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PARIS-JOURDAN SCIENCES ÉCONOMIQUES

48, Bd JOURDAN – E.N.S. – 75014 PARIS
TÉL. : 33(0) 1 43 13 63 00 – FAX : 33 (0) 1 43 13 63 10
www.pse.ens.fr

A ranking method based on handicaps

Gabrielle DEMANGE¹

April 12, 2012

Abstract

Ranking systems are becoming a fundamental tool in many areas, including the Web environment for ranking pages. PageRank, the most popular method, is based on the links stemming from a page and defined so as to be immune to their inflation, a property called intensity-invariance. This paper introduces and characterizes a new intensity-invariant method, called the handicap-based method. The method assigns not only a ranking to the items but also weights to the experts on which the ranking is based. Ranking and weights form an equilibrium for a relationship based on the notion of handicaps. The method is, in a sense made precise in the paper, the counterpart of the counting method in environments that require intensity-invariance.

Keywords ranking, scores, invariant method, peers' method, handicap, scaling matrix.

JEL D71, D89.

1 Introduction

The use of rankings is becoming pervasive in many areas, including the Web environment for ranking pages and academia for ranking journals. Most popular ranking methods are based on observed behaviors such as hyperlinks or citations. The underlying premise is that these behaviors reflect preferences: interpreting hyperlinks toward a Web page or citations toward an article as positive votes, a ranking method aggregates these votes. Taking this point of view, the counting method, which ranks items according to the total number of cites they receive, provides a natural benchmark. The counting method, however, is not the most popular method, beaten by the 'invariant' method, which serves as a basis to PageRank of Google (Brin and Page 1998) or to some academic journals' rankings. There are indeed arguments against the counting method, as detailed below, and, furthermore, these arguments vary with the context under consideration and cannot be all solved by a single method. In such a case, the axiomatization approach, which aims to characterize methods through properties or 'axioms', provides a useful guide for choosing a method. This is the approach followed here. The paper introduces a new method, called the *handicap-based method*, and characterizes it through some properties.

¹PSE-EHESS, address 48 bd Jourdan, 75014 Paris, France e-mail demange@pse.ens.fr. Part of the material was in the working paper entitled 'Collective attention and ranking methods' WP 2010-42, PSE.

I consider ranking problems described by a set of ‘items’ to be ranked and a set of ‘experts’, who provide statements on the items. For example, in a ranking of journals based on citations, journals are both the items to be ranked and the experts; a statement of a journal (as an expert) is derived from the number of citations by its articles towards articles published in all journals. Similarly, in a ranking of Web pages based on the link structure, the statement of a page (as an expert) is given by its links towards other pages. Here again the items to be ranked -the Web pages- coincide with the experts, a property sometimes referred to as a ‘peers’ system. The framework of this paper also allows for distinct sets of experts and items. As explained in the paper, it can be applied for example to the choice of the allocation of a budget to different categories. A ranking method assigns to each experts’ statements a ranking of the items. Such a ranking is described by the relative *scores* assigned to the items, namely a ranking is cardinal and defined up to a multiplicative scalar.

To understand how the handicap-based method is built, it is first useful to view a method as assigning simultaneously scores to the items and weights to the experts. The ranking writes as a weighted combination of the experts’ statements and, furthermore, the scores and the weights form some equilibrium relationship. Current methods are built that way, though not always explicitly. The counting method, which ranks items according to their received totals, simply assigns identical weights to the experts whatever their statements. The invariant method, alluded to above, determines which pages are influential on the basis that a page is influential if it is heavily cited by other influential pages. This generates a loopback between the scores and the weights. By its very definition, the invariant method looks for scores and weights that form an equilibrium of the process of equalization between scores and weights.

The method introduced in this paper is supported by an equilibrium relationship built on handicaps. Observe that rankings and handicaps are strongly related, specifically can be seen as the inverse of each other. Since the purpose of handicaps is to adjust the items’ marks so as to equalize their ‘strength’, the handicap of i is half that of ℓ if i can be said to be twice as good as ℓ , that is, if i ’s score is the double of that of ℓ . In a handicap-based ranking, handicaps equalize the strengths between items and experts’ weights equalize the handicap points assigned by the experts. These relationships define a unique ranking under some conditions stated in the paper. Furthermore, the ranking can be computed using a well-known procedure of matrix scaling, called RAS method or iterative proportional fitting procedure.

The method is characterized by three properties. A main property is *intensity-invariance*. The property has been introduced in the environments in which the ‘intensity’ of statements is not controlled, as in the first two examples above, where neither the number of citations per article nor the number of links from a page are restricted *a priori*. In the Web environment for example, PageRank deflates a link from a site by the total number of links from that site. Factoring out the intensity of outward links avoids pages to increase their score by inflating the number of links. (There are other justifications to intensity-invariance, as explained in the paper). A second important property is called *homogeneity*. In our framework, the (relative) values taken by the scores, and not only the order, matter. Saying that the score of i is twice the score of k should mean that i is ‘twice as good’ as k . In this interpretation, start with some statements and multiply the statement for i

relative to other items by an identical factor for each expert. The homogeneity property requires the relative score of i to be improved by that factor. The third property is *uniformity*², according to which items are not discriminated when they all receive the same totals: the method assigns equal scores when the counting method does.

The handicap-based method is the only method that is intensity-invariant, homogeneous, and uniform. Furthermore, the handicap-based method can be seen as the counterpart of the counting method in environments that require intensity-invariance. Indeed, the counting method, which is not intensity-invariant, is characterized by properly extended versions of homogeneity and uniformity.

This paper is related to recent studies that have characterized ranking methods based on citations. In the peers' context, several axiomatizations of the invariant method have been provided (Palacios-Huerta and Volij 2004, Slutzki and Volij 2006, and Altman and Tennenholtz 2008). In a different context, Woeginger (2008) provides an axiomatization of the h -index, a much used method for ranking researchers. The method relies on the number of citations received by each researchers' paper, independently of the citations origin. The paper is also related, albeit loosely, to the concern on incentive compatibility, which has been recently investigated in the peers' context. Altman and Tennenholtz (2008) provide an impossibility theorem. Clippel, Moulin, and Tideman (2008) introduce a requirement in the spirit of non-manipulability (the score of an entity must not be affected by its own citations) and display a family of methods that satisfy it. This paper does not consider incentive compatibility, though intensity-invariance prevents a specific form of manipulation.

The paper is organized as follows. Section 2 presents ranking methods, defines some properties, and gives examples. Section 3 introduces the handicap-based method under the assumption of strictly positive statements; it provides a characterization for both the handicap-based and counting methods. Section 4 investigates settings in which statements can be nil, a typical situation in the Web environment in which statements are limited to the presence or absence of a link. Most proofs are given in Section 5.

2 Ranking methods

2.1 The framework

Let N be the set of items to be ranked. Items can be individuals, journals, articles, political parties. Let M be the set of experts. Experts can be pundits, or voters for example; in the judgment by peers, an expert is an item and the two sets N and M coincide. In the following, an element of N is called an *item* and an element of M an *expert*, keeping in mind the different interpretations. The cardinality of N is denoted by n and that of M by m .

Experts provide some statements on which the ranking of the items will be based. Experts' statements are described by a $n \times m$ *statement matrix* $\pi = (\pi_{i,j})$, in which j 's column represents j 's statement over N . Given these statements, one seeks for a ranking that assigns to each item i

²Intensity-invariance and uniformity have been considered in the literature by various authors, as those cited later on. Homogeneity in the sense used here seems to be new.

a non-negative number r_i , called the *score* of i . The aim of the ranking is to provide the relative strength of the n items. This means that not only the order of the scores matters but also their values up to a multiplicative constant. Thus, normalizing the sum of the scores to 1, a ranking of N is given by a vector \mathbf{r} in the simplex Δ_N : $\Delta_N = \{\mathbf{r} = (r_i) \in \mathbb{R}^n, r_i \geq 0, \sum_i r_i = 1\}$.

A method assigns a ranking to each feasible statement matrix. Let \mathcal{S} denote the set of feasible statements matrices. Formally,

Definition 1 *Given N, M , and a set \mathcal{S} of feasible statements matrices, a ranking method F assigns to each matrix $\boldsymbol{\pi}$ in \mathcal{S} a ranking $\mathbf{r} = F(\boldsymbol{\pi})$ in Δ_N .*

For example, the **counting** method assigns scores proportional to the total number of received statements:

$$r_i = \frac{\pi_{i+}}{\sum_{\ell} \pi_{\ell+}} \text{ for each } i \text{ where } \pi_{i+} = \sum_{j \in M} \pi_{i,j}. \quad (1)$$

More sophisticated methods are introduced in Section 2.3.

Statement matrices are restricted in this section to be positive, that is each $\pi_{i,j}$ is positive, but additional restrictions may be relevant. Two main feasible sets will be considered. One, denoted by \mathcal{P} , is the full set of positive matrices. The other, denoted by \mathcal{R} , is the set of positive matrices for which each column sums to 1. In that case, an expert's statement is called *relative*. Let us illustrate the problem in different contexts.

1. Ranking journals based on citations. The two sets of items and experts N and M coincide, both given by the set of journals under consideration, and statements are the number of citations by articles in journal j towards articles published in journal i . To be more precise, let $C_{i,j}$ be the total number of cites from j to i in a relevant period. Cites are normalized to account for the total number n_j of articles in j : the matrix $\boldsymbol{\pi}$ is given by $\pi_{i,j} = \frac{C_{i,j}}{n_j}$ where $\pi_{i,j}$ is the average number of citations per article in j to articles in i . The total of column j is the average total number of citations in an article of j . This total may differ across journals.
2. Representation problem. N is a set of categories and M is a set of electoral bodies. In the political domain for example, a category is a political party and an electoral body represents a constituency, say a district. In the context of a scientific association, a category may represent a specialty and the electoral bodies be defined by geographical areas. The problem is to assign voting weights to the categories based on the votes of the electoral bodies. Hence statements are the number of votes from j to i . The simplest and more natural method is to count the total of the votes obtained by each category. One may however contemplate other methods and distinguish the votes cast by distinct experts.
3. Budget allocation. The problem is to allocate a budget to different issues (transport, sanitation, education,...) as a result of the desiderata of some representatives. Here the items to be ranked are the issues, the experts are the representatives, and a representative's statement is given by the *shares* of the budget she would like to allocate to each issue (possibly not knowing the

budget total).³ By definition, each expert's statement is restricted to sum to one. Hence each statement is in \mathcal{R} .

4. Ranking Web pages based on the link structure. The two sets of items and experts N and M coincide, both given by the set of 'relevant' pages and the method defines a ranking of the pages that is based on the links within N . Hence the statement $\boldsymbol{\pi}$ is the adjacency matrix of the Web network: it has $\pi_{i,j}$ equal to 1 if page j points to i and 0 otherwise. Note that here the matrix has many zeros because many pages are not pointing to each other.

Such a binary representation also arises in settings where items and experts differ, and in which an expert is asked to name the items he finds acceptable. An expert's statement is described by the set of items he cites, represented by the vector of 1 and 0 indicator of the set. An expert could also state more general statements, not necessarily binary but with the possibility of assigning a zero to an item, meaning that it is not acceptable. The possibility of zeros will be analyzed in Section 4.

Finally, statements can be interpreted in stochastic terms, where $\pi_{i,j}$ represents the probability of expert j choosing i .

Notation $\mathbf{1}_N$ denotes the vector in \mathfrak{R}^N whose components are equal to 1, and $\mathbf{e}_N = \frac{1}{N}\mathbf{1}_N$ the ranking that assigns equal scores to items.

Given a finite set I and a vector \mathbf{x} in \mathfrak{R}^I , $dg(\mathbf{x})$ denotes the diagonal $I \times I$ matrix with x_i as the i -th element on the diagonal.

Given a matrix $\boldsymbol{\pi} = (\pi_{i,j})$, π_{i+} denotes the total in row i , $\pi_{i+} = \sum_{j \in M} \pi_{i,j}$, and π_{+j} the total in column j , $\pi_{+j} = \sum_{i \in N} \pi_{i,j}$.

2.2 Some properties

Let us start by describing natural properties that one may want a method to satisfy. Intensity-invariance, uniformity, and exactness appear in the literature under various names.

Intensity-invariance Intensity-invariance requires the ranking not to be affected by a multiplicative scaling of a column. Justifications in different contexts are provided below.

Definition 2 A method F defined on \mathcal{P} is **intensity-invariant** if $F(\boldsymbol{\pi}') = F(\boldsymbol{\pi})$ for $\boldsymbol{\pi}'$ the matrix obtained from $\boldsymbol{\pi}$ by multiplying column j by a positive μ_j . Equivalently

$$F(\boldsymbol{\pi}dg(\boldsymbol{\mu})) = F(\boldsymbol{\pi}) \text{ for each positive } m\text{-vector } \boldsymbol{\mu} = (\mu_j), \text{ each } \boldsymbol{\pi} \text{ in } \mathcal{P}. \quad (2)$$

The first statement requires (2) to be true for vector $\boldsymbol{\mu}$ whose components are all equal to 1 but for one component. Condition (2) follows for any vector $\boldsymbol{\mu}$ by multiplying iteratively each column of matrix $\boldsymbol{\pi}$ by μ_j and applying the property.

³This is in the spirit of the experience of Participatory Budgeting implemented in Porto Alegre, in which citizens set priorities to the different issues.

Basically, an intensity-invariant method is characterized by its behavior on matrices whose column totals are fixed. To see this, let a positive value for each column's sum, c_j for column j . Given a matrix $\boldsymbol{\pi}$, scale each of its column so that it meets the required total. Such a matrix is equal to $\boldsymbol{\pi}dg(\boldsymbol{\mu})$ where μ_j satisfies $\mu_j\pi_{+j} = c_j$. Intensity-invariance implies $F(\boldsymbol{\pi}) = F(\boldsymbol{\pi}dg(\boldsymbol{\mu}))$. Hence, intensity-invariant F is characterized by its values on the set of matrices whose column sums are fixed by \mathbf{c} . Conversely, intensity-invariant methods can be built on any method, possibly non intensity-invariant, by considering such a restriction and properly extending it. Start with a method F and choose a positive value for each column's sum, c_j for column j . Given a matrix $\boldsymbol{\pi}$, scale each of its column so that it meets the required total and apply F to the scaled matrix. Formally, define $[F]_{\mathbf{c}}$ as follows for each $\boldsymbol{\pi}$: $[F]_{\mathbf{c}}(\boldsymbol{\pi}) = F(\boldsymbol{\pi}dg(\boldsymbol{\alpha}))$ where $\alpha_j\pi_{+j} = c_j$. $[F]_{\mathbf{c}}$ is intensity-invariant since the scaled matrix of $\boldsymbol{\pi}dg(\boldsymbol{\mu})$ is the same as that of $\boldsymbol{\pi}$.

Observe that $[F]_{\mathbf{c}}$ varies with \mathbf{c} (except if F is intensity-invariant itself, in which case $[F]_{\mathbf{c}}$ coincides with F whatever \mathbf{c}). This construction shows that intensity-invariance is not related with any notion of fairness. Take for instance the method F to be anonymous with respect to the experts, namely, the ranking to be independent on a permutation of the experts' indices. In that case, choosing equal sums for each expert amounts to treat them *a priori* equally, and choosing distinct sums treats them differentially. Let us illustrate this with two examples.

For ranking journals or Web pages, sums are often chosen to be equal. In the case of journals, recall that $\pi_{i,j}$ is the average number of references of an article from j to i . Choosing equal sums for each journal means that the ranking depends only on the proportion of the cites made by an article in j that are received by i . As a result, a proportional increase in the statements by j , keeping the shares received by each journal unchanged, has no impact on the ranking. In particular, the ranking is not influenced by distinct citations' practice across journals or fields.⁴ In the case of Web pages, intensity-invariance implies that a link from a page is divided by the number of links from that page. Intensity is 'factored out'. A justification, relevant for the invariant method described in the next section, is that factoring out intensity avoids that a page improves its score simply by multiplying the pages it points to.

In the representation problem, an expert is the electoral body of a district whose statements are given by the number of votes to the parties in that district. Intensity-invariance requires the final representation to be independent of the turnout in the districts. One may want to assign columns' totals that are not proportional to the sizes of the electoral bodies. This allows for example to 'over'-represent a body with small population and 'under'-represent one with large population.

In the sequel, we work with matrices whose column sums equal to 1, namely matrices in \mathcal{R} , and with the associated intensity-invariant version, $[F]_{\mathbf{1}}$, denoted simply as $[F]$.⁵ Specifically let $[\boldsymbol{\pi}]$ be

⁴Indeed the average number of cites per article varies across fields. See Palacios-Huerta and Volij (2004) for an analysis of the impact of cite intensities on the ranking of economic journals.

⁵By doing so, no intensity-invariant methods are neglected; the method $[F]_{\mathbf{c}}$ associated with columns sums \mathbf{c} is the intensity-invariant version of the method $F_{\mathbf{c}}$ defined by $F_{\mathbf{c}}(\boldsymbol{\pi}) = F(\boldsymbol{\pi}dg(\mathbf{c}))$. By definition, $[F_{\mathbf{c}}](\boldsymbol{\pi}) = F(\boldsymbol{\pi}dg(\boldsymbol{\mu}))$ where $\mu_j\pi_{+j} = c_j$ or, equivalently, $\boldsymbol{\pi}dg(\boldsymbol{\mu}) = [\boldsymbol{\pi}]dg(\mathbf{c})$. Hence $F_{\mathbf{c}}(\boldsymbol{\pi}) = F([\boldsymbol{\pi}]dg(\mathbf{c}))$ which is equal to $[F]_{\mathbf{c}}(\boldsymbol{\pi})$ by definition.

the matrix in \mathcal{R} associated to π :

$$[\pi]_{i,j} = \frac{\pi_{i,j}}{\pi_{+j}} \text{ for each } i, j. \quad (3)$$

The intensity-invariant version of F denoted simply as $[F]$ is defined by

$$[F](\pi) = F([\pi]) \text{ for each } \pi \in \mathcal{P}. \quad (4)$$

Finally, note that any method that is defined on \mathcal{R} , such as in the budget example, can be extended (in a unique way) to an intensity-invariant method on \mathcal{P} . So we will assume methods to be defined on \mathcal{P} .

Uniformity and exactness The next properties bear on some specific matrices, hereafter called row-balanced. Row-balanced matrices constitute a kind of 'neutral' statements because there is no rationale for distinguishing between items if experts are not discriminated *a priori*: rows obtain identical totals or, put equivalently, rows obtain equal scores under the counting method (the counting method simply assigns to each item a score equal to the average experts statements, as made precise below). Formally, a matrix is said to be *row-balanced* if each row receives the same total.

Definition 3 A method F is **uniform** on \mathcal{S} if $F(\pi) = \mathbf{e}_N$ for all row-balanced π in \mathcal{S} .

A method is **exact** on \mathcal{S} if $F(\pi) = \mathbf{e}_N$ for π in \mathcal{S} implies that π is row-balanced.

A method is uniform if it assigns equal scores to each row-balanced feasible statement matrix. Exactness asks the converse property that items obtain equal scores only if they receive identical totals.

Clearly the intensity-invariant version of a uniform (resp. exact) method on \mathcal{P} is uniform (resp. exact) on \mathcal{R} since $[F](\pi) = F([\pi])$ for π in \mathcal{R} . However $[F]$ may not be uniform (or exact) on \mathcal{P} . Furthermore, there is no intensity-invariant method that is both uniform and exact on \mathcal{P} . To see this, start with a row-balanced matrix π with distinct columns. Its ranking is \mathbf{e}_N by uniformity. The distinct columns can be multiplied by some factors so as to obtain a matrix that is not row-balanced. Intensity-invariance requires the ranking of this adjusted matrix to be \mathbf{e}_N , which contradicts exactness.

Homogeneity The homogeneity property is very natural but has not yet been considered in the literature, as far as I know.⁶ For cardinal rankings, the relative values taken by the scores, and not only their orders, matter. Saying that the score of i is twice the score of k should mean that i is 'twice as good' as k . In this interpretation, starting with a matrix and multiplying all the statements to i by a factor, i 's relative position should be multiplied by the same factor. This is the homogeneity property. I spell out the property for the two situations in which absolute or relative statements matter, referred respectively as to a-homogeneity or r-homogeneity. Start with absolute statements on \mathcal{P} .

⁶The homogeneity axiom introduced in Palacios-Huerta and Volij (2004) differs from here: their axiom bears on a given matrix that has two proportional rows whereas ours bears on two distinct matrices.

Let $\boldsymbol{\pi}$ be in \mathcal{P} and $\mathbf{r} = F(\boldsymbol{\pi})$. Let $\boldsymbol{\pi}'$ be the matrix obtained from $\boldsymbol{\pi}$ by multiplying a row, say row 1, by ρ_1 . Homogeneity of F requires $F(\boldsymbol{\pi}')$ to be the ranking proportional to $(\rho_1 r_1, r_2, \dots, r_n)$. In matrix form, this writes as: $F(dg(\boldsymbol{\rho})\boldsymbol{\pi})$ is the ranking proportional to $dg(\boldsymbol{\rho})F(\boldsymbol{\pi})$ where $\boldsymbol{\rho}$ is the vector with first component equal to ρ_1 and all others equal to 1. The property is required for each row i , so iteration yields the following equivalent definition.

Definition 4 *A method is **a-homogeneous** if multiplying the statements on item i by a positive scalar, ρ_i , multiplies i 's score relative to other items by ρ_i . Equivalently, for each $\boldsymbol{\pi}$ in \mathcal{P} and positive n -vector $\boldsymbol{\rho} = (\rho_i)$, $F(dg(\boldsymbol{\rho})\boldsymbol{\pi})$ is the ranking proportional to $dg(\boldsymbol{\rho})F(\boldsymbol{\pi})$.*

Clearly, the counting method is a-homogeneous. To illustrate the property, let us consider the journals' example. Assume that between two periods, the citations received by a journal, say 1, from *each* journal increase by say ten per cent, and all others are unchanged. The property of a-homogeneity requires the score of journal 1 to increase by ten percent relatively to others, hence to become $\frac{(1+0.1)r_1}{\sum_i r_i + 0.1r_1} = \frac{1.1 \cdot r_1}{1+0.1r_1}$.

Homogeneity on relative statements requires the same behavior when a factor modifies the relative statements of an item.

Definition 5 *A method is **r-homogeneous** if multiplying the statements on i relative to other items by a positive scalar ρ_i multiplies i 's score relative to other items by ρ_i . Equivalently, for each $\boldsymbol{\pi}$ in \mathcal{R} and positive n -vector $\boldsymbol{\rho} = (\rho_i)$, $F([dg(\boldsymbol{\rho})\boldsymbol{\pi}])$ is the ranking proportional to $dg(\boldsymbol{\rho})F(\boldsymbol{\pi})$.*

In the budget example, statements are in \mathcal{R} . Consider two communities with the same number of experts. Their statements differ only by the fact that each expert in the first community assigns ten percent more to education relative to other issues than in the second. The property of r-homogeneity requires the share devoted to education to be ten percent larger relative to other issues in the first community than in the second; if the education share is 20% in the second community for example, it is roughly 21.5% in the first.

To sum up, a-homogeneity considers the absolute increase in items' citations whereas r-homogeneity considers their relative increase. The two distinctions do not matter for intensity-invariant methods, as stated in the next lemma, proved in the appendix.

Lemma 1 *Let F be intensity-invariant. F is a-homogeneous if and only if it is r-homogeneous.*

So, for intensity-invariant methods, we can simply refer to homogeneity. The homogeneity property however is lost when intensity is factored out. Specifically, let method F be a-homogeneous but not intensity-invariant. The intensity-invariant version $[F]$ may not be homogeneous (in whatever sense), as is illustrated with the counting method.

The counting method defined by (1) is a-homogeneous but the intensity-invariant version is not. In the following example, $\boldsymbol{\pi}'$ is obtained by multiplying the first row of $\boldsymbol{\pi}$ by 2:

$$\boldsymbol{\pi} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \boldsymbol{\pi}' = \begin{pmatrix} 4 & 2 \\ 1 & 2 \end{pmatrix}, [\boldsymbol{\pi}'] = \begin{pmatrix} 4/5 & 1/2 \\ 1/5 & 1/2 \end{pmatrix}.$$

The intensity-invariant version of the counting method assigns equal scores, $(1/2, 1/2)$, to $\boldsymbol{\pi}$ but $(13/20, 7/20)$ to $\boldsymbol{\pi}'$ (and to $[\boldsymbol{\pi}']$) instead of $(2/3, 1/3)$ as required by homogeneity. The reason is that expert 1 likes relatively more item 1 than expert 2 does, hence its total increases more than that of expert 2. Thus, when normalizing $\boldsymbol{\pi}'$, the adjustment on expert 1's statements is larger than on 2's. This explains why count of shares of item 1 is less than doubled relative to that of item 2.

2.3 Examples: the Invariant and Hits methods

This section introduces well-known methods that differ from the counting one. Whereas the counting method treats experts equally, these methods treat them differently according to their statements.

The next two methods are the Liebowitz-Palmer method⁷ and its intensity-invariant version, called the invariant method. The sets of items and experts coincide ($N = M$). The methods are 'peers' methods based on the premise that the statements made by a peer as an expert should be weighed by the received score as an item. This induces a loopback effect: a score of an item is defined as proportional to the sum of the received shares weighted by the experts' scores. Specifically the method looks for \mathbf{r} in Δ_N that satisfies

$$\text{for some positive } \lambda, r_i = \lambda \sum_{j \in N} \pi_{i,j} r_j \text{ for each } i. \quad (5)$$

Equations (5) say that the positive vector \mathbf{r} is an eigenvector of matrix $\boldsymbol{\pi}$. By Perron-Frobenius theorem on matrices with positive elements, such an eigenvector exists, and is unique up to a scalar. (5) and the method is well defined.

The invariant method is the intensity-invariant version of the Liebowitz-Palmer method. For a normalized matrix, the largest eigenvalue is equal to one.⁸ The invariant ranking of $\boldsymbol{\pi}$ is the unique \mathbf{r} in the simplex that satisfies

$$r_i = \sum_{j \in N} [\pi]_{i,j} r_j \text{ for each } i. \quad (6)$$

The method is intensity-invariant, uniform, exact, but not homogeneous.

The Hits method, introduced by Kleinberg (1999), also ranks Web pages on the basis of their link structure. Given a relevant set of pages, N , the Hits method defines a ranking of these pages, based on the links within N . Thus, as for the invariant method, the two sets of items and experts coincide. The method distinguishes two weights for each 'page', one associated with the relevance or authority of a page, the other with the adequacy of a page to point towards the relevant pages. The first set of weights defines the ranking, which should help users to find the relevant pages. The second set of weights identifies the pages -called 'hubs'- that are important because they point to relevant pages

⁷This terminology refers to the work of Liebowitz and Palmer (1984), who use an approximation of the method for ranking economic journals. The methods or some variants have been (re)defined and used in various contexts: in sociology by Katz (1953) and Bonacich (1987), in academics for ranking journals by Pinski and Narin (1976), and on the Web for ranking pages using the link structure between pages by Brin and Page (1998).

⁸Recall that the eigenvalues of a matrix and of its transpose are identical. The set of equations $\sum_i \pi_{i,j} = 1$ for each j implies that $\mathbf{1}_N$, the n -vector with components equal to 1, is a positive eigenvector of the transpose of $\boldsymbol{\pi}$ with eigenvalue 1.

(but might be not useful to Internet users). Specifically the method assigns the ranking \mathbf{r} and the experts weights \mathbf{q} in Δ_N that satisfy

$$r_i = \sum_j \pi_{i,j} q_j \text{ for each } i \text{ and } q_j = \lambda \sum_i \pi_{i,j} r_i \text{ for each } j \quad (7)$$

for some positive λ . As argued by Kleinberg (1999), hubs and authorities exhibit a *mutually reinforcing relationship*: a good authority is a page that is pointed to by many good hubs, a good hub is one that points to many good authorities. The method is well defined: In matrix form, (7) writes as $\mathbf{r} = \boldsymbol{\pi} \mathbf{q}$ and $\mathbf{q} = \lambda \tilde{\boldsymbol{\pi}} \mathbf{r}$ where $\tilde{\boldsymbol{\pi}}$ is the transpose of $\boldsymbol{\pi}$. Thus $\mathbf{r} = \lambda \boldsymbol{\pi} \tilde{\boldsymbol{\pi}} \mathbf{r}$ and $\mathbf{q} = \lambda \tilde{\boldsymbol{\pi}} \boldsymbol{\pi} \mathbf{q}$. The 'authority' weights \mathbf{r} and the 'hub' weights \mathbf{q} are well defined as respectively the normalized principal eigenvectors of the positive matrices $\boldsymbol{\pi} \tilde{\boldsymbol{\pi}}$ and $\tilde{\boldsymbol{\pi}} \boldsymbol{\pi}$.⁹ The method is uniform, exact, but not a-homogeneous.

The invariant and the Hits methods, by their very definitions, assign not only a ranking but also 'weights' to experts in such a way that the ranking writes as the combination of the statements according to these weights (as can be seen from (6) or (7)). This is also true for the counting method: the experts' weights are all equal, equal to $1/m$, whatever the statements.

3 The handicap-based method

Bearing on the notion of handicaps, this section introduces a new method based on a different equilibrium relationship between items' scores and experts' weights. It first provides a definition and characterization. Then a characterization of the counting method is provided that shows the similarities between the handicap-based and counting methods. Statements are assumed here to be all positive. Null entries are considered in Section 4.

3.1 Definition and properties

The purpose of handicaps is to equalize the strengths between items. Handicaps and scores may be seen as inversely related: saying that the handicap of i is twice that of ℓ means that the score of i is half that of ℓ . So handicaps are assigned to a ranking \mathbf{r} by the relationship $h_i = 1/r_i$. The handicap-based method is based on an equilibrium relationship between handicaps and experts' weights : it looks for handicaps that equalize items' weighted counts and for experts' weights that equalize the distributed handicaps across experts. Proposition 1 states that this leads to a well defined method.

Proposition 1 *Given a positive matrix $\boldsymbol{\pi}$, there is a unique $\mathbf{r} = (r_i)$ in Δ_N such that*

$$\sum_j (\pi_{i,j} q_j) \frac{1}{r_i} = 1 \text{ for each } i \text{ where } \sum_i \left(\frac{\pi_{i,j}}{r_i} \right) q_j = \frac{n}{m} \text{ for each } j. \quad (8)$$

The handicap-based method H assigns to each matrix this unique ranking \mathbf{r} .

H is intensity-invariant; furthermore it is uniform, exact, and homogeneous.

⁹ $1/\lambda$ is the dominant eigenvalue of $\boldsymbol{\pi} \tilde{\boldsymbol{\pi}}$ and is not equal to 1 in general even for a normalized $\boldsymbol{\pi}$.

Defining handicaps as $h_i = 1/r_i$, (8) states that handicaps and weights exhibit a mutual relationship: according to the left hand side, handicaps equalize items' weighted counts and according to the right hand side, experts' weights equalize their distributed handicap 'points' defined as $\sum_i (\frac{\pi_{i,j}}{r_i})$. The advice of an expert who assigns large values to items with high handicaps may be considered as unduly represented and its weight is decreased in proportion.

There is a fanciful interpretation of the method that helps to understand why the experts weights need to be adjusted. In a ballroom, a matchmaker \mathcal{M} wants to give each person equal chances to dance (take $n = m$). A single couple dances each time. \mathcal{M} proposes a match between a man and a woman, which the woman can refuse. Woman j accepts i with frequency $\pi_{i,j}$. The matchmaker has access to statistical results on the number of dances performed by each person, but does not know the preferences nor keep track of the partners accepted or refused by a woman. So \mathcal{M} chooses to form a pair according to some process that depends on the man and the woman in the pair but is independent of the pair's history: \mathcal{M} chooses the pair (i, j) with probability $\alpha_i \beta_j$ where α and β are distribution probabilities respectively over men and women.

In order to give equal chances to men, \mathcal{M} must propose more often men who tend to be refused more often. But by doing so, he has also to perform a bias in the selection process on women: otherwise those who like *relatively more* the men with low handicaps will face dancers they often tend to reject, hence will dance less often than those women who like *relatively more* the men with high handicaps. The term 'relatively' is important because in absolute terms men with low handicaps tend to be more accepted. To make this point clear, let all the women share exactly the same acceptance rates, $\pi_{i,j} = r_i$ for a common \mathbf{r} . Then choosing men with the frequency proportional to the inverse of the ranking \mathbf{r} and women with equal frequencies gives equal chance to each person to dance. When acceptance rates differ, drawing the men with the frequency proportional to the handicap vector (inverse of the ranking \mathbf{r}) and the women with frequency equal to the weight vector \mathbf{q} gives equal chance to each person to dance. This can be seen by writing (8) with the normalized handicaps $\alpha_i = \frac{h_i}{\sum h_i}$ so as to obtain frequencies (and using $n = m$)

$$\sum_j \pi_{i,j} q_j \alpha_i = f \text{ for each } i \text{ and } \sum_i \pi_{i,j} \alpha_i q_j = f \text{ for each } j \text{ where } f = \frac{1}{\sum h_i}. \quad (9)$$

The proof of existence of a ranking solution to (8) relies on the following observation. The method can be seen as searching for a ranking and experts' weights so that the statements appropriately adjusted yield a balanced matrix. Indeed (8) requires the matrix \mathbf{p} of general element $p_{i,j} = \frac{1}{r_i} \pi_{i,j} q_j$ obtained from π by multiplication of its rows by the items' handicaps and of its columns by the experts' weights, is $(1, n/m)$ -balanced:

$$\sum_j p_{i,j} = 1 \text{ for each } i \text{ and } \sum_i p_{i,j} = n/m \text{ for each } j. \quad (10)$$

The problem of adjusting a given matrix π by multiplication of its rows and its columns by some numbers so as to meet some totals constraints is a standard matrix scaling problem, known as the RAS model in economics or the log-fitting model in statistics. Relying on these results the matrix

\mathbf{p} is unique. It remains to show that the multipliers are uniquely defined when \mathbf{r} is in the simplex.¹⁰

The handicap-based ranking and the associated experts' weights can be computed through an iterative process,¹¹ similar to the iterative scaling algorithm used in the RAS model (Bacharach 1965). Let $\boldsymbol{\pi}$ be a normalized matrix. The procedure starts by assigning the counting ranking to $\boldsymbol{\pi}$ and equal weights to experts, $\mathbf{q}^0 = \mathbf{e}_M$. If the handicap points distributed by j as $\sum_i \frac{1}{r_i^0} \pi_{i,j}$, are identical across j , the process stops. Otherwise, some experts assign more handicap points than others and experts' weights are adjusted down or up so as to equalize the distributed points across experts:

$$\left(\sum_i \frac{1}{r_i^0} \pi_{i,j}\right) q_j^1 = \lambda^1 \text{ for each } j$$

where the value λ^1 is chosen so as to have the q_j^1 summing to 1. Thus, if the matrix with general element $\frac{1}{r_i^0} \pi_{i,j} q_j^1$ is row-balanced, \mathbf{r}^0 is the handicap-based ranking supported by experts weights equal to \mathbf{q}^1 . Otherwise, the procedure starts over again, alternating row-scaling and column-scaling:

$$\frac{1}{r_i^\tau} \left(\sum_j \pi_{i,j} q_j^\tau\right) = 1 \text{ for each } i \text{ and for some } \lambda^{\tau+1}, \left(\sum_i \frac{1}{r_i^\tau} \pi_{i,j}\right) q_j^{\tau+1} = \lambda^{\tau+1} \text{ for each } j. \quad (11)$$

The process can be shown to converge to normalized vectors (\mathbf{r}, \mathbf{q}) that satisfy (8).¹²

3.2 Characterization of the handicap-based and the counting methods

The next proposition provides a characterization of the handicap-based method.

Proposition 2

(a) *The handicap-based method is the only ranking method that is uniform on \mathcal{R} , intensity-invariant, and homogeneous.*

(b) *The handicap-based method is the only ranking method that is exact on \mathcal{R} , intensity-invariant, and homogeneous.*

The next proposition provides a characterization of the counting method.

Proposition 3

(a) *The counting method is the only method that is a -homogeneous and uniform on \mathcal{P} .*

(b) *The counting method is the only method that is a -homogeneous and exact on \mathcal{P} .*

Comparing the two characterizations of the handicap-based and counting method shows that their main difference stems from intensity-invariance. Put differently, the handicap-based method is adequate for relative statements and the counting method for absolute ones.

¹⁰The approaches differ in that the object of interest of the RAS method is the final adjusted matrix \mathbf{p} whereas we are interested in the (relative) values of the adjustment on rows and columns so as to define the handicaps and experts' weights.

¹¹Similarly, the rankings and experts' weights assigned by the invariant and the Hits methods can be obtained as the equilibrium values of an iterative process, see respectively Amir (2002) and Kleinberg (1999).

¹²The iterative scaling algorithm defines alternatively multipliers for rows and columns so as to meet required constraints on the rows' and columns' totals. Here the totals on rows are met at every odd period and the vectors are required to belong to the simplex.

Stochastic statements Statements in \mathcal{R} can be interpreted in stochastic terms: $\pi_{\cdot j} = (\pi_{i,j})$ represents the probability with which expert j chooses i . This interpretation allows us to understand better the differences and similarities between the handicap-based and the counting methods. Both methods assign equal scores to a balanced matrix. We compare how a distortion on a balanced matrix affects the ranking assigned by the methods.

Start with a balanced matrix \mathbf{p} . Now 'distort' \mathbf{p} by a positive vector \mathbf{r} in a uniform way across experts as follows. Multiply row i by r_i and normalize: the distribution of the drawings of j becomes $\pi_{i,j} = \frac{r_i p_{i,j}}{\sum_k r_k p_{k,j}}$. Thus, the chances for i to be drawn relative to ℓ have been multiplied by r_i/r_ℓ for each expert. By homogeneity, \mathbf{r} is the handicap-based ranking of $\boldsymbol{\pi}$ (with weights $q_j = \sum_k p_{k,j} r_k$ as follows from the expression of $\boldsymbol{\pi}$). The counting method instead assigns to i its expected count $\frac{1}{m} \sum_j \pi_{i,j}$. Though the chances for i to be drawn relative to ℓ have been multiplied by r_i/r_ℓ for each expert, the expected counts are not in general.

Comparison with the invariant method The invariant method is intensity-invariant and exact but not homogeneous. The following example shows that its behavior with respect to changes in statements may present serious drawbacks in some contexts (I choose null diagonal elements to show that the behavior is not due to self-citations). Let $\boldsymbol{\pi}$ and $\boldsymbol{\pi}'$ be

$$\boldsymbol{\pi} = \begin{pmatrix} 0 & 1/3 & 1 \\ 1 & 0 & 1/3 \\ 1/3 & 1 & 0 \end{pmatrix} \quad \boldsymbol{\pi}' = \begin{pmatrix} 0 & 10/19 & 10/13 \\ 9/10 & 0 & 3/13 \\ 1/10 & 9/19 & 0 \end{pmatrix},$$

The invariant ranking for balanced matrix $\boldsymbol{\pi}$ is $(1/3, 1/3, 1/3)$ and for $\boldsymbol{\pi}'$ is (approximately) $(0.38, 0.395, 0.225)$. Comparing the statements in the two matrices, those for 1 relative to 2 are 10/9 times larger in $\boldsymbol{\pi}'$ than in $\boldsymbol{\pi}$, and those for 2 relative to 3 are 3 times larger. That is, $\boldsymbol{\pi}' = [dg(\boldsymbol{\rho})\boldsymbol{\pi}]$ for $\boldsymbol{\rho} = (10/9, 1, 1/3)$. Thus, although the statements on 1 relative to 2 have increased, each one multiplied by $\rho_1 = 10/9$, the score of 2 is now larger than 1's.

This non-monotone behavior is explained by the dual role of the scores in the invariant method and the induced loopback effect. At $\boldsymbol{\pi}$, item 1 receives a high evaluation from expert 3. As item 3 gets less support (the scaling down by $\rho_3 = 1/3$), its weight as an expert is decreased as well. Thus, the high evaluation from 3 counts less and this produces a negative effect on item 1 counterbalancing the positive effect due to ρ_1 . As for item 2, it receives an high evaluation from 1 at $\boldsymbol{\pi}$. So it benefits indirectly both from the increase in 1's assessments and the decrease in 3's when they result in a larger score for 1 and a smaller for 3 than the initial 1/3. This explains why the score of 2 not only increases but ends up larger than 1's score.

Of course, for some values of ρ_1 larger than 1 and ρ_3 smaller than 1, 1's score may end up larger than 2's. As should be clear from the above argument, there are effects possibly in opposite directions due to the double role of the scores in the invariant method. The final order of the scores depends on the relative intensity of these various effects, which, in turn, depends on the matrix and the values of the ρ_i .

4 Extending the handicap-method

So far the handicap-based method has been defined for positive statements. In some settings, the positivity assumption is too strong. In the budget example (Section 2), an expert may not want to allocate any funds to a particular item. In the matchmaking example (after Proposition 1, a woman may want to refuse a particular man. In the Web setting, in the absence of any further information than the link structure, the statement of a page is binary, made of 0 or 1. Such a binary representation also arises in voting in which an expert is asked to name the items he finds acceptable, without having the possibility to state intensity. More generally, an expert might assign a zero to an item, meaning that this item is not acceptable or not evaluated.

This section investigates how to extend the handicap-based method to non-negative matrices. Uniqueness is important because it relates to the continuity of the method with respect to perturbation of the matrices.

Given a statement matrix $\boldsymbol{\pi}$, $\pi_{i,j} \geq 0$, we assume that $\boldsymbol{\pi}$ has no null row nor null column (these could be deleted). Let us introduce some notation. Let $I(j) = \{i | \pi_{i,j} > 0\}$ be the set of items cited by j . For J subset of M , $I(J) = \cup_{j \in J} I(j)$ is the set of items cited by at least one expert in J . Consider the bipartite graph G with sets of nodes N and M where (i, j) is an edge for the i in N and j in M with positive $\pi_{i,j}$. In the peers' setting in which the items are also the experts, the two sets N and M are replica of each other so that the bipartite graph distinguishes the two roles as expert or item for each element. G is *items-connected* if each pair of items is linked by a path: for each pair of items i, ℓ , there is a sequence alternating items and experts, starting at i and ending at ℓ : $i = i_0, j_0, i_1, \dots, i_t, j_t, \dots, j_k, i_{k+1} = \ell$ such that each expert j_t in the sequence cites the two adjacent items i_t and i_{t+1} .

Since each expert assigns a positive statement on one item at least, there is a path between any two elements, item or expert : G is *connected* as well.¹³

We look for statement matrices for which a handicap-based ranking as defined by (8) is well defined, that is, exists and is unique. Under (8) the matrix \boldsymbol{p} defined by $p_{i,j} = \frac{1}{r_i} \pi_{i,j} q_j$ is balanced and has exactly the same null cells as $\boldsymbol{\pi}$. Hence a necessary condition for a handicap-based ranking to exist is easy to understand. There must exist a balanced matrix that has exactly the same null entries as $\boldsymbol{\pi}$. This existence problem can be put as a flow problem solved by supply-demand conditions, as described in the next proposition.

Proposition 4 *Given a non-negative matrix $\boldsymbol{\pi}$, there is a unique ranking $\mathbf{r} = (r_i)$ in Δ_N such that (8) holds:*

$$\sum_j (\pi_{i,j} q_j) \frac{1}{r_i} = 1 \text{ for each } i \text{ where } \sum_i (\frac{\pi_{i,j}}{r_i}) q_j = \frac{n}{m} \text{ for each } j$$

if and only if the bipartite graph G associated to $\boldsymbol{\pi}$ is items-connected and

$$\frac{1}{m} |J| < \frac{1}{n} |I(J)| \text{ for any strict subset } J \text{ of } M. \tag{12}$$

¹³To obtain a path between two experts for example take two items that are cited respectively by each one, a path between the two experts is obtained by taking the path linking the two items (if they differ), and adding the link between each item and the expert who cites it.

The items-connectedness of graph G ensures the uniqueness of the ranking. This is a natural condition. If G is not items-connected, then there are two disjoint sets of experts who cite two disjoint sets of items: these experts do not share any common interest. In such a situation, it makes sense that no unique ranking can reflect the statements of all experts.

Conditions (12) require that the proportion of items cited by a given subgroup of experts should exceed their proportion in the set of experts. Cites should be sufficiently 'disseminated' so as to allow for the adjustment as required by (8). Taking the complementary inequalities (12) are equivalent¹⁴

$$|I| < \frac{n}{m}|J(I)| \text{ for any strict subset } I \text{ of } N. \quad (13)$$

These conditions require that, for each subset of items, the proportion of experts who cite that subset should exceed the proportion of this subset in the whole set of items.

To get an intuition for these conditions, observe that a balanced matrix which has null elements for each pair where the element in $\boldsymbol{\pi}$ is null corresponds to a flow in the bipartite graph G such that each j sends $1/m$ unit and each node i in N receives $1/n$ units. The existence of such a flow is solved by well-known supply-demand conditions, which are the same inequalities as in (12) but only required weakly. Requiring the inequalities to be strict ensure that the flow can be made strictly positive on each link (if not, the flow can be perturbed by having a small enough amount going through a link without flow and finding a cycle without creating zero).

The invariant or the Hits methods also require some conditions to be uniquely defined. The matrix $\boldsymbol{\pi}$ or $\boldsymbol{\pi}\tilde{\boldsymbol{\pi}}$ should admit a unique ranking associated to the largest eigenvalue, as ensured by the irreducibility of the matrix.

The items-connectedness of G is equivalent to the irreducibility of $\boldsymbol{\pi}\tilde{\boldsymbol{\pi}}$. Recall that a non-negative square matrix \mathbf{a} is irreducible if for each pair (i, ℓ) there is an integer t such that the (i, ℓ) element of the t -product matrix $\mathbf{a}^{(t)}$ is positive. The element (i, ℓ) of $\mathbf{a} = \boldsymbol{\pi}\tilde{\boldsymbol{\pi}}$ writes $\sum_j \pi_{i,j}\pi_{\ell,j}$, hence is positive if and only if there is an expert who cites both i and ℓ . Thus $a_{i,\ell}^{(t)}$ is positive if there is a path with t experts linking i to ℓ in the bipartite graph G . Hence the items-connectedness ensures that $\boldsymbol{\pi}\tilde{\boldsymbol{\pi}}$ has a unique positive normalized eigenvector, thanks to Perron-Frobenius theorem. So the Hits ranking is uniquely defined (without the need of a perturbation of the matrix). Using a similar argument, the hubs' (or experts) weights are also well defined: any two experts are linked so that the matrix $\tilde{\boldsymbol{\pi}}\boldsymbol{\pi}$ is irreducible.

For the invariant method, the ranking is uniquely defined when the matrix $\boldsymbol{\pi}$ is irreducible. This condition differs from the irreducibility of $\boldsymbol{\pi}\tilde{\boldsymbol{\pi}}$, as shown by the following example. There are three items/experts; 1 cites 2, 2 cites 3, and 3 cites 1. The matrix¹⁵ $\boldsymbol{\pi}$ is irreducible but $\boldsymbol{\pi}\tilde{\boldsymbol{\pi}}$ is null (each expert cites a different item).

¹⁴Given a strict subset I , let $J = M - J(I)$. If J is empty, (13) is met. If not, apply (12) to J : $\frac{1}{m}(m - |J(I)|) < \frac{1}{n}|I(J(I))|$. By definition, the experts cited by $M - J(I)$ do not cite any items in I . So $I(J(I)) \subset N - I$, which implies $|I(J(I))| \leq n - |I|$, and finally $|I| < \frac{n}{m}|J(I)|$.

¹⁵ $\boldsymbol{\pi} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

To end this extension, observe that the homogeneity axiom does not make sense in the 0-1 setting case, (the Web graph) (except if one is ready to work with stochastic statements). Altman and Tennenholtz (2008) provides an axiomatization of the invariant ranking in the 0-1 setting, but as far as I know it is the only one. (Those in Palacios-Huerta and Volij (2004) and Slutzki and Volij (2006) rely on axioms that pertain to statements that are not restricted to 0-1 matrices.)

Concluding remarks This paper has introduced a new method that is adequate for some environments, either because the ‘intensity’ of statements is not controlled, or because, by construction, statements are relative evaluations. The handicap-based method is, in a sense made precise in the paper, the counterpart of the counting method in these environments.

There are at least two possible developments. First there is a need for a further analysis of ranking methods in the binary setting, as in the Web. Second, viewing a method as assigning scores to items and weights to experts in an equilibrium relationship provides a useful tool for designing alternative methods.

5 Proofs

Proof of Lemma 1 By intensity-invariance,

$$F([\boldsymbol{\pi}]) = F(\boldsymbol{\pi}) \text{ and } F([dg(\boldsymbol{\rho})\boldsymbol{\pi}]) = F(dg(\boldsymbol{\rho})\boldsymbol{\pi}). \quad (14)$$

Let F be a-homogeneous: $F(dg(\boldsymbol{\rho})\boldsymbol{\pi})$ is proportional to $dg(\boldsymbol{\rho})F(\boldsymbol{\pi})$ for any $\boldsymbol{\pi}$ in \mathcal{P} , in particular for $\boldsymbol{\pi}$ in \mathcal{R} . Thus $F([dg(\boldsymbol{\rho})\boldsymbol{\pi}])$, which is equal to $F(dg(\boldsymbol{\rho})\boldsymbol{\pi})$, is proportional to $dg(\boldsymbol{\rho})F(\boldsymbol{\pi})$: F is r-homogeneous.

To show the converse, let F be r-homogeneous and $\boldsymbol{\pi}$ be in \mathcal{P} . Observe that $[dg(\boldsymbol{\rho})\boldsymbol{\pi}] = [dg(\boldsymbol{\rho})[\boldsymbol{\pi}]]$. Hence $F(dg(\boldsymbol{\rho})\boldsymbol{\pi}) = F([dg(\boldsymbol{\rho})[\boldsymbol{\pi}]])$. Since $[\boldsymbol{\pi}]$ is in \mathcal{R} , homogeneity on \mathcal{R} implies that $F([dg(\boldsymbol{\rho})[\boldsymbol{\pi}]])$ is proportional to $dg(\boldsymbol{\rho})F([\boldsymbol{\pi}])$. Using $F([\boldsymbol{\pi}]) = F(\boldsymbol{\pi})$, we finally obtain that $F(dg(\boldsymbol{\rho})\boldsymbol{\pi})$ is proportional to $dg(\boldsymbol{\rho})F(\boldsymbol{\pi})$. This proves that F is a-homogeneous. ■

Proof of Proposition 1. The proof of existence and uniqueness of H involves two steps.

The first step shows the existence of \mathbf{r} in the simplex that satisfies (8). As said in the text, this is equivalent to the fact that matrix \mathbf{p} with general element $p_{i,j} = \frac{1}{r_i}\pi_{i,j}q_j$ is $(1, n/m)$ -balanced. The proof relies on the known result about matrix scaling: there is a unique $(1, n/m)$ -balanced matrix \mathbf{p} that is obtained from $\boldsymbol{\pi}$ by multiplication of its rows and its columns by some numbers. A simple proof relies on a convex program. We recall the argument here for positive matrices. Consider the program (ln denotes the natural logarithm):

$$\begin{aligned} \mathcal{P} : \text{minimize}_{\mathbf{p}} \quad & \sum_{i,j} p_{i,j} [\ln(\pi_{i,j}/p_{i,j}) - 1] \text{ over the } \mathbf{p} = (p_{i,j}) > 0 \\ \text{subject to} \quad & \sum_j p_{i,j} = 1 \text{ for each } i \text{ and } \sum_i p_{i,j} = n/m \text{ for each } j. \end{aligned} \quad (10)$$

The program is convex with a strictly convex objective function and a feasible set with a non empty relative interior. Hence the solution \mathbf{p} is unique, characterized by the first order conditions on the

Lagrangian: There are multipliers α_i and β_j associated respectively to the constraints (10) on the totals of row i and column j such that

$$\ln(\pi_{i,j}) - \ln(p_{i,j}) = \alpha_i + \beta_j. \quad (15)$$

Set $r_i = \exp(\alpha_i)$ and $q_j = \exp(-\beta_j)$, (15) writes $p_{i,j} = \frac{1}{r_i} \pi_{i,j} q_j$. Plugging these expressions into the constraints (10) give (8). It remains to show that \mathbf{r} can be chosen to be in the simplex. The multipliers $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are defined up to an additive constant, i.e., $(\alpha_i + c)$, $(\beta_j - c)$ satisfy (15) for any c if $\boldsymbol{\alpha}, \boldsymbol{\beta}$ does (this is due to the fact the linear system (10) is formed with linearly dependent equations). Thus there is a vector \mathbf{r} , $r_i = \exp(\alpha_i + c)$, that belongs to the simplex for an appropriate value of c .

The second step proves that \mathbf{r} is unique. Given \mathbf{r} that satisfies (8), the matrix \mathbf{p} defined by $p_{i,j} = \frac{1}{r_i} \pi_{i,j} q_j$ satisfies the constraints (10). Furthermore, taking the log of $p_{i,j} = \frac{1}{r_i} \pi_{i,j} q_j$, the first order conditions (15) are met for $\alpha_i = \ln(r_i)$, $\beta_j = -\ln(q_j)$. Hence \mathbf{p} is the unique solution to \mathcal{P} . So if there are two rankings that satisfy (8), \mathbf{r} and \mathbf{r}' , the corresponding values (α_i) (β_j) and (α'_i) (β'_j) satisfy (15) for the same matrix \mathbf{p} . Taking the difference yields

$$(\alpha_i - \alpha'_i) + (\beta_j - \beta'_j) = 0 \text{ for each } i, j. \quad (16)$$

Hence $\alpha'_i = \alpha_i + c$ for some scalar c . There can be only one value for c so that \mathbf{r} defined by $r_i = \exp(\alpha_i)$ belongs to Δ_N . This proves that H is a well defined method. ■

Let us now prove the properties of H

H is intensity-invariant. Let $\boldsymbol{\pi}'$ be obtained from $\boldsymbol{\pi}$ by multiplying column j of $\boldsymbol{\pi}$ by μ_j . Letting \mathbf{q}' be the vector obtained from \mathbf{q} by dividing q_j by μ_j , the vectors \mathbf{r} and \mathbf{q}' satisfy (8) for $\boldsymbol{\pi}'$. By the uniqueness result proved previously, $H(\boldsymbol{\pi}')$ is equal to \mathbf{r} .

H is uniform on \mathcal{R} . Let matrix $\boldsymbol{\pi}$ in \mathcal{R} be row-balanced. It satisfies $\sum_j \pi_{i,j} = m/n$ for each i and $\sum_i \pi_{i,j} = 1$ for each j . Hence, the conditions

$$\sum_j (\pi_{i,j} q_j) \frac{1}{r_i} = 1 \text{ for each } i \text{ where } \sum_i \left(\frac{\pi_{i,j}}{r_i} \right) q_j = \frac{n}{m} \text{ for each } j \quad (8)$$

are satisfied by taking equal scores and equal experts' weights, that is, each r_i equals to $1/n$ and each q_j equals to $1/m$. (Equivalently $\mathbf{p} = \boldsymbol{\pi} \frac{n}{m}$ satisfies (10).) Hence, by the uniqueness result, $H(\boldsymbol{\pi}) = \mathbf{e}_N$.

H is exact on \mathcal{R} . Let $H(\boldsymbol{\pi}) = \mathbf{e}_N$ for a matrix $\boldsymbol{\pi}$ in \mathcal{R} . We need to show that $\boldsymbol{\pi}$ is row-balanced. Applying $\mathbf{r} = \mathbf{e}_N$ to the second equation in (8) yields that the weight vector \mathbf{q} satisfies $(\sum_i \pi_{i,j}) q_j = 1/m$ for each j . This implies $q_j = 1/m$ since $\boldsymbol{\pi}$ is in \mathcal{R} . Plugging $r_i = 1/n$ and $q_j = 1/m$ for each i, j into the first set of equations of (8) we obtain that each row's total is equal to m/n : the matrix $\boldsymbol{\pi}$ is row-balanced.

H is homogeneous. Let $\boldsymbol{\pi}' = dg(\boldsymbol{\rho})\boldsymbol{\pi}$ for a positive vector $\boldsymbol{\rho}$. By the definition of H , we have $\pi_{i,j} = \frac{r_i p_{i,j}}{q_j}$ where \mathbf{p} is a $(1, n/m)$ -balanced matrix. Hence $\pi'_{i,j} = \rho_i \pi_{i,j} = \frac{\rho_i r_i p_{i,j}}{q_j}$ so that \mathbf{p} is obtained from $\boldsymbol{\pi}'$ by dividing each row i by $\rho_i r_i$ and multiplying each column j by q_j . By the uniqueness result stated in the first part of Proposition 1, this implies that the handicap-based

ranking \mathbf{r}' associated to $\boldsymbol{\pi}'$ is the vector in Δ_N proportional to $(\rho_i r_i)$. Thus, $H(\boldsymbol{\pi}') = (\frac{\rho_i r_i}{\sum_i \rho_i r_i})$, namely multiplying the rows of $\boldsymbol{\pi}$ by some vector multiplies the scores in the same proportions: this proves homogeneity. ■

Proof of Proposition 2.

Proof of (a). Let method F be uniform on \mathcal{R} , intensity-invariant and homogeneous.

F and H coincide if they coincide on the set \mathcal{R} of relative statement matrices since both methods are intensity-invariant. Given $\boldsymbol{\pi}$ consider $\mathbf{r} = H(\boldsymbol{\pi})$, \mathbf{q} , and the balanced matrix \mathbf{p} associated by the handicap-based method, $p_{i,j} = \frac{q_j \pi_{i,j}}{r_i}$. By uniformity on \mathcal{R} , $F(\mathbf{p}) = \mathbf{e}_N$. Let $\mathbf{p}' = dg(\mathbf{r})\mathbf{p}$. Homogeneity implies $F(\mathbf{p}') = \mathbf{r}$. The normalized matrix of \mathbf{p}' is $\boldsymbol{\pi}$: since matrix $\boldsymbol{\pi}$ is in \mathcal{R} , $\sum_i \pi_{i,j} = 1$ for each j , which writes $\sum_i p_{i,j} r_i = q_j$ thus $[\mathbf{p}'] = \boldsymbol{\pi}$. By intensity-invariance $F([\mathbf{p}']) = F(\mathbf{p}') = \mathbf{r}$, which finally gives $F(\boldsymbol{\pi}) = H(\boldsymbol{\pi})$, the desired result.

Proof of (b). Let method F be exact on \mathcal{R} , intensity-invariant and homogeneous.

Given $\mathbf{r} = F(\boldsymbol{\pi})$, divide each row i by r_i so as to obtain matrix $\boldsymbol{\pi}' = dg(1/r_1, \dots, 1/r_n)\boldsymbol{\pi}$. Homogeneity implies that the scores are equalized: $F(\boldsymbol{\pi}') = \mathbf{e}_N$ thanks to Lemma 1. By intensity-invariance of F we have $F([\boldsymbol{\pi}']) = F(\boldsymbol{\pi}') = \mathbf{e}_N$. Now, exactness on \mathcal{R} implies that $[\boldsymbol{\pi}']$ is row-balanced, hence each row sums to m/n . Since $[\boldsymbol{\pi}']_{i,j} = \frac{\pi_{i,j}}{r_i \sum_\ell \pi_{\ell,j}/r_\ell}$, this writes

$$\sum_j \frac{\pi_{i,j}}{r_i \sum_\ell \pi_{\ell,j}/r_\ell} = \frac{m}{n} \text{ for each } i,$$

$$\text{or } r_i = \sum_j \pi_{i,j} q_j \text{ for each } i, \text{ where } \frac{1}{q_j} = \frac{m}{n} \sum_\ell \frac{\pi_{\ell,j}}{r_\ell} \text{ for each } j.$$

Thus \mathbf{r} and \mathbf{q} satisfy (8). Since \mathbf{r} is in Δ_N , \mathbf{r} is equal to $H(\boldsymbol{\pi})$, the desired property. ■

Proof of Proposition 3.

Proof of (a). Clearly the counting method is a-homogeneous and uniform on \mathcal{P} . To show the reverse, let method F satisfies these properties. Given a matrix $\boldsymbol{\pi}$, divide each row i by its total $\pi_{i+} = \sum_{j \in M} \pi_{i,j}$ and denote by $\boldsymbol{\pi}'$ the obtained matrix: $\boldsymbol{\pi}' = dg(1/\pi_{1+}, \dots, 1/\pi_{n+})\boldsymbol{\pi}$. By the a-homogeneity of F , the ranking assigned by F to $\boldsymbol{\pi}'$ is obtained by dividing each component i of $F(\boldsymbol{\pi})$ by π_{i+} and normalizing: for some positive λ , $F_i(\boldsymbol{\pi}') = \lambda F_i(\boldsymbol{\pi})/\pi_{i+}$ for each i . Since $\boldsymbol{\pi}'$ is row-balanced $F(\boldsymbol{\pi}') = \mathbf{e}_N$. This yields that $F_i(\boldsymbol{\pi})/\pi_{i+}$ is constant across i : $F(\boldsymbol{\pi})$ is the counting ranking of $\boldsymbol{\pi}$, the desired result.

Proof of (b). Clearly the counting method is a-homogeneous and exact on \mathcal{P} . To show the reverse, let method F satisfy these properties. Given $\mathbf{r} = F(\boldsymbol{\pi})$, divide each row i by r_i so as to obtain matrix $\boldsymbol{\pi}' = dg(1/r_1, \dots, 1/r_n)\boldsymbol{\pi}$. F is a-homogeneity implies that the scores are equalized: $F(\boldsymbol{\pi}') = \mathbf{e}_N$ thanks to Lemma 1. Exactness implies that $\boldsymbol{\pi}'$ is row-balanced: for some positive λ , $\sum_j \pi'_{i,j} = \lambda$ for each i . Hence $\sum_j \pi_{i,j} = \lambda r_i$ for each i : \mathbf{r} is the counting ranking of $\boldsymbol{\pi}$, the desired result. ■

Proof of Proposition 4 Under the stated conditions, the proof of Proposition 1 extends to a matrix $\boldsymbol{\pi}$ with some null elements as follows. The objective in \mathcal{P} takes the sum over the (i, j) for

which $\pi_{i,j}$ is positive, namely the (i, j) in G :

$$\begin{aligned} \mathcal{P} : \underset{\mathbf{p}}{\text{minimize}} \quad & \sum_{i,j \in G} p_{i,j} [\ln(\pi_{i,j}/p_{i,j}) - 1] \text{ over the } \mathbf{p} = (p_{i,j}) \geq 0 \\ \text{subject to} \quad & \sum_j p_{i,j} = 1 \text{ for each } i \text{ and } \sum_i p_{i,j} = n/m \text{ for each } j. \end{aligned} \quad (10)$$

Under conditions (12), the feasible set defined by (10) has a nonempty interior. Hence the Kuhn and Tucker theorem applies. A solution \mathbf{p} is associated to multipliers α_i and β_j such that

$$\ln(\pi_{i,j}) - \ln(p_{i,j}) = \alpha_i + \beta_j, \text{ for each } (i, j) \in G \quad (17)$$

Following the same arguments as in Proposition 1, setting $r_i = \exp(\alpha_i)$ for each i up to a multiplicative constant ensures the existence of \mathbf{r} that satisfies (8).

Let us consider uniqueness. Let \mathbf{r} a solution to the conditions (8) on a handicap-based ranking. \mathbf{r} is associated to a balanced matrix \mathbf{p} and the values defined by $\alpha_i = \ln(r_i)$, $\beta_j = -\ln(q_j)$ satisfy (17). Hence $\mathbf{p}, \boldsymbol{\alpha}, \boldsymbol{\beta}$ satisfy the Kuhn and Tucker conditions associated to \mathcal{P} . Since the program \mathcal{P} is strictly convex a solution \mathbf{p} is unique. Thus, if there are two rankings solutions to (8), \mathbf{r} and \mathbf{r}' , taking the difference in (17) for their corresponding values yields

$$(\alpha_i - \alpha'_i) + (\beta_j - \beta'_j) = 0 \text{ for each } (i, j) \in G. \quad (18)$$

This implies that along a path linking two items the values $(\alpha_i - \alpha'_i)$ are all equal. Using the same argument as for Proposition 1, the uniqueness of \mathbf{r} follows if G is connected: $\alpha'_i - \alpha_i$ is constant across all i , and there can be only one \mathbf{r} defined by $r_i = \exp(\alpha_i)$ that belongs to Δ_N . On the other hand, if G is not connected, there are no links between the values on each component and uniqueness fails. ■

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