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## Euler, infinitesimals and limits

Giovanni Ferraro

**Abstract.** This paper examines the Eulerian notion of infinitesimal or evanescent quantity and compares it with the modern notion of limit and non-standard analysis concepts. The Eulerian infinitesimal, when interpreted using the conceptual instruments available to modern mathematics, seems to be a fluid mixture of different elements, a continuous leap from a vague idea of limit to a confused notion of infinitesimal. However, while the modern notions of limit, hyperreal numbers etc. derive from precise definitions or constructions and are developed within non-equivocal contexts, Euler did not define the notion of evanescent quantity, either explicitly or implicitly: it is substantially a 'common notion', directly derived from the experience of physical and geometric quantities, which can diminish until they vanish. Infinitesimals are fictions, useful instruments for dealing with finite quantities, and thus Euler could state that the algorithm of the calculus did not concern differentials but functions. The vague and imprecise Eulerian notions can be reformulated in modern terms only by introducing extraneous elements into Euler's work and changing the logical and conceptual framework of his mathematics.

### 1. Introduction

In this paper I investigate the Eulerian notion of evanescent or infinitesimal quantity with the aim of contributing to the understanding of the foundations of calculus in the eighteenth century.<sup>1</sup> When interpreted in the light of modern theories such as real analysis or non-standard analysis, the Eulerian notion appears to be an extremely confused mixture of different elements which are sometimes irreducible to each other. Nevertheless, the comparison with modern theories is problematic since the latter use concepts (such as set, real number, structure) which do not exist in eighteenth-century mathematics, which was based on the classic notion of quantity. The modern notions of limit, hyperreal numbers etc. derive from precise definitions or constructions and are inserted in non-equivocal contexts. Euler, however, does not define the notion of evanescent quantity, either explicitly or implicitly: it is substantially a 'common notion', directly derived from the experience of physical and geometric quantities, which can diminish until they vanish. Rather than making use of modern concepts, the Eulerian use of infinitesimals can be better understood if they are thought of as 'fictions', represented by symbols with which one operated in analogy with 'real' quantities. Unlike modern mathematics, Euler dealt with two species of objects, which differ ontologically: actually existing quantities (in the sense that these were conceived of as the abstract but immediate representations of physical objects) and fictitious quantities. Actually existing quantities were expressed in the calculus by means of functions and thus Euler could think that infinitesimal or differential (Euler *de facto* reduced differentials to infinitesi-

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<sup>1</sup> I have dealt elsewhere with other aspects of the Eulerian foundations of the calculus. in particular the notions of the sum of a series and of a function (see [Ferraro 2000a and 2000b]).

mals)<sup>2</sup> were only an instrument for dealing with functions and that the algorithm of the calculus did not transform differentials into differentials but functions into functions.

This paper was originated from the reading of some recent papers, such as [Laugwitz ...], [McKinzie-Tuckey 1997] and [Pourciau 2001]. In these writings the authors recast the early procedures directly in terms of the modern foundation of analysis or interpret the earlier results in terms of modern theory of non-standard analysis and understand the results in the light of this later context. I felt necessity to present a different point of view and to show that the attempt to specify Euler's vague and ambiguous notions by applying modern concepts is only possible if elements are added which essentially alien to it, and thus transforming Eulerian mathematics into something wholly different. Of course, I am not stating that eighteenth-century mathematics had to be investigated without considering modern theories. Modern concepts are essential for understanding eighteenth-century notions and why these led to meaningful results, even when certain procedures, puzzling from the present views, were used. However, eighteenth-century analysis had its own principles, different from those of modern analysis: a unproblematic translation of certain chapters in the history of mathematics into modern terms tacitly assumes that the same logical and conceptual framework guiding work in modern mathematics also guided work in past mathematics.

## 2. Infinitesimals, infinites, evanescent quantities and infinitely increasing quantities

In his [1755] Euler claimed the absolute correctness of the calculus: "geometric rigour escapes even the slightest error".<sup>3</sup> These words echo Newton's and Berkeley's statements: "errors are not to be disregarded in mathematics, no matter how small".<sup>4</sup> Based on this assertion, Berkeley, in his famous *The Analyst*, argued that the calculus, in contrast to other branches of mathematics, was not an exact theory. He did not cast any doubt upon the usefulness of the calculus in solving many problems of physics or geometry; nevertheless, he believed that it did not possess solid foundations. In particular, he criticised the principle of cancellation of higher-order differentials (in other words, the rule according to which  $d^n x + d^m x$  was equal to  $d^n x$  if  $n < m$ ). In Berkeley's opinion, by writing  $d(x^2) = (x+dx)^2 - x^2 = 2xdx + dx^2 = 2xdx$ , firstly one considered  $dx \neq 0$ , then one let  $dx=0$ : thus there was a negation of the first hypothesis and an elementary logical principle was violated. The same criticism had been made of the method of first and last reasons: indeed, in the expression  $((x+h)^n - x^n)/h$ , one first considered a finite increase  $h \neq 0$ , but one then let  $h=0$  and obtained  $nx^{n-1}$  [Berkeley 1734, sections 13-18]. Berkeley thought that the calculus achieved correct results only thanks to a compensation of mis-

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<sup>2</sup> The differential was regarded as an infinitesimal obtained by making the increment  $\omega$  of a variable quantity tend to zero (see Euler [1755, pp. 5-6 and p.84]). Euler often simply treats differentials and infinitesimals as the same thing (for instance, see [Euler 1755, p.70]).

<sup>3</sup> "[R]igor geometricus etiam a tantillo errore abhorret" [Euler 1755, 6].

<sup>4</sup> "Errores quam minimi in rebus mathematicis non sunt contemnendi" [Newton 1704, 334]. In §.4 of [1734] Berkeley emphasised: "It is said that the minutest errors are not to be neglected in mathematics" and, in §.9, he quoted Newton's Latin words.

takes. This justification was considered by Euler to be insufficient: he observed that by ignoring infinitely small quantities, but not noughts, one could still commit extremely serious errors. Neither could the correctness of the calculus be based on an exhibition of examples in which differential calculus and elementary geometry reached the same results [Euler 1755, 6]. In his writings, Euler asserted the independence of the calculus from geometry as a result of which one could not use geometry to establish a solid basis of the analysis.

The crux of the question lay in knowing what meaning to attribute to the equation  $a+dx=a$ . The exactness of mathematics required, according to Euler, that the differential  $dx$  should be precisely equal to 0: simply by assuming that  $dx=0$ , the outrageous attacks on the calculus would be shown to lack any basis. Those who considered infinitesimal quantities as different from nothing could effectively be accused of ignoring geometric rigour and so the deductive process based on such assumptions were really in doubt [Euler 1755, 6].

Nevertheless, it is difficult to base the calculus on differentials understood as zeros: indeed, if we examine the definitions of infinitesimal that Euler offers in the third chapter of *Institutiones calculi differentialis*, it can immediately be seen that they cannot be reduced to the mere statement that an infinitesimal is zero. In [1755] Euler asserts: "There is no doubt that every quantity can be diminished until it vanishes completely and is reduced to nothing. But an infinitely small quantity is simply an evanescent quantity and therefore actually equal to zero."<sup>5</sup> Immediately afterwards, he argues that an infinitesimal can also be defined as a quantity which is less than any assignable quantity (in more modern terms, less than any finite quantity), since a quantity which is less than any assignable quantity is necessarily equal to zero [Euler 1755, 69].

The statement that an infinitesimal  $\omega$  or a differential  $dx$  are equal to 0 can be understood in two ways:  $\omega$  is 'numerically' equal to zero (this justifies the principle of cancellation) or else  $\omega$  is a variable quantity tending towards zero' (this justifies the existence of different symbols which denote the different way of tending towards zero). From a modern point of view, the difference between the two notions is such that Euler's argument is incomprehensible. Let us imagine, for example, that we subdivide a given quantity  $q$  into two equal parts and then divide one of the halves again, and so on. We have a quantity  $q$  that vanishes; according to Euler's definition, it is an infinitesimal. If we denote the act of vanishing of  $q$  by  $\omega$  and observe that it can analytically expressed as  $q2^{-n}$  or, more simply, as  $2^{-n}$  (by assuming  $q=1$ ); therefore  $\omega=2^{-n}$  is an infinitesimal  $\omega=2^{-n}=0$ .

The meaning of  $\omega=2^{-n}=0$  is unclear: it may be a way (a rather infelicitous one) of denoting a limit (therefore  $2^{-n}=0$  would mean  $\lim_{n \rightarrow \infty} 2^{-n}=0$ ), or it may symbolise a equation valid for a process which occurs a really infinite number of times (therefore  $2^{-n}=0$  would mean  $st(2^{-n})=0$ ,<sup>6</sup> where  $n$  is an infinite number).<sup>7</sup>

<sup>5</sup> "Nullum [...] est dubium, quin omnis eousque diminui queat, quoad penitus evanescat atque in nihilum abeat. Sed quantitas infinite parva nil aliud est nisi quantitas evanescens ideoque revera erit =0." [Euler 1755, 69].

<sup>6</sup> Of course, the symbol  $st(a)$  denotes the standard part of the hyperreal number  $a$ .

<sup>7</sup> It certainly does not mean that  $2^{-n}$  is approximately equal to 0 because this is contrary to a literal and substantial interpretation of Euler (see also section 4). The exactness of mathematics makes it necessary that  $a+2^{-n}=a$  when  $n=\text{infinity}$ , is a precise equation.

The definition of the infinitesimal as a quantity smaller than any assignable quantity  $\varepsilon$  is vague and does not clarify the question: this leads to the relation  $\omega = 2^{-n} < \varepsilon$ , but this relation could be understood as  $2^{-n} < \varepsilon$ , for every  $\varepsilon$  and for a suitable  $n$  depending on  $\varepsilon$ , or else as  $2^{-n} < \varepsilon$ , for every  $\varepsilon$  and for  $n$  as a fixed (and therefore infinite) number.

these definitions contain three statements which he believes to be equivalent, while today's are considered different. Precisely: a) an infinitesimal is a quantity that vanishes; a) an infinitesimal is actually equal to zero; c) an infinitesimal is less than any finite quantity.

The Eulerian definitions seem to confuse various notions (limit, infinitesimal, value of a variable) which, from a modern viewpoint, are clearly different. It might be thought that this situation is due to an imperfect formulation of the definitions themselves: however, as we shall see in several examples, it corresponds to the ambiguous procedures of Eulerian analysis, which is actually composed of an inextricable interplay of elements which today appear as different ones.

These definitions presuppose knowledge of the notions of set, real numbers, function in the modern sense, continuity, etc.: it is such concepts that enable real analysis to give a clear and rigorous formulation of the intuitive notion of one quantity approaching another. However, Euler did not have the mathematical concept of set, nor the theory of real numbers nor the modern notion of function. He based the calculus on the notion of quantity. This assumes many different meanings in Euler's writings, which now we examine.

Quantity was usually defined as that which could be increased or diminished. Euler often referred to this traditional definition. It is evident that is not satisfactory, if we compare with a modern mathematical definition: it is even tautological: to increase means to make the quantity more, to decrease, to make the quantity less.<sup>8</sup> As many concepts of 18<sup>th</sup> century mathematics we cannot understand it only by a mere analysis of the term used in the definition: it refers to physical or abstract objects. by referring to idea that the definition express. This idea is developed at different levels of abstraction by Euler.

First, quantity is an attribute of certain objects (that can change without changing the essence of these objects). It is an empirical notion of quantity. In this sense, a quantity presupposes the material and is always a 'quantity of a substance': thus in [1755], Euler considered the trajectory of a bullet. He observed that it was determined by four quantities (the amount of gunpowder, the angle of fire, the range, and the time).

This 'empirical' idea of quantity was the classical Greek idea of quantity. It presupposes the material, or, at least, an idealization of the material. Thus a number was a quantity insofar as a sensible or ideal object that included (sensible or ideal, monads) parts. And a geometrical line was a quantity insofar as an abstraction practised upon real objects.

The classical distinction between continuous and discrete quantity was based upon this empirical concept. Discrete quantity is made up of discontinuous parts, namely there is no common boundary at which they join (For example: two fives make ten, but the two fives have no common boundary, but are separate: ha un

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<sup>8</sup> Neither the matter is different for the other classic definition:

The terms part and whole are not defined.

senso solo se si pensa ai numeri come oggetti materiali). Continuous quantity is that which consists of parts having position in reference to each other, so that the limit of the one is the limit of the next.

After a long elaborazione, in the 18<sup>th</sup> century this 'empirical' idea of quantity is affacciata da the notion of analytical quantity: it was general quantity, namely the property that all the quantity have in common (namely the propriety of essere aumentata e diminuita). Usually by analytical quantity one intended abstract continuous quantity.<sup>9</sup> General quantity was an abstract idea (a abstraction of the second level, operando su notion as lines, that were already abstract). It was treated by a symbolic notation that highlights the relations between different quantities (distinte only by their symbol  $x, y, \dots$ ). The notion of quantity as a quantity of a 'material object' was not abbandonata, but it was confinata alla geometry: indeed a geometrical quantity always referred to a sensible representation in a diagram.

La relazione tra quantità analitica e geometrica rimane molto forte: la quantità analitica aveva un modello nella quantità geometrica. When Euler operated upon variables  $x, y, \dots$  he operated upon symbols, which expressed quantities and whose archetype was the segment of stright line. Anche abstract quantity is the crucial notion of the calculus, but it was not precisely defined, as in modern axiomatic. Il concept of quantity was intuitive, in quanto not espresso interamente a parole e formalizzato.<sup>10</sup>

Not only did quantity become abstract, but also number. Indeed, he did not think the numbers as a number of things, but as an abstract entity, without a material support. This concept was widely accepted after Descartes. For instance, Newton ...

A number is what all the different groups formed by the same discrete quantity of objects have in common.

It is to note that non ha più senso dire che: "two fives make ten, but the two fives have no common boundary, but are separate", if one thinks ten and five as abstract notions. When Euler described the discrete quantity, he described it as a finite sequence of numbers  $1, 2, 3, \dots$  and not as a single number.

In this way the notion of a number and that of quantity begin to be differentiated. A number is not a quantity but a specific determination of quantity. We can distinguish between quantity and quantum. The first is a general propriety of objects (an amount, a quantity of stones), considered only in their capacity of being increased or decreased, the latter measured a specific amount of objects.

Thus a fixed segment was a quantity only if it was considered as a variable segment that increased or diminished remaining practically the same: a quantity was an intrinsically variable object. It could be determined or

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<sup>9</sup> Mentre la empirical quantity was studied only by means of specific cases (lines in a diagram, given numbers in a calculation), astract quantity was studied in general.

<sup>10</sup> In nessuno caso il discorso analitico poteva essere una semplice deduzione da assiomi, perché questi non c'erano in senso moderno.

Assiomatizzare è come trasformare la tradizione orale in legge scritta. Negli assiomi (in senso moderno) vi è tutto il contenuto e si procede sulla base della lettera degli assiomi.

Nel 1700 c'erano principi, non del tutto definiti, ed una serie di applicazione dei principi, da cui si doveva partire per ottenere modalità con cui procedere. La vaghezza è elemento essenziale della matematica pre-assiomatica.

could assume a given value; nevertheless, the calculus referred to indeterminate quantities, subject to possible variations, whether increases or decreases.<sup>11</sup>

Ciò dà una certa asimmetria alla nozione di quantità. Il discreto è ridotto alla sequenza di numeri. Non così il continuo. Ciò corrispondeva al fatto che il segmento era un oggetto unico non un insieme di punti.

Ogni singola determinazione del continuo poteva essere approssimata da successioni di razionali e dava luogo a un numero irrazionale: questo rimane sostanzialmente il rapporto di quantità fissate. (Un numero irrazionale era noto quando era data la legge con cui si costituivano le infinite entità che lo componevano). Euler sembra pensare a singole, specifici numeri irrazionali. Il continuo aveva una sua specificità, che non sembrava riducibile a numeri. In questo senso i numeri erano solo un modo di trattare il continuo. Quindi la quantità analitica non era neanche (interamente) numerica.

In conclusione: La quantità analitica è un oggetto, simbolicamente rappresentato da lettere  $x, y, \dots$ , che si può immaginare assumere differenti determinazioni, variando con continuità nel passare tra una determinazione a un'altra. Assegnata una determinazione della quantità essa può essere sempre aumentata o diminuita. Variare con continuità significa variare come un segmento. Assumere una determinazione significa assegnare un valore numerico alla variabile oppure rappresentare la variabile come una linea o altro oggetto geometrico

Ancora un'osservazione sul continuo: non essendo il continuum un insieme di punti è impossibile distinguere un segmento aperto o chiuso, o meglio un segmento è sempre chiuso con il suo estremo. This question non ha senso, se il continuo non è pensato per punti. Ma è foriera di molte delle difficoltà della matematica settecentesca.

La stessa cosa succede alla quantità analitica: quando passa da una determinazione  $x_1$  a una determinazione  $x_2$  è impossibile dire se assume oppure no i valori estremi.

Infine va detto che una funzione era una relazione tra quantità variabili, non definita puntualmente.

In such a situation, Euler cerca di inserire 'evanescent quantity'. Even though Euler based the calculus on this notion, he never offered a definition, not even a vague or imprecise one, of it.<sup>12</sup> it was only possible to

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<sup>11</sup> It is well-known that eighteenth-century mathematicians subdivided quantities into constants and variables. However Euler explained: "Primum [...] hic calculus circa quantitates variabilis versatur: etsi enim omnis quantitas sua natura in infinitum augeri et diminui potest; tamen dum calculus ad certum quoddam institutum dirigitur, aliae quantitates costanter eandem magnitudinem retinere concipiuntur, aliae vero per omnes gradus auctionis ac diminutionis variari: ad quam distinctionem notandam illae quantitates constantes, hae vero variabiles vocari solent; ita ut *hoc discrimen non tam in rei natura, quam in quaestionis, ad quam calculus refertur, indole sit positum.*" [Euler 1755, 3, my emphasis]. As an illustration, Euler considered the trajectory of a bullet. He observed that it was determined by four quantities (the amount of gunpowder, the angle of fire, the range, and the time); however, each of them could be conceived of as a variable or constant according to circumstances. In the *Introductio*, he defined: "Quantitas constans est quantitas determinate, perpetuo eodem valorem servans" [Euler, 1748, 17]. A quantity was variable by itself, however, it could be determinate and fixed (if the problem required this) and the it was considered as a constant.

refer to a vague intuition of a process of whereby a quantity A approaches a quantity B. In contrast to other eighteenth century mathematicians (e.g., d'Alembert [Diff] and [Lim]), Euler generally avoided even the use of the term 'limit'.<sup>13</sup> Furthermore, he never considered the general case of a variable quantity A which goes to any finite quantity B, but only variables that vanish or endlessly increase.<sup>14</sup>

The idea of an evanescent quantity seems to be more a notion borrowed from the natural world than from mathematical notions. The mental image that may be associated with it is that of a physical entity (such as the quantity of gunpowder, in the initial example of the *Institutiones calculi differentialis*) which we can consider to an increasingly smaller extent or else, still remaining within the field of mathematics, the image of a segment which increasingly diminishes until it becomes a single point and disappears as a segment.

Euler's use of the evanescent quantity corresponds to Euclid's use of the 'common notion' in the *Elements*: a part is smaller than the whole. This is not an axiom in the modern sense of the term but a general and rather vague principle where use is made of the terms 'part', 'smaller' and 'whole' without their meaning being defined (not even implicitly). In the same way, Euler regarded the fact that a quantity vanishes or increases infinitely as simply part of the idea of quantity and not in need of further explanation or clarification: it is a notion which forms part of the knowledge of every human being.<sup>15</sup> (mathematical educato)

Now the question is how this intuition of an evanescent quantity and the correspondent one of a quantity increasing beyond any bound can be made mathematically operativa. Qualche esempio può allora aiutarci a capire. In section 114 of the *Introductio in analysin infinitorum* (the first section of chapter V) Euler stated that if  $a$  is a number greater than one and  $\omega$  and  $\psi$  are infinitesimals (or rather such small fractions that they are almost equal to nothing), then it follows that

$$(1) \quad a^\omega = 1 + \psi.$$

Euler justifies (1) by making a vague reference to what he had stated in the preceding chapter IV: "Indeed, from the preceding chapter, it was established that if  $\psi$  was not an infinitely small number, then neither could  $\omega$  be an infinitely small number."<sup>16</sup> In reality, in chapter IV, exponential and logarithmic functions had been

<sup>12</sup> Euler prende la notion of evanescent quantity from Newton, who have based upon it the famous twelve mathematical lemmas of the *Principia*

<sup>13</sup> As far as I am aware, only in the preface to *Institutiones calculi differentialis* [Euler 1755, 7] (see footnote 60) did Euler use 'limit' to mean "approaching to a limit".

<sup>14</sup> Euler seems to believe that every other case of nearing a limit can be included within these two cases. Thus, in order

to calculate the value of  $\frac{x^\alpha - x^{\alpha+m\beta}}{1 - x^\beta}$  at the point  $x=1$  and  $\frac{\pi^2}{6n(n-1)} + \frac{1}{n(n-1)^2} - \frac{(2n-1)\left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}\right)}{n^2(n-1)^2}$  for  $n=1$ ,

Euler took  $x=1-w$  and  $n=1+a$ , respectively (see [Euler 1732-33, 44] and [Fuss 1843, 2:229-231]).

<sup>15</sup> given the many debates argued and the many pages written about the nature of the infinite in the 18th century, perhaps it was not "common knowledge" what was behind Euler's notion of evanescent quantity.

<sup>16</sup> "Quia est  $a^0=1$ , atque crescente exponente ipsius  $a$  simul valor potestatis augetur, si quidem  $a$  est numerus unitate major; sequitur si esponens infinite parum cyphram excedat, potestatem ipsam quoque infinite parum unitatem esse su-



introduced without making any mention of infinitesimals [Euler 1748, 1:103-105]. He suggested the idea that the difference between  $a^{z_1}$  and  $a^{z_2}$  of the exponential function  $a^z$  might be made equal to a tiny finite quantity, provided that  $z_1$  and  $z_2$  are taken very close together (in other words, at a tiny finite distance).<sup>17</sup> In the following chapter this idea was expressed by (1), or rather by means of explicit use of the language of infinitesimals, as if *one could move finite quantity, small as desired, to infinitesimals in an immediate and natural way, without an adequate theoretical construction*. Vice versa, one could move from infinitesimals to sequences of numbers that becomes very small. Indeed, Euler continued by assuming that the infinitesimal  $\omega$  is equal to  $k\omega$  and that therefore  $a^\omega = 1+k\omega$  and

$$(2) \quad \omega = \log_a(1+k\omega).$$

To prove that  $k$  depends on the base  $a$  of the logarithm, he put  $a=10$  and  $k\omega=1/1000000$  and found that  $\omega=0.00000043429$  and  $k=2.30258$  [Euler 1748, 1:122]. This example, according to Euler, was sufficient to show that  $k$  is a finite number and that, by changing the base of the logarithm, the result changes. Euler assigned a finite value to the infinitesimal: his reasoning implied that  $k\omega$  can assume increasingly small values and that, by reducing the value  $k\omega$ , one obtained a larger number of exact decimal figures of  $k=\log 10$  (the exact value of  $k$  can be obtained when  $\omega$  was evanescent).

Lo stesso discorso vale per l'infinitely large. Thus, in the following paragraph 118 of chapter V, Euler multiplied (2) by  $i$  and obtains  $i\omega = \log_a(1+k\omega)^i$ . He observed that the greater the value of  $i$ , the more  $(1+k\omega)^i$  exceeded one: assuming that  $i$  is an infinite number, the value of  $(1+k\omega)^i$  could be any number greater than one [Euler 1748, 1:125]. He used  $i$  as an infinite number, but the properties of the infinite number  $i$  derived from the fact that the finite number  $i$  increased beyond all limits (and so became infinite): *it is not easy to separate the infinite number  $i$  from the process of growth of a finite variable  $i$* .

One can also note that, when  $i$  is finite, it is necessary to suppose that  $\omega$  is finite so that  $(1+k\omega)^i$  grows beyond all limits; if  $\omega$  is an infinitesimal (as Euler has assumed from the beginning of section 118),  $(1+k\omega)^i$  remains near to one. Euler is perfectly aware of this: indeed, immediately afterwards, at p.125 of the *Introductio in analysin infinitorum*, he posed  $(1+k\omega)^i = 1+x$  and stated that  $i$  must be finite in order to be  $x$  a finite quantity. It is reasonable to suppose that Euler considered  $\omega$  as a small but finite quantity and then imagined that he could make it vanish (which precisely corresponds to the fact that  $\omega$  is an evanescent quantity).

There is tantissimi casi di questo tipo in Euler. It seems that for Euler the use of assignable quantities or inassignable quantities sia only a question of language and one can move from a language to another without constraints. Euler immagina una riducibilità (teorica) dell'uno all'altro. Vedremo che in realtà ci sono casi in cui ciò non avviene. However, that is the starting's point of Euler. In this sense he stated that an infinitesi-

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peraturam. Sit  $\omega$  numerus infinite parvus, seu fractio tam exigua, ut tantum non nihilo sit aequalis,  $\omega$  erit  $a^\omega = 1+\psi$ , existente  $\psi$  quoque numero infinite parvo. Ex praecedente enim capite constat, nisi  $\psi$  esset numerus infinite parvus, neque  $\omega$  talem esse posse." [Euler 1748, 1:122].

<sup>17</sup> Indeed, the idea underlying Euler's construction is that the value of  $a^z$ , when  $z$  is an irrational number, could be obtained by considering rational approximations of the irrational number  $z$ . He did not take into consideration, for example, the case where  $a < 0$ , because in this case the function proceeded by jumps.

mal is an evanescent quantità and an infinite is a quantity increasing beyond any bound. Basandosi su tale nozione, Euler cerca di giustificare un'ampliamento della nozione di uguaglianza che coinvolga anche infiniti e infinitesimi. Let us consider the relation

$$(8) \quad \frac{c}{a} + \frac{c}{a+b} + \dots + \frac{c}{a+(i-1)b} = C + \frac{b}{c} \log(a+ib)$$

where  $i$  is an infinite number and  $C$  is a finite constant.

Euler pointed out that, if  $i$  is infinite,  $\log i$  and  $\log(a+ib)$  are an infinite times smaller than any power of  $i$ . Justification for the fact that the logarithm is a smaller infinite with respect to  $i^n$ , for every  $n$ , can be found in (1778). It is based on dell'Hospital's rule, which considers the infinite number  $i$  as a variable (see Bos(1974, 84-87)]. The relation (8) essentially states that the two quantities behave in the same way when  $i$  increases unlimitedly. The following theorem, for example, should be interpreted in this way:

the sum of the series of the reciprocals of prime numbers  $\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \dots$  is infinitely large but is nevertheless

infinitely smaller than the harmonic series  $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots$ , the sum of the first is equal to the

logarithm of the sum of the second. Since  $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots$  is equal to  $\log \infty$ , one has

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \dots = \log(\log \infty). \text{ [Euler 1737, 242-243].}$$

Comparing infinite and infinitesimal quantities meant for Euler assessing their behaviour when they simultaneously increased unlimitedly or vanished. This idea is at the basis of the distinction between arithmetic and geometric equations, by which Euler aims to provide an improved mathematical explanation of the equation between infinitesimals ( $0=0$ ) and between infinite numbers ( $\infty=\infty$ ).

In [1755, 70 and 74] Euler stated that, given two quantities  $a$  and  $b$ , the equality  $a=b$  can be understood in an arithmetic sense (in other words,  $a=b$ , if  $a-b=0$ ) and in a geometric sense ( $a=b$ , if  $a/b=1$ ). The arithmetic equality coincides with the geometric one for finite quantities but the situation is different for infinite numbers and infinitesimals. For infinitesimals, the arithmetic equation, which is always verified, does not imply a geometric one: in Euler's rudimentary symbolism,  $0=0$  does not imply  $0/0=1$ .<sup>18</sup> For infinite numbers, the geometric equation does not imply an arithmetical one, that is  $\infty/\infty=1$  does not imply  $\infty=\infty$ .

By using this distinction, Euler believed he could justify the principle of cancellation. He observes that  $a \pm ndx = a$  ( $n$  being any number) is true in that it is not only verified in an arithmetical sense ( $(a \pm ndx) - a = ndx$ , being  $ndx=0$ ) but also in a geometric sense. Indeed, one obtains  $(a \pm ndx)/a = 1$  and this means that the infinitesimals vanish before any finite quantity. The situation is analogous for higher order infinitesimals. Euler observes that "the infinitely small quantity  $dx^2$  vanishes before  $dx$ ", since the quantities  $dx+dx^2$  and  $dx$  (both evanescent:  $dx+dx^2=dx=0$ ) go to zero in the same way  $(dx+dx^2):dx=1+dx=1$ . More generally, if  $m < n$ , then

<sup>18</sup> In (1755), Euler shows how the laws of elementary arithmetic can be adapted to the case  $0=0$ ... Naturally the example does not show us how  $0:0$  becomes exactly equal to a certain  $n$ . This can only be understood by studying the zero as the limit of variable.

$adx^m + bdx^n$  was equal to  $adx^m$  because the arithmetical and geometrical equalities ( $adx^m + bdx^n = dx^m = 0$  and  $\frac{adx^m + bdx^n}{adx^m} = 1 + \frac{a}{b} dx^m = 1$ ) were verified.

We could briefly say: if  $A=B+C$ , and if  $C$  goes to zero before  $B$  (in other words,  $A/(B+C)=1$ ), then  $A=B$ . In modern terms,  $A$  and  $B$  are asymptotically equal<sup>19</sup> or have the same asymptotic behaviour.<sup>20</sup>

Nevertheless, while it is now clear what is meant by the ratio  $\psi/\xi$  between infinitesimal and infinite quantities  $\psi$  and  $\xi$ , it is not at all clear how it is possible to move from asymptotic equations to algebraic equations, or rather how the quantities  $\psi$  and  $\xi$ , which are asymptotically equal, can be manipulated as though they were equal numbers.<sup>21</sup>

Naturally there is no intention of attributing to Euler modern asymptotic notions or the definition of the Peano derivative. I merely noted that Eulerian notions possessed many facets which, with the hindsight of modern knowledge, we can appreciate but which were hazy to Euler: thus the idea of evanescent quantity contained subtleties that lead us to think of the standard notion of limit or of modern infinitesimals or of asymptotic processes or generalised limit concepts. I would not be surprised if traces of other modern notions are found in Eulerian infinitesimals. Nevertheless, one cannot, in my opinion, speak of confusion between different aspects, since Euler could not confuse notions that did not yet exist. Instead, he refers to a primordial idea of approaching (which we can term 'proto-limit') from which later mathematics has derived various modern notions.

Finally, *Perchè un infinitesimo è 0? Non distingue il limite dall'operazione. L'immagine geometrica o fisica del corpo veloce che raggiunge la sua posizione finale.*

Prima di continuare la nostra analisi è opportuno sgomberare il campo da alcune errate interpretazioni.

### 3. Infinite and infinitesimals quantities

Some scholars have proposed interpretations in cui Euler's notion was translated into modern terms, in particular in any version of non-standard analysis. For the above discussion, it would be highly surprising that present day foundational notions can be used in reference to 18th-century mathematics. However, the presence of this paper makes appropriate a wider discussion of this possibility. There are, in effect, many cases in which Euler's use of evanescent seems to fit very-well to modern formulation. Let us first consider

the following example. In his [1748], Euler observed that  $a^{i\omega} = (1 + k\omega)^i = \sum_{r=0}^{\infty} \binom{i}{r} (k\omega)^r$  for any  $i$ . He assumed  $x$  to be a finite number and posed  $i=x/\omega$ , where  $\omega$  is an infinitesimal. This yielded

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<sup>19</sup>

<sup>20</sup> In particular, the idea of the approach of two quantities, which we can see here in operation, rather than the usual definition of limit, makes us think of the following generalised definition:....

<sup>21</sup> It should be noted that Euler did not necessarily regard the quantities  $\psi$  and  $\xi$  as functions but as variable quantities.

$a^x = \left(1 + \frac{kx}{i}\right)^i = \sum_{r=0}^{\infty} \binom{i}{r} \left(\frac{kx}{i}\right)^r$ . For an infinitely large numbers  $i$ ,  $\frac{i-1}{i}=1$ ,  $\frac{i-2}{i}=1$ ,  $\frac{i-3}{i}=1$ , ..., and therefore the development of  $a^x$  is

$$(E) \quad a^x = \sum_{r=0}^{\infty} \frac{1}{r!} (kx)^r.$$

In this proof, Euler considers  $\omega$  and  $i$  as infinitesimal and infinite numbers respectively; nevertheless, he justifies the relation  $\frac{i-1}{i}=1$  by stating that “however much larger the number that we substitute for  $i$ , the more the value of the fraction  $(i-1)/i$  comes closer to unity. Therefore, if  $i$  is a larger number than any assignable one, the fraction  $\frac{i-1}{i}$  equals the unity.”<sup>22</sup>

The equation  $\frac{i-1}{i}=1$  was thus obtained by first interpreting  $i$  as a finite variable and then as infinite number.

It was an approximate equation if  $i$  was finite; nevertheless, when  $i$  was an infinite number, it needed to be considered as an exact equation, equivalent to the statement that the *infinitesimal was precisely zero*.

The difference between Eulerian infinitesimals and non-standard analysis is therefore clear. Nowadays a hyperreal number  $a$  has a standard part  $b$ , from which it differs by an infinitesimal (in symbolic form,  $b=\text{st}(a)$  or  $a \approx b$ ). In particular, one obtains  $\text{st}\left(\frac{i-1}{i}\right)=1$ , an equation which we can regard as the non-standard version of

$\lim_{i \rightarrow \infty} \frac{i-1}{i} = 1$ . According to Euler, however, the form  $a+dx=a$  had to be an exact equation, not an approximate

one: his objective was precisely to eliminate approximations such as  $\frac{i-1}{i} \approx 1$ . He refused the idea of basing

the calculus on infinitesimals as really existent entities since they implied errors of approximation. For instance, in chapter VII of *Institutionum calculi integralis* [1768-70, 1:183-184], Euler expressed the integral

$y(x) = \int_a^x X(x) dx$  in the form

$$(6) \quad y(x) = \int_a^x X(x) dx = b + a(X(a) + X(a+\alpha) + X(a+2\alpha) + \dots + X(n+\alpha)), \text{ where } x = a + n\alpha.$$

He observed that (6) could be assumed as the definition of an integral, taking  $\alpha$  as an infinitesimal. However, (6) is precise only if one considers infinitesimals as nothing: if, however, infinitesimals are not considered nothing, then (6) offers a useful rule for the approximate calculation of integrals (and, effectively, this formula is included in the chapter of the treatise dedicated to this subject). According to Euler, the concept of the integral as the sum of an infinite number of infinitesimals is similar to the concept of lines as aggregates

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<sup>22</sup> “Cum autem  $i$  sit numerus infinite magnus, erit  $\frac{i-1}{i}=1$ ; pater enim, quo maior numerus loco  $i$  substituitur, eo proprius

valorem fractionis  $\frac{i-1}{i}$  ad unitatem esse accessurum; hinc si  $i$  sit numerus omni assignabili maior, fractio quoque  $\frac{i-1}{i}$

ipsam unitatem adaequabit.” [Euler 1748, 124].

of infinite points. Both can be admitted (or better tolerated), as long as there is reference to the true principles of the calculus and geometry,<sup>23</sup> thus avoiding every kind of sophism (see [Euler 1768-1770, I:185]. In other words, it is possible to take into account non-null infinitesimals but they constitute only an imprecise and approximate version of the notion of infinitesimal, which nevertheless has useful applications.<sup>24</sup>

There are other reasons that make the assimilation of Eulerian calculus into non-standard analysis problematic.<sup>25</sup> Today infinitesimal and infinitely large numbers are used in non-standard analysis as elements of a set  ${}^*\mathbf{R}$ : they are elements a rich and well-organised algebraic structure which encodes how a sequence approaches a limit. A possible construction of  ${}^*\mathbf{R}$  is the following. Let  $m$  be a finitely additive measure on the set  $\mathbf{N}$  of the positive integers such that: for all  $A \subset \mathbf{N}$ ,  $m(A)$  is 0 or 1;  $m(A)=0$  if  $A$  is finite,  $m(\mathbf{N})=1$ .<sup>26</sup> Now, consider the equivalence relation  $\sim$  on the set  $\mathbf{S}$  of the all the sequence of numbers:

$$\{a_n\} \sim \{b_n\} \text{ iff } m\{n: \{a_n - b_n = 0\}\} = 1$$

The set  ${}^*\mathbf{R}$  of the hyperreal numbers is defined by  ${}^*\mathbf{R} = \mathbf{S} / \sim$ . The classes of equivalence of the sequence  $\{0\}$ ,  $\{1/n\}$ ,  $\{1/n^2\}$  are elements of  ${}^*\mathbf{R}$  and are examples of infinitesimals; the classes of equivalence of the sequence  $\{n\}$ ,  $\{n^2\}$ ,  $\{n^3\}$  are three examples of infinite numbers.

The intuitive idea of approaching a limit is at the basis of the construction of  ${}^*\mathbf{R}$ ; however, in no case can a demonstration concerning objects of  ${}^*\mathbf{R}$  refer to such an intuition but only to the axioms and the definitions which enable us to arrive at  ${}^*\mathbf{R}$ . The same is true for the calculation of the limits in  $\mathbf{R}$ , which entirely depends on the definition of limit and the construction of the set  $\mathbf{R}$ .

Viewed from the perspective of the refined concepts of the twenty-first century, the way in which Euler codifies the idea of approaching is naïve and over-simplified: an infinite number is simply an short way of indicating a variable that goes to infinity while an infinitesimal is a short way of indicating a variable that goes to zero. The rules that Euler uses upon infinite and infinitesimal numbers constitute an immediate extrapolation of the behaviour of a finite variable  $i$  tending to  $\infty$  or to 0: what is wholly missing is the complex construction of  ${}^*\mathbf{R}$  and the assumptions upon which it is based. I ho data vari examples above. Here is another, used by Laugwitz. For instance, in *De progressionibus harmonicis observationes*, given the equality

$1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{i} = \log(1+i) + \gamma$ , where  $\gamma$  is the constant named today after Euler and Mascheroni, Euler derived:

$$\log n = \log \frac{ni+1}{i+1} = \log(ni+1) - \log(i+1) = \left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{ni}\right) - \left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{i}\right)$$

<sup>23</sup> In the case of integration, the integral is regarded as an antidifferential.

<sup>24</sup> Naturally, if one imagines  $\alpha$  as an evanescent quantity (that is, finite and therefore making  $\alpha$  tend towards 0), it might be possible to follow a definition of an integral which is acceptable from a Eulerian point of view. However, Euler is not interested in such a possibility and regards the definition of an integral as an antidifferential as satisfactory.

<sup>25</sup> On the difference between the seventeenth and eighteenth-century calculus and non-standard analysis, see [Bos 1974, 81-86].

<sup>26</sup> For a proof of the existence of this measure, see [Lindstrøm 1988, 84-85].

$$= \left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}\right) - 1 + \left(\frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n}\right) - \frac{1}{2} + \left(\frac{1}{2n+1} + \frac{1}{2n+2} + \dots + \frac{1}{3n}\right) - \frac{1}{3} + \dots$$

By specialising, he had  $\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \dots$ ,  $\log 3 = 1 + \frac{1}{2} - \frac{2}{3} + \frac{1}{4} - \frac{1}{5} + \frac{2}{6} - \frac{1}{7} + \dots$ , ... [Euler 1734-35a, 90].

There is no doubt that Euler uses infinitely large numbers but it is difficult to point to what distinguishes the infinite number  $i$  from a short way of denoting finite sums for which one wants the limit for  $i \rightarrow \infty$ .<sup>27</sup>

One could object that this argument cannot be applied to certain recent writing of Laugwitz and McKinzie. These authors have employed new notions of infinitesimals that, unlike Robinson's non-standard analysis, do not require an highly modern theoretical basis in logic and models theory. They admit that Robinson's hyperreals do not vindicate Euler and that the continuum can not be understood in such sophisticated ways in Euler's time and propongono "a more naïve approach". [McKinzie and Tucky....] However, my opinion is that my criticism coinvolge anche questi autori. Consider Laugwitz [1980, 1986] "Its starting point is a generalization of field extension. A symbol  $\Omega$  ... is adjoined to the real numbers. If a formula  $F(n)$  containing the variable  $n$  (for natural numbers) is true for all sufficiently large  $n$ , then  $F(\Omega)$  is defined to be true in extended theory... This theory had been modeled after ideas of Leibniz and Euler." [Laugwitz 1987, 273].

McKinzie and Tuckey have in a recent paper reinterpreted several Euler's results. It is true that their theory reduced some hypotheses of Robinson's theory (for instance, they refer only elementary functions), however they used some tools, among which "the more or less axiomatic introduction of infinite and infinitesimal numbers". In a footnote, they explain: "In modern terms, the Eulerian continuum is an ordered field; the natural numbers are a subset of the real numbers, ... The fundamental equations or axioms are exactly what one would use to axiomatize the mathematics necessary for basic algebra and trigonometry, with additional assumption that there is an infinite natural number. By the field axioms, this implies the existence of infinitesimals." [...]

These commentators use a continuum made by numbers, set theory (at least in a naïve form), axiomatic method: these is very far from Eulerian methods. E poi evitano di far riferimento al formalismo Euleriano. Usando materiali different, they obtain a result that solo assomiglia a quello di Euler. Thus uno dei punti forti è l'osservazione che there is a gap in Euler's derivation of (E). In effect, however we interpret this step, it is clear that it is not enterly justified and that Euler was probably aware of it. He knew the sum of infinite infinitesimals (or evanescent quantities) could be different from zero. La questione non può essere affrontata come oggi, assicurandoci a priori che ogni passaggio è lecito. Euler è interessato a trovare la forma dello

<sup>27</sup> in other words a short way of saying that

$$\begin{aligned} \log n &= \lim_{i \rightarrow \infty} \log \frac{ni+1}{i+1} = \lim_{i \rightarrow \infty} \{\log(ni+1) - \log(i+1)\} = \lim_{i \rightarrow \infty} \left\{ \left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{ni}\right) - \left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{i}\right) \right\} \\ &= \lim_{i \rightarrow \infty} \left\{ \left(1 + \frac{1}{2} + \dots + \frac{1}{n}\right) - 1 + \left(\frac{1}{n+1} + \dots + \frac{1}{2n}\right) - \frac{1}{2} + \left(\frac{1}{2n+1} + \dots + \frac{1}{3n}\right) - \frac{1}{3} + \dots + \left(\frac{1}{(i-1)n+1} + \dots + \frac{1}{in}\right) - \frac{1}{i} \right\}. \end{aligned}$$

sviluppo, l'espressione formale  $\sum_{r=0}^{\infty} \frac{1}{r!} (kx)^r$ . La convergenza di quest'ultima è verificabile a posteriori, ma è

anche considerata garantita dalle procedure usate, estensione infinitaria di regole valide al finito.

Euler did not see gaps, because he operava formally. The gaps were born of the fact that we require a justification of every equality as a numerical equality.

On the other side, who wants tradurre Euler in modern theory of infinite numbers neglected some remarkable uses of infinite quantity in Euler.

Consider the following example. In the chapter VII of his *Institutiones calculi differentialis*, Euler obtained

$$(Mac) \quad S(x) = 1^n - 2^n + 3^n - 4^n + \dots + (-1)^{x+1} x^n = (-1)^{x+1} \left( \frac{1}{2} x^n + \sum_{m=1}^s (-1)^{m+1} \binom{n}{2m-1} \frac{(2^{2m} - 1) B_{2m}}{2m} x^{n-2m+1} \right) + C$$

where  $n \geq 0$ ,  $s = [(n+1)/2]$  is the integral part of  $(n+1)/2$ ,  $B_n$  are the Bernoulli numbers<sup>28</sup> and the constant  $C$  is determined by the condition  $S(0)=0$ . This constraint implies that if  $n$  is even, then  $C=0$ ; if  $n=0$ , then  $C=1/2$ ; if

$n$  is greater than 0 and is odd, then  $C = (-1)^s \frac{(2^{n+1} - 1) B_{n+1}}{n+1}$ .<sup>29</sup>

Consequently  $1^n - 2^n + 3^n - 4^n + \dots + (-1)^{x+1} x^n$  can be expressed as  $C + (-1)^x f(x)$ , where  $C$  is a constant and  $f(x)$  is appropriate function of the index  $x$ . Euler thought that one could make  $(-1)^x f(x)$  equal to zero for  $x=\infty$  since "if  $x$  is an infinite number, which is neither even nor odd, these considerations [he intends the alternating sequence of + and - in  $+(-1)^x f(x)$ ] had to end and, therefore, the sum of the ambiguous terms should be rejected. The sum of the series to infinity are now given by the constant term".<sup>30</sup> In other terms, this means  $(-1)^\infty = 0$ <sup>31</sup> and the sum are

$$(7) \quad 1^n - 2^n + 3^n - 4^n + \dots = C = (-1)^{[n/2]} \frac{(2^{n+1} - 1) B_{n+1}}{n+1}.$$

Equation (7) cannot be expressed by means of real analysis or non-standard analysis. Thus the use of such theories to reformulate Eulerian calculus distorts the meaning, on the one hand, transforming fictions into 'true' numbers; on the other hand, it only manages to obtain a part of Euler's results. Reformulating (7) in terms that are acceptable today means using the theory of summability. This is at the very least problematic. Indeed, summability presupposes a conception of mathematics as a set of theories syntactically derived from axioms and definitions. The usual notion of a sum of a series, of the limit of a sequence and function, of an improper integral is not considered intrinsic to the actual nature of mathematical objects (and thus the only possible one); on the contrary, it is assumed that different and arbitrary definitions of sum, limit and improper

<sup>28</sup> There are several definition of Bernoulli numbers  $B_r$ . Here I refer to the following:  $\frac{t}{e^t - 1} = 1 + \sum_{r=1}^{\infty} (-1)^{[r/2]+1} \frac{B_r}{r!} t^r$

( $|t| < 2\pi$ ,  $[x]$  is the integral part of  $x$ ),  $B_0=1$ , which is closer to Eulerian use.

<sup>29</sup> Euler's derivation of this result is illustrated in [Goldstine 1977, 131-135].

<sup>30</sup> [Euler 1755, 384]. Euler's words remember Leibniz's in a famous letter to Wolff: with regard to infinite number, "paris imparisque jura confunditur" [Leibniz 1713, 269].

<sup>31</sup> If one work fomally, as Euler did, it is clear that  $+(-1)^\infty$  involves  $\cos \infty = 0$  (being  $e^{i\pi} = -1$ ).

integral can be given. However, Euler thought that alternative theories based on alternative definitions of sum and limit could not exist.

I do not underestimate the extent to which modern mathematical commentators have contributed to an understating of Euler's analysis.

The question sollevata dalle moderne interpretazioni è storiograficamente e philosophically interesting.

We have a certain result A, this result is valid today, However we do not accept its derivation. Cioè noi oggi abbiamo un risultato A1, that we see to be the same that A.

What hanno in A, versione Euler, ed A1 (versione odierna)?

Se si ammette la matematica come struttura formale, A e A1 sono irriducibili.

Ma la matematica non è una mera struttura formale, il sistema logico-deduttivo è sempre accompagnata a un'interpretazione. Proprio per questo una matematica formale può riconoscere certe procedure non formali. A e A1 possono essere usate per descrivere una certa classe di fenomeni.

Gli oggetti della matematica hanno un'esistenza non platonica ma nel mondo esterno (qualsiasi cosa esso significhi). Sono schemi di interpretazione del reale. Se A si può riformulare in A1 significa che il nuovo schema S1 di cui fa parte A1 (anche se non include S) è stato costruito per interpretare gli stessi fenomeni oppure una classe più estesa di fenomeni.

Data l'enorme differenza tra i punti di partenza è sorprendente che la (E) sia ancora valida. Sicuramente oggi la leggiamo in modo diverso. Per quanto astratta quella formula interpreta data reali (esistono fenomeni che si comportano come l'esponenziale).

A) I fenomeni descritti dalla matematica euleriana sono descritti dalla matematica successiva (ci può essere qualche perdita marginale).

B) Ci sono fenomeni che matematica euleriana non riesce a descrivere e che la matematica successiva riesce.

I fenomeni descritti sono fenomeni matematici: ossia aspetti isolati dai fenomeni reali (sensibili).

#### **4. Infinitesimal and infinite large numbers as fictitious quantities**

Rather than making use of sophisticated twentieth century concepts, the Eulerian use of infinite numbers<sup>32</sup> and infinitesimals can be better understood if one resorts to the Leibnizian notion of 'fiction': infinite numbers and infinitesimals are fictions of variable quantities, represented by symbols with which one operates in analogy with 'real' quantities. Leibniz had also made use, in certain occasions, of this notion in order to justify

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<sup>32</sup> Euler stated: "[E]tiam si quis neget in mundo numerum infinitum revera existere, tamen in speculationibus mathematicis saepissime occurrunt questiones, ad quas, nisi numerus infinitus admittatur, responderi non posset" [Euler 1755, 69]. In his writings Euler is ready to admit the existence of infinity; nevertheless, it is not strictly necessary. Euler justifies the need to introduce the use of infinity for numbers by observing that the sum of the numbers  $1+2+3+4+\dots$  is larger than any finite quantity and therefore does not exist among finite quantities: this is the sense in which it is infinite. Given the non-existence of the object  $1+2+3+\dots$ , however, he behaves as though one could pretend that it existed.



infinitesimals; he used it as the basis for using imaginary quantities and other entities introduced in mathematics.<sup>33</sup>

Un'attenta analisi dell'infinito come quantità fittizia in Leibniz come opposta agli indivisibili si trova in Knobloch.

Now I shall try to clarify this notion. Above, we saw that in analysis, Euler consider quantity as an abstraction derived from real entities. It was implicit that quantity was finite and non null. It is however possible to imagine a quantity increasing beyond any bound or vanishing.

Leibniz's idea was to consider these (lo svanire) as new ideal entity, which is called fictitious quantity.

One could argue that the true quantities are ideal entity. In effect this is true. However, real quantities are originated from 'empirical quantities', they possono essere compresi by the consideration of some of the empirical oggetti which have originated them, without need of considering any fictitious quantities. Real quantities are the objects of mathematics. Fictitious quantities could be understood by referring to some real quantity. (si possono comprendere vari livelli di fictitious). Infinite quantities can be understood by referring to real quantity.

Esse avevano the following caratteristiche:

1) esse scaturivano dalle quantità reale e non erano introdotte arbitrariamente, ma erano fondate sul reale (indirettamente); In questo senso A fiction was not an invention and mathematics is not a game.

2) Fictitious quantity is a name or symbol, that non esiste nella realtà, solo nel linguaggio della matematica. In questo senso esso modi di parlare, abbreviazioni del discorso: ESEMPIO.

A fictitious quantity was a short way to denote a property of 'true' quantities; "It does not matter whether they appears in nature or not, because they allow abbreviations for speaking, for thinking, for discovering, and for proving." [Knobloch 1999, 95]. This means that, even if a fictitious quantity did not exist, it was connected with real quantities. They assumono l'esistenza, come enti ideali o ipotetici, del risultato di operazione come il crescere oltre ogni limite, il diminuire oltre ogni limite, il levare una quantità maggiore da una minore, che erano prive di senso sulla quantità reale e tuttavia potevano essere utili in determinati contesti.

3) Moreover, a fiction presupposes the existence of real objects to which it relates by imitating them. The mathematics of fictions distinguishes between true and false objects; the latter express qualities or characteristics of true ones. Today, infinite numbers and infinitesimals, as elements of  $^*R$ , are not ontologically different from other elements of  $^*R$ , which all derive from the construction of  $^*R$ . According to Euler, an infinitesimal quantity was not an actual quantity in the true sense of the term but a short way of indicating how a 'true' quantity vanishes. Analogously, in modern mathematics, 0, as an element of the set  $(0, 1, 2, \dots)$ , is a number that exists in the same way as the other numbers 1, 2, 3...; In Euler's opinion, however, 0 was the symbol that represented the absence of quantity; it was not a quantity in the true sense of the term but a fiction that could be treated as a quantity.<sup>34</sup> There was an ontological difference between the fictions and the 'real' objects of Eulerian mathematics, a difference which does not exist in modern mathematics.

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<sup>33</sup> Leibnizian notion of fiction has often discussed in the secondary literature. For instance, see [Bos 1974, 54-57], Giorello [1992, 148-151], [Horváth 1982 and 1986]. Here is ...

<sup>34</sup> See (1748, p...), where the zero is placed at the same level as imaginary numbers, with a different degree of reality compared to natural, fractional and irrational numbers. At this point it is clear that when Euler refers to evanescent quan-

4) a fictitious quantity was una quantità: This means that they were objects that could be treated mathematically by an analogical extension of the rule valid for true quantity. [Knobloch 1999, 95] observes. "The decisive aspect is that they are quantities, fictive ones certainly, because they were introduced by a fiction, but quantities nevertheless."

According to Euler, se l'infinito è una quantità, esso può essere sempre aumentato, come è proprio della quantità: di conseguenza egli considera quantità come  $\infty+1$ . Lo stesso dicasi dei differenziali. Inoltre come la quantità reale è determinata da un numero, anche quantità infinita può essere determinata by means of numbers. Questo può essere equivoco: l'infinito è una quantità (abbreviazione di quantità crescente oltre limite); numero infinito che misura come la quantità cresce oltre ogni limite.

Naturalmente, questa concezione apre molti problemi: come si fa a distinguere una finzione ben fondata da una che non lo è? In che modo va adoperato l'analogia, ossia quali sono i limiti che l'estensione di tale leggi ammette?

Qui la concezione di Euler sembra differenziarsi in modo notevole da quella leibniziana. Per Leibniz, "le grandezze fittizie sono ausiliarie per l'arte di trovare la soluzione e non per la dimostrazione." [Breger ??? 82]. Questo significa che il significato ultimo è da cercarsi nella geometria, che funziona da controllo a posteriori della validità del risultato. Per Euler, l'analisi è autonoma e deve trovare in se stessa le sue giustificazioni. Il problema delle quantità fittizie assume nuova rilevanza: e la geometria non può supportare l'analisi, non è il metodo di controllo. La giustificazione del metodo deve essere interna. La sua risposta è: gli infinitesimi  $dx$ ,  $dy$  sono entità fittizie collegate al reale dal fatto di denotare come una variabile tende a zero; l'unico modo di misurare ciò è di studiarne il rapporto, il rapporto è una quantità reale.

Lo stesso per gli infiniti, sono reali solo in quanto è reale il loro rapporto.

La fondazione euleriana ha il problema della relazione tra fittizio  $df=f(x+dx)-f(x)$  e reale  $df/dx$ .

L'analisi studia la quantità generale, anche tutte le sue fittize generalizzazioni.

La quantità reale ci fornisce le funzioni, analiticamente espresse e le operazioni con cui vanno trattate.

Abbiamo quantità  $x, y, \dots$  alcune funzioni  $x+y, \dots$ , alcune regole per operare<sup>35</sup> su di esse  $\log(xy)=\log x+\log y \dots$

Le funzioni sono studiate per la loro natura formale e mediante operazioni formali: formale vuol dire che esse nascono dall'esprimere una quantità tramite una forma (analitica), soggetta ad operazioni determinate, e studiarle senza condizioni. Ossia si studia come una forma si trasforma in un'altra.

Si ottengono relazioni di forma, che specializzate possono avere un senso in termini di quantità reali o di numeri, ma altrimenti rimangono come relazioni in sé come la (E), o che possono avere un senso come relazione tra altre quantità reali.

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tity, in the sense of a quantity that vanishes, we are dealing with something which differs ontologically from the modern idea of tending towards zero.

<sup>35</sup> Le operazioni ammesse sono: quelle algebriche, trascendentali elementari (ripetute anche infinite volte, il che dà luogo alle serie), ma anche infinitesimali (principio di cancellazione).

L'analisi si configura come una struttura sintattica, in cui però gli oggetti le regole operative non sono inventati ad arbitrio, ma hanno un carattere necessitante.<sup>36</sup> In questo senso la matematica del settecento non crea i suoi oggetti, può solo manipolarli liberamente.<sup>37</sup> Tutti i risultati sono accettabili salvo cercarne un'interpretazione, se non esiste una quantitativa.

Nowadays, every object is introduced into mathematics by means of an appropriate definition, although it is arbitrary in principle. For example, if  $n$  and  $a$  are positive integers, the power  $a^n$  denotes the object  $a^n = \underbrace{aa \dots a}_n$  ( $n$  times). The objects  $a^0$ ,  $a^1$ ,  $a^r$  ( $r$  real),  $a^\infty$ ,  $a^{-\infty}$  do not come under this definition and are therefore defined separately. In the eighteenth century no mathematician would have stated: 'I define  $a^0$  by letting  $a^0=1$ '. Instead he would have explained that  $a^0$  is equal to 1 because  $a^0 = a^{n-n} = a^n : a^n = 1$ . The objects  $a^0$ ,  $a^1$ ,  $a^r$  ( $r$  real),  $a^\infty$ ,  $a^{-\infty}$  were used not because they were mathematically defined objects but as fictions that were useful for reasoning: their meaning could not be arbitrary but derived 'naturally' from the meaning of  $a^n$ . Given this context, it is not surprising that Euler thought of using the symbol  $(-1)^\infty$  without defining it: if he had behaved differently it would have been anomalous with respect to standard practice. He was definitely aware that a clear piece of intuition in terms of infinitely growing or evanescent quantities corresponded to objects such as  $2^\infty$  or  $2^{-\infty}$  while this did not occur with  $(-1)^\infty$ . Nevertheless, he thought he could provide an *a posteriori* justification for  $(-1)^\infty$ ,<sup>38</sup> in the same way that the equation  $a^0=1$  was justified *a posteriori* rather than being defined *a priori*. Such a justification in the case of  $(-1)^\infty$ , rather than being linked to the fairly confused idea of infinite parity,<sup>39</sup> was based on the conscious acceptance of the calculus as algebraic formalism and of the principle of generality in algebra. However, it is not my intention in this article to examine this point in detail (even though I will briefly return to it in the last paragraph). I simply intend to emphasise the point that Euler's use of infinity also includes (7) and that a translation of the vague Eulerian notions in the precise terminology of non-standard analysis eliminates aspects that were regarded as unitary.

## 5. The calculus as the calculus of functions

In his [1748], Euler introduced functions as autonomous objects in analysis and constructed an organic theory of these objects. Functions became the heart of mathematics and Euler claimed also that functions the subject-matter of the calculus and that differentials were mere tools for dealing with function. In *De usu functionum discontinuarum in Analysisi*, Euler stated:

<sup>36</sup> Ognuna di queste regole ha origine dalla trattazione della quantità reale, ma poi l'oggetto di applicazione è la quantità generale, cioè sono applicate indipendentemente dalla loro giustificazione.

<sup>37</sup> Serie divergenti e immaginari hanno un senso solo all'interno della struttura sintattica, come collegamento della struttura in se.

<sup>38</sup> It should be noted that not all 18<sup>th</sup> century mathematicians accepted the idea of assigning a meaning to  $(-1)^\infty$ . For instance, ....

<sup>39</sup> The fact that this idea was far from fundamental to Euler's thought can be deduced from what he states in (1754-55, section 4).

“differential calculus is not concerned with investigating the quantity of differentials, which is nothing, but with defining their mutual ratio, which had a determinate quantity in any case. It certainly investigates not so much the differential  $dy$  of the function  $y$  but its ratio with the differential  $dx$  (p.11-12)”.

He subsequently made it clear that the value of the fraction  $dy/dx$  gives rise, for any possible case, to a determinate quantity which can be considered a new function of  $x$ . Similar statements can be found in [Euler 1755 and 1768]. According to Euler, given the function, for example  $y=x^3$  whose differential is  $dy=3x^2dx$ , the real object of the calculus was the study of the differential coefficient (in the example,  $3x^2$ ) and not of the differential  $3x^2dx$ . The algorithm of the calculus transformed functions into functions and not differentials into differentials: this represents a rather innovative viewpoint which became part of the general rewriting of analysis as a theory of functions and separated itself from the tradition of Leibniz and Bernoulli.

In the preface to (1755) and in the *De usu*, Euler provided examples where the calculation of the differential coefficient seems to look forward to the modern definition of the derivative. In (1755, p.5), he considered the function  $y=x^2$  and observed that  $y(x+\omega):\omega=(2x+\omega):1$  (being  $y(x+\omega)$  the increment of  $2x\omega+\omega^2$  of  $x^2$ ), so that the smaller  $\omega$  is, the smaller is  $(2x\omega+\omega^2):\omega$  becomes close to  $2x:1$ , even though it actually reaches  $2x:1$  only when the increment has completely vanished.

The similarity with the modern notion of derivative is striking, but it must not deceive us. In chapters 3 and 4 of [1755], where the rules of the calculus are formulated,<sup>40</sup> the introduction of differential coefficients was considerably more tortuous. Indeed, Euler tried to include the analysis of infinites organically into the method of finite differences, considering it as a special case of this method “which occurs when the differences, which were previously supposed to be finite, are taken infinitely small.” [Euler 1755, 84]. The idea of limit, which had already been expressed with sufficient clarity in the preface to (1755), was thus absorbed by the notion of infinitely small.<sup>41</sup>

In chapter 1, Euler defined the finite differences of any order<sup>42</sup> and had set out the rules of the sum and the product of finite differences [1755, 16-20]. He then calculated the differences of algebraic functions and exponential, logarithmic, trigonometric functions. For example, in [Euler 1755, 28-29], by using the series expansion of the logarithm

$$(9) \quad \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

Euler obtained

<sup>40</sup> In the preface to (1755) and in the *De usu*, Euler briefly discusses the nature of calculus, restricting himself to the examples mentioned here.

<sup>41</sup> In his [1972], Bos stated that the definition of the analysis of infinite in chapter of the *Institutiones calculi differentialis* “is rather at variance with” his remark quoted in footnote ??, “a contradiction which shows that his arguments about the infinitely small did not really influence his presentation of the calculus” [Bos 1972, 68-69]. I think that one can see a contradiction in the *Institutiones* only if, unlike Euler, one distinguishes between limits and infinitesimals.

<sup>42</sup> Set  $y^{(n)} = y(x+n\omega)$ , for a nonnegative integer  $n$ , and  $y=y^{(0)}$ , Euler defined  $\Delta y = y^{(1)} - y$ ,  $\Delta^2 y = y^{(2)} - y^{(1)}$ ,  $\Delta^m y = \Delta^{m-1} y^{(1)} - \Delta^{m-1} y$ ,  $\Delta^m y^{(n)} = \Delta^{m-1} y^{(n+1)} - \Delta^{m-1} y^{(n)}$ , for  $m > 1$  and  $n > 0$ .

$$(10) \quad \Delta y = y^{(1)} - y = \log(x+\omega) - \log x = \log\left(1 + \frac{\omega}{x}\right) = \frac{\omega}{x} - \frac{\omega^2}{2x^2} + \frac{\omega^3}{3x^3} - \frac{\omega^4}{4x^4} + \dots$$

On the basis of the study of these functions (which constituted the universe of the most commonly studied functions), he stated that the difference  $\Delta y$ , for every function  $y$ , could be expressed in the form

$$(11) \quad \Delta y = P\omega + Q\omega^2 + R\omega^3 + S\omega^4 + \dots$$

Analogously he asserted that higher-order differences could be written in the form:

$$(12) \quad \Delta^2 y = P\omega^2 + Q\omega^3 + R\omega^4 + \dots,$$

$$(13) \quad \Delta^3 y = P\omega^3 + Q\omega^4 + R\omega^5 + \dots, \text{ etc.}$$

Differential calculus originated by letting  $\omega = dx$  infinitesimal in the (11): in this way  $\Delta y$  also became an infinitesimal and, by ignoring higher order infinitesimals (which vanish before  $\omega$ ) one obtained  $\Delta y = P\omega$  or, in a different notation, could be written as  $dy = Pdx$ . Euler thought he had proved that the differential  $dy$  of a function could be expressed by means of a certain function  $P$  of  $x$  multiplied by the differential  $dx$  and that the relationship between  $dx$  and  $dy$  was finished, since  $dy:dx = P:1$ . [Euler 1755, 86].

If one takes into account the examples given in the preface and the *De usu*, he could have defined the differential coefficient as

$$\frac{\Delta y}{\omega} = \frac{P\omega + Q\omega^2 + R\omega^3 + S\omega^4 + \dots}{\omega} = P + Q\omega + R\omega^2 + S\omega^3 + \dots$$

for  $\omega$  as an evanescent quantity. However, Euler preferred to give a direct definition of the differential  $dy$  of the function  $y(x)$  (by means of  $\Delta y = P\omega + Q\omega^2 + R\omega^3 + S\omega^4 + \dots$ ) and only an indirect definition the differential coefficient.<sup>43</sup> For example, in order to determine the differential of  $y = x^n$ , he considered

$$dy = y^{(1)} - y = (x+dx)^n - x^n = nx^{n-1}dx + \frac{n(n-1)}{1 \cdot 2}x^{n-2}dx^2 + \dots$$

By neglecting higher order infinitesimals, which vanish before  $dx$ , he had  $dx^n = nx^{n-1}dx$ .

Similarly, since  $dy = \log(x+dx) - \log x = \log\left(1 + \frac{dx}{x}\right)$ , he derived

$$(14) \quad dy = \frac{dx}{x} - \frac{dx^2}{2x^2} + \frac{dx^3}{3x^3} - \frac{dx^4}{4x^4} + \dots$$

By applying (9) in formula (14), the terms  $dx^n/nx^n$ , for  $n > 1$ , vanish in comparison with  $dx/x$ , and he obtained  $d(\log x) = dy = dx/x$  [Euler 1755, 122].

It should be observed that this presents a rather peculiar situation. Even though the calculus dealt with finite quantities  $dy/dx$ , in the actual construction of the calculus, he considered  $dy/dx$  as the real ratio of the differentials  $dy$  and  $dx$ . Moreover, the symbols  $dy$  and  $dx$  have a meaning as variable quantities tending to zero or evanescent quantity, but this idea was not developed. Euler used differentials on the basis of the fact that the differential  $dy$  of the function  $y(x)$  could be considered as the increment of  $y(x)$  when  $x$  has an infinitesimal increment  $dx$  [Euler 1755, 85], namely on the basis of the formula  $dy = y(x+dx) - y(x)$ . Thus, in the calculation of the differential of  $x^n$ , there is no reference to limit process and evanescent quantity, in contrast to the example presented in the preface to *Institutiones* and in *De usu*. Despite these ambiguities, Euler really did place differential coefficients at the centre of attention and he made a systematic attempt to reduce the

<sup>43</sup> Euler's approach seems to be an attempt to fit Newtonian conception in the Leibnizian tradition. For instance, ...

calculus to a calculus of finite quantities: this is particularly clear in the case of functions with several variables and in the elimination of higher order infinitesimals.

In (1755) the differential of a function  $V(x,y,z)$  in the variables  $x,y,z$  was introduced by Euler by letting  $dV=V(x+dx,y+dy,z+dz)-V(x,y,z)$ , without making further mention of finite increments tending to zero. By analysing two examples, he observed that  $dV$  can be expressed as  $dV=pdx+qdy+rdz$  where  $p,q,r$  are functions of  $x,y,z$ ... He also noted that if  $y$  and  $z$  were taken as constants then  $dy=0$ ,  $dz=0$ , and  $dV=pdx$ . Similarly, if  $x$  and  $y$  were constants then  $dx=0$ ,  $dy=0$ , and  $dV=rdz$ ; if  $x$  and  $z$  were constants, then  $dx=0$ ,  $dz=0$  and  $dV=qdy$ . Consequently,  $dV$  is obtained by calculating the differentials of  $V$  supposing, on each occasion, that two of the variables are constant [Euler 1755, 144-146].

Euler then demonstrated the theorem of mixed differentials which, in the case of functions with two variables, could be formulated as follows: if  $dV=Pdx+Qdy$  then the differential of  $P$  for variable  $y$  and constant  $x$  and the differential of  $Q$  for variable  $x$  and constant  $y$  are equal [Euler 1755, 153-154].<sup>44</sup> Subsequently [Euler 1755, 156-157], he posed  $dP=rdy$  (constant  $x$ ) and  $dQ=qdx$  (constant  $y$ ), and observed that  $dPdx=rxdy$  and  $dQdx=qxdy$ . Since the mixed differentials are equal, he had  $r=q$ . Only at this point Euler decided to introduce a symbolism to indicate the functions  $r$  and  $q$  in a convenient and unambiguous way. He denoted  $r$  by

means of the symbol  $\left(\frac{dP}{dy}\right)$ , which meant *the differential of  $P$  for variable  $y$  and constant  $x$*  (that is, consider-

ing  $P$  as a function of the single variable  $y$ ) divided by  $dy$ . Similarly  $\left(\frac{dP}{dx}\right)$  indicated *the differential of  $Q$  for variable  $x$  and constant  $y$* .<sup>45</sup> Therefore the condition that linked the finite quantities  $P$  and  $Q$  in the differential

$dV=Pdx+Qdy$  can be expressed as  $\left(\frac{dP}{dy}\right)=\left(\frac{dP}{dx}\right)$ .

In his treatises, Euler systematically used the coefficients  $\left(\frac{dP}{dy}\right)$  and this constituted an essential contribution to the transformation of the calculus with multiple variables into a calculus of finite quantities. Nevertheless, the underlying problems and ambiguities seem evident: to an even greater extent than for functions with one variable, the notion of differential  $dV$  appears to be hard to reconcile with the meaning of evanescent quantity and, *de facto*, the symbols  $\left(\frac{dP}{dy}\right)$  and  $\left(\frac{dP}{dx}\right)$  were introduced as ratios of differentials.

<sup>44</sup>Euler first published in [1734-35b]. An hand-written version was published in [Engelsman 1984, 205-213]. This proof is well-known, it is therefore not illustrated here (for instance, see [Engelsman 1984, 128-130]).

<sup>45</sup> "Brevitas, gratia autem hoc autem capite quantitates  $r$  et  $q$  ita commode denotari solent, ut  $r$  indicetur per  $\left(\frac{dP}{dy}\right)$ , qua scriptura designatur  $P$  ita differentiari, ut sola  $y$  tanquam variabilis tractetur atque differentiale istud per  $dy$  dividatur; sic enim prodibit quantitas finita  $r$ . Simili modo significabit  $\left(\frac{dP}{dx}\right)$  quantitatem finitam  $q$ , quia hac ratione indicatur functionem  $Q$  sola  $x$  posita variabili differentiari tumque differentiale per  $dx$  dividi habere." [Euler 1755, 157].

The situation does not differ significantly from that regarding the elimination of higher order differentials. These were regarded as particularly problematic from the origins of the calculus and were, for example, the object of an attack by Nieuwentijdt upon Leibniz. In chapter IV of the *Institutiones*, Euler [1755, 84 and 88] stated that higher order differentials derived from higher order finite differences  $\Delta^n y = P\omega^n + Q\omega^{n+1} + R\omega^{n+2} + \dots$  putting  $\omega = dx$  in the same way in which first differentials derived from  $\Delta y = P\omega + Q\omega^2 + R\omega^3 + \dots$  posing  $\omega = dx$ . For example, as regards the second differentials, the terms  $Q\omega^3, R\omega^4 \dots$  of (12) vanished before  $P\omega^2$  and, therefore,  $d^2 y = Pdx^2$ , where  $dx^2$  was the square of  $dx$ . Naturally,  $d^2 y$  was equal to 0 while the ratio between  $d^2 y$  and  $dx^2$  was finite and equal to  $P:1$  [Euler 1755, 88].

At §.126 of the first part of *Institutiones*, Euler stated that the second differential was simply the differential of the first differential (since the second difference was simply the difference of the first difference) [Euler 1755, 88-89]. Nevertheless, stating that  $d^2 y$  was simply  $d(dy)$  led to the fundamental question as to what is meant by differentiating  $dy$ . The question was far from clear. In reality, it involved making something become evanescent when it was already evanescent. It is also worth pointing out that, if it is true that the finite second difference  $\Delta^2 y$  is by definition equal to  $\Delta(\Delta y)$ , it does not appear at all obvious that the second differential, defined by means of (12), must coincide with the differential of  $dy$ .

Furthermore, given a function  $y=y(x)$ , whose first and second differentials are  $dy=pdx$  and  $d^2 y=qdx^2$ , the problem arises of establishing the connection between  $p$  and  $q$ . Having used (12) to introduce second differentials, there is no *a priori* assurance that there exists a simple relationship between  $p$  and  $q$ . In order to determine such a relationship, Euler observed that one could set  $dp=qdx$  (since all the differentials of functions possessed this form) and that  $ndp=nqdx$ , where  $n$  represented any constant quantity.<sup>46</sup> If one then posed  $n=dx$  (therefore  $dx$  is constant), one obtained  $pdx=qdx$ . Remembering that  $dy=pdx$  and  $dp=qdx$ , one obtained  $d^2 y=d(pdx)=qdx^2$ : the second differential of  $y$  had a finite relationship with  $dx^2$ , which coincided with the differential coefficient of  $p$  [Euler 1755, 89].<sup>47</sup>

In this proof demonstration, Euler assumed that  $dx$  was a constant (non null) and that the second differences and the higher order differences were null. In order to justify this assertion, he assumed that the variable quantity  $x$  received equal increments, or rather that the sequence of values  $x, x^{(1)}=x+dx, x^{(2)}=x+2dx, \dots, x^{(n)}=x+ndx$  was assigned to the variable  $x$ .<sup>48</sup> In this way, however,  $d^2 x$  was constantly equal to zero (in Euler's terminology, the higher order differentials  $d^2 x, d^3 x, d^4 x, \dots$  were *per se* equal to zero): they vanished in a different sense from  $dx$  and the intuitive meaning of a differential understood as an evanescent variable disappeared. Euler therefore attempted to provide a different interpretation of the differential according to  $d^2 x$ . If one applied (12) to the function  $y=x$ , then  $P=Q=R=\dots=0$ , that is  $d^2 y=d^2 x=0dx^2+0dx^3+\dots$  and

<sup>46</sup> Euler justifies this step by appealing finite differences; however, its extension to differentials is not a source of further difficulties.

<sup>47</sup> Naturally, the reasoning can be repeated; if  $dq=rdx$ , then  $d^3 y=dq=rdx^3$ , if  $dr=tdx$ , then  $d^4 y=dr=tdx^4, \dots$ : therefore the higher order differentials of  $y$  can be calculated one after another by differentiating  $p, q, r, t$ , etc.

<sup>48</sup> It is possible to think the sequence  $x^{(n)}=x+ndx$  as a sequence  $x^{(n)}=x+n\omega$ , where the increment  $\omega$  is initially finite and subsequently made evanescent;  $dx=\text{constant}$  means that the increment diminishes in the same way for all the points  $x^{(n)}=x+ndx$ . Whatever value is assigned to  $dx$  tending uniformly to zero, we will always have  $d(dx)$  constantly = 0.

$P=Q=R=\dots=0$ : it was possible to state that the second differential  $d^2x$  vanished before  $dx^2$  and all the higher powers of  $dx$  [Euler 1755, 88]. Therefore,  $d^2x$  was both zero *per se* and a relative zero (namely,  $d^2x$  was actually equal to zero and was a quantity that vanished before  $dx^2$ ,  $d^3x$ ,  $d^4x$ ,...) A similar interpretation can be given for  $d^3x$ ,  $d^4x$ , ... .

The aim of these complex and occasionally rather unclear considerations is to prove that the second differential coefficient is obtained by differentiating the first coefficient: it is natural to ask ourselves why Euler did not directly define the second differential as  $d^2y=qdx^2$ , where  $q=dy/dx$ . Yet even in the preface to *Institutiones*, Euler seemed to suggest precisely this definition for  $d^2y$ . Indeed, he had observed that since infinitesimals were equal to zero, the higher order differentials were never considered *per se* but in relation to each other. More precisely, given a function  $y=f(x)$ , whose differential coefficient is a certain function  $p$ , the second differentials were obtained by considering the ratio of increment of the function  $p$  with other increments and the symbols of the differentials serve only to give a convenient representation of certain finite quantities [Euler 1755, 8].

Instead, in chapter IV of the *Institutiones*, Euler made use of the complex construction described above to introduce higher order differential coefficients: he seemed to be worried about the confrontation with the Leibnizian tradition and to demonstrate how Leibniz' and Bernoulli's differential calculus was capable of being translated into the new calculus of functions and their differential coefficients.

The Leibnizian calculus was not based on functions but on curves analytically expressed by an equation  $f(x,y)=0$ . The independent variable was not chosen *a priori* and therefore it was not established *a priori* that  $dx$  was a constant. The notion of function is based rather on a clear distinction between dependent and independent variable: the entire Eulerian construction, beginning with the formulae of finite differences, is made using the hypothesis that  $x$  is the independent variable.<sup>49</sup> Euler emphasised that the increments of  $x$  are taken as constants (that is,  $x$  is the independent variable and  $d^2x=0$ ) and that he is studying the variability of functions in relation to  $x$ . In the case that constant increments were not assumed, the second differentials have a more complicated expression.

Indeed, if  $dx$  was not constant, then the differential of  $p(x)dx$  was derived by considering the increment  $dx$  of  $x$  and  $d^2x$  of  $dx$ . Put  $p(x+dx)=p(x)+q(x)dx$ , he had

$$d^2y=p(x+dx) \cdot (dx+d(dx)) - pdx = (p+qdx) \cdot (dx+d^2x) - pdx = pdx + pdx^2 + qdx^2 + qdxdx^2 - pdx = pdx^2 + qdx^2 + qdxdx^2.$$

Since  $qdx^2$  vanished before  $qdx^2$ , he obtained  $d^2y=px^2+qdx^2$  [Euler 1755, 88-90].

In chapter IV Euler restricted himself to observing that nothing can be said with certainty about second differentials if  $dx$  is not assumed to be constant: there is no explanation about how  $d^2x$  can be reduced in this

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<sup>49</sup> In chapter 1, Euler had already noted that, as regards finite differences, the first difference  $\Delta y=y(x_1)-y(x)=y(x+\omega)-y(x)$  was not influenced by the sequence  $x_i$ , while the second differences changed according to the nature of  $x_n$ . Indeed, in general it is  $\Delta^2 y=\Delta^{(1)}y-\Delta y=y(x_2)-2y(x_1)+y(x)$ , and, for  $x_n=x+n\omega$ ,  $\Delta^2 y=\Delta^{(1)}y-\Delta y=y(x+2\omega)-y(x+\omega)-y(x+\omega)+y(x)=y(x+2\omega)-2y(x+\omega)+y(x)$ .

If one did not consider a sequence in arithmetic progression, the result was different. These remarks are also valid for differentials. The question of choice of the progression of variables and of the indeterminacy of higher order differentials in the eighteenth-century calculus is treated in detail by Bos (see, in particular [Bos 1974, 25-31]).



case to the theory of evanescent quantities. He dealt with the question more thoroughly in chapter VIII, where he applied the rules of differentiation to functions such as  $V=(dx^2+dy^2)^{1/2}$  and  $V=ydy/dx$  by considering  $dy$  and  $dx$  as variables. For instance, he found that the differential of  $V=(dx^2+dy^2)^{1/2}$  was  $dV = \frac{dx d^2x + dx d^2y}{\sqrt{dx^2 + dy^2}}$

and that the differential of  $V=ydy/dx$  was  $dV = dx + \frac{yd^2x}{dy} - \frac{ydx d^2y}{dy^2}$  [Euler 1755, 167-168].

Euler [1755, 168] observed that the formulae containing higher order differentials have a vague meaning since they do not possess any determined value *per se* but assume values which vary according to which differential is taken as constant. For example, given the expression  $\frac{yd^2x + xd^2y}{dxdy}$ , if one considered  $dx$  as a constant then the differential was  $\frac{xd^2y}{dxdy}$ , while if  $dy$  was a constant, then one obtained a different result

$$\frac{yd^2x}{dxdy}.$$

Furthermore, Euler believed that higher order differentials did not have an effective use in analysis [Euler 1755, 263]: they can be eliminated due to the differential coefficients  $p, q, r, \dots$ , (defined by the relations  $dy=pdx$ ,  $dp=qdx$ ,  $dq=rdx, \dots$ ); namely, putting  $d^2y=qdx^2$ ,  $d^3y=dq=rdx^3, \dots$ . For instance, Euler stated that  $\sqrt{dx^2 + dy^2}$  as a constant was often found in the applications of the calculus. Set  $dy=pdx$  and  $dp=qdx$ , he

obtained  $dx\sqrt{1+p^2}=\text{constant}$ ,  $d^2x\sqrt{1+p^2} + \frac{pqdx^2}{\sqrt{1+p^2}} = 0$  and  $d^2x = -\frac{pqdx^2}{1+p^2}$ . He also derived

$d^2y=qdx=qdx^2+pd^2x=qd^2x - \frac{p^2qdx^2}{1+p^2} = \frac{qdx^2}{1+p^2}$ . Similarly one could derive  $d^3x$ ,  $d^4x, \dots$ ,  $d^3y$ ,  $d^4y, \dots$  [Euler 1755, 178].<sup>50</sup>

At this point, a fundamental clarification must be made in order to understand Eulerian calculus. Differentiation is not a punctually defined operation;<sup>51</sup> in other words, the differential is not defined at a specific point  $x_0$  of the domain of the function. Euler never considers limitations to the domain of variables: such a concept is completely absent from his mathematics and he only considers general or abstract quantities (see, for instance, (1755, 15 and 83). Differentiation was regarded as a global operation that involved the entire analytical expression  $y(x)$  and was aimed at determining another analytical expression  $p(x)$ . The statement that the calculus transformed functions into functions should be understood in the sense that calculus transformed analytical expressions (functions in the eighteenth century use of the term) into analytical expressions. Since this article is principally concerned with offering an analysis of the notion of differential, I will not provide a detailed examination of this aspect, which is an integral part of the eighteenth century concept of function (see [Ferraro 2000a]), and instead will merely offer two examples.

<sup>50</sup> On Euler's program to eliminate higher-order differentials from analysis, see [Bos 1984, 66-77].

<sup>51</sup> Today a derivative is locally defined in the sense that, given a point  $x_0$  of the domain of a function  $f(x)$ , the derivative is defined at the point  $x_0$ .

Another example concerns the differential of the logarithm function. In [1755], once he had established that  $d(\log x) = dx/x$ , Euler also did not hesitate to apply this formula to the case where the variable is imaginary, without making any distinction between real and imaginary variable. Thus he found the differential

$dy = \frac{dx}{\sqrt{1-x^2}}$  of the function  $y = \frac{1}{\sqrt{-1}} \log \left( x\sqrt{-1} + \sqrt{1-x^2} \right)$  and asserted that the differential of an imaginary quantity could be real [Euler 1755, 124]. More explicitly, in [Euler 1749] he had stated: "Car, comme ce calcul roule sur des quantités variables, c'est-à-dire sur des quantités considérées en général, s'il n'étoit pas vrai généralement qu'il tût  $d.lx=dx/x$ , quelque quantité qu'on donne à  $x$ , soit positive ou negative, ou même imaginaire, on ne pourrait jamais se servir de cette règle, la vérité du calcul differential étant fondée sur la généralité des règles qu'il renferme."<sup>52</sup>

The only, and in any case partial, exception to the approach to the calculus as an algorithm transforming analytical expressions into analytical expressions concerns the treatment of equations of partial derivatives. Without going into detail, it can be stated that, in order to resolve this type of equation, Euler also considered functions which did not have an analytical expression which he called discontinuous functions. He denoted the differential of a discontinuous function  $f:y$  by the symbol  $dyf':y$  [Euler 1768-70, III:69]), he took the geometric meaning of a function into account and stated that if  $f(x)$  represented a curve then  $f'(x)$  was the slope of the tangent whereas, if  $f(x)$  was interpreted as an area then the differential coefficient  $f'(x)$  was a curve. There was no adequate development of this idea: Euler never actually dealt with such functions but merely stated that it was possible to define the differential coefficient and that these formed part of the solution to the equations of partial derivatives.<sup>53</sup>

## 6. Conclusion

This article has attempted to highlight some aspects of the Eulerian foundations of the calculus. It has been observed that Eulerian calculus is not based on the notion of the set nor that of real number but on that of quantity. Only natural numbers and rational numbers were considered numbers in their own right. Irrational numbers, zero and imaginary numbers were ways of dealing with quantities; they were useful fictions for mathematics. The notions of evanescent quantity and indefinitely increasing quantity, when interpreted using the conceptual instruments available to modern mathematics, seem to be an ambiguous mixture of different elements, a continuous leap from a vague idea of limit to a confused notion of infinitesimal. In reality, Euler does not confuse the modern notions of limit and the modern concept of infinitesimal: he simply did not possess such notions, but merely a primordial idea (directly derived from the physical world) of bringing a variable quantity close to a certain limit. The application of modern concepts to Euler's vague and ambiguous notions results in transforming Eulerian mathematics into something different.

It has also been said that Euler, in principle, conceived the calculus as the calculus of finite quantities, having as an object not the differentials  $dy, dx, \dots$  but the differential coefficients  $dy/dx$ ; nevertheless, the first order

<sup>52</sup> "For, as this calculus concerns variable quantities, that is quantities considered in general, if it were not generally true that  $d(\log x) = dx/x$ , whatever value we give to  $x$ , either positive, negative or even imaginary, we would never be able to make use of this rule, the truth of the differential calculus being founded on the generality of the rules it contains" [15, 143-144].

<sup>53</sup> On this subject, see [Truesdell 1960] and [Ferraro 2000a].

differentials not only serve to introduce differential coefficients but, as fictions, can be used per se and play an important role in the calculus. Moreover, the differential coefficient  $p(x)$  of a function  $f(x)$  was not punctually defined, as in the case of the modern derivate (that is, given the function  $f(x)$ , subject to a specific condition, to a number  $x_0$  belonging to an interval  $I$  is associated with an appropriately defined value  $p(x_0)$  but globally transformed analytical expressions into analytical expressions.

Euler's successors<sup>54</sup> developed certain concepts of his foundation of the calculus and rejected others. For example, Lagrange rejected the use of evanescent quantities and infinitely large quantities; however, he accepted the attempt to give a systematic translation of the calculus of differentials into a calculus of finite quantities as well as the use of rules applied globally to analytical expressions and not locally to functions in the modern sense. Shortly afterwards, Cauchy rejected this latter aspect of Eulerian theory, but accepted the fundamental idea that the algorithm of the calculus consisted in determining the differential coefficient. Cauchy's use of the idea of limit was explicit but the use of infinitesimals and infinite numbers persisted provided that they were effectively capable of being reduced to limits (accordingly a case such as  $(-1)^\infty$  was excluded).

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<sup>54</sup> In this article, I have not dealt in detail with the interpretations given to the calculation of the Eulerian zeros in the eighteenth century. I will simply mention the fact that Lagrange considered the foundations of the calculus of Euler and d'Alembert as analogous and contrasted both with the Leibnizians.

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$$1 + \frac{\alpha\beta}{1 \cdot \gamma}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1 \cdot 2 \cdot \gamma(\gamma+1)}x^2 + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1 \cdot 2 \cdot 3 \cdot \gamma(\gamma+1)}x^3 + \text{etc.}, \text{Werke, Bd.3, 124-162.}$$

