65), valid under non-satiation.

Let  $LF$  be the subset of agents who are least favored, in the sense that

$$LF = \{i : p' \mathbb{E} \tilde{z}_i = \min_j p' \mathbb{E} \tilde{z}_j\},$$

so that from Lemma 4 no one outside  $LF$  envies them. Then

**Lemma 5.** *At a qualified solution of (1), for all  $i$  in  $LF$ ,  $z_i$  maximizes  $v_i(z)$  under the budget constraint  $p'(z - \mathbb{E} \tilde{z}_i) \leq 0$ .*

*As a consequence, the net trade of the agents in  $LF$  is non random. For all  $i$  in  $LF$ ,*

$$z_i = \mathbb{E} \tilde{z}_i.$$

**Proof:** From Lemma 4, for all  $i$  in  $LF$  and all  $j$  not in  $LF$

$$V_j(\tilde{z}_j) > V_j(\tilde{z}_i),$$

which implies  $\lambda_{ji}$  is zero.

Therefore considering only the net trades of the agents in  $LF$  and using the fact that the constraints (1a) are linear, the program can be rewritten as the maximization of

$$\sum_{i \in LF} a_i n_i V_i(\tilde{z}_i) - \sum_{i \in LF} n_i p' \mathbb{E} \tilde{z}_i,$$

subject to the incentive constraints among  $LF$

$$V_i(\tilde{z}_i) \geq V_i(\tilde{z}_j) \quad \text{for all } i, j \in LF, i \neq j,$$

and outside  $LF$

$$V_i(\tilde{z}_i) \geq V_i(\tilde{z}_j) \quad \text{for all } i \in LF, j \notin LF.$$

Now the maximum feasible utility for  $i$ , knowing that  $i$  belongs to  $LF$ , is obtained by selecting  $z_i = z_i^*$  as stated in the Lemma. By construction it satisfies the incentive constraints among  $LF$  members. Also it satisfies the final incentive constraints, since otherwise one would have  $v_i(\mathbb{E} \tilde{z}_i) \geq V_i(\tilde{z}_i) \geq V_i(\tilde{z}_j) > v_i(z_i^*)$ , which would imply  $p' \mathbb{E} \tilde{z}_i > p_i^* = p' \mathbb{E} \tilde{z}_i$ , a contradiction.  $\square$

**Corollary 1.** *At a solution of (1) the vector  $p$  has all its components strictly positive.*

**Lemma 6.** *Consider an agent  $j$  not in  $LF$  at a qualified solution of (1). Then she satisfies one of the two following properties:*

1. *either there is an agent  $i$ , whose net trade has a lower expected value ( $p'\mathbb{E}\tilde{z}_j > p'\mathbb{E}\tilde{z}_i$ ), who envies her:*

$$V_i(\tilde{z}_i) = V_i(\tilde{z}_j).$$

2. *or  $z_j$  maximizes  $v_j(z)$  under the budget constraint  $p'(z - z_j) \leq 0$ .*

**Proof:** We show that 2. holds whenever 1. does not. Suppose that for all  $i$  such that  $p'\mathbb{E}\tilde{z}_j > p'\mathbb{E}\tilde{z}_i$  we have

$$V_i(\tilde{z}_i) > V_i(\tilde{z}_j).$$

Then, from Lemma 4 all the incentive constraints featuring  $\tilde{z}_j$  and agents  $i$  such that  $p'\mathbb{E}\tilde{z}_i \neq p'\mathbb{E}\tilde{z}_j$  are slack. From the expression of the Lagrangian, then  $z_j$  maximizes  $v_j(z)$  under the budget constraint  $p'(z - z_j) = 0$ .  $\square$

Note that in the case where there is a single good,  $z$  in  $\mathbb{R}$ , all agents  $j$  not in  $LF$ , if any, have a random allocation. Otherwise,  $z_j > z_{LF}$  would imply that all the agents in  $LF$  would prefer  $z_j$  to their own allocation.

## E The example with isoelastic preferences

### E.1 Properties of the nonrandom optimum

We have to consider separately the cases where  $\ell_1 > \bar{\ell}$ , and  $\ell_1 \leq \bar{\ell}$ , and in the latter to distinguish the situations where the incentive constraints of type 1 agents bind or not.

#### E.1.1 $\ell_1 > \bar{\ell}$ .

The program reduces to the maximization of type 1 utility

$$\frac{c_1^{1-\alpha_1}}{1-\alpha_1} - \frac{1}{2}\ell_1^2$$

subject to the feasibility constraint where type 2 consumption is reduced to zero

$$c_1 = \ell_1 + \frac{n_2 \bar{\ell}}{n_1},$$

and the inequality  $\ell_1 > \bar{\ell}$ .

There are two cases, depending on whether the last inequality binds or not (this is an open inequality, so that strictly speaking, it cannot bind: we shall denote the value of  $\ell_1$  at the boundary as  $\bar{\ell}_+$ , to denote a number above  $\bar{\ell}$  but close to it).

1) The constraint binds. Then

$$\ell_1 = \bar{\ell}_+ \quad c_1 = \frac{\bar{\ell}_+}{n_1}.$$

2) The constraint does not bind. The program is a simple concave program, with first order condition

$$c_1^{-\alpha_1} = \ell_1.$$

It is easy to check that the unique solution in  $\ell_1$  is decreasing in  $\bar{\ell}$ , and equal to  $\bar{\ell}$  when  $\bar{\ell} = n_1^{\frac{\alpha_1}{1+\alpha_1}}$ .

Therefore for  $\bar{\ell} \geq n_1^{\frac{\alpha_1}{1+\alpha_1}}$ , the constraint binds and the utility of type 1 agents is equal to

$$\frac{1}{1-\alpha_1} \left( \frac{\bar{\ell}_+}{n_1} \right)^{1-\alpha_1} - \frac{1}{2} \bar{\ell}_+^2.$$

For large enough  $\bar{\ell}$ , this is negative, and therefore cannot be the optimum, since  $c_1 = \ell_1 = 0$  is always feasible.

### E.1.2 $\ell_1 \leq \bar{\ell}$ .

The first order conditions of the problem are

$$(n_1 + \lambda_{12})c^{-\alpha_1} - pn_1 - \lambda_{21}c^{-\alpha_2} = 0, \quad (21)$$

$$-pn_2 + (\lambda_{21} - \lambda_{12})c^{-\alpha_2} = 0, \quad (22)$$

$$-n_1\ell_1 + n_1p - \lambda_{12}\ell_1 = 0. \quad (23)$$

The feasibility constraint reduces to

$$c = n_1\ell_1 + n_2\bar{\ell}.$$

**The incentive constraint of agent 1 does not bind.** When one ignores the incentive constraint (15) of agent 1, i.e.  $\lambda_{12} = 0$ , the optimization problem is concave, and its solution is characterized by the first-order conditions. Eliminating  $\lambda_{21}$  and  $p$  yields

$$F(c) = n_1 c^{-\alpha_1} - \frac{c}{n_1} + \frac{n_2 \bar{\ell}}{n_1} = 0.$$

The function  $F$  is decreasing from  $+\infty$  to  $-\infty$ , when  $c$  increases from 0 to  $\infty$ , so that it has a single zero, say  $c^*$ . Both agents have the same consumption  $c^*$  such that  $F(c^*) = 0$ .

Summing up (21) and (22) gives  $n_1 c^{-\alpha_1} = p$ . Substituting  $p$  into (21) shows that the corresponding value of  $\lambda_{21}$ ,  $\lambda_{21}^*$ , is equal to  $n_1 n_2 c^{*(\alpha_2 - \alpha_1)}$ .

We have to check that the incentive constraint that we ignored,  $v_1(c_1) \geq v_1(c_2)$ , is satisfied. Since both agents have the same consumptions, this reduces to the inequality  $\ell_1 \leq \bar{\ell}$ . From the feasibility constraint this is equivalent to  $c \leq \bar{\ell}$ , or  $F(\bar{\ell}) = n_1 \bar{\ell}^{-\alpha_1} - \bar{\ell} \leq 0$ . Therefore, (15) is not binding if and only if  $\bar{\ell} \geq n_1^{1/(1+\alpha_1)}$ , which justifies the presentation in the text.

**The incentive constraint of agent 1 binds.** Finally, it is possible that both incentive constraints bind. Then the allocation is given by  $\ell_1 = \bar{\ell}$  and  $c_1 = c_2 = \bar{\ell}$ . Summing up (21) and (22) gives  $p = n_1 \bar{\ell}^{-\alpha_1}$ . Substituting  $p$  into (23) yields  $\lambda_{12} = n_1(n_1 \bar{\ell}^{-1-\alpha_1} - 1)$ . Finally, from (22),  $\lambda_{21} - \lambda_{12} = n_1 n_2 \bar{\ell}^{\alpha_2 - \alpha_1}$ . In this case, using the expressions for  $\lambda_{21}$  and  $\lambda_{12}$  derived above, the Hessian becomes

$$H = \begin{pmatrix} n_1^2 c^{-\alpha_1 - 1} [(\alpha_2 - \alpha_1) c^{-\alpha_1 - 1} - \alpha_2] & 0 & 0 \\ 0 & -n_1^2 c^{-\alpha_1 - 1} & 0 \\ 0 & 0 & n_1^2 c^{-\alpha_1 - 1} \left[ -(\alpha_2 - \alpha_1) c^{-\alpha_1 - 1} + \alpha_2 - \frac{\alpha_1}{n_1} \right] \end{pmatrix}.$$

The shape of the Hessian is similar to that described in the text where only (16) binds: the sum of the top and bottom eigenvalues is negative. But it is unlikely that the top eigenvalue be positive, which would require a small value of  $\bar{\ell}$  so that  $(\alpha_2 - \alpha_1) \bar{\ell}^{-\alpha_1 - 1} > \alpha_2$ .

## E.2 The global random optimum: Proof of Lemma 3

We first write the conditions that have to be satisfied by a global optimum, where agent 2 is the least favored agent and receives a non random transfer,

while agent 1 may get a random allocation. We note that labor supply does not enter the utility function of agent 2, so that  $\ell_1$  is not an argument of the incentive constraint: the Lagrangian is concave in  $\ell_1$ , which is non random at the optimum. Randomness, if any, can only bear on the consumption of agent 1.

Supposing that  $c_1$  is random. From Lemma 2 all the values  $c_1^s$  attained with positive probability must be such that

$$G(c) = n_1 \frac{c^{1-\alpha_1}}{1-\alpha_1} - pn_1 c - \lambda_{21} \frac{c^{1-\alpha_2}}{1-\alpha_2} = \mu = \max_x G(x). \quad (24)$$

This implies the first order condition

$$n_1 c^{-\alpha_1} - pn_1 - \lambda_{21} c^{-\alpha_2} \leq 0, \quad (25)$$

with equality if  $c > 0$ . Given a positive  $(p, \lambda_{21})$ , with  $\alpha_1 < \alpha_2$ , the function  $G(c)$  at the left hand side of (24), whose second derivative has the same sign as  $\lambda_{21}\alpha_2 - n_1\alpha_1 c^{\alpha_2-\alpha_1}$ , is convex in the interval  $[0, \bar{c}]$  and concave for  $c > \bar{c}$ , where  $\bar{c} = (\lambda_{21}\alpha_2/n_1\alpha_1)^{1/(\alpha_2-\alpha_1)}$ . It is equal to zero at the origin, with slope  $-\infty$ . Hence, the only possible consumptions which may constitute the support of the lottery offered to agent 1 are 0 and the positive local maximum  $c$  solution to (25), when it exists. To have a random optimum, we must have these two consumption levels in the support of the lottery. Evaluated at the origin, (24) gives  $\mu = 0$ . We shall note  $\pi$ ,  $0 < \pi < 1$ , the probability of the consumption of agent 1 being equal to  $c_1$ , different from 0.

We can write the system of equations defining the random optimum as follows. The random consumption of type 1 satisfies

$$\begin{aligned} n_1 \frac{c_1^{1-\alpha_1}}{1-\alpha_1} - pn_1 c_1 - \lambda_{21} \frac{c_1^{1-\alpha_2}}{1-\alpha_2} &= 0, \\ n_1 c_1^{-\alpha_1} - pn_1 - \lambda_{21} c_1^{-\alpha_2} &= 0. \end{aligned}$$

The first order conditions corresponding to the derivatives of the Lagrangian with respect to  $c_2$  and  $\ell_1$  are:

$$\begin{aligned} \lambda_{21} c_2^{-\alpha_2} - pn_2 &= 0 \\ pn_1 &= n_1 \ell_1 \end{aligned}$$

Feasibility takes the form

$$n_1\pi c_1 + n_2c_2 = n_1\ell_1 + n_2\bar{\ell},$$

while the binding incentive constraint can be written as

$$c_2 = \pi^{\frac{1}{1-\alpha_2}} c_1$$

**Step 1.** Eliminate  $c_2$  and  $\ell_1$  using

$$c_2 = \pi^{\frac{1}{1-\alpha_2}} c_1 \quad n_1\ell_1 = pn_1$$

One gets:

$$\begin{aligned} n_1 \frac{c_1^{1-\alpha_1}}{1-\alpha_1} - pn_1c_1 - \lambda_{21} \frac{c_1^{1-\alpha_2}}{1-\alpha_2} &= 0 \\ n_1c_1^{-\alpha_1} - pn_1 - \lambda_{21}c_1^{-\alpha_2} &= 0 \\ \lambda_{21}\pi^{-\frac{\alpha_2}{1-\alpha_2}} c_1^{-\alpha_2} - pn_2 &= 0 \\ \left( n_1\pi + n_2\pi^{\frac{1}{1-\alpha_2}} \right) c_1 &= pn_1 + n_2\bar{\ell}. \end{aligned}$$

**Step 2.** Eliminate  $p$  through

$$pn_2 = \lambda_{21}\pi^{-\frac{\alpha_2}{1-\alpha_2}} c_1^{-\alpha_2} \Leftrightarrow pn_1 = \frac{n_1}{n_2} \lambda_{21}\pi^{-\frac{\alpha_2}{1-\alpha_2}} c_1^{-\alpha_2}$$

One gets

$$\begin{aligned} n_1 \frac{c_1^{\alpha_2-\alpha_1}}{1-\alpha_1} - \frac{n_1}{n_2} \lambda_{21}\pi^{-\frac{\alpha_2}{1-\alpha_2}} - \frac{\lambda_{21}}{1-\alpha_2} &= 0 \\ n_1c_1^{\alpha_2-\alpha_1} - \left( 1 + \frac{n_1}{n_2} \pi^{-\frac{\alpha_2}{1-\alpha_2}} \right) \lambda_{21} &= 0 \\ \left( n_1\pi + n_2\pi^{\frac{1}{1-\alpha_2}} \right) &= \frac{n_1}{n_2} \lambda_{21}\pi^{-\frac{\alpha_2}{1-\alpha_2}} c_1^{-1-\alpha_2} + n_2\bar{\ell}c_1^{-1}. \end{aligned}$$

**Step 3.** The difference between the two first equations gives

$$\lambda_{21} = n_1 \frac{\alpha_1}{1-\alpha_1} \frac{1-\alpha_2}{\alpha_2} c_1^{\alpha_2-\alpha_1}$$

Reintroducing  $\lambda_{21}$  into the first and third equations leads to (since  $\alpha_2 > \alpha_1$ , the term between brackets on the right hand side of (26) is positive)

$$\pi = \left[ \frac{n_2}{n_1} \left( \frac{1 - \alpha_1}{\alpha_1} \frac{\alpha_2}{1 - \alpha_2} - 1 \right) \right]^{-\frac{1 - \alpha_2}{\alpha_2}} \quad (26)$$

$$-n_1\pi - n_2\pi^{\frac{1}{1 - \alpha_2}} + \frac{n_1}{n_2} n_1 \frac{\alpha_1}{1 - \alpha_1} \frac{1 - \alpha_2}{\alpha_2} \pi^{\frac{-\alpha_2}{1 - \alpha_2}} c_1^{-1 - \alpha_1} + n_2 \bar{\ell} c_1^{-1} = 0. \quad (27)$$

Now  $\pi < 1$  is equivalent to

$$\frac{1 - \alpha_1}{\alpha_1} \frac{\alpha_2}{1 - \alpha_2} - 1 > \frac{n_1}{n_2},$$

which gives the formula of Lemma 3. The left hand side of (27) is decreasing in  $c_1$  from  $+\infty$  to a negative number, and consequently has a single root. Since the expression is increasing in  $\bar{\ell}$ , the root is also an increasing function of  $\bar{\ell}$ .

It remains to make sure that the incentive constraint of agent 1 is satisfied:

$$\pi \frac{c_1^{1 - \alpha_1}}{1 - \alpha_1} - \frac{\ell_1^2}{2} \geq \frac{c_2^{1 - \alpha_1}}{1 - \alpha_1} - \frac{\bar{\ell}^2}{2}, \quad (28)$$

where  $\pi$  is given by (10),  $c_1$  satisfies (27) and  $\ell_1$  follows from the feasibility constraint

$$[n_1\pi + n_2\pi^{\frac{1}{1 - \alpha_2}}]c_1 = n_1\ell_1 + n_2\bar{\ell}.$$

Using (27), the feasibility constraint is equivalent to

$$\ell_1 = \frac{n_1}{n_2} \frac{\alpha_1}{1 - \alpha_1} \frac{1 - \alpha_2}{\alpha_2} \pi^{\frac{-\alpha_2}{1 - \alpha_2}} c_1^{-\alpha_1}.$$

Substituting  $c_2$  and  $\ell_1$  as functions of  $c_1$  into (28) yields

$$\pi \left( 1 - \pi^{\frac{\alpha_2 - \alpha_1}{1 - \alpha_2}} \right) \frac{c_1^{1 - \alpha_1}}{1 - \alpha_1} - \frac{1}{2} \left( \frac{n_1}{n_2} \frac{\alpha_1}{1 - \alpha_1} \frac{1 - \alpha_2}{\alpha_2} \pi^{\frac{-\alpha_2}{1 - \alpha_2}} c_1^{-\alpha_1} \right)^2 + \frac{\bar{\ell}^2}{2} \geq 0.$$

Since  $c_1$  is increasing in  $\bar{\ell}$ , the three terms of the above inequality are increasing in  $\bar{\ell}$ , so that the inequality is equivalent to

$$\bar{\ell} \geq L,$$

for some  $L$ , possibly zero, which depends on the parameters of the model.  $\square$