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Contagion in financial networks: A threat index

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Abstract This paper proposes to measure the spill-over effects that cross-liabilities generate on the magnitude of default in a system of financially linked institutions. Based on a simple model and an explicit criterion -the aggregate debt repayments- the measure is defined for each institution, affected by its characteristics and links to others. These measures -one for each institution- summarize relevant information on the interaction between the liabilities structure and the shocks to resources and they can be useful to determine optimal intervention policies. The approach is illustrated to evaluate the consolidated foreign claims of 10 EU countries.

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1 Introduction

An intricate web of claims and obligations ties together the balance sheets of a wide variety of institutions, especially in the financial sector between banks, hedge funds, and various intermediaries. Some argue that these ties have played a large role in the dissemination of the financial crisis of 2007-2008. As such, cross-liabilities are an important concern for both the financial institutions and the regulators and there is a general call for addressing their role in the risk of the system, the so-called 'systemic' risk.[1] This paper proposes to measure the spill-over effects that cross-liabilities generate on the magnitude of default in a system of financially linked entities. Based on an explicit criterion -the aggregate debt repayments- the measure is defined for each institution, affected by its characteristics and links to others. These measures -one for each institution- summarize relevant information on the interaction between the liabilities structure and the shocks to resources. They can be useful to a regulator to determine how to inject cash during a liquidity crisis, to the safe

[1] The new framework proposed by the Basel committee (Basel III) identifies some 'systemically important financial institutions' (SIFI) from which higher standards are required. The SIFI are mainly determined by their size.

institutions to decide which debts they should write-off in priority, or to evaluate the impact of raising capital before the occurrence of default. The approach is illustrated on the consolidated foreign claims of 10 EU countries.

The risk considered in this paper comes from the default of institutions on their cross-liabilities, as occurs during liquidity crises. The capacity of an institution to repay its liabilities depends not only on its revenues from outside activities -its operating cash-flow- but also on the reimbursements on its claims, calling for a *joint* determination of the repayments that reflects the full extent of the propagation of defaults. In this paper, repayments are endogenous, determined by the clearing mechanism due to Eisenberg and Noe (2001) (hereafter EN). At clearing repayment ratios, each institution in default reimburses as much as it can given others' repayments and limited liability.[2] The aggregate repayments are thus endogenously determined as a function of the cash-flows of all institutions.

The impact of an institution on the system is measured by the variation in these aggregate repayments following a decrease in the cash-flow of the institution. When the institution does not default, the variation is null. When it defaults, the variation includes not only the decrease in its repayment following the decrease in its cash-flow but also the decrease caused by the propagation of defaults, the *spill-over effects*. The variation is simple to compute if the institution's cash-flow decrease is moderate enough to leave unchanged the set of defaulting institutions. In that case, the variation is proportional to a term called hereafter *threat index*. The index of defaulting i counts all the chains of successive defaulting creditors starting at i, discounted by liabilities' proportions.[3] This implies that the threat indices of defaulting institutions depend on the financial health of their (direct and indirect) creditors. Since repayment ratios depend on the health of their debtors, threat indices and repayment ratios may not be aligned.

The default of an institution is interpreted here as temporary, not as bankruptcy.[4] For example, creditors agree to postpone unpaid debts, as within the network of firms described in Kiyotaki and Moore (1997).[5] In the financial sector, this situation corresponds to an illiquidity episode, in which all institutions are solvent (and it is commonly known) but possibly default due to liquidity constraints. The clearing mechanism organizes these delays in an orderly way, thereby avoiding costly liquidation of projects or run phenomena. Though temporary, defaults have an impact when activ-

[2]Limited liability can also interpreted as a liquidity constraint, see Section 3.

[3]It is thus given by a centrality 'Katz-Bonacich' index on the network of defaulting banks (see the related literature).

[4]Alternatively, if we assume as Elliott et al. (2013) that an institution is bankrupt when its ratio falls below a threshold, the analysis applies to the situations where the cash-flows are large enough so that each ratio is above the threshold.

[5]In their model, firms are symmetric and the shocks are macro-economic, affecting all cash-flows identically. The ratio is the same for all so that the clearing mechanism is simplified and there is a single multiplier, the same for all firms.

ities are linked to the total reimbursed amounts (as developed in a two-period model in Section 3). In such a case, the intervention of a regulator or a change in the institutions' agreements to improve the reimbursement flows are justified.

The analysis has several policy implications. First, an injection of cash, by increasing institutions' cash-flows, results in an increase in the activity. To obtain the maximal improvement, the injection must account for the spill-over effects across institutions. When the amount of cash is moderate, the optimal policy is characterized by the threat indices: Cash should be injected into the defaulting institutions with the largest threat index, which are not necessarily those with the lowest repayment ratio. Similarly, the optimal policy of postponing or writing-off part of the claims of the safe institutions on the defaulting ones is determined by the threat indices.

Second, the lack of information on the bilateral liabilities between financial institutions is a concern to regulators. How valuable is this information? To address this question, I compare the optimal injection strategy, which assumes complete information, with the benchmark in which the regulator knows the total liabilities and claims of each institution but not the bilateral ones. The value of information is defined as the improvement in the aggregate repayments reached by the optimal policy under complete information over the benchmark. When the injected amount is moderate, the value is proportional to the difference between the maximal threat index and its average over the defaulting institutions. This difference, hence the value of information, is driven by the asymmetry in the liabilities' proportions of the defaulting institutions to each other.

Third, the impact of increasing capital levels can be assessed by considering an ex ante stage before the cash-flows are realized and the clearing ratios are determined. The value of raising the capital of an institution at the margin is the expectation of its threat index. The value is increasing in both the institution's default probability and the expected spill-over effects conditional on its default. The latter are increasing in the (random) size of the defaulting set when the institution defaults. Even if cash-flows are independent, the cross-liabilities induce positive correlations between defaults, which in turn increase the conditional spill-over effects hence the values of raising the capital. Also, the size of an institution has an ambiguous effect on the value of raising its capital, reflecting the two roles a large institution may have: on one hand, its large reimbursements serve as a cushion to its defaulting creditors when it does not default, on the other, its default causes more defaults.

Finally, to illustrate the approach, I use consolidated foreign claims between 10 EU countries[6] and compute expected ratios and threat indices. The results uncover interesting features that are not

[6]The data is available from the Bank of International Settlements. The same data on 6 EU countries is used by Elliott et al. (2013) in a model with cost to default.

easy to detect from the liabilities structure because of its asymmetry and heterogeneity.

The literature on financial contagion is growing very fast. It investigates various channels of contagion and spill-over effects in a financial system, through the cross-liabilities as here, or through correlation in cash-flows, fire sales, panic phenomena amplified by asymmetric information and mis-coordination. I review the most related works to this paper.

Empirical studies have examined the potential for contagion in calibrated interbank markets, as reviewed by Upper (2011). Recent studies, often based on simulations, aim to assess the impact of cross-holdings on contagion[7] (Nier et al. 2007, Gai and Kapadia 2010, Elliott et al. 2013, Glasserman and Young 2015, Acemoglu et al. 2015) and to identify the strength of the risk-sharing versus risk-spreading effects of the liabilities. These works measure the risk of the system by the expected contagion size (say, the expected number of defaulting banks) triggered by a bank picked at random. Given the observed heterogeneity of financial institutions, both in their size and connections, it is important to assess the spill-over effects initiated by a *given* institution, as is performed by the measure introduced in this paper.

Several approaches have been followed to measure the risk of a particular institution while accounting for the interdependencies within the system. Most of the proposals consider the interdependencies induced by correlated portfolios and amplification phenomena due to fire sales and neglect the impact of the cross-liabilities. One approach extends the standard banks' risk indicators -VaR, expected shortfall- by conditioning on systemic events, which are defined as those where a stock index falls below a threshold (the CoVaR measure in Adrian and Brunnermeier 2011 and the Marginal Expected shortfall in Acharya et al. 2011 or Brownee and Engle 2010). These measures are based on a reduced form and cannot distinguish between the initiation and contagion effects.

An alternative approach takes the perspective of the management of the risk in a banking system. Viewing a regulator as owning a portfolio (composed of short put options) on banks due to its role as a lender of last resort, Lehar (2005) assesses the risk of that portfolio and the contribution of each bank to that risk. My approach is similar in the sense that the impact of an institution on the system is defined in reference to an aggregate indicator, the aggregate repayments. Greenwood et al. (2015) assess the vulnerability of banks in a model where the channel of contagion comes from fire sales induced by active asset management. Incorporating the two channels of contagion, cross-liabilities as here and fire sales, would give a threat index that would depend not only on the liabilities structure but also on the similarity in holdings and the sensitivity of prices to sales. Gouriéroux et al. (2012), based on the equilibrium EN model, consider the impact of a shock, say in

[7] A more economic approach has been initiated by Allen and Gale (2000) and Freixas et al. (2000).

stock prices, on the equilibrium and the number of defaults, thus with a different criteria than here. Finally, Cont et al. (2010) define the `contagion' index of a bank as the expected loss in all banks' tier 1 capital induced by the bankruptcy of that bank and estimate these indices on the Brazilian interbank network. The default mechanism differs from here since defaulting banks are bankrupt and inflict losses to their creditors defined by an exogenous and small recovery rate; instead in this paper, default is gradual and temporary.

The intervention policies considered here are adapted to liquidity crises encountered by solvent institutions. Other policies have been investigated in the context of bankruptcy, when the default of an institution involves additional costs. An important issue is then to identify conditions under which the institution should be bailed out. Rogers and Veraart (2013) consider the incentive for the stockholders of a pool of banks to rescue a failing bank. Building on that model, Alter et al. (2014) estimate the impact of different rules of reallocation of capital on a measure of bankruptcy losses in the German interbank market. Though the externalities differ from here, they find that capital rules based on Katz-Bonacich centrality indices perform better than other measures. Another intervention policy investigated recently is to help the strongest banks by providing them liquidity under a financial crisis. Such a policy can been justified in order to limit fire sales from the solvent but illiquid ones (Diamond and Rajan 2011), to help them buy the assets of bankrupt banks (Acharya and Yorulmazer 2008) or to limit panic phenomena (Choi 2014).

Finally, the paper relates to the large literature that studies the interactions and externalities channeled through a network of connections. Firstly, the threat index provides an assessment of a position in a network, and, as such, is related to the power or centrality indices introduced in the sociological literature by Katz (1953) or Bonacich (1987). These indices depend on an `attenuation' parameter that captures the importance of indirect links. The approach here differs in an important way since it is based on an explicit criterion. As a result, both the relevant network (the sub-network of the defaulting institutions) and the importance of indirect links are endogenous. Secondly, intervention policies have been investigated in games in which individual actions generate externalities channelled through a network. In a criminal network for example, the 'key player' to remove, the one whose arrest triggers the largest decrease in global criminal activity, may not be the most active one (see Ballester et al. 2005). A similar result holds in our model since the most threatening institutions are not necessarily those with the lowest repayment ratios.

The paper is organized as follows. Section 2 presents the model and the EN clearing mechanism, analyzes how the aggregate repayments vary with the cash-flows, and defines the threat indices. Section 3 examines intervention policies -cash injection and writing-off of claims-, and computes the value of the information on the cross-liabilities. Section 4 takes an ex ante point of view and introduces the value of raising capital. Section 5 illustrates the approach on the consolidated foreign claims for 10 EU countries and Section 6 concludes. Section 7 gathers the proofs.

2 A model with defaults

There are n institutions in a financially linked system, say banks, hedge funds, and various intermediaries in a financial system, or countries in an integrated market as considered in Section 5. Denote $N = \{1, \cdots, n\}$. Institutions hold claims on each other that arrive at maturity and have the same priority. The liability of i towards j is denoted by ℓ_{ij}. Institutions are endowed with a positive amount of resources to honor these debts, their *operating cash-flow*, denoted by z_i for i. The n-vector $\boldsymbol{z} = (z_i)$ and the $n \times n$ matrix $\boldsymbol{\ell} = (\ell_{ij})$ where ℓ_{ii} is null for each i summarize the relevant data.

No assumption is made on $\boldsymbol{\ell}$ as the liabilities' pattern depends on the situation under consideration. In payment systems, liabilities are often both ways, reflecting common clienteles for example. In long term arrangements, some patterns are more directed, such as the ones described in the Austrian banking system, with almost a pyramidal structure (see for example Upper and Worms 2004).

The next section describes the EN mechanism by which the claims are liquidated and Section 2.2 analyzes how the aggregate payments vary with the cash-flows.

2.1 The clearing mechanism

Claims are liquidated according to the clearing mechanism described by Eisenberg and Noe (2001). The capacity for institutions to repay their debts depends on the cash-flow levels $\boldsymbol{z}$, the mutual liabilities $\boldsymbol{\ell}$, and the repayments they effectively receive. The default of an institution is interpreted here as temporary, not as bankruptcy. Default can be partial, meaning that an institution in difficulty pays a fraction of its liability, the same to each creditor since liabilities have the same priority. The fraction is called *repayment ratio* or simply ratio.

Let us denote by ℓ_i^* i's total liabilities: $\ell_i^* = \sum_j \ell_{ij}$. Start by assuming that all institutions fully repay their debts to i. In that case, the total cash-flow available to i is equal to $z_i + \sum_j \ell_{ji}$. Due to the limited liability of stockholders, i will fully repay its debts only if this amount is larger than ℓ_i^*. Otherwise i defaults; i is said to initiate default, since it defaults even when its claims are fully reimbursed. Default possibly propagates due to the unpaid liabilities. The clearing mechanism determines the outcome of propagation by specifying all repayment ratios, θ_i between 0 and 1 for each i, based on two simple rules, limited liability and creditors' priority over stockholders.

Given $\boldsymbol{\theta} = (\theta_i)$, i's *total cash-flow*, denoted by $a_i(\boldsymbol{\theta})$, is the sum of its operating cash-flow plus the claims repayments by other institutions:

$$a_i(\boldsymbol{\theta}) = z_i + \sum_j \theta_j \ell_{ji}. \tag{1}$$

Under limited liability, i's repayments $\theta_i \ell_i^*$ are constrained to be less than $a_i(\boldsymbol{\theta})$ since stockholders cannot be forced to add cash. Equivalently, i's net-worth defined as

$$e_i(\boldsymbol{\theta}) = z_i + \sum_j \theta_j \ell_{ji} - \theta_i \ell_i^* \tag{2}$$

must be non-negative. The clearing mechanism basically requires institutions to reimburse their debts as much as possible under limited liability.[8] Formally

Definition 1 *Given $(\boldsymbol{z}, \boldsymbol{\ell})$, a vector $\boldsymbol{\theta} = (\theta_i) \in [0,1]^n$ is said to be a clearing ratio if it satisfies for each i*

(limited liability): $a_i(\boldsymbol{\theta}) \geq \theta_i \ell_i^*$ (equivalently $e_i(\boldsymbol{\theta}) \geq 0$)

(priority of creditors over stockholders): either $\theta_i = 1$ or $a_i(\boldsymbol{\theta}) = \theta_i \ell_i^*$ i.e. $e_i(\boldsymbol{\theta}) = 0$.

If no institution initiates default, then $\mathbb{1}$ (the n-vector of 1) is a clearing ratio since $e_i(\mathbb{1}) \geq 0$ for each i. Otherwise there is surely default. As shown by EN, the existence of a clearing ratio vector follows from the complementarities between the ratios, according to which the capacity of an institution to repay its debts is increasing in others' repayment ratios. Complementarities imply that there is a greatest (in each component) ratio vector under which limited liability is satisfied for each institution. It is easy to see that this ratio is a clearing ratio.[9] Furthermore, it is the unique clearing vector.[10] Creditors' priority can thus be seen as forcing the clearing ratio vector to maximize the payment of *each* institution within the system under the limited liability condition. As a result, the clearing vector maximizes any function that is increasing in the ratios.

[8] According to the definition, an institution with no liabilities towards others has a ratio equal to 1. This convention allows us to treat indebted and non-indebted institutions similarly.

[9] Due to complementarities, the set of $\boldsymbol{\theta}$ in $[0,1]^n$ for which limited liability is satisfied, $e_i(\boldsymbol{\theta}) \geq 0$ for each i, has a greatest element. This element is a clearing ratio. By contradiction, if creditors' priority is not satisfied, then both $\theta_i < 1$ and $e_i(\boldsymbol{\theta}) > 0$ hold for some i. Under a small increase in θ_i, i's net-worth remains positive and others' net-worths can only improve: the desired contradiction.

[10] As shown by EN, uniqueness holds because the cash-flows are assumed to be all strictly positive. This weak assumption simplifies the presentation and does not change substantially the results: In case of multiple clearing vectors, the greatest clearing vector is the solution to program $\mathcal{P}$ and net-worth levels are identical at all clearing vectors.

2.2 Aggregate repayments and threat indices

This section studies how aggregate repayments vary with the institutions' cash-flows and defines the threat indices. The results provide tools useful to the analysis on interventions on the system.

2.2.1 Aggregate repayments V

The clearing ratio maximizes the aggregate repayments, as we have just seen. Formally it solves

$$\begin{aligned} \mathcal{P}(\boldsymbol{z}) \quad &: \quad \max_{\boldsymbol{\theta},\mathbf{0}\leq\boldsymbol{\theta}\leq\mathbf{1}} \sum_{i\in N} \theta_i \ell_i^* \\ \theta_i \ell_i^* - \sum_j \theta_j \ell_{ji} &\leq z_i \text{ for each } i. \end{aligned} \tag{3}$$

Let $V(\boldsymbol{z})$ denote the value of the program $\mathcal{P}(\boldsymbol{z})$. The liabilities structure also influences repayments, as investigated in Section 3.3, but ℓ is omitted as an argument in V to simplify notation.

The remaining of the section analyzes how aggregate repayments V vary with the cash-flows. These variations summarize relevant information on the interaction between the liabilities structure and shocks on the cash flows, as illustrated in Section 5 on European cross-claims. Furthermore, in some contexts, a 'regulator' -a governmental regulator, the organizer of a payment system, or the pool of the institutions themselves- is concerned with these aggregate repayments and may intervene to improve them, as studied in Section 3.

The approach extends to alternative criteria, provided they are increasing in the ratios. It does not extend to an objective pertaining to stockholders' aggregate net worth due to the fact that default does not entail additional costs to them:[11] Summing net-worth values (1) over i, $\sum_{i\in N} e_i(\boldsymbol{\theta})$ is equal to the aggregate cash-flow, $\sum_{i\in N} z_i$ because the repayments within N cancel out. In case of default, the clearing mechanism performs transfers between the stockholders. The institutions that initiate default, those for which the value $e_i(\mathbb{1})$ is negative, finally end up with a null net-worth without the need for their stockholders to add cash. All the other institutions end up with a net-worth that is smaller than $e_i(\mathbb{1})$, the level would be no default.[12] Basically, partial default on liabilities plays the role of a 'buffer' to stockholders.

[11]The analysis thus clearly differs from those that interpret default (whatever level) in EN model as bankruptcy and consider stockholders' incentives to intervene as e.g. Rogers and Veraart (2013).

[12]This is obvious for those that do not default as they fully reimburse their debt but their total cash-flow is decreased. For those that do not initiate default but end up defaulting, their net-worth is null hence smaller than $e_i(\mathbb{1})$.

2.2.2 Properties of V and threat indices

Let us fix the terminology. Institution i is said to be *safe* if its net-worth is positive at the clearing ratio. At least one institution is safe since the sum of net-worth levels is equal to the positive aggregate cash-flow. i is said to be *defaulting* if its ratio is strictly lower than 1; since i is surely indebted, $\ell_i^* > 0$, i's *liabilities shares* are defined by $\pi_{ij} = \frac{\ell_{ij}}{\ell_i^*}$ for each j. Finally, i is at the *boundary* if its net-worth is null and its ratio is equal to 1: $e_i(\boldsymbol{\theta}) = 0$ and $\theta_i = 1$. Observe that typically no institution is at the boundary: If there is, a small perturbation in $\boldsymbol{z}$ (or $\boldsymbol{\ell}$) makes either net-worth strictly positive or its ratio strictly smaller than 1.

Next proposition analyzes how aggregate repayments V vary with the cash-flows and defines the threat indices. Notation is standard: Given a n-vector $\boldsymbol{x}$ and D a subset of N, $\boldsymbol{x}_D$ denotes $(x_i)_{i\in D}$, $\boldsymbol{x}_{-i}$ denotes $\boldsymbol{x}_{N-\{i\}}$ and $(x_i, \boldsymbol{x}_{-i})$ denotes $\boldsymbol{x}$.

Proposition 1 *The function V is piece-wise linear and concave. V is differentiable at each $\boldsymbol{z}$ for which no institution is at the boundary. Given the set of defaulting institutions D, the derivative vector $\boldsymbol{\mu} = (\frac{\partial V}{\partial z_i})$ is null outside D and $\boldsymbol{\mu}_D$ is the unique solution to*

$$\mu_i = 1 + \sum_{j\in D} \pi_{ij}\mu_j \text{ for each } i \text{ in } D. \tag{4}$$

μ_i is called i's threat index. The clearing ratio is non-decreasing and convex in $\boldsymbol{z}$, the default set and the threat indices are non-increasing in $\boldsymbol{z}$. Furthermore V is sub-modular: For each i

$$z_i' \geq z_i \text{ and } \boldsymbol{z}'_{-i} \geq \boldsymbol{z}_{-i} \text{ imply } V(z_i', \boldsymbol{z}'_{-i}) - V(z_i, \boldsymbol{z}'_{-i}) \leq V(z_i', \boldsymbol{z}_{-i}) - V(z_i, \boldsymbol{z}_{-i}). \tag{5}$$

The function V is thus well-behaved. Assuming no institution at the boundary, the generic situation, its derivative (the threat index) depends on $\boldsymbol{z}$ through the set of defaulting institutions D only, as can be seen from (4). Therefore V is linear over the set of cash-flows $\boldsymbol{z}$ that lead to the same set D and the kinks arise at cash-flows for which there is an institution at the boundary.

A marginal decrease of one unit in i's cash-flow decreases the repayments by μ_i, hence the term 'threat' index for μ_i (considering an increase instead, the index can be interpreted as a credit multiplier as well). Due to the envelope theorem, μ_i is the multiplier associated to the net-worth constraint (3) at points where V is differentiable. Thus, the index of a safe institution is null. Indeed its repayments are maximal and unchanged by a small variation of its cash-flow, hence no change in V occurs. For the defaulting institutions, the indices follow expression (4) by applying standard complementarity relationships between the repayment ratios (the solutions to $\mathcal{P}$) and the threat indices (the solutions to its dual).[13] The threat indices of defaulting institutions are jointly

[13] Their uniqueness -equivalent to the differentiability of V- is not straightforward without further assumptions on $\boldsymbol{\ell}$. See Section 3.1 and Lemma 1 in Section 7.

determined: the index of defaulting i is the sum of 1 -one unit in i's cash-flow induces i to repay one unit of its debt- and a term that depends on the indices of i's creditors that are themselves defaulting. This additional term thus represents the *spill-over effects* due to liabilities. It will be investigated more closely in Section 3.1, which also considers the case where an institution is at the boundary.

The monotony properties for the ratios and the threat indices relate to the complementarity between ratios: increasing the cash-flow of one institution not only (weakly) increases its ratio but also those of the other institutions. As a result, the set of defaulting institutions and the threat indices decrease. Sub-modularity extends this to non-marginal variations of the cash-flows. Equation (5) compares the incremental values in V due to an increase in i's cash-flow (from z_i to z_i') when the other institutions' cash-flows are either $\boldsymbol{z}_{-i}$ (the right-hand side) or the larger $\boldsymbol{z}'_{-i}$ (the left-hand side). When the defaulting set is the same set D for the cash-flows $\boldsymbol{z}$ and $\boldsymbol{z}'$, V is linear for the cash-flows larger than $\boldsymbol{z}$ and smaller than $\boldsymbol{z}'$: the two incremental values are equal to $\mu_i(z_i' - z_i)$ where μ_i is the index of i given D. When the defaulting set changes, sub-modularity says that the incremental value in V due to an increase in i's cash-flow is larger the lower other institutions' cash-flows are. As shown in the proof, this is due to the fact that the decreases in default and in spill-over effects due to an increase in i's cash-flow are larger the weaker other institutions are.

Example 1 Consider a chain of intermediaries who collect funds for institution 1: Each i, $1 < i \leq n$, lends to $i-1$. Clearing ratios are computed recursively starting from 1. Since 1 has no claims on other institutions, its repayment ratio is determined by its cash-flow as the minimum of 1 and z_1/ℓ_1^*. Now, since 2 receives payments from 1 only, its total cash-flow is known; θ_2 is thus determined as the minimum of 1 and $(z_2 + \theta_1 \ell_1^*)/\ell_2^*$. The computation proceeds until reaching institution n.

Once the default set is known, and assuming no boundary institution,[14] threat indices are computed recursively, starting from institution n. Since n has no creditors, its threat index is either null (n does not default) or equal to 1 (n defaults). The computation proceeds: at step i, if i is safe, set μ_i to zero; otherwise, since $i+1$ is i's unique creditor, expression (4) yields $\mu_i = 1 + \mu_{i+1}$ where μ_{i+1} has been determined at the previous step and. It follows that the threat index of a defaulting institution is simply equal to 1 plus the number of its consecutive (direct and indirect) defaulting creditors. Clearly, the default ratios, $1 - \theta_i$, and the threat indices may be in different orders. ∎

Similar recursive computations can be performed in the reverse situation in which all the liabilities point toward the top or in pyramidal structures (as for a 'conglomerate' in which each institution borrows funds from its direct subordinates). In a general network with cycles, a recursive compu-

[14]If there are boundary institutions, the computation can be performed by considering them either as defaulting or safe, as explained in the next section.

tation is not possible. The clearing vector can be computed by solving the linear program $\mathcal{P}$ or by using the algorithm defined by EN, which exploits the complementarities structure (see Section 3.1). Once the set D is known, the threat indices are computed by solving the linear system (4).

Comparing clearing ratios and threat indices Let us write down the conditions satisfied by the clearing ratio and threat index vectors, assuming no boundary institution. They are respectively of the form $(\mathbb{1}_{N-D}, \boldsymbol{\theta}_D)$ and $(\mathbf{0}_{N-D}, \boldsymbol{\mu}_D)$ where D denotes the default set. The clearing ratio satisfies the system of linear inequalities that says that each net-worth, $z_i - [\ell_i^* - \sum_{j\in D}\theta_j\ell_{ji} - \sum_{j\in S}\ell_{ji}]$ for i, is positive for those i not in D and is null for those in D. Considering these nullity conditions on D, and dividing them by ℓ_i^*, one obtains that $\boldsymbol{\theta}_D$ and $\boldsymbol{\mu}_D$ respectively solve the linear systems

$$\text{for each } i \in D: \theta_i = \sum_{j\notin D}\pi_{ji} + \sum_{j\in D}\theta_j\pi_{ji}\frac{z_i}{\ell_i^*}, \text{and for each } i \in D: \mu_i = 1 + \sum_{j\in D}\pi_{ij}\mu_j.$$

The backward-forward relationships between clearing ratios and threat indices that arise in the case of a debt chain are present in general in the form of dual relationships. Whereas the distress of i as measured by its repayment ratio depends on the distress of its debtors (through the π_{ji}), the threat that i poses on the payment system depends on the threat of its creditors (through the π_{ij}). Also, the repayment ratios are affected by the precise values taken by the cash-flows whereas the indices depend on them only through the default set (see Section 3.1 for an explanation). This explains why the ratios and threat indices of the defaulting institutions are not necessarily aligned.

3 Interventions to increase aggregate payments

This section examines two types of intervention when the objective is to improve aggregate repayments during a default episode. The first intervention is conducted by a regulator who injects cash into institutions, whereas the second one relies on the safe institutions, which either inject cash or write-off part of their claims. In each case, the threat indices play a major role in determining the optimal policies. Additional results on the threat indices (their determinants and a comparison with the order of defaults in EN algorithm) are given and the value of knowing the liabilities structure is assessed.

Let us first consider two settings in which the objective to improve aggregate repayments is justified. The first setting has financial institutions in a two-period model. At the current date 1, institutions are solvent but possibly default due to liquidity constraints. Defaults, though not costly to them, have a negative impact on their loans to the economy, which justifies intervention. Specifically, at date 1, banks receive their operational cash-flows and interbank claims arrive at maturity. They also hold illiquid assets that will be realized in the future and long run debts, denoted respectively by Z_i and L_i for i. Each bank's total net worth is positive (solvency condition). Though, because raising additional financing on top of their cash-flows or liquidating long run projects would be costly, institutions are subject to liquidity constraints specified by their cash-flows.[15] Interbank claims are cleared according to the EN mechanism. Default is temporary and the unpaid obligations are not written off. The balance sheets before and after the clearing are given

$$\left[\begin{array}{c|c} assets & liabilities \\ Z_i & L_i \\ \sum_j \ell_{ji} & \ell_i^* \\ z_i & E_i \end{array}\right] \qquad \left[\begin{array}{c|c} assets & liabilities \\ Z_i & L_i \\ \sum_j (1-\theta_j)\ell_{ji} & (1-\theta_i)\ell_i^* \\ z_i & \\ \sum_j \theta_j \ell_{ji} - \theta_i \ell_i^* & E_i \end{array}\right]$$

Figure 1: i's balance sheet before and after the clearing

in Figure 1. Initially, i's initial total asset is $A_i = Z_i + \sum_j \ell_{ji} + z_i$ and i's equity (total net worth), $E_i = A_i - (L_i + \ell_i^*)$, is positive. The clearing has two effects on i's asset: First, the value of i's interbank claims becomes equal to the non-reimbursed amounts (since they are not written off), second, the cash is modified by the payments made to i diminished of those made by i. The net effect of the claims reimbursed to i is null so that i's total asset becomes equal to $A_i - \theta_i \ell_i^*$. The liabilities, $L_i + \ell_i^*$, are diminished by i's reimbursements and become equal to $L_i + (1-\theta_i)\ell_i^*$. Equity is unchanged, still equal to positive E_i, due to the fact that the non-reimbursed debts are not written off.

Default, though temporary, has nevertheless an impact if the loans to the economy are decreasing in the non-reimbursed claims. This arises for example if banks adjust their activity by targeting a fixed leverage ratio[16] λ, with $\lambda > 1$. Such a behavior may be induced by a regulatory upper-bound leverage ratio, when banks always choose the maximal activity compatible with that bound. According to the target rule, i offers new loans d_i defined by $A_i - \theta_i \ell_i^* + d_i = \lambda E_i$ (i's equity is not affected by d_i). Thus there is a substitution between i's loans to the economy and its non-reimbursed debt. Aggregating over all banks, the substitution between the loans to the economy and the non-reimbursed interbank claims justifies to intervene so as to improve aggregate repayments.

[15] Starting from Diamond and Dybvig (1983), the distinction between liquidity and solvency has been much studied in the banking literature, see e.g., Acharya et al. (2011). Even if banks' solvency is common knowledge, runs can arise as a rational phenomena. Here runs are excluded by the clearing mechanism, because it picks the maximal repayments across institutions (under equal priority) while avoiding costly premature liquidation.

[16] Such a behavior is described and documented by Adrian and Shin (2010).

If the cash is reimbursed at the next period, the injection is temporary, akin to lending.

The second setting is a production economy with a network of firms. The objective is justified by credit constraints as previously and the effect of time on production, drawing on Kiyotaki and Moore (1997). At a previous date 0, firm i ordered forward some units to j to be delivered at date 1 against ℓ_{ij} dollars. The gross return to production is ρ: if at date 1 i buys the units he ordered to each supplier, the value of i's production at a future date will be ρ times the total cost, i.e., $\rho\ell_i^*$. At date 1, firms have available cash, z_i for i, and receive payments for the units they deliver. They are credit-constrained, unable to raise finance from outside investors. If i defaults against his suppliers, then he does so on a pro-rata basis and scales down his orders to $\theta_i\ell_{ij}$ for each j, triggering possible downsize for them. The clearing mechanism thus specifies the maximal scales of production, given the available resources and the credit constraints. The value of total production will be $\rho\sum_i \theta_i\ell_i^*$, i.e., proportional to the realized orders' total. To improve production, one can consider for example a decrease in the price of the inputs paid at date 1 by defaulting i; such a decrease is similar to a writing-off of part of i's debt as described in Section 3.3 and can be analyzed similarly.

3.1 Cash injection policies

Consider a regulator who is endowed with an amount of cash, denoted by m, to be injected into the institutions during a default episode. The regulator's objective is to maximize aggregate payments. A *feasible injection policy* is described by a non-negative n-vector $\boldsymbol{x} = (x_i)$ whose total $\sum_i x_i$ is less than or equal to m (budget equation). x_i represents the amount received by i, which changes z_i into $z_i + x_i$. As a result, aggregate payments are changed from $V(\boldsymbol{z})$ to $V(\boldsymbol{z} + \boldsymbol{x})$.

An injection policy $\boldsymbol{x}$ is said to be *optimal* if it maximizes $V(\boldsymbol{z} + \boldsymbol{x})$ over the set of feasible strategies. i is said to be a 'target' if it receives a positive amount, $x_i > 0$. The optimal injection policies are characterized by the threat indices when m is small enough, using Proposition 1.

Proposition 2 *A marginal injection of cash is optimal if it targets the institutions with the largest threat index. The same strategy is optimal for a larger amount provided it is moderate enough to keep the defaulting set unchanged. The increase in V is equal to $\mu^{max}m$ where μ^{max} denotes the maximal threat index.*

The policy is especially simple when the injected amount is moderate since there is no need to modify the targets. These targets may not be the institutions with the largest default ratios, since ratios and threat indices can be in different order. The targets may not be the institutions with a large 'size' either (the size can be measured in different ways, for instance by the liabilities total or the ratio of this total to the cash-flow). As clear from the expression (4), the threat index of defaulting

i is determined by its liabilities' shares (not by their levels) towards its defaulting creditors. Hence the defaulting institutions with the largest index are not necessarily those with the largest size.[17] To understand better the determinants of the threat indices of defaulting institutions, let us write the relationships (4) in matrix form[18]

$$(\mathbb{I}-\boldsymbol{\pi})_{D\times D}\boldsymbol{\mu}_D = \mathbb{1}_D. \tag{6}$$

where $\mathbb{I}$ denotes the identity matrix. As shown in Lemma 1 in Section 7, the matrix $(\mathbb{I}-\boldsymbol{\pi})_{D\times D}$ is invertible[19] with an inverse expressed as an infinite sum: $(\mathbb{I}-\boldsymbol{\pi})^{-1}_{D\times D} = \mathbb{I}_{D\times D}+\boldsymbol{\pi}_{D\times D}+\boldsymbol{\pi}^{(2)}_{D\times D}+ ... + \boldsymbol{\pi}^{(p)}_{D\times D} + ...$ where $\boldsymbol{\pi}^{(p)}$ denotes the product of $\boldsymbol{\pi}$ by itself p times. We thus obtain from (6)

$$\boldsymbol{\mu}_D = \mathbb{1}_D + \boldsymbol{\pi}_{D\times D}\mathbb{1}_D + \boldsymbol{\pi}^{(2)}_{D\times D}\mathbb{1}_D + ... + \boldsymbol{\pi}^{(p)}_{D\times D}\mathbb{1}_D + ... \tag{7}$$

which also writes for each i in D as

$$\mu_i - 1 = \sum_{j\in D}\pi_{ij} + \sum_{j\in D, k\in D}\pi_{ij}\pi_{jk} + \cdots + \sum_{j_k\in D, k=1\cdots p}\pi_{ij_1}\pi_{j_1j_2}\pi_{j_2j_3}..\pi_{j_{p-1}j_p} + \cdots \tag{8}$$

Recall that $\mu_i - 1$ measures the spill-over effects generated by a unit of cash injected in defaulting i, provided there is no change in status (this is possible by scaling the unit since no institution is at the boundary). These effects are decomposed into a sequence of reimbursements that correspond to the terms on the right hand side of (8). Since defaulting i must use the received unit to reimburse its creditors, each i's creditor j receives the share π_{ij} of this unit, entirely used for reimbursement by those in default. This generates a first additional payment equal to i's *cumulated liabilities share* toward D, $\sum_{j\in D}\pi_{ij}$, the first term on the right hand side of (8). By the same argument, each of the π_{ij} units received by defaulting i's creditor j generates $\sum_{k\in D}\pi_{jk}$ extra units of payments. Summing over all defaulting creditors of i, the second additional reimbursement equals $\sum_{j\in D}\pi_{ij}(\sum_{k\in D}\pi_{jk})$, or $\sum_{k\in D}(\sum_{j\in D}\pi_{ij}\pi_{jk})$. Iterating, the additional indirect impact along a

[17] This property is true at the liquidation stage given the realized default set. One may suspect that a large institution, more precisely one with large liabilities, induces important losses and defaults among its creditors when it defaults. If true, its index is likely to be large when its default. This issue is addressed in Section 4 and the illustration of Section 5.

[18] There is a slight abuse of notation if some institutions are not indebted since then the matrix $\boldsymbol{\pi}$ is not defined on $N\times N$; it suffices then to consider the relative liabilities shares within indebted institutions.

[19] For a complete liability structure ($\ell_{ij} > 0$ for each distinct i and j) the property follows from well known results on diagonally dominant matrices since then the total of each row of $\boldsymbol{\pi}_{D\times D}$ is strictly smaller than 1 (D is a strict subset of N). For an incomplete liability structure, the proof relies on the fact that any subset of D has necessarily creditors outside the subset: Since the z_is are positive and the net worth of each defaulting institution is null, some payments must go out from the subset. This implies that, for some integer p, the total of each row of $\boldsymbol{\pi}^{(p)}_{D\times D}$ is strictly less than 1, hence the dominant eigenvalue of $\boldsymbol{\pi}_{D\times D}$ is smaller than 1. Without an outside creditor for each subset, invertibility may fail, as shown in the appendix.

path of p institutions, each one defaulting and indebted to its successor, is given by the p-th term in (8). This process explains why the indices are determined by the liability *shares* (not the absolute liabilities) within the set D and furthermore do not depend upon cash-flows' levels: The priority of creditors triggers automatic payments that are entirely determined by the liability shares of the recipient defaulting institutions.[20]

The cumulated liability shares to D, the first term in (8), are a primary determinant of the indices but they are not the only ones. The 'long-run' effects of injection are measured by the additional terms in (8) for p large, as made precise in Lemma 3 in Section 7. These terms converge to a dominant eigenvector $\boldsymbol{v}$ of $\boldsymbol{\pi}_{D\times D}$, with a null v_i if i is not involved in cycles. The dominant eigenvalue ρ_D measures the persistence of the flow of repayments in D. The larger ρ_D is, the more important the long-run effects are, i.e., the closer the threat index to a dominant eigenvector. This is illustrated in the following example, where the order of the cumulated shares and indices differ.

Example 2 There are 7 defaulting institutions, 1 to 7. Figure 2 represents the liabilities of $D = \{1, .., 7\}$, in which an arrow from i to j represents a liability from i to j. Each i in D has thus a creditor outside D. The liabilities of an institution are all equal so that the shares within D are: $\pi_{1i} = 1/5$ for $i = 2, 5, 6, 7$, $\pi_{2i} = 1/4$ for $i = 1, 3, 4$, $\pi_{3i} = 1/3$ for $i = 2, 4$, $\pi_{4i} = 1/3$ for $i = 2, 3$ and all others are null. Consider the indices in D. Since 5, 6 and 7 are not indebted to defaulting institutions, their indices are equal to 1. The other indices are approximately $\mu_1 = 2.2$, $\mu_2 = 3.09$, $\mu_3 = \mu_4 = 3.03$; observe that they are not ordered as the cumulated shares to D:1 has the largest share, $4/5$, but the smallest index. This is explained by the role of the the indirect liabilities in the process following an injection of one unit into an institution. Though 1 transfers the largest amount to the defaulting institutions, $4/5$, only 3/5 is transferred to 5,6,7, which transfer the received amount entirely outside D. Thus, after two 'rounds', only the liabilities between $1, 2, 3, 4$ matter with a network depicted on the right of Figure 2. This is reflected by the fact that dominant eigenvectors of $\boldsymbol{\pi}_{D\times D}$, proportional to $(0.1672, 0.5298, 0.5879, 0.5879, 0, 0, 0)$, have null components for $i = 5, 6, 7$. Furthermore, others are in a different order than the shares to D. The value ρ_D is equal to 0.634. ■

The process following cash injection can be used to understand the case where some institutions are at the boundary. When an institution, say i, is at the boundary, an increase in i's cash flow has to

[20] An alternative interpretation of expression (6) is in stochastic term. Using that $\boldsymbol{\pi}$ is a transition matrix since the sum $\sum_j \pi_{ij}$ is equal to 1, interpret π_{ij} as the probability of reaching j from i. Start with i in D and stop the process as soon as a safe institution is reached. Since $\rho_D < 1$, the process stops for sure. In that interpretation, the element i, j of the matrix $\boldsymbol{\pi}^{(p)}_{D\times D}$ is the probability of reaching j from i in p steps while staying all along in D and μ_i is the average number of times where the process stays in D when it starts at i. Probability techniques can thus be useful.

Figure 2: Index versus cumulated shares

be distinguished from that of a decrease: An increase has no impact on payments because i already repays its debt: $\frac{\partial V}{\partial z_i^+} = 0$, whereas a decrease triggers i's default; thus the left derivative $\frac{\partial V}{\partial z_i^-}$ is at least equal to 1. (Using the same argument as above, it can be computed by taking for D the set of institutions that are defaulting or at the boundary.) This explains why the value function V is not differentiable.[21]

Relationships with centrality indices Threat indices have a flavor of the Katz-Bonacich (KB) centrality indices. Given a network with incidence matrix $\boldsymbol{g}$, the vector of KB indices is defined as $(\mathbb{I} - a\boldsymbol{g})^{-1}\boldsymbol{g}\mathbb{1}$ where a is an `attenuation' parameter, which captures the importance of indirect links. In the next example, threat indices coincide, up to a linear transformation, with KB indices on the sub-network linking defaulting institutions. Despite this similarity, our approach differs in an important way: The threat index is based on an objective. This explains why both the subset of relevant nodes and the attenuation parameter are endogenous, respectively determined by the subset of defaulting institutions and the number of creditors per institution.

Example 3 Let each positive liability have an identical level and each institution have the same number of creditors, say p, hence the same total liabilities. The matrix $\boldsymbol{\pi}$ is thus proportional to the incidence matrix of the liabilities network: $\boldsymbol{\pi} = \frac{1}{p}\boldsymbol{g}$ where $g_{ij} = 1$ has 1 if ℓ_{ij} is positive and 0 otherwise. Given defaulting set D, $\boldsymbol{\mu}_D$ is equal to $\left((\mathbb{I} - \frac{1}{p}\boldsymbol{g})_{D\times D}\right)^{-1}\mathbb{1}_D$, which is a linear transformation[22] of the KB index associated to the network $\boldsymbol{g}_{D\times D}$ and attenuation parameter $1/p$. Therefore, the more numerous creditors each institution has, the smaller the attenuation factor and the more dissipated the impact of default is along a chain of creditors. ■

Threat indices and the order of default in EN algorithm Let us compare the order given by

[21] By the same argument, V is not differentiable with respect to the cash-flow of a defaulting j if there is a chain of defaulting creditors from j to a boundary institution.

[22] Multiplying $\mathbb{I} = (\mathbb{I} - ag) + ag$ by $(\mathbb{I} - a\boldsymbol{g})^{-1}$, one obtains $(\mathbb{I} - a\boldsymbol{g})^{-1}\mathbb{1} = \mathbb{1} + a(\mathbb{I} - a\boldsymbol{g})^{-1}\boldsymbol{g}\mathbb{1}$.

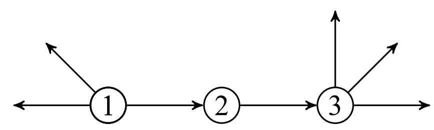

Figure 3: Index versus order of default in EN algorithm

the threat indices with the order suggested by EN using an algorithm to compute the ratios. The algorithm starts by setting all ratios equal to 1 and computes net-worth levels. If all net-worths are non-negative, $\mathbb{1}$ is the clearing ratio. Otherwise, $e_i(\mathbb{1}) < 0$ for at least one i. In that case, the algorithm adjusts sequentially the ratios downward. Several adjustments may be necessary for a given institution when there are cycles, contrary to the case of a chain. EN suggest that the first step at which an institution fails in the algorithm indicates its fragility. These levels are not necessarily in the same order as the threat indices, as illustrated by the following example.

Example 4 There are three defaulting institutions, 1, 2, 3, with liabilities of equal size represented in Figure 3. Assume that 1 is the only one to initiate default (this is possible for some values of the cash-flows). In EN algorithm, 1 defaults first, 2 second and 3 third. Since 3 is indebted outside the defaulting set, $\mu_3 = 1$. Thus $\mu_2 = 2$ and $\mu_1 = 4/3$ (because 2/3 of it liabilities are toward non-defaulting institutions) and it is more beneficial to inject a moderate amount of cash into 2 rather than 1. This is due to the fact that 2/3 of the cash injected in 1 ends up outside the defaulting set. ∎

3.2 The value of information

The optimal injection strategy uses information on the liabilities structure. In the absence of such information, a natural strategy is a *uniform strategy*, which injects the same amount in each defaulting institution: $x_i = m/d$ for i in D where d is the number of elements in D and 0 otherwise. As shown at the end of this section, a uniform strategy is indeed optimal when there is a lack of data on bilateral exposures. When there is complete information and the optimal policy can be implemented, the value of information can be defined as the improvement in the aggregate repayments over the uniform policy. This value is computed in the following proposition.

Proposition 3 *Let the injected amount m be allocated uniformly to the d defaulting institutions. If D does not change, the increase in V is equal to $(\frac{1}{d}\sum_{i\in D}\mu_i)m$. Thus, the loss with respect to the optimal strategy is proportional to the difference $(\mu^{max} - \frac{1}{d}\sum_{i\in D}\mu_i)$.*

The proof follows straightforwardly from Proposition 1 using thatthe marginal increase in V due to an injection is equal to μ_i. The value of information, for moderate amounts, is thus proportional to the difference between the maximal index and the average index over the defaulting set. In Example 2, the maximal index is equal to 3.09 and the average one to 2.05, so that the value is quite large, roughly equal to m.

When the cumulative liability shares differ, the indices differ as well[23] and the value of information is positive. Note that the indices may be in a different order than the cumulated shares because of chain effects (see e.g. Example 4) so that the information on the cumulated shares is not enough to implement the optimal policy. When the cumulative liability shares within D are equal, the value of information is null was shown in the next example.

Example 5 Let the cumulative liability shares within D be equal: The sum $\sum_{j\in D}\pi_{ij}$ is equal to σ_D for each i in D. In that case, the indices in D are all equal: $\mu_i = \frac{1}{1-\sigma_D}$ for i in D. The value of information is null and any moderate injection into D is optimal. The injection benefit itself may be large if σ_D is close to 1, that is, if defaulting institutions are mainly indebted between themselves. ■

When there is a lack of data on bilateral exposures, the *log-fitting* method is used to estimate the missing data given the available information (see for example Upper and Worms 2004 and Elsinger, Lehar, and Summer 2004). Let us consider the situation in which the total amount of liabilities and total amount of loans are known but there is no specific information on bilateral exposures. In that case, the estimated i's liabilities shares are independent of i, equal to the overall proportions of the loans, $p_j = \frac{\sum_{i\in N}\ell_{ij}}{\sum_{i,k\in N}\ell_{ik}}$ for j. This implies that the cumulated shares within a default set D are estimated all equal to $\sigma_D = \sum_{j\in D}p_j$. Hence the threat indices are equal as well (see example 5), equal to $\frac{1}{1-\sigma_D}$: A uniform strategy is optimal and the information on bilateral liabilities is null. Therefore, the log-fitting model surely underestimates the value of information.[24]

3.3 Alternative policies: Transfers across institutions

This section examines alternative policies based on transfers within institutions, without external intervention. A first policy is similar to the injection policy considered previously but the cash is injected by the safe institutions instead of a regulator. An alternative policy asks the creditors to an institution in difficulty to write-off part of their claims on that institution. A writing-off of part of the claim from safe j to defaulting i amounts to a decrease in ℓ_{ij}.

For each policy, the precise contribution on each safe institution does not matter as long as the safe institutions remain safe: For the injection policy, only the total injected amount matters.

[23]The indices are equal when $\boldsymbol{\mu}_D$ is equal to $\mu^{max}\mathbb{1}_D$. From (6), this is equivalent to $\mu^{max}(\mathbb{I}-\boldsymbol{\pi})_{D\times D}\mathbb{1}_D = \mathbb{1}_D$, which writes as $(\mu^{max}-1)\mathbb{1}_D = \mu^{max}\boldsymbol{\pi}_{D\times D}\mathbb{1}_D$; hence the cumulated shares within D are equal.

[24]It also underestimates the benefit of an injection if μ^{max}, the maximal index of the 'true' matrix, is larger than $\frac{1}{1-\sigma_D}$, the index of the estimated matrix, which is the most likely situation.

For the writing-off policy, the impact on the repayments of a writing-off of part of i's liabilities only depends on the total reduction in i's debt, not on the precise amount written off by each safe creditor. As a result, the optimal policies described in the following proposition do not specify the contributions. (Of course, these contributions could be made precise ex ante, so as to satisfy some fairness or participation constraints.)

Proposition 4 *A transfer of one unit from the safe institutions to the defaulting ones is optimal if it targets those with the largest index.*

A marginal writing-off by safe institutions of one unit of their claims on defaulting i leads to increase V by $\theta_i(\mu_i - 1)$. It is thus optimal first to write-off part of the claims of the institutions with the largest $\theta_i(\mu_i - 1)$.

According to the proposition, writing-off part of the claims of the safe creditors of a defaulting institution i results in an increase in the overall payments if i is also indebted to another defaulting institution ($\mu_i > 1$), and leaves them unchanged otherwise. That repayments never decrease is not obvious because i reimburses less to safe institutions due to the writing-off of their claims. However, i's total reimbursements are unchanged since i defaults; thus a larger proportion is allocated towards defaulting creditors, which triggers further repayments. The larger μ_i, the more beneficial the impact on subsequent creditors, as in the case of a cash injection; the larger θ_i, the more important the amount redistributed towards defaulting institutions.

4 Ex ante value of raising capital

So far, the analysis has taken place at the liquidation of the liabilities. This section evaluates the impact of an increase in the requirement on capital at an ex ante stage. Specifically, it considers the situation in which the increase in capital has to be invested in a risk-free asset, keeping risky investments and liabilities unchanged.

Initially, institutions expect possibly random cash-flows from their investments, $\widetilde{z}_i$ for i, where $\widetilde{}$ indicates the random variable before its realization. Let V be the payment value associated to the cross-liabilities. The aggregate repayments will be $V(\boldsymbol{z})$ if the realized value for the cash-flows is $\boldsymbol{z}$, hence the expected repayments are equal to $E[V(\widetilde{\boldsymbol{z}})]$. Consider an increase in capital, k_i for i, invested in a risk-free asset with null net return (to simplify notation); i's cash-flow is changed into $\widetilde{z}_i + k_i$ and the expected repayments into $E[V(\widetilde{\boldsymbol{z}} + \boldsymbol{k})]$. It follows that raising i's capital by 1 (marginal) unit raises expected repayment by $E[\mu_i(\widetilde{\boldsymbol{z}})]$. This defines the marginal value of i's capital, denoted by v_i. The following proposition provides an expression for these values. $D(\boldsymbol{z})$

denotes the set of defaulting institutions given $\boldsymbol{z}$. $\mathbb{P}$ denotes the probability distribution of $\widetilde{\boldsymbol{z}}$ and $\mathbb{P}[i \in D(\boldsymbol{z})]$ the probability of the set $\{\boldsymbol{z} \text{ s.t. } i \in D(\boldsymbol{z})\}$.

Proposition 5 *The marginal value of i's capital, $v_i = E[\mu_i(\widetilde{\boldsymbol{z}})]$, is equal to $\mathbb{P}[i \in D(\boldsymbol{z})](1+\alpha_i)$, where α_i equals the expectation of the spill-over effects conditional on i defaulting:*

$$\begin{aligned}\alpha_i &= \sum_j \pi_{ij}\mathbb{P}[j \in D(\boldsymbol{z})|i \in D(\boldsymbol{z})] + \sum_{j,k} \pi_{ij}\pi_{jk}\mathbb{P}[j \textit{ and } k \in D(\boldsymbol{z})|i \in D(\boldsymbol{z})] + \\ &+ \sum_{j,k,l} \pi_{ij}\pi_{jk}\pi_{kl}\mathbb{P}[j, k \textit{ and } l \in D(\boldsymbol{z})|i \in D(\boldsymbol{z})]...\end{aligned}$$

The marginal value of i's capital is thus (not surprisingly) increasing in its probability of default.[25] It is also increasing in the expectation of the spill-over effects conditional on i defaulting, α_i, hence in the weakness and default of i's creditors, direct or indirect, when i is defaulting. The spill-over term is thus increasing in the correlation in defaults between i and its creditors. A positive correlation in defaults is induced by a positive correlation in the cash-flows, as results from similar investment portfolios, as studied in papers referred in the introduction. It is also induced by the cross-liabilities themselves, even if the cash-flows are independent, as shown in the next corollary.

Corollary 1 *Let $\widehat{\pi}_{ij} = \pi_{ij}\mathbb{P}[i \in D(\boldsymbol{z})]$ for each i and j, and*[26] $\widehat{\boldsymbol{\mu}} = (\mathbb{I} - \widehat{\boldsymbol{\pi}})^{-1}\mathbb{1}$.

If the defaults are independent, then $v_i = \mathbb{P}[i \in D(\boldsymbol{z})]\widehat{\mu}_i$.

If the cash-flows $\tilde{z}_i$ are independent, then the defaults are positively correlated, i.e., for each subset S of N $\mathbb{P}[S \subset D(\boldsymbol{z})] \geq \Pi_{i\in S}\mathbb{P}(i \in D(\boldsymbol{z}))$, and $v_i \geq \mathbb{P}[j \in D(\boldsymbol{z})]\widehat{\mu}_i$ for each i.

$\widehat{\boldsymbol{\mu}}$ is a threat index computed as if the default set is the whole set N and each share π_{ij} is multiplied by the creditor's default probability $\mathbb{P}[j \in D(\boldsymbol{z})]$. Under independent defaults, the marginal value of i's capital is equal to its probability of default multiplied by $\widehat{\mu}_i$.[27] Though, defaults are not independent but positively correlated when cash-flows are independent, due to the liabilities which 'propagate' default. The effect is to increase the marginal value.

Impact of the size There is a debate about the impact of the size of the institutions on systemic risk (for a recent analysis and the references therein, see De Jonghe, Diepstraten, Schepens 2015). To investigate this issue, I compute here the marginal values of capital for institutions that differ only in their size, meaning that they are 'proportional': the distribution of their cash-flows are equal up

[25]Note that an institution that never initiates default may nevertheless defaults due to default on its loans. This is reminiscent of the distinction put forward by Tarashev, Borio, and Tsatsaronis (2010) as the 'participation' to default.

[26]The matrix $\widehat{\boldsymbol{\pi}}$ is positive with a dominant eigenvalue strictly smaller than 1 since the probabilities of default are not all equal to 1. Thus $\mathbb{I} - \widehat{\boldsymbol{\pi}}$ is invertible with a positive inverse.

[27]Under independent defaults, for each i: $\alpha_i = \sum_{j\in N} \widehat{\pi}_{ij} + \sum_{j\in N, k\in N} \widehat{\pi}_{ij}\widehat{\pi}_{jk} + \cdots +$, hence $\alpha_i = \widehat{\mu}_i$.

to a scale factor and their liabilities and loans are proportional with the same scale factor. Formally institution 1 is proportional to institution 2 with scale factor λ if the distribution of 1's cash-flow $\widetilde{z}_1$ is the same as that of $\lambda\widetilde{z}_2$, and 1's liabilities and loans toward institutions other than 2 are λ times those of 2, and 1's liability toward 2 is λ times 2's liability toward 1. Thus 1's and 2's liabilities' shares towards other institutions are equal, $\pi_{1,i} = \pi_{2,i}$ for $i > 2$ and $\pi_{1,2} = \pi_{2,1}$.

Furthermore, to isolate the size effect, let us assume that 1 and 2 cash-flows are 'conditionally proportional ', i.e., that a low realization of 1's cash-flow relative to 2 's (or the reverse) is independent of others' cash flows. Formally, given (z_1, z_2), denote $(z_1^\sigma, z_2^\sigma) = (\lambda z_2, z_1/\lambda)$: 1 and 2 cash-flows are exchanged and adjusted to the size. Thus, $\widetilde{z}_1^\sigma$ is distributed as $\lambda\widetilde{z}_2$, hence as $\widetilde{z}_1$ by assumption; similarly $\widetilde{z}_2^\sigma$ and $\widetilde{z}_2$ are identically distributed. The cash-flows of institutions 1 and 2 are said to be *conditionally proportional* if the distributions of $(\widetilde{z}_1, \widetilde{z}_2)$ and $(\widetilde{z}_1^\sigma, \widetilde{z}_2^\sigma)$ conditionally on $\boldsymbol{z}_{-12}$ are identical. Of course, 1 and 2 cash-flows are conditionally proportional if all cash flows are independent.

Proposition 6 *Let institutions 1 and 2 differ in size, with 1's size λ times 2's size for $\lambda > 1$. Assume their cash-flows to be conditionally proportional. Then*

$$E[\mu_1(\widetilde{\boldsymbol{z}})|z_1 \leq \lambda z_2] \geq E[\mu_2(\widetilde{\boldsymbol{z}})|z_1 \geq \lambda z_2] \text{ and } E[\mu_2(\widetilde{\boldsymbol{z}})|z_1 \leq \lambda z_2] \geq E[\mu_1(\widetilde{\boldsymbol{z}})|z_1 \geq \lambda z_2]. \quad (9)$$

The marginal value of capital is equal to the mean of the two conditional expected values ($v_i = \frac{1}{2}(E[\mu_i(\widetilde{\boldsymbol{z}})|z_1 \leq \lambda z_2] + E[\mu_i(\widetilde{\boldsymbol{z}})|z_1 \geq \lambda z_2])$ for each i) so the values differ but cannot a priori ordered. The reason is that the default set depend on whether (z_1, z_2) or (z_1^σ, z_2^σ) is realized. Thus, though the distributions of (z_1, z_2) and (z_1^σ, z_2^σ) are identical conditional on others' cash flows, the expected threat indices depend on which institution is relatively weaker. Specifically, the larger institution has a larger impact on default and this works in two directions. The impact is harmful when 1 is relatively weaker than 2: For (z_1, z_2) with $z_1 \leq \lambda z_2$, the default set can only be larger than in the symmetric situation (z_1^σ, z_2^σ) where 2 is weaker than 1 (see the proof): This explains the left-hand-side inequality of (9). Conversely, the impact of the size is beneficial when 1 is relatively stronger than 2: For (z_1, z_2) with $z_1 \geq \lambda z_2$ the default set can only be smaller than in the symmetric situation (z_1^σ, z_2^σ): This gives the right-hand-side inequality of (9). Though the marginal values of capital cannot be compared, that of institution 1 is larger if 1 and 2 have few chances to default together: In that case, 1 defaults only if $z_1 < \lambda z_2$ and 2 only if $z_1 > \lambda z_2$. The marginal values are thus equal to half the expressions on the left-hand side inequality of (9). In that case, the institution with the larger size should have a relatively larger capital.

5 European cross-liabilities

This section illustrates the method on the consolidated foreign claims for 10 EU countries: Austria, Belgium, France, Germany, Greece, Ireland, Italy, the Netherlands, Portugal, Spain. These claims are those of reporting banks in one country on debt obligations of another country. The data is available from the Bank of International Settlements. The exercise can be viewed as a thought experiment in which all these claims are liquidated. How much will a country reimburse assuming that each country allocates a fraction of its GDP to the liquidation of its foreign claims? (Since the claims are consolidated, the default of a country, in particular the default on its governmental bonds, is internalized among its residents. This suppresses the corresponding spill-over effects.)

Liquidation is based on a fraction of the GDPs at the time of liquidation. Specifically, given the level of GDP_i in country i at the year of reference, i's 'cash-flow' available allocated to liquidation is given by

$$\widetilde{z}_i = \widetilde{k}_i GDP_i$$

Ex ante, the factor k_i is perceived as random, reflecting the uncertainty on growth rate and the future GDP level or on political factors such as the resistance to reimburse within the country. In the reported simulations, the reference year is 2008 and the simplest assumptions on $\widetilde{k}_i$ are taken: they are log-normally and independently distributed, with the same mean m and standard error v with $m = 1/2$ and $v = 1/8$. Simulations with 1000 draws are run for three years, 2009, 2010 and 2011. To concentrate on the impact of cross-liabilities only, the same reference year is used for the three years so that $\widetilde{z}_i$ follows the same distribution: The changes linked to default are only due to changes in the cross-liabilities.

Let us first consider the situation in which all the z_is are equal to their expected value, here half of GDP_i. Belgium and Ireland initiate default, since their 'net-worths' are negative as can be seen from Table 1. Their default does not propagate, i.e., no other country defaults at the clearing ratios, so Table 2 only reports the ratios and threat indices for Belgium and Ireland. Their indices only depend on the liabilities' shares between them. Belgium's index is much smaller than Ireland's one due to the asymmetry in their liabilities: Belgium's liabilities share to Ireland is much smaller than that of Ireland to Belgium. More generally, liabilities are very asymmetric, and this plays an important role in the results.

Table 1: Net-worths at the mean values. Unit: 100 millions of dollars

	$Aust$	$Belg$	$France$	$Germany$	$Greece$	$Ireland$	$Italy$	$Netherlands$	$Port$	$Spain$
2009	335	−905	23278	18028	153	−1346	6001	5727	142	3424
2010	543	−43	19257	16733	668	−417	7776	4956	526	4838
2011	598	−318	18043	16764	866	−1150	8751	4847	594	5842

Table 2: The clearing ratio and threat index at the mean values for Belgium and Ireland

	$Belg$	$Ireland$	$Belg$	$Ireland$
	θ	θ	μ	μ
2009	0.776	0.664	1.014	1.162
2010	0.97	0.845	1.015	1.171
2011	0.868	0.494	1.001	1.155

Let us consider now the effect of the shocks on k_i. A country initiates default when its cash-flow, $k_i GDP_i$, is smaller than its net liabilities (the difference between total liabilities and loans) $\ell_i^* - \sum_j \ell_{ji}$, or equivalently when $k_i < \rho_i$ where ρ_i is equal to the ratio of i's net liabilities to GDP. The ρ_is are reported in Table 3.

Table 3: Net liabilities to GDP ratio ρ_i

	$Aust$	$Belg$	Fra	$Germ$	$Greece$	$Ireland$	$Italy$	$Neth$	$Port$	$Spain$
2009	0.3989	0.7282	−0.5623	−0.0915	0.4541	1.2104	0.1995	−0.3114	0.4465	0.2733
2010	0.3363	0.5110	−0.3788	−0.0490	0.2991	0.7200	0.1106	−0.2022	0.3015	0.1797
2011	0.3198	0.5803	−0.3234	−0.0500	0.2397	1.1071	0.0617	−0.1867	0.2757	0.1132

A country with a negative ratio never initiates default. This is the case for France, Germany, and the Netherlands. We will see that they never default through contagion as well. A country with a positive ratio initiates default with the probability that the lognormal distribution with mean 1/2 and standard error 1/8 is less than ρ_i. They are reported in Table 4. The decrease in the probabilities in 2010 is due to the general decrease in the net liabilities and the ρ_is for the vulnerable countries between 2009 and 2010.

Table 4: Probability to initiate default

	$Aust$	$Belg$	$Greece$	$Ireland$	$Italy$	$Port$	$Spain$
2009	0.4856	0.8183	0.5663	0.9561	0.1303	0.5559	0.2643
2010	0.3805	0.6378	0.3126	0.8135	0.0201	0.3170	0.0988
2011	0.3507	0.7096	0.2016	0.9414	0.0015	0.2688	0.0220

Table 5 gives the expectation of the clearing ratios, unconditional and conditional on the default of the country, threat indices, the spill-over index α_i defined in Proposition 5 and the number of defaults. France, Germany, and the Netherlands never default so they are excluded from the table. Some general features are as follows. The clearing ratios at the mean are smaller than the expected ratios given in Table 2 for Belgium and Ireland (this follows from the convexity of the clearing ratio, established in Proposition 1). Comparing the frequency of default ($N/1000$) with the probability to initiate default, we see that Greece and Ireland do not suffer from contagion, but this is not true for the other countries. Finally, the net liabilities over the years change substantially, and mostly decrease. This explains the decrease both in defaults and in the spill-over index α_i. The expected threat indices a fortiori decrease.

Let us consider now the countries. Ireland is by all accounts the weakest country: it has the largest probability of default, smallest expected repayment ratio, largest expected threat index and largest spill-over index. It cuts its liabilities in 2010 (from 404 249 to 270 618) and cuts its loans in 2011 (from 134 266 to 17 710). This explains why its frequency of default decreases in 2010 and increases in 2011; the payment ratio moves in the opposite direction. Belgium's default is also substantial but its spill-over index is the smallest one. Inspecting its liabilities explains this: it is mostly indebted towards France, Germany and the Netherlands, which never default. Portugal instead has the largest spill-over index in 2009. This is due to the fact that more than half Portugal's liabilities shares are towards Spain, which is is fragile since its frequency of default is almost 30 %. Spain's spill-over index on the other hand is rather small, partly due to the fact that its liabilities share towards Portugal is less than 7%. A similar analysis holds for Austria. Almost half of Austria's liabilities are due to Italy, which is fragile in 2009. Thus a simultaneous default of both Austria and Italy (resp. Portugal and Spain) makes Austria's (resp. Portugal) threat index large; furthermore, the occurrence of a simultaneous default is larger than it would be without such a concentration in liabilities. This structure also explains the decrease in Portugal's and Austria's spill-over effects in 2010 and 2011 though their defaults are still substantial. These decreases are due to Spain and Italy, whose default sharply decrease thanks to cut in their liabilities (their net liabilities per GDP is divided by a third for Italy and a half for Spain between 2009 and 2010, see Table 3).

The analysis thus uncovers interesting features that are not easily seen from the 10 by 10 net liabilities matrices nor from aggregate net liabilities because these matrices are highly asymmetrical and change overtime in a non-uniform way.

Table 5: Expected ratios, indices, and numbers of default.

$$
2009\begin{pmatrix}
 & Aust & Belg & Greece & Ireland & Italy & Port & Spain \\
\mathbb{E}\theta & 0.901 & 0.717 & 0.804 & 0.641 & 0.982 & 0.852 & 0.947 \\
\mathbb{E}\mu & 0.594 & 0.881 & 0.658 & 1.168 & 0.181 & 0.759 & 0.339 \\
\mathbb{E}_{|def}\theta & 0.806 & 0.668 & 0.655 & 0.625 & 0.882 & 0.757 & 0.813 \\
\alpha & 0.136 & 0.031 & 0.149 & 0.217 & 0.128 & 0.218 & 0.144 \\
N & 509 & 854 & 568 & 958 & 151 & 612 & 284
\end{pmatrix}
$$

$$
2010\begin{pmatrix}
 & Aust & Belg & Greece & Ireland & Italy & Port & Spain \\
\mathbb{E}\theta & 0.934 & 0.84 & 0.915 & 0.799 & 0.999 & 0.942 & 0.987 \\
\mathbb{E}\mu & 0.421 & 0.686 & 0.329 & 0.954 & 0.026 & 0.393 & 0.11 \\
\mathbb{E}_{|def}\theta & 0.831 & 0.759 & 0.72 & 0.752 & 0.923 & 0.831 & 0.861 \\
\alpha & 0.045 & 0.018 & 0.064 & 0.164 & 0.077 & 0.107 & 0.104 \\
N & 390 & 663 & 303 & 812 & 19 & 341 & 94
\end{pmatrix}
$$

$$
2011\begin{pmatrix}
 & Aust & Belg & Greece & Ireland & Italy & Port & Spain \\
\mathbb{E}\theta & 0.938 & 0.767 & 0.948 & 0.474 & 1.0 & 0.94 & 0.998 \\
\mathbb{E}\mu & 0.381 & 0.77 & 0.203 & 1.089 & 0.003 & 0.346 & 0.027 \\
\mathbb{E}_{|def}\theta & 0.828 & 0.694 & 0.724 & 0.443 & 0.93 & 0.813 & 0.919 \\
\alpha & 0.018 & 0.003 & 0.046 & 0.152 & 0.055 & 0.032 & 0.07 \\
N & 362 & 761 & 187 & 944 & 3 & 323 & 24
\end{pmatrix}
$$

6 Concluding remarks

This work represents a contribution to our understanding of the impact of cross-liabilities on a system. It could be extended for example by incorporating fire sales, which would make the cash-flows depend on the defaults. Extending the analysis to allow for derivatives is much more challenging. For instance, if institutions hold CDS on each other, the reimbursements are no longer complements because an institution may benefit from the default of another one. Multiple and non-comparable ratios might then clear the market, raising selection issues and making it difficult to predict the impact of an intervention.

Another interesting direction of research is to study regulatory tools based on the threat index, extending the analysis in Section 4 on the ex ante value of capital. The threat index reflects an externality posed by a defaulting institution on others' debt repayments. Since this externality depends on the safety of its creditors, the institution has no incentive to assess it properly when choosing its liabilities. Regulatory tools based on the threat index could improve incentives. The analysis should be addressed by taking an ex ante perspective in a model incorporating the liabilities and investment decisions.

7 Proofs

Proof of Proposition 1

1. **The properties of** V. They follow from the computation of the multipliers to the constraints in $\mathcal{P}$. We may assume all banks to be indebted, since non-indebted banks are surely safe with null indices. Writing the constraint $\theta_i \leq 1$ as $\theta_i \ell_i^* \leq \ell_i^*$, the program $\mathcal{P}(\boldsymbol{z})$ is equivalent to:

$$\mathcal{P}'(\boldsymbol{z}) \quad : \quad \max_{\boldsymbol{\theta}} \sum_i \ell_i^* \theta_i$$

$$0 \leq \theta_i \ell_i^* \leq \ell_i^* \text{ for each } i \tag{10}$$

$$\theta_i \ell_i^* - \sum_j \theta_j \ell_{ji} \leq z_i \text{ for each } i \tag{11}$$

and the value of program $\mathcal{P}'$ is V. When the multipliers are unique, the derivative of V with respect to z_i is given by the multiplier μ_i associated to the i-th constraint (11). Let us denote by λ_i the multiplier associated to i's upper bound constraint (10). The program $\mathcal{P}'(\boldsymbol{z})$ has a finite solution: the feasible set is non-empty (it contains $\boldsymbol{\theta} = \mathbf{0}$ since $\boldsymbol{z}$ is positive) and is compact. From well known results on linear programming, the multipliers associated to the constraints are the solutions to the dual program of $\mathcal{P}'(\boldsymbol{z})$, and furthermore, the values of the primal and dual coincide. The dual program of $\mathcal{P}'(\boldsymbol{z})$ is (the proof is standard and left to the reader):

$$\min_{(\lambda,\mu)\geq 0} \sum_i \mu_i z_i + \sum_i \lambda_i \ell_i^* \text{under} (\lambda_i + \mu_i - 1)\ell_i^* - \sum_j \ell_{ij}\mu_j \geq 0 \text{ for each } i. \tag{12}$$

We now show that the constraints (12) are binding. By contradiction, suppose $(\lambda_i + \mu_i - 1)\ell_i^* - \sum_j \ell_{ij}\mu_j > 0$ for some i. If $\lambda_i > 0$, then λ_i can be decreased (since it does not affect the other constraints) and the objective is decreased, a contradiction. Similarly, if $\mu_i > 0$ then μ_i can be decreased (since it can only relax the other constraints), a contradiction again. Thus both λ_i and μ_i are null. But then $-\ell_i^* - \sum_j \ell_{ij}\mu_j > 0$ is impossible. This proves

$$(\lambda_i + \mu_i - 1)\ell_i^* - \sum_j \ell_{ij}\mu_j = 0 \text{ for each } i. \tag{13}$$

Let us assume that there are no boundary institutions. i is either safe with a strict inequality in (11) or it defaults and $\theta_i < 1$. The slackness conditions imply: $\mu_i = 0$ for i not in D and $\lambda_i = 0$ for i in D. Plugging these values into equations (13) for each i in D, we obtain $\mu_i \ell_i^* - \sum_{j\in D} \ell_{ij}\mu_j = \ell_i^*$: this proves (4). The fact that the system (4) has a unique solution, which is furthermore positive, follows from Lemma 1.

2. **The clearing ratio is non-decreasing and convex in $\boldsymbol{z}$.** Given $\boldsymbol{z}$, let us say that ratio $\boldsymbol{\theta}$ in $[0,1]^n$ is feasible if each net-worth of is non-negative. Recall that the clearing ratio given $\boldsymbol{z}$, denoted

by $\boldsymbol{\theta}(\boldsymbol{z})$, is the greatest feasible ratio given $\boldsymbol{z}$. Let $\boldsymbol{z}'$ with $\boldsymbol{z}' \geq \boldsymbol{z}$. $\boldsymbol{\theta}(\boldsymbol{z})$ is feasible for $\boldsymbol{z}'$, hence is surely less than the greatest feasible ratio given $\boldsymbol{z}'$, which is $\boldsymbol{\theta}(\boldsymbol{z}')$: This proves $\boldsymbol{\theta}(\boldsymbol{z}) \leq \boldsymbol{\theta}(\boldsymbol{z}')$. The convexity of $\boldsymbol{\theta}$ follows the same type of argument: Given two vectors $\boldsymbol{z}$ and $\boldsymbol{z}'$ and λ in $[0, 1]$, the ratio $\lambda\boldsymbol{\theta}(\boldsymbol{z}) + (1-\lambda)\boldsymbol{\theta}(\boldsymbol{z}')$ is feasible for $\lambda\boldsymbol{z} + (1-\lambda)\boldsymbol{z}'$, thus lower than $\boldsymbol{\theta}(\lambda\boldsymbol{z} + (1-\lambda)\boldsymbol{z}')$.

3. **The default set and the threat index are non-increasing in $\boldsymbol{z}$.** Let $\boldsymbol{z}' \geq \boldsymbol{z}$. The monotony of the default set follows from that of the ratio. As for the indices, consider $\boldsymbol{\mu}_D$ and $\boldsymbol{\mu}_{D'}$ where $D = D(\boldsymbol{z})$ and $D' = D(\boldsymbol{z}')$. They satisfy (7) hence it suffices to show that $\boldsymbol{\pi}^{(p)}_{D'\times D'}(ij) \leq \boldsymbol{\pi}^{(p)}_{D\times D}(ij)$ for each i and j both in D'. $\boldsymbol{\pi}^{(p)}_{D\times D}(ij)$ is determined as follows: for each path of p elements in D linking i to j, compute the product of the $\boldsymbol{\pi}$ entries over the edges of the path; $\boldsymbol{\pi}^{(p)}_{D\times D}(ij)$ is the sum of these products over all the paths. $\boldsymbol{\pi}^{(p)}_{D'\times D'}(ij)$ is determined in the same way by considering the paths included in D' instead of D. There are less such paths since D' is a subset of D: the inequality $\boldsymbol{\pi}^{(p)}_{D'\times D'}(ij) \leq \boldsymbol{\pi}^{(p)}_{D\times D}(ij)$ follows.

4. **The sub-modularity of V.** Sub-modularity can be proved by considering vectors that differ in a single component j : Changing successively each component of $\boldsymbol{z}_{-i}$ to that of $\boldsymbol{z}'_{-i}$ produces the required inequality. The sub-modularity for two vectors $\boldsymbol{z}_i$ and $\boldsymbol{z}'_i$ with identical components $\boldsymbol{z}_{-ij}$ except for j writes:

$$V(z'_i, z'_j, \boldsymbol{z}_{-ij}) - V(z_i, z'_j, \boldsymbol{z}_{-ij}) \leq V(\boldsymbol{z}'_i, z_j, \boldsymbol{z}_{-ij}) - V(z_i, z_j, \boldsymbol{z}_{-ij}) \text{ for } z'_i \geq z_i, z'_j \geq z_j.$$

For a differentiable function V, the sub-modularity is satisfied if the partial derivative $\frac{\partial V}{\partial z_i}$ is non-increasing in z_j, as can be seen from the following expression:

$$V(z'_i, z_j, \boldsymbol{z}_{-ij}) - V(z_i, z_j, \boldsymbol{z}_{-ij}) = \int_{z_i}^{z'_i} \frac{\partial V}{\partial z_i}(t, z_j, \boldsymbol{z}_{-ij})dt. \tag{14}$$

Here V is not differentiable at all points but the partial derivative of V exists almost everywhere: $\frac{\partial V}{\partial z_i}(t, z_j, \boldsymbol{z}_{-ij})$ is given by the unique multiplier μ_i at $(t, z_j, \boldsymbol{z}_{-ij})$ when there is no boundary bank, which is the case but for a finite number of points as t runs in the interval (z_i, z'_i). The integral expression (14) is thus still valid and V is sub-modular if for each i the multiplier μ_i at $(t, z_j, \boldsymbol{z}_{-ij})$ is a non-decreasing function of z_j, for j distinct from i, given t and $\boldsymbol{z}_{-ij}$ when there is no boundary bank. This follows from point 2 since the vector $\boldsymbol{z}(t) = (t, z_j, \boldsymbol{z}_{-ij})$ increases as t increases. ■

Proof of Proposition 2. The optimal strategy directly follows from the fact that the function V is concave and $\frac{\partial V}{\partial z_i} = \mu_i$ for each i at each $\boldsymbol{z}$ for which no bank is at the boundary. Furthermore, V is linear for all the cash values above $\boldsymbol{z}$ that lead to the same set D, hence the optimal increase in V is equal to $\mu^{max}m$. ■

Lemma 1 *Let D be the default set at a clearing ratio vector. Any subset of D has a creditor outside D and the dominant eigenvalue ρ_D of $\boldsymbol{\pi}_{D\times D}$ is smaller than 1. Thus the matrix $(\mathbb{I} - \boldsymbol{\pi})_{D\times D}$ is* *invertible, with a positive inverse given by the converging infinite sum:* $(\mathbb{I}-\boldsymbol{\pi})^{-1}_{D\times D} = \mathbb{I}_{D\times D} + \boldsymbol{\pi}_{D\times D} + \boldsymbol{\pi}^{(2)}_{D\times D} + ... + \boldsymbol{\pi}^{(p)}_{D\times D} + ...$ *where* $\boldsymbol{\pi}^{(p)}$ *denotes the product of* $\boldsymbol{\pi}$ *by itself* p *times.*

Proof of Lemma 1 Let D be the set of defaulting banks at a clearing ratio. To show that $(\mathbb{I}-\boldsymbol{\pi})_{D\times D}$ is invertible with an inverse given by the infinite sum $\mathbb{I}_{D\times D} + \boldsymbol{\pi}_{D\times D} + \boldsymbol{\pi}^{(2)}_{D\times D} + ... + \boldsymbol{\pi}^{(p)}_{D\times D} + ...$, we prove that an iterate of the matrix $\boldsymbol{\pi}_{D\times D}$ has all its rows totals smaller than 1: $\boldsymbol{\pi}^{(p)}_{D\times D}\mathbb{1}_D \ll \mathbb{1}_D$. The result then follows from standard results on productive matrices: the spectral radius of $\boldsymbol{\pi}^{(p)}_{D\times D}$ is strictly smaller than 1 hence also that of $\boldsymbol{\pi}_{D\times D}$.

Each bank in D has a null net-worth , so, from Lemma 2 below, each subset of D has an outside creditor. Interpret $\boldsymbol{\pi}$ as a transition matrix in which element π_{ij} is the probability of reaching j from i. The (i,j) element of the matrix $\boldsymbol{\pi}^{(q)}_{D\times D}$ gives the probability of reaching j from i in q steps along a path included in D. Hence the sum $\sum_{j\in D}\pi^{(q)}_{D\times D}(i,j)$ is the probability of the paths of length q that start from i and are included in D. Such a sum is non-increasing in q since a path included in D of length $q+1$ has necessarily its first q elements included in D.[28] So, if the inequality $\sum_{j\in D}\pi^{(q)}_{D\times D}(i,j) < 1$ holds for q, then it holds for all larger values than q. Thus, $\boldsymbol{\pi}^{(p)}_{D\times D}\mathbb{1}_D \ll \mathbb{1}_D$ holds if, for each i in D, there is q for which $\sum_{j\in D}\pi^{(q)}_{D\times D}(i,j) < 1$.

By contradiction, assume that for some i in D we have $\sum_{j\in D}\pi^{(q)}_{D\times D}(i,j) = 1$ for each q. All the paths starting from i are included in D. Let C be composed with all the elements that can be reached from i. By construction, C has no outside creditor and is included in D, hence the net-worth of each element in C is null. Applying Lemma 2 gives the desired contradiction.[29] ■

Lemma 2 *Let* T *be a non-empty subset of* N *such that each* i *in* T *has null net-worth and a positive cash-flow:* $e_i(\boldsymbol{\theta}) = 0$ *and* $z_i > 0$. *Then* T *is not the whole set* N *and has a creditor in* $N - T$.

Proof The lemma is proved by considering the following equality which holds for any ratio $\boldsymbol{\theta}$ and non-empty subset T of N:

$$\sum_{i\in T} e_i(\boldsymbol{\theta}) = \sum_{i\in T} z_i + \sum_{i\in T, j\notin T} \theta_j \ell_{ji} - \sum_{i\in T, j\notin T} \theta_i \ell_{ij}. \tag{15}$$

[28]Formally $\sum_{j\in D}\pi^{(q+1)}_{D\times D}(i,j) = \sum_{j\in D}(\sum_{k\in D}\pi^{(q)}_{D\times D}(i,k)\pi(k,j))$ which is equal, exchanging sums, to $\sum_{k\in D}\pi^{(q)}_{D\times D}(i,k)[\sum_{j\in D}\pi(k,j)]$. Since for each k the term inside the square bracket is not larger than 1 we finally obtain $\sum_{j\in D}\pi^{(q+1)}_{D\times D}(i,j) \le \sum_{k\in D}\pi^{(q)}_{D\times D}(i,k)$, the desired inequality.

[29]The lemma may not hold for any subset of defaulting banks. Let T be made of indebted banks and $\boldsymbol{\pi}_{T\times T} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Since the row total of bank 3 is null, bank 3 has a creditor outside T, hence T has a creditor outside T. The matrix $(\mathbb{I}-\boldsymbol{\pi})_{T\times T}$ is not invertible because $\boldsymbol{x} = (1,1,0)$ satisfies $\boldsymbol{x} = \boldsymbol{x}\boldsymbol{\pi}_{T\times T}$. One checks that T cannot be a default set: since banks 1 and 2 have no creditor outside $\{1,2\}$, surely θ_1 or θ_2 is equal to 1 at a clearing ratio.

(15) is proved by summing net-worth values over T since the repayments within T cancel out. It says that the aggregate net-worth of the banks in T is equal to their aggregate cash-flow plus the net payment from banks outside T.Now consider T and $\boldsymbol{\theta}$ for which $e_i(\boldsymbol{\theta}) = 0$ and $z_i > 0$ each i in T. (15) applied to T implies $\sum_{i\in T} z_i + \sum_{i\in T, j\notin T} \theta_j \ell_{ji} - \sum_{i\in T, j\notin T} \theta_i \ell_{ij} = 0$. Since $\sum_{i\in T} z_i$ is strictly positive, it must be that $\ell_{ij} > 0$ for a pair i in T and j not in T. ∎

Lemma 3 *Assume $\boldsymbol{\pi}_{D\times D}$ to be primitive: there is a positive q for which its q-th power $\boldsymbol{\pi}^{(q)}_{D\times D}$ is positive. Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be normalized left and right dominant eigenvectors of $\boldsymbol{\pi}_{D\times D}$. Then, given a moderate injection strategy $\boldsymbol{x} = (x_i)$, the additional flow received by j at step p becomes equivalent to $\rho^p_D(\frac{\sum_i u_i x_i}{\sum_i u_i v_i})v_j$ when p increases to infinity, hence the flows become proportional to $\boldsymbol{v}$. As a result, choosing $x_i > 0$ only for i with maximal u_i maximizes the flow at each j in the long run.*

Proof If the matrix is primitive, by an extension of Perron-Frobenius theorem, the sequence of matrices $\boldsymbol{\pi}^{(p)}_{D\times D}/\rho^p_D$ converges to the matrix with i,j element equal to $\frac{u_i v_j}{\boldsymbol{u}.\boldsymbol{v}}$. From (7), the amount that reaches j along a chain of p defaulting institutions when 1 unit is injected into i is the element i,j of the matrix $\boldsymbol{\pi}^{(p)}_{D\times D}$, hence becomes equivalent to $\rho^p_D \frac{u_i v_j}{\boldsymbol{u}.\boldsymbol{v}}$. The flow associated to a moderate strategy $\boldsymbol{x}$ is obtained by linearity. ∎

Proof of Proposition 4 The proof relies on the computation of $\frac{\partial V}{\partial \ell_{ij}}$. Thanks to the envelope theorem, and assuming no boundary institution, the payment function V is differentiable with respect to ℓ_{ij} with a derivative given by $\frac{\partial V}{\partial \ell_{ij}} = \theta_i(1 - \mu_i + \mu_j)$. The result follows. ∎

Proof of Proposition 5 Using that the threat index of a safe institution is null, we can write (4) as

$$\mu_i(\boldsymbol{z}) = \mathbb{1}_{D(\boldsymbol{z})}(i)[1 + \sum_j \pi_{ij}\mu_j(\boldsymbol{z})\mathbb{1}_{D(\boldsymbol{z})}(j)]\mathbb{1}_{D(\boldsymbol{z})}(i).$$

Taking the expectation over $\widetilde{\boldsymbol{z}}$, we obtain

$$v_i \quad = \quad \mathbb{P}[i \in D(\boldsymbol{z})] + \sum_j \pi_{ij}\mathbb{P}[i, j \in D(\boldsymbol{z})] + \sum_{j,k} \pi_{ij}\pi_{j,k}\mathbb{P}[i, j \text{ and } k \in D(\boldsymbol{z})] + ...$$

which gives the expression for v_i by factoring out $\mathbb{P}[i \in D(\boldsymbol{z})]$. ∎

Proof of Corollary 1 Assume cash-flows to be independent. Let us show that $\mathbb{P}[S \subset D(\boldsymbol{z})] \geq \Pi_{i\in S}\mathbb{P}(i \in D(\boldsymbol{z}))$ for each subset S of N. By induction, it suffices to prove

$$\text{For each } S \subset N, k \in N, k \notin S: \mathbb{P}[S \cup k \subset D(\boldsymbol{z})] \geq \mathbb{P}[S \subset D(\boldsymbol{z})]\mathbb{P}[k \in D(\boldsymbol{z})] \qquad (16)$$

Write

$$\mathbb{P}[S \subset D(\boldsymbol{z})] = \mathbb{P}[S \cup k \subset D(\boldsymbol{z})] + \mathbb{P}[S \subset D(\boldsymbol{z}) \text{ and } k \notin D(\boldsymbol{z})]. \qquad (17)$$

$k \notin D(\boldsymbol{z})$ is equivalent to the fact that θ_k is equal to 1, its maximal value. Due to the fact that ratios are increasing in each component of $\boldsymbol{z}$ and that cash-flows are independent, it follows that

$$\mathbb{P}[S \subset D(\boldsymbol{z}))|k \notin D(\boldsymbol{z})] \leq \mathbb{P}[S \subset D(\boldsymbol{z}))|k \in D(\boldsymbol{z})].$$

which writes as $\frac{\mathbb{P}[S \subset D(\boldsymbol{z}) \text{ and } k \notin D(\boldsymbol{z})]}{(1-\mathbb{P}[k \in D(\boldsymbol{z})])} \leq \frac{\mathbb{P}[[S \cup k \subset D(\boldsymbol{z})]}{\mathbb{P}[k \in D(\boldsymbol{z})]}$. Plugging this inequality into (17) we obtain $\mathbb{P}[S \subset D(\boldsymbol{z})] \leq \frac{\mathbb{P}[S \cup k \subset D(\boldsymbol{z})]}{\mathbb{P}[k \in D(\boldsymbol{z})]}$, which is (16).

The lower bound $\mathbb{P}[i \in D(\boldsymbol{z})]\widehat{\mu}_i$ to v_i immediately follows from the expression of v_i. ∎

Proof of Proposition 6 If 1 and 2 are not indebted, then their indices are null and (9) is straightforward. Let us assume that 1 and 2 are indebted. Fix $\boldsymbol{z}_{-12}$. The proof uses two steps.

Step 1. We have $\theta_1(\boldsymbol{z}) \leq \theta_2(\boldsymbol{z})$ and $\mu_1(\boldsymbol{z}) \geq \mu_2(\boldsymbol{z})$ for $z_1 \leq \lambda z_2$; conversely, $\theta_1(\boldsymbol{z}) \geq \theta_2(\boldsymbol{z})$ and $\mu_1(\boldsymbol{z}) \leq \mu_2(\boldsymbol{z})$ for $z_1 \geq \lambda z_2$.

Let $e_i(\boldsymbol{z}, \boldsymbol{\theta})$ denote i's net-worth at ratios $\boldsymbol{\theta}$. Using the proportionality of liabilities and loans:

$$\begin{aligned} e_1(\boldsymbol{z}, \boldsymbol{\theta}) &= z_1 + \lambda \sum_{j \geq 2} \theta_j \ell_{j2} + \lambda \theta_2 \pi_{1,2} - \lambda \theta_1 \ell_2^* \\ &= \lambda e_2(\boldsymbol{\theta}, \boldsymbol{z}) + z_1 - \lambda z_2 - \lambda(\theta_1 - \theta_2)\ell_2^*. \end{aligned} \tag{18}$$

Let $z_1 \leq \lambda z_2$. By contradiction, assume $\theta_1(\boldsymbol{z}) > \theta_2(\boldsymbol{z})$. Equation (18) implies $e_1(\boldsymbol{z}, \boldsymbol{\theta}(\boldsymbol{z})) < \lambda e_2(\boldsymbol{z}, \boldsymbol{\theta}(\boldsymbol{z}))$. However, since surely $\theta_2(\boldsymbol{z}) < 1$, 2 defaults and $e_2(\boldsymbol{z}, \boldsymbol{\theta}(\boldsymbol{z})) = 0$. We thus obtain $e_1(\boldsymbol{z}, \boldsymbol{\theta}(\boldsymbol{z})) < 0$, a contradiction. As for the indices, if 2 does not default, its index is null, so that surely $\mu_1(\boldsymbol{z}) \geq \mu_2(\boldsymbol{z})$. If 2 defaults, we have just shown that 1 surely defaults. Given that the liability shares of the two banks are equal, their indices are equal, hence again $\mu_1(\boldsymbol{z}) \geq \mu_2(\boldsymbol{z})$.

The proof does not use that λ is larger than 1. Hence it works in the opposite case where $z_1 \geq \lambda z_2$ replacing λ by $1/\lambda$

Step 2. Consider $\boldsymbol{\theta}^\sigma(\boldsymbol{z})$ obtained from the clearing ratio $\boldsymbol{\theta}(\boldsymbol{z})$ for $\boldsymbol{z}$ by exchanging θ_1 and θ_2. For $z_1 \leq \lambda z_2$, $\boldsymbol{\theta}(\boldsymbol{z}^\sigma) \geq \boldsymbol{\theta}^\sigma(\boldsymbol{z})$ and $\boldsymbol{\mu}(\boldsymbol{z}^\sigma) \leq \boldsymbol{\mu}^\sigma(\boldsymbol{z})$ (that is, when 1 is weak, permuting the cash-flows (up to the scale) can only increase the clearing ratio and decrease the index).

Let $z_1 \leq \lambda z_2$. To show $\boldsymbol{\theta}(\boldsymbol{z}^\sigma) \geq \boldsymbol{\theta}^\sigma(\boldsymbol{z})$, it suffices to prove that $\boldsymbol{\theta}^\sigma(\boldsymbol{z})$ is feasible given $\boldsymbol{z}^\sigma$, i.e., that the net-worth of each bank is non-negative. To simplify notation, let e_i denote i's net-worth given $\boldsymbol{z}$ and $\boldsymbol{\theta}(\boldsymbol{z})$: $e_i = e_i(\boldsymbol{z}, \boldsymbol{\theta}(\boldsymbol{z}))$ and e_i' i's net-worth given $\boldsymbol{z}^\sigma$ and $\boldsymbol{\theta}^\sigma(\boldsymbol{z})$: $e_i' = e_i(\boldsymbol{z}^\sigma, \boldsymbol{\theta}^\sigma(\boldsymbol{z}))$. We have $e_i \geq 0$ for each i and want to prove $e_i' \geq 0$ for each i.

Consider the banks other than 1 or 2. Their cash-flows are equal under $\boldsymbol{z}$ and $\boldsymbol{z}^\sigma$ and the permutation does not affect their repayment ratios. Thus net-worths e_i and e_i' only differ though 1 and 2's reimbursements to i, equal to $(\lambda\theta_1(\boldsymbol{z}) + \theta_2(\boldsymbol{z}))\ell_{2i}$ at $\boldsymbol{\theta}(\boldsymbol{z})$ and to $(\lambda\theta_2(\boldsymbol{z}) + \theta_1(\boldsymbol{z}))\ell_{2i}$ at $\boldsymbol{\theta}^\sigma(\boldsymbol{z})$.The latter is not smaller than the former, since $\theta_1(\boldsymbol{z}) \leq \theta_2(\boldsymbol{z})$ from Step 1 and $\lambda > 1$ by assumption. Thus $e_i' \geq e_i$; this proves $e_i' \geq 0$ for each $i > 2$.

Consider banks 1 and 2. From (18), 1's net-worth at $(\boldsymbol{z}^{\sigma}, \boldsymbol{\theta}^{\sigma}(\boldsymbol{z}))$ is λ times that of bank 2 at $(\boldsymbol{z}, \boldsymbol{\theta}(\boldsymbol{z}))$: $e_1' = \lambda e_2$. Similarly $e_2' = e_1/\lambda$. Thus both e_1' and e_2' are non-negative.

We now conclude. From Step 2, $(\mu_1(\boldsymbol{z}) - \mu_2(\boldsymbol{z}^{\sigma}))\mathbb{1}_{z_1 \leq \lambda z_2} \geq 0$ for each realization $\boldsymbol{z}_{-12}$. This implies $E[(\mu_1(\widetilde{\boldsymbol{z}}) - \mu_2(\widetilde{\boldsymbol{z}}^{\sigma}))\mathbb{1}_{\widetilde{z}_1 \leq \lambda \widetilde{z}_2}] \geq 0$. By the conditional independence assumption, $E[\mu_2(\widetilde{\boldsymbol{z}}^{\sigma})\mathbb{1}_{\widetilde{z}_1 \leq \lambda \widetilde{z}_2}] = E[\mu_2(\widetilde{\boldsymbol{z}})\mathbb{1}_{\widetilde{z}_1 \geq \lambda \widetilde{z}_2}]$ We thus obtain the first inequality in (9). Similarly $E[(\mu_2(\widetilde{\boldsymbol{z}}) - \mu_1(\widetilde{\boldsymbol{z}}^{\sigma}))\mathbb{1}_{\widetilde{z}_1 \leq \lambda \widetilde{z}_2}] \geq 0$ and $E[\mu_1(\widetilde{\boldsymbol{z}}^{\sigma})\mathbb{1}_{\widetilde{z}_1 \leq \lambda \widetilde{z}_2}] = E[\mu_1(\widetilde{\boldsymbol{z}})\mathbb{1}_{\widetilde{z}_1 \geq \lambda \widetilde{z}_2}]$ imply the second inequality. ∎

References

Acemoglu D., Ozdaglar A. and Tahbaz-Salehi A. (2015) "Systemic Risk and Stability in Financial Networks", *Am. Economic Rev.*, 105(2), 564-608.

Acharya V.V., Pedersen L., Philippon T. and Richardson M. (2012) ``Measuring Systemic Risk", CEPR, Discussion Paper 8824.

Acharya V. V., Shin H. S., and Yorulmazer, T. (2011) ``Crisis resolution and bank liquidity", *Rev. Financial Stud.* 24(6), 2166-205.

Acharya V. V., and Yorulmazer T. (2008) ``Cash-in-the-market pricing and optimal resolution of bank failures", *Rev. Financial Stud.* 21(6) 2705-2742.

Adrian T. and Brunnermeier M. K. (2011) ``CoVaR", NBER Working Paper No. 17457.

Adrian T. and Shin H. S. (2010) ``Liquidity and leverage", *J. Financial intermediation*, 19(3), 418-437.

Allen F. and Gale D. (2000) ``Financial Contagion", *The J. of Political Economy*, 108(1), 1-33.

Alter, A., Craig, B., and Raupach, P. (2014) ``Centrality-based capital allocations", (No. 14-237). IMF.

Ballester C., Calvo-Armengol A., and Zenou Y. (2006) ``Who's Who in Networks. Wanted: The Key Player", *Econometrica*, 74(5), 1403-1417.

Bonacich P. (1987) ``Power and Centrality: A Family of Measures", *Am. J. of Sociology*, 92(5), 1170-1182.

Brownlees, C. and Engle R. (2010) ``Volatility, Correlation, and Tails for Systemic Risk Measurement", DP Stern Business School.

Choi D. B. (2014) ``Heterogeneity and stability: bolster the strong, not the weak", *Rev. of Financial Stud.*, 27(6), 1830-1867.

Cont, R., Moussa, A. and Santos E.E. (2010) ``Network structure and systemic risk in banking systems", Available at SSRN 1733528.

De Jonghe, O., Diepstraten, M. and Schepens, G. (2015) ``Banks size, scope and systemic risk: What role for conflicts of interest?", *J. of Banking and Finance*, 61, S3-S13.

Diamond, D. W. and Dybvig P. H. (1983) ``Bank runs, deposit insurance, and liquidity", *J. of Political Economy*, 401-419.

Diamond D. W. and Rajan R. G. (2011) ``Fear of fire sales, illiquidity seeking, and credit freezes", *Quarterly J. of Economics*, 126(2), 557-591.

Eisenberg L. and Noe T. H. (2001) ``Systemic Risk in Financial Systems", *Management Sci.*, 47(2), 236-249.

Elliott M., Golub B. and Jackson M. (2014) ``Financial networks and contagion", *Am. Economic Rev.*, 104(10) pp. 3115-3153.

Elsinger H., Lehar A. and Summer M. (2004) ``Risk Assessment for Banking Systems", *Management Sci.*, 52(9), 1301-1314.

Freixas X., Parigi B. M. and Rochet J.-C. (2000) ``Systemic Risk, Interbank Relations, and Liquidity Provision by the Central Bank", *J. of Money, Credit and Banking*, 32(3), 611-638.

Gai P. and Kapadia S. (2010) ``Contagion in Financial Networks", *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sci.*, 2401-2423.

Glasserman P. and Young H.P. (2015) ``How likely is contagion in financial networks" ?, *J. of Banking and Finance*, 50, 383-399.

Gouriéroux C., Héam J-C and Monfort A. (2012) ``Bilateral Exposures and systemic solvency risk", *Canadian J. of Economics*, (45)4, 1273-1309.

Greenwood R., Landier A. and Thesmar D. (2015) ``Vulnerable banks", *J. of Financial Economics* 115 (3), 471-485.

Katz L. (1953) ``A New Status Index Derived from Sociometric Analysis", *Psychometrika*, 18(1), 39-43.

Kiyotaki, N. and Moore, J (1997) ``Credit chains", WP Princeton University.

Lehar, A. (2005) ``Measuring systemic risk: A risk management approach", *J. of Banking and Finance*, 29(10), 2577-2603.

Nier E., Yang J., Yorulmazer T. and Alentorn A. (2007) ``Network models and financial stability", *J. Economic Dynamics and Control*, 31(6), 2033-2060.

Rogers L. C. and Veraart L. A. (2013) ``Failure and rescue in an interbank network", *Management Sci.*, 59(4), 882-898.

Tarashev N. A., Borio C. E. and Tsatsaronis K. T. (2010) ``Attributing Systemic Risk to Individual Institutions", BIS Working Paper No. 308.

Upper C. and Worms A. (2004) ``Estimating Bilateral Exposures in the German Interbank Market: Is there a Danger of Contagion?", *European Economic Rev.*, 48(4), 827-849.

Upper, C. (2011) ``Simulation methods to assess the danger of contagion in interbank markets", *J. Financial Stability*, 7(3), 111-125.