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The sovereign default puzzle: Modelling issues and lessons for Europe*

Daniel Cohen† Sébastien Villemot‡

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Abstract

Why do countries default? This seemingly simple question has yet to be adequately answered in the literature. Indeed, prevailing modelling strategies compel the to choose between two unappealing model features: depending on the cost of default selected by the modeler, either the debt ratios are too high and the probability of default is too low or the opposite is true. In view of the historical evidence that countries always default after a crisis, we propose a novel approach to the theory of debt default and develop a model that matches the key stylized facts regarding sovereign risk.

Keywords: Sovereign debt, Lévy stochastic processes

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1 Introduction

Europe has recently been hit by a sovereign debt crisis which has caused three of its members to be ousted from financial markets. Those three countries, Greece, Ireland and Portugal, had to ask for the support of the other eurozone countries to refinance their debt. Additionally, in the case of Greece the eventual implementation of a nominal haircut of more than 50% was decided. In response to this unexpected crisis, Europe decided to impose a much stricter budgetary discipline, aiming for a near zero deficit rule. How did the eurozone suddenly become so vulnerable to sovereign risk? Is Europe overreacting by imposing budget constraints that are too restrictive?

Sovereign debt crisis specialists have been asked for answers. Trying to understand why some countries default is the theme of a large body of literature. Reinhart et al. (2003) have investigated the nature of what they called the "debt intolerance" of many countries over the centuries. Greece is certainly one of these, having already defaulted many times over the past two centuries. The key paradox of the academic literature however, is that it is actually very hard to satisfactorily fit the data on default probabilities and debt levels. Work by Aguiar and

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Gopinath (2006) or Arellano (2008) for instance struggled with the fact that a debt-to-GDP ratio in excess of only 5% could trigger a default within reasonably calibrated models. These papers have, on the surface, trivialized the problem, as almost any level of debt seems to create a risk of default.

These difficulties led Rogoff (2011) to argue that the narrative approach to debt default, as exemplified in Reinhart and Rogoff (2009), does a better job at understanding default than simulated models. This is clearly a provocative statement. Barring a calibrated model, how can one think about what the proper debt levels should be? Further, how can one rationalize the eurozone policymakers' attempts to set safe debt levels in order to avoid another crisis?

In previous models of sovereign risk, default is a costly decision that the country weighs against the alternative of repaying its debt. From a modeler's perspective, the following trade off arises. Either the cost of default is high, in which case high debt-to-GDP ratios can be sustained at the expense of a low frequency of default, as countries don't default when the costs are high. Or the cost of default is set at low levels, in which case the frequency of default can fit the data, but the sustainable debt levels become abnormally low; this is the outcome of most calibrated models today.

In this paper, we revisit existing sovereign debt models and amend them in order to get predictions which better fit real world debt levels. The key motivation of our analysis comes from the following observation of which the Greek crisis is one illustration: countries usually do not want to default unilaterally. In fact, as well documented by the Inter-American Development Bank (2007) and Levy-Yeyati and Panizza (2011), in all cases of sovereign debt crises but one, the "decision" to default was never really a decision of the country: it came after the crisis already took place. The only case of a "strategic default" is Ecuador in 2009. This leads us to the following new modelling assumption. In our model, the sequencing of events is inverted: the crisis begins before the decision to default has been taken. Think of a bank panic or a temporary collapse of a key industrial sector. In these "trembling times," the cost of default becomes lower as the financial panic or the economic meltdown already happened. Default does add extra costs, but lower than those which would have be borne in "normal times." With this distinction, we show that we can simultaneously model high levels of debt with a high frequency of default.

The paper is organized as follows. In section 2, we provide a brief overview of recent debt models, then convey an intuitive outline of our contribution. In section 3, relying on the key insights of the theory of Lévy processes which allows one to split output into a Brownian and a Poisson process, we develop a simplified model. We interpret "trembling times" as shocks generated by Poisson jumps; in our model, they are those which have the potential to generate default. We demonstrate that Brownian shocks instead do not have the property of triggering default events. In a continuous time setting, we show that an optimizing social planner should always absorb Brownian shocks so as to avoid default. This allows us to discriminate among two key causes of debt crises. One is the failure to adjust in real time to a smooth shock, the solution to which being to have a more efficient monitoring of intra-annual deficit. The second is the challenge of a discontinuous shock, which is where the core problem comes from. We argue that previous models' difficulties in replicating default owe much to the lack of understanding of this distinction. In section 4, we then simulate the full-fledged version of the model using the standard assumptions of emerging economies. In section 5, we recast the model in the European setting and draw some policy conclusions for the eurozone. Section 6 concludes.

2 Calibrating sovereign debt models

Calibrated models of sovereign debt owe much to the papers by Aguiar and Gopinath (2006), Arellano (2008) and Mendoza and Yue (2011), which followed in the (earlier) tradition of Eaton and Gersovitz (1981), Cohen and Sachs (1986) and Bulow and Rogoff (1989).

Their framework is as follows. An indebted country can decide to default, paying as a consequence a (lump-sum) cost. This cost sets the upper limit of debt, as lenders attempt to monitor the risks. These models have successfully reproduced key business cycle correlations regarding aggregate spending and balance of payments in particular. The problem encountered by these models however, is that they meet great difficulties in calibrating reasonable debt thresholds and probabilities of default at the same time. Table 1 summarizes the key results obtained by several recent papers along these dimensions.

Table 1: Overview of mean debt-to-GDP ratios and default probabilities in the literature

Paper	Main feature	Debt-to-GDP mean ratio (%, annual)	Default probability (%, annual)
Arellano (2008)	Non-linear default cost	1	3.0
Aguiar and Gopinath (2006)	Shocks to GDP trend	5	0.9
Cuadra and Sapriza (2008)	Political uncertainty	2	4.8
Fink and Scholl (2011)	Bailouts and conditionality	1	5.0
Yue (2010)	Endogenous recovery	3	2.7
Mendoza and Yue (2011)	Endogenous default cost	6	2.8
Hatchondo and Martinez (2009)	Long-duration bonds	5	2.9
Benjamin and Wright (2009)	Endogenous recovery	16	4.4
Chatterjee and Eyigungor (2011)	Long-duration bonds	18	6.6

Most papers report the debt-to-GDP ratio using GDP measured at a quarterly frequency; we choose to use GDP measured at an annual frequency, since this is the convention used by policymakers and in the policy debate. For Aguiar and Gopinath (2006), we report results for their model II (with shocks to GDP trend). For Arellano (2008) and Aguiar and Gopinath (2006), the reported values come from Hatchondo et al. (2010) who resimulate these models using more precise numerical techniques. For Hatchondo and Martinez (2009), the reported values are those obtained for their λ parameter equal to 20%.

Before discussing the results of these papers, one should note the improbably high discount factor that some models have to rely on to sustain their equilibrium. For example, Yue (2010) and Aguiar and Gopinath (2006) set respective values of 0.72 and 0.8 for the (quarterly!) discount factor. This high impatience helps to generate frequent defaults and a desire to hold debt, but it is unrealistically high, even when accounting for political instability. Others, like

Arellano (2008), Benjamin and Wright (2009) or Chatterjee and Eyigungor (2011) use values close to 0.95, which is more realistic. We will use this last value in our simulations (see section 4.4).

In order to fit the conventional wisdom of markets and international financial institutions, one would want a model that could predict:

- *Threshold debt levels in the vicinity of 40% of yearly GDP.* The mean debt-to-GDP ratio in 2009 was 42% across countries, according to our computations using World Bank (2010). Note that World Bank (2004) classifies as "severely indebted" countries with a debt-to-GNI ratio above 80%, and as "moderately indebted" countries with a ratio above 48%: our target of 40% is therefore in the lower end of the range of interest; in section 5.3 we show how to reach higher levels of debts.

- *Annual default probabilities in the range of 3%.* Yue (2010) reports that the average default rate of Argentina since 1824 is 2.7%. Benjamin and Wright (2009) estimate an average default rate across countries of 4.4% for the period 1989–2006. In the data collected by Cohen and Valadier (2011) over the period 1970–2007, which includes "soft defaults" such as IMF loans, an even higher probability of default of about 7% is documented. We stick to the 3% target preferred by most papers, to make the comparison easier.

Even though many papers listed in Table 1 reach the target in terms of default probabilities, they all fail with respect to the sustainable debt ratios; the two best results along this dimension are Benjamin and Wright (2009) and Chatterjee and Eyigungor (2011) who respectively reach debt levels of 16% and 18% of yearly GDP.[1]

We now turn to the task of proposing a quantitative sovereign debt model that matches the two stylized facts regarding debt levels and default probabilities, with a minimal departure from the canonical model. Our modifications hinge on the following arguments:

1. The cost of default in the models found in the literature is too low to be true. Based on historical averages they typically assume that it usually takes two and a half years of financial autarky to pay for the consequences of a default. We revise upwards this cost by adding one first trick: post default countries are rarely debt free. As documented by Sturzenegger and Zettelmeyer (2007) and Cruces and Trebesch (2011), creditors do capture a recovery value of debt after default. Cohen (1992) also showed that post-Brady recovery values were quite significant in the eighties. Even in the most celebrated default incident, Argentina, creditors clawed back about one third of their claims. By taking into account the post-default recovery value, we significantly raise the upper limits of debt. Although the point is often acknowledged in the literature,[2] it has seldom been theorized

[1]Note that Benjamin and Wright (2009) argue that the historical average of the yearly debt-to-GDP ratio over their data set is precisely 18%. They choose their calibration in order to match that target and are able to do so using a relatively low value for the output cost of default. Their model may therefore be able to reach higher levels of debt while still keeping the output cost at a reasonable value, but we did not check that.

[2]See, for example, Hatchondo and Martinez (2009, footnote 15).

or calibrated in previous models (Yue (2010) and Benjamin and Wright (2009) are two notable exceptions).

2. In framing our model, we add another critical ingredient, building on the theoretical inspiration of the Lévy stochastic processes. These processes can be roughly defined as the generalization of random walks to continuous time. More precisely, any stochastic process in continuous time with stationary and independent increments is a Lévy process. The Lévy-Itô decomposition states that any Lévy process is essentially the sum of two components: a Brownian process and a compound Poisson process. As we shall demonstrate, Brownian processes do not have the ability to generate defaults. Instead they function as in deterministic models; whatever the cost of default, the corresponding probability of default is zero. Default must depend on exogenous shocks, creating discrete jumps in the wealth of a nation. Such shocks are well-represented by the Poisson process.

3. This insight allows us to add the critical change that we alluded to in the introduction, namely that crises almost always precede the decision to default, rather than the other way round, as assumed by the literature. We use the Poisson component as generating the "trembling times" during which a transitory crisis hits the country. They correspond to the episodes when default becomes possible.

3 A Lévy driven model of default

In this section we develop a very stylized model of sovereign default to demonstrate that the properties of these models dramatically change with the type of stochastic process assumed for output. Our discussion is based on the theory of Lévy processes, that we briefly introduce below. Building on this intuition, in section 4 we will present a quantitative sovereign debt model designed to match the quantitative targets identified in section 2.

3.1 Lévy processes

3.1.1 Definition and key properties

A Lévy process is a stochastic process that has stationary and independent increments.[3] It is the generalization in continuous time of random walks in discrete time. As the Lévy-Itô decomposition shows, a Lévy process is the sum of three terms: a Brownian process with deterministic drift; a compound Poisson process; and a third term which intuitively represents an infinite sum of infinitesimally small jumps. We ignore the third term since it is more a mathematical curiosity, and thus consider a process which is simply the sum of a Brownian process with drift and of a compound Poisson process.

In order to simplify the presentation, we shall consider a discrete time approximation of this process, calling h the time horizon that we shall shrink to zero in the analysis. We first examine

[3] In addition to these two basic properties, there are also technical regularity conditions. See for example Applebaum (2004) for more details.

the two limiting cases at hand.

3.1.2 The Brownian case

A first simple case is when the law of motion of (the log of) output corresponds (asymptotically) to a discrete time version of a Brownian process:

$$Q_{t+h} = \begin{cases} e^{\sigma\sqrt{h}} Q_t & \text{with probability } \frac{1}{2} + \frac{\mu}{2\sigma}\sqrt{h} \\ e^{-\sigma\sqrt{h}} Q_t & \text{with probability } \frac{1}{2} - \frac{\mu}{2\sigma}\sqrt{h} \end{cases}$$

As h goes to zero, this process converges towards a geometric Brownian process of "percentage drift" μ and "percentage volatility" σ.

3.1.3 The Poisson case

The second simple case is when the law of motion of (the log of) output corresponds (asymptotically) to a discrete time version of a compound Poisson process:

$$Q_{t+h} = \begin{cases} Q_t & \text{with probability } e^{-p_0 h} \\ k\,\tilde{m}_t\, Q_t & \text{with probability } 1 - e^{-p_0 h} \end{cases}$$

where $\tilde{m}_t$ is a stationary process. For the purpose of our economic analysis, we shall assume that the support of $\tilde{m}_t$ is included in the interval $(0, 1)$, and therefore represents a "malus:" with an infinitesimal probability, the country loses a non infinitesimal amount of output. The term $k = \frac{p_0 h}{1 - e^{-p_0 h}}$ is a technical artefact of the discretization,[4] and it goes to 1 as h goes to 0.

As h goes to zero, Q_t converges towards a geometric compound Poisson process. More precisely, $\ln Q_t$ converges towards a compound Poisson process whose rate is p_0 and whose jump size distribution equals the stationary distribution of $\tilde{m}_t$.

3.1.4 General form

A discrete time approximation of a Lévy process can be embedded in the following model:

$$Q_{t+h} = \begin{cases} e^{\sigma\sqrt{h}} Q_t & \text{with probability } \left(\frac{1}{2} + \frac{\mu}{2\sigma}\sqrt{h}\right) e^{-p_0 h} \\ e^{-\sigma\sqrt{h}} Q_t & \text{with probability } \left(\frac{1}{2} - \frac{\mu}{2\sigma}\sqrt{h}\right) e^{-p_0 h} \\ k\,\tilde{m}_t\, Q_t & \text{with probability } 1 - e^{-p_0 h} \end{cases}$$

[4] To be precise, k corresponds to the expectation of the number of shocks of the continuous Poisson process during a period of length $p_0 h$, conditional on the fact that there is at least one shock in this interval.

3.2 Financial markets

3.2.1 Financial environment

The world financial markets are characterized by an instantaneous, constant, riskless rate of interest r. Lenders are risk-neutral and subject to a zero-profit condition by competition. We further suppose that debt is short-term and needs to be refinanced every year.

The timing of events unfolds as follows. First assume that the country has incurred a debt obligation D_t due at time t, and has always serviced it in full in previous years. At the beginning of period t, the country learns the value of its output Q_t. It then either defaults on its debt or reimburses it. If the debt is reimbursed in full, the country can contract a new loan, borrowing L_t, which must be repaid at time $t+h$ in the amount of D_{t+h}. Note that the implicit instantaneous interest rate is equal to $\frac{\log(D_{t+h}/L_t)}{h}$. Such financial agreements being concluded, the country eventually consumes, in the event it services its debt in full:

$$C_t = Q_t + L_t - D_t.$$

Alternatively, in the event of a debt crisis the country may default (see below). This occurs when output is too low to allow the country to service its debt. Call π_{t+h} the probability of default at time $t+h$, from the perspective of date t.

The zero-profit condition for creditors may be written as:

$$L_t e^{rh} = D_{t+h}(1 - \pi_{t+h}) \tag{1}$$

Note that we have assumed that in case of default, the investors recover nothing. We will relax this assumption further in the paper.

3.2.2 Default

At any time t, a country that has accumulated a debt D_t may decide to default upon it. When it does so, we assume that the country suffers a penalty $\lambda \in [0, 1)$ on output as a consequence of the crisis. This penalty is captured by no one and is therefore a net social loss. We call Q_t^d the post-penalty value of income (which we distinguish from output) and for the time being simply write:

$$Q_t^d = (1 - \lambda)Q_t.$$

As another cost, we assume that the country is subject to financial autarky, being unable to borrow again later on.[5] We then write consumption as:

$$C_t^d = Q_t^d = (1 - \lambda)Q_t.$$

[5] A milder form of a sanction would be, more realistically, that the country is barred from the financial market for some time only, as in Aguiar and Gopinath (2006). We explore this less demanding assumption in the model of section 4.

3.3 Preferences and equilibrium

3.3.1 Preferences

The decision to default or to stay on the financial markets involves a comparison of two paths that implies expectations over the entire future. In order to address this problem, we assume that the country seeks to solve:

$$J^*(D_t, Q_t) = \max_{\{C_{t+kh}\}_{k\geq 0}} \mathbb{E}_t \left\{ \sum_{k=0}^{\infty} e^{-\rho kh} u(C_{t+kh}) \right\}$$

where ρ is the instantaneous rate of preference for the present. D_t can be negative if the country builds up foreign assets. We assume that utility is isoelastic, of the form:

$$u(C_t) = \frac{C_t^{1-\gamma}}{1-\gamma}$$

We call:

$$J^d(Q_t) = \mathbb{E}_t \left\{ \sum_{k=0}^{\infty} e^{-\rho kh} u(Q_{t+kh}^d) \right\}$$

the post-default level of utility, which becomes by definition independent of debt, and to which the country is nailed down in case of servicing difficulties. If it were to stay current on its debt obligation, the country would obtain:

$$J^r(D_t, Q_t) = \max_{L_t, D_{t+h}} \left\{ u\left(Q_t + L_t - D_t\right) + e^{-\rho h} \mathbb{E}_t[J^*(D_{t+h}, Q_{t+h})] \right\}$$

subject to the zero profit condition (1).

When comparing how much it can get by staying on the markets and the post-default level of welfare, the country chooses its optimum level:

$$J^*(D_t, Q_t) = \max\{J^r(D_t, Q_t), J^d(Q_t)\}$$

3.3.2 Recursive equilibrium

We define a recursive equilibrium in which the government does not have commitment and in which the various agents act sequentially.

The aggregate state of the model is $s = (\delta, D, Q)$, where δ is past credit history (equal to 1 if country is barred from financial markets, 0 otherwise), D is the stock of debt due in the current period (necessarily equal to zero if $\delta = 1$) and Q is current GDP.

Definition 1 (Recursive equilibrium in Lévy model) *The recursive equilibrium for this economy is defined as a set of policy functions for the (i) government's default decision $\tilde{\delta}'(s)$; (ii) government's decision for tomorrow's debt holding $\tilde{D}'(s)$; and (iii) investor's supply of borrowing $\tilde{L}(s, D')$ such that:*

- *taking as given the investor's policy function, the default decision* $\tilde{\delta}'(s)$ *and decision for tomorrow's debt holding* $\tilde{D}'(s)$ *satisfy the government optimization problem:*

$$\tilde{\delta}'(s) = \begin{cases} 1 & \textit{if } \delta = 1 \textit{ (default in the past) or } J^d(Q) > J^r(D,Q) \textit{ (default now)} \\ 0 & \textit{otherwise} \end{cases}$$

$$\tilde{D}'(s) = \begin{cases} \arg\max_{D'} \left\{ u(Q - D + \tilde{L}(s,D')) + e^{-\rho h}\mathbb{E}_Q[J^*(D',Q')] \right\} & \textit{if } \tilde{\delta}'(s) = 0 \\ 0 & \textit{otherwise} \end{cases}$$

where:

$$\begin{aligned} J^r(D,Q) &= \max_{D'} \left\{ u(Q - D + \tilde{L}(s,D')) + e^{-\rho h}\mathbb{E}_Q[J^*(D',Q')] \right\} \\ J^d(Q) &= u((1-\lambda)Q) + e^{-\rho h}\mathbb{E}_Q J^d(Q') \\ J^*(D,Q) &= \max\{J^r(D,Q), J^d(Q)\} \end{aligned}$$

- *taking as given the government's default decision function, the investor's policy function* $\tilde{L}(s,D')$ *satisfies the zero profit constraint:*

$$\tilde{L}(s,D') = e^{-rh}\left[1 - \mathbb{E}_Q\tilde{\delta}'(\delta,D',Q')\right]D'$$

Note that the formulation for lending decision by the investors prevents multiple equilibria in the interest rate, as noted by Chamon (2007).

3.3.3 Equilibrium in the Brownian case

Let us first investigate the nature of the equilibrium in the Brownian case. In this section we assume that output follows a discretized version of the geometric Brownian motion, as described in section 3.1.2. We have the following result:

Proposition 2 *In the Brownian case, if* $h < \frac{1}{\left(\frac{\mu}{\sigma}+4\sigma\right)^2}$, *only two cases are possible (for a given initial value of the debt-to-GDP ratio):*

- *the country immediately defaults;*
- *the country never defaults (whatever the future path of output).*

Proof. See the appendix. ■

In other words, either the debt is already too high and the country immediately defaults, or it will never do so. The intuition is straightforward: because of the continuous nature of growth, the country can always adjust to shocks and the creditors monitor it. Brownian noise is not different from deterministic fluctuations.

One empirical question that this result points to is whether, in the real world, decisions are indeed taken continuously. The Greek case provides an instructive example. When Prime Minister Papandreou took office, he realized that the deficit he inherited was much larger than he originally thought. Having been given the wrong information in the beginning inevitably delayed the right policy choices on his part. Perhaps, the lag in evaluating the situation is responsible for the crisis. We return to this question in the empirical analysis that we submit below. One can nevertheless compute the length of the time window h^* during which a policymaker can prevent crises triggered by Brownian shocks. For reasonable parameterizations of the model, one has a time window of about one quarter (this would be roughly the case when μ/σ is near one). For more volatile economies, say when $\mu = 2\%$ and $\sigma = 3\%$ in quarterly frequency, the time window is about 5 months. A paradox here is that the more volatile an economy is, the more time a policymaker has to react to the shocks.

3.3.4 Equilibrium in the Poisson case

Let us now investigate the nature of the equilibrium in the Poisson case. In this section we assume that output follows a discretized version of the geometric compound Poisson process, as described in section 3.1.3. We arrive at the following result:

Proposition 3 *In the Poisson case, the probability of default between dates t and $t+1$ is inferior to $1 - e^{-p_0}$.*

Proof. By lemma 8, default never happens in the good state of nature. So the probability of not having a default between dates t and $t+h$ is superior to $e^{-p_0 h}$. Therefore the probability of not having a default between dates t and $t+1$ is superior to e^{-p_0} (using the independence of growth shocks between periods). ■

There are cases in which the probability of not having a default at each period is exactly equal to e^{-p_0}. Consider the extreme case where the country totally ignores the future ($\rho = 0$). The default threshold (expressed as a debt-to-GDP ratio) is clearly $d^* = \lambda$. Since, by lemma 8, default never happens in the good state of nature, the country will always choose the maximum debt level conditional to not defaulting in the good state (*i.e.* $D_{t+1} = \lambda Q_t$). This means that the country will default in the bad state, and therefore the probability of default at each period is equal to the probability of moving to the bad state: e^{-p_0}.

The comparison between the Brownian and Poisson cases is straightforward. When the economy is smooth, countries can continuously adjust their debt levels and never default. Obviously, when the economy is disrupted by a Poisson shock, default becomes a possibility of probability p_0 (per unit of time).

3.3.5 Comparison with Aguiar and Gopinath (2006)

Aguiar and Gopinath (2006) suggested an interesting line of reasoning, which can be summarized as follows: in emerging countries, growth rates (not output) are highly volatile. When growth is expected to be high, this raises the willingness to borrow (as fast growth raises debt by raising the permanent income) and therefore it also raises the risk of a debt overhang. Yet the outcome is less satisfactory than expected as shown in Table 1: despite very high discount factors, the debt levels remain quite low. Aguiar and Gopinath (2006) fail to recognize that the volatility of the growth rate is not enough to trigger the risk of default; what is really needed is a discontinuous jump in the parameters that switch the probability of default.

3.4 Numerical results

In order to get a better understanding of the properties of the models presented in this section, we perform numerical exercises with calibrated versions of those models and analyze the sensitivity of the results to the period duration h. Table 2 shows the calibration that we use for this exercise.

Table 2: Calibration of discretized Lévy models

Risk aversion	γ	2
Discount rate	ρ	log(0.8)
Riskless interest rate	r	log(1.01)
Loss of output in autarky (% of GDP)	λ	0.5%
Drift of Brownian process	μ	1%
Volatility of Brownian process	σ	2.2%
Period size for which Brownian and Poisson are observationally equivalent	h_0	4

Quarterly frequency.

For the Brownian case (section 3.1.2), we set $\sigma = 2.2\%$, $\mu = 1\%$. The corresponding threshold for h under which defaults are impossible in this model, as given by proposition 2, is $h^* \simeq 3.4$. For the Poisson case, we slightly modify the process described in section 3.1.3 to allow for a positive trend. We set:

$$Q_{t+h} = \begin{cases} g^+(h)Q_t & \text{with probability } e^{-p_0 h} \\ g^-(h)Q_t & \text{with probability } 1 - e^{-p_0 h} \end{cases}$$

where g^+ and g^- are functions of h which we define below. Note that we don't allow the jump size to be stochastic, since we don't need it for the purpose of this exercise. We calibrate the Poisson process so that:

- for some specific value $h = h_0$, the Brownian and Poisson are observationally equivalent (*i.e.* same probabilities for up and down moves, same jump sizes);
- for all values of h, the two processes have the same average growth;
- the magnitude of the output loss in case of a bad Poisson shock does not depend on h.

These constraints translate into the following relationships, which identify p_0, $g^+(h)$ and $g^-(h)$:

$$
\begin{aligned}
e^{-p_0 h_0} &= \frac{1}{2} + \frac{\mu}{2\sigma}\sqrt{h_0} \\
\forall h,\ e^{-p_0 h} g^+(h) + (1 - e^{-p_0 h}) g^-(h) &= \left(\frac{1}{2} + \frac{\mu}{2\sigma}\sqrt{h}\right) e^{\sigma\sqrt{h}} + \left(\frac{1}{2} - \frac{\mu}{2\sigma}\sqrt{h}\right) e^{-\sigma\sqrt{h}} \\
\forall h,\ g^-(h) \frac{p_0 h}{1 - e^{-p_0 h}} &= e^{-\sigma\sqrt{h_0}} \frac{p_0 h_0}{1 - e^{-p_0 h_0}}
\end{aligned}
$$

Finally, the value chosen for h_0 is one year.

Table 3 reports the results from the numerical simulations of these two models for various values of h, which range approximately from one year to one week.

Table 3: Moments of discretized Lévy processes for various period durations

Period duration (h, in quarters)	4	2	1	0.33
Discretized Brownian process				
Default threshold (debt-to-GDP, quarterly, %)	48.4	51.9	68.8	79.3
Default probability in 10 years (%)	35.7	0.0	0.0	0.0
Discretized Poisson process				
Default threshold (debt-to-GDP, quarterly, %)	48.4	47.7	47.6	47.5
Default probability in 10 years (%)	35.1	34.6	34.3	40.0

The processes are parametrized as described in section 3.4. The solution to the detrended model is approximated using value function iteration on a grid of 25 points for the debt-to-GDP ratio. Moments are obtained by averaging over 1,000 simulations of a length of 10 years.

One can see that proposition 2 is verified empirically: for $h < h^*$, the Brownian model has zero default; conversely, the Poisson case still exhibits defaults as h goes to zero.

4 The full-fledged model

Building on the ideas presented in the previous section, we now construct a new model of sovereign debt. It shares the core features of the model presented by Aguiar and Gopinath (2006) and incorporates a few key ingredients which enable it to perform better with respect to default frequency and average debt-to-GDP ratio.

4.1 The stochastic process

4.1.1 General form

We assume that the output of the country is described by the following stochastic process:

$$\frac{Q_t}{Q_{t-1}} = g_t = e^{y_t} + z_t$$

where y_t and z_t are two stochastic processes. The y_t process is a standard auto-regressive process:

$$y_t - \mu_y = \rho_y(y_{t-1} - \mu_y) + \varepsilon_t \qquad \varepsilon_t \rightsquigarrow \mathcal{N}(0, \sigma_y^2) \qquad \mu_y = \log(\mu_g) - \frac{\sigma_y^2}{2(1-\sigma_y)^2}$$

It is such that $\mathbb{E}(e^{y_t}) = \mu_g$. This component of the growth process is identical to the process assumed by Aguiar and Gopinath (2006). From the perspective of the previous section, it embodies the "Brownian" component of the growth process. We call it the *B component* of the process. The second term, z_t, embodies the "Poisson" component of the growth process. We call it the *P component*. It evolves according to the following law of motion:

$$z_t = \rho_z z_{t-1} + \eta_t$$

where the behavior of the innovation η_t depends on the state of the economy, which we now describe.

1. So long as the **country has not defaulted**, we assume that the economy is in either of the following two states of nature: "normal times" or "trembling times."

   - In "normal times," the innovation follows:

     $$\eta_t = \begin{cases} 0 & \text{with probability } 1-p \\ -\mu_z & \text{with probability } p \end{cases} \tag{2}$$

     This variable embodies the risk of a low-probability but violent negative confidence shock, in the spirit of the Poisson process. We assume that when this shock occurs, in addition to the negative growth shock, the economy moves to the "trembling times" state: the markets have lost confidence, this lack of confidence has real negative consequences, but the markets are willing to revise their judgment if the country behaves responsibly and does not default.

   - The "trembling times" correspond to a situation where the markets have doubts concerning the economy's strength. These doubts have real negative consequences for the economy, but there is a possibility for confidence to be restored and for real negative effects to be reverted. In other words, the crisis is "reversible." In that case, the innovation follows:

     $$\eta_t = \begin{cases} 0 & \text{with probability } 1-q \\ \mu_z & \text{with probability } q \end{cases} \tag{3}$$

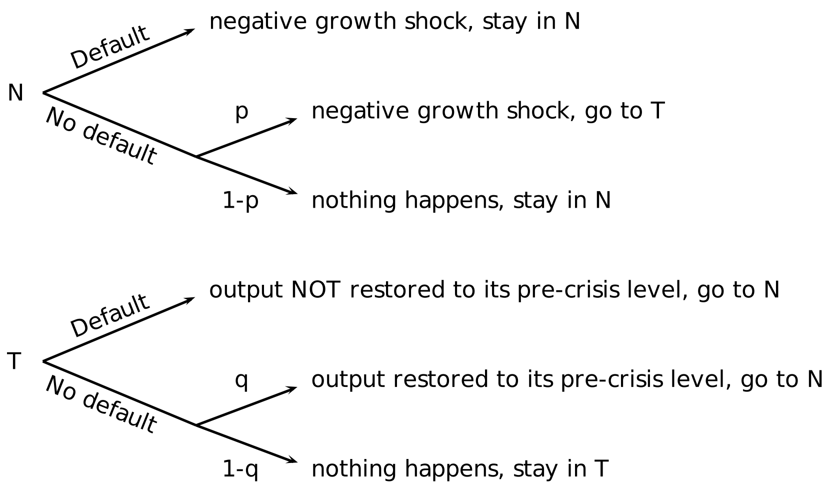
     The idea here is that the country can recover from the tremor of the confidence shock with a probability q (which is typically much higher than p). Once the country moves out of the confidence crisis, growth is "restored" to its pre-crisis level and the economy returns to “normal times.”

2. **Following a default**, the economy evolves as follows:

   - If the country defaults during “normal times,” then the country suffers the same negative growth shock that it would have undergone in case of a confidence crisis (*i.e.* $\eta_{n+1} = -\mu_z$). However after default it remains in “normal times,” which means that the output loss is not reversible (since the country has already defaulted).
   - If the country defaults during “trembling times,” then it loses the ability to restore output to pre-crisis levels. The country does not suffer an additional negative growth shock (it has already undergone one), but the economy returns to “normal times,” which means that it can no longer expect that positive news may end the crisis. Since the country has defaulted, the doubts of the investors about the strength of the economy have been confirmed and there is no reason to revert the shock.

In other words, a confidence shock acts like a “trembling hand” event: it shakes the economy for a while. If during such an episode the country defaults, then the trembling shock becomes permanent and no recovery takes place. When instead a country defaults while being in good times, default creates on its own a confidence shock from which the economy does not recover (except for the fact that the growth loss dies out naturally over time). The whole process is summarized in Figure 1.

Figure 1: Law of motion of the economy



N is “normal times”, T is “trembling times”.

It should be noted here that the process assumed for the output shares some features with the Markov-switching model for GDP introduced by Hamilton (1989). Like Hamilton’s, our model has an underlying state variable corresponding to the current regime, and the mean of the growth rate is different across regimes. But there are two critical differences. First, in our model, the growth rate is only temporarily lowered in the "trembling" state, even if the economy stays in that state for a long time (because z_t is a mean reverting process), while in Hamilton's model, the growth rate is permanently lowered as long as the economy is in the bad state. More importantly, the switch between the two regimes is partly endogenous in our model (it can be triggered by a default decision), while it is entirely exogenous in Hamilton's model.

4.2 The other costs of default

Beyond the output costs that we just described, a defaulting country is subject, as in the previous models, to the following costs. First, creditors manage to inflict, on top of the trembling shock, a penalty λ (which they do not monitor). Furthermore, they impose financial autarky on the debtor, at least for a while, so that:

$$C_t^d = Q_t^d = (1-\lambda)Q_t$$

as long as the country is in default.

Following Arellano (2008) and others we assume that a defaulting country can return to financial markets after a while. We call x the probability of a settlement at a given period. When the settlement occurs, the penalty λ is lifted (but not the effect of the trembling shock). Once a settlement is reached, debt is not canceled; it is only written down to a level consistent with the post-default level of output and the historical data on post-default haircuts (see more on this in section 4.4). We call V the settlement value of post default debt.

The pair (x, V) (the duration of financial autarky after a default and the post-default recovery value) have been modeled by Yue (2010) and Benjamin and Wright (2009) as the endogenous outcome of a bargaining process following a default. Contrary to these authors, we assume that the recovery value is not a function of past debt. The idea is simply that, after a default, prior commitments become irrelevant.

Also note that if the recovery value V were too high (for example greater or equal to the debt threshold above which the country defaults), then the resulting model would be conceptually equivalent to a model where no settlement is ever reached after a default (*i.e.* where $x = 0$).[6]

4.3 The equilibrium

We define a recursive equilibrium in a similar fashion as in section 3.3.2. The government does not make decisions under commitment, and the various agents act sequentially in response to a state as defined below.

Definition 4 (State of the economy) *The state of the economy is:*

$$s = (\delta, \theta, D, y, z, Q)$$

[6] Note that in our framework the two options are not strictly equivalent because, in the case of a high V, the country will repeatedly pay the negative growth shock that occurs after a default. In the $x = 0$ case, the country pays the negative growth shock only once.

where δ is past credit history (equal to 1 if the country is barred from financial markets, 0 otherwise), θ is equal to N in "normal times" and T in "trembling times," D is the stock of debt due in the current period (necessarily equal to zero if $\delta = 1$), y and z characterize current GDP growth as described in section 4.1, and Q is current GDP level.

Starting from a state $s = (\delta, \theta, D, y, z, Q)$, the next state $s' = (\delta', \theta', D', y', z', Q')$ is defined by the following law of motion:

- The default history evolves according to the corresponding policy function (see definition 5):

$$\delta' = \tilde{\delta}'(s)$$

- θ' and z' are jointly determined according to the following table (which is the mathematical reformulation of Figure 1):

	$\tilde{\delta}'(s) = 0$ (Repayment)	$\tilde{\delta}'(s) = 1$ (Default)
$\theta = N$ (Normal)	$\begin{cases} \theta' = N, z' = \rho_z z & \text{with prob. } 1-p \\ \theta' = T, z' = \rho_z z - \mu_z & \text{with prob. } p \end{cases}$	$\theta' = N, z' = \rho_z z - \mu_z$
$\theta = T$ (Trembling)	$\begin{cases} \theta' = T, z' = \rho_z z & \text{with prob. } 1-q \\ \theta' = N, z' = \rho_z z + \mu_z & \text{with prob. } q \end{cases}$	$\theta' = N, z' = \rho_z z$

- The B component of growth evolves according to the law described in section 4.1:

$$y' = \mu_y + \rho_y(y - \mu_y) + \varepsilon' \qquad \varepsilon' \rightsquigarrow \mathcal{N}(0, \sigma_y^2)$$

- GDP is augmented by its growth rate, as described in section 4.1:

$$Q' = Q(e^{y'} + z')$$

- The level of debt evolves according to the corresponding policy function (see definition 5):

$$D' = \tilde{D}'(s)$$

Definition 5 (Recursive equilibrium) *The recursive equilibrium for this economy is defined as a set of policy functions for (i) the government's default decision $\tilde{\delta}'(s)$, (ii) the government's decision for tomorrow's debt holding $\tilde{D}'(s)$, and (iii) the investor's supply of lending $\tilde{L}(s, D')$ such that:*

- *Taking as given the investor's policy function, the default policy function, $\tilde{\delta}'(s)$, and decision for tomorrow's debt holding, $\tilde{D}'(s)$, satisfy the government optimization problem:*

$$\tilde{\delta}'(s) = \begin{cases} 1 & \textit{if } \delta = 1 \textit{ (default in the past) or } J^{d,\theta}(y,z,Q) > J^{r,\theta}(D,y,z,Q) \textit{ (default now)} \\ 0 & \textit{otherwise} \end{cases}$$

$$\tilde{D}'(s) = \begin{cases} \arg\max_{D'} \{u(Q - D + \tilde{L}(s, D')) & \textit{if } \tilde{\delta}'(s) = 0 \textit{ and } \theta = N \\ \qquad +\beta\, \mathbb{E}_y\big[(1-p)J^{*,N}(D', y', \rho_z z, Q') & \\ \qquad\qquad +p\, J^{*,T}(D', y', \rho_z z - \mu_z, Q')\big]\} & \\ \arg\max_{D'} \{u(Q - D + \tilde{L}(s, D')) & \textit{if } \tilde{\delta}'(s) = 0 \textit{ and } \theta = T \\ \qquad +\beta\, \mathbb{E}_y\big[(1-q)J^{*,T}(D', y', \rho_z z, Q') & \\ \qquad\qquad +q\, J^{*,N}(D', y', \rho_z z + \mu_z, Q')\big]\} & \\ 0 & \textit{otherwise} \end{cases}$$

where:

$$\begin{aligned} J^{r,N}(D,y,z,Q) &= \max_{D'} \{u(Q - D + \tilde{L}(s,D')) + \beta\, \mathbb{E}_y\big[(1-p)J^{*,N}(D', y', \rho_z z, Q') \\ &\qquad +p\, J^{*,T}(D', y', \rho_z z - \mu_z, Q')\big]\} \\ J^{r,T}(D,y,z,Q) &= \max_{D'} \{u(Q - D + \tilde{L}(s,D')) + \beta\, \mathbb{E}_y\big[(1-q)J^{*,T}(D', y', \rho_z z, Q') \\ &\qquad +q\, J^{*,N}(D', y', \rho_z z + \mu_z, Q')\big]\} \\ J^{d,T}(y,z,Q) &= u((1-\lambda)Q) + \beta\, \mathbb{E}_y \Big[(1-x)J^{d,T}(y', \rho_z z, Q') + x\, J^{r,N}(V(Q'), y', \rho_z z, Q')\Big] \\ J^{d,N}(y,z,Q) &= u((1-\lambda)Q) + \beta\, \mathbb{E}_y \Big[(1-x)J^{d,T}(y', \rho_z z - \mu_z, Q') \\ &\qquad\qquad +x\, J^{r,N}(V(Q'), y', \rho_z z - \mu_z, Q')\Big] \\ J^{*,\theta}(D,y,z,Q) &= \max\{J^{r,\theta}(D,y,z,Q), J^{d,\theta}(y,z,Q)\} \end{aligned}$$

and $V(Q')$ is the recovery value after a default (as a function of GDP)

- *Taking as given the government's default policy function, the investor's policy function $\tilde{L}(s, D')$ satisfies the zero profit constraint:*

$$(1+r)\tilde{L}(s, D') = D' + \mathbb{E}_{y,\theta}\Big\{\tilde{\delta}'(0, \theta', D', y', z', Q')(V(Q') - D')\Big\}$$

where θ' and z' evolve according to the law of motion outlined above.

4.4 Calibration and simulation results

4.4.1 Benchmark calibration

Table 4 shows our benchmark calibration for the model. We use a quarterly frequency. Several parameters are set to the same value as in model II of Aguiar and Gopinath (2006): the risk aversion γ, the world riskless interest rate r, the loss of output in autarky λ, the probability of settlement after a default x, and the parameters of the B component of the growth process μ_g, σ_y, ρ_y. For the discount factor β, we use a substantially higher value than Aguiar and Gopinath (2006); our value still amounts to a 20% time rate preference in annualized terms, but such a level is arguably plausible for an impatient and debt hungry country.

The other parameters are more difficult to calibrate, since they do not directly appear in other models in the literature. The closest analogs come from regime switching models à la Hamilton (1989). We discuss this literature below when we come to European responses to the crisis (section 5). But clearly we do not want to map business cycles into risks of default. The "trembling times" that we have in mind are certainly less frequent than mere recessions as they are associated with significant disruptions of economic activity. Bearing this in mind, we set the probability p of entering the "trembling times" to 6% in annualized terms. This is above the unconditional historical probability of a sovereign default, since we anticipate that in some cases the country will successfully go through the crisis without defaulting. The value of 6% is also below the probability to enter into a recession in regime switching based models.[7] The probability q of exogenously leaving the "trembling times" is calibrated so that, on average, being hit by a trembling shock leads to a default half of the time; in section 4.4.2 we run sensitivity analysis exercises regarding this critical parameter. The P component of growth is calibrated so that, when the shock hits, the annualized growth rate goes down by 4 percentage points, and the effect of the shock is halved after 3 quarters. The debt recovery value after a default is calibrated so that the average haircut is 40%, as in the historical data (Benjamin and Wright, 2009; Cruces and Trebesch, 2011).

Table 4: Benchmark calibration

Risk aversion	γ	2
Discount factor	β	0.95
World riskless interest rate	r	1%
Probability of settlement after default	x	10%
Loss of output in autarky (% of GDP)	λ	2%
Probability of entering "trembling times"	p	1.5%
Probability of exiting "trembling times"	q	5%
Recovery value (% of yearly GDP)	V	25%
Size of trembling shock in the P component of growth	μ_z	1%
Auto-correlation of the P component of growth	ρ_z	0.8
Standard deviation of the P component of growth	σ_y	3%
Auto-correlation of the B component of growth	ρ_y	0.17
Mean gross growth rate (ignoring the P component)	μ_g	1.006

Quarterly frequency.

The model is solved using the endogenous grid method described in Villemot (2012). Table 5

[7] For example, Altug and Bildirici (2010) estimate that the annualized probability of entering a low-growth state between 1982 and 2009 is 22% for Mexico and 45% for Argentina. Over a longer period these estimates would have probably been lower, since the 80s and 90s were particularly difficult times for these countries.

reports the moments of the model simulated with the benchmark calibration. Like most models in the quantitative sovereign debt literature, our model is able to replicate some important stylized facts of the business cycle in emerging countries, such as a counter-cyclical current account, counter-cyclical spreads and a more volatile consumption than output. But unlike most models in the literature, it is able to sustain both a realistic frequency of defaults and a realistic indebtment level (close to 40% of *annual* GDP).

Table 5: Moments of benchmark model

	Benchmark	With no Poisson ($p = 0$)
Rate of default (%, per year)	2.50	0.26
Mean debt output ratio (%, annualized)	38.17	46.82
$\sigma(Q)$ (%)	4.45	4.42
$\sigma(C)$ (%)	6.04	6.89
$\sigma(TB/Q)$ (%)	2.63	3.47
$\sigma(R_s)$ (%)	0.57	0.18
$\rho(C, Q)$	0.92	0.89
$\rho(TB/Q, Q)$	-0.41	-0.49
$\rho(R_s, Q)$	-0.60	-0.41
$\rho(R_s, TB/Q)$	0.64	0.90

Parameters of the model are set to their benchmark values as in Table 4. The solution to the detrended model is computed using the endogenous grid method described in Villemot (2012). The policy function are interpolated using a cubic spline on a 3-dimensional grid of 10 points for y, z and the debt-to-GDP ratio. Moments are obtained by averaging over 500 simulated series of 1,500 points each, the first 1,000 of which are discarded. Q is GDP, C is consumption, TB/Q is trade balance over GDP, R_s is the spread. GDP, consumption, trade balance and spread are detrended with an HP filter of parameter 1600.

In the last column of this table we also report the moments of the model when all parameters are set to the benchmark calibration except for the probability p of a trembling shock coming from the P component of growth, which is set to zero. One can see that in this configuration defaults almost disappear. This shows the importance of the Poisson shock in this class of models. As another consequence, the volatility of spreads almost goes to zero since there is almost no risk of default.

4.4.2 Sensitivity analysis

We first investigate the sensitivity of the results to the probability q of exiting the "trembling times." As Figure 2 shows, three ranges appear. When q is high, no default ever takes place: the trembling shock is expected to be short lived, the country will not destroy the recovery with a default. At the other extreme, when q is low, the shock "pre-pays for the default." Although the country would not do it by on its own, the default now becomes the cheap option. In the intermediate case, the choice being made depends on when and how the shock occurs. When the economy is on a positive streak, default can be avoided, when instead the economy is already down, then default becomes more palatable.

Figure 2: Default probability as a function of probability q of exiting the "trembling times"

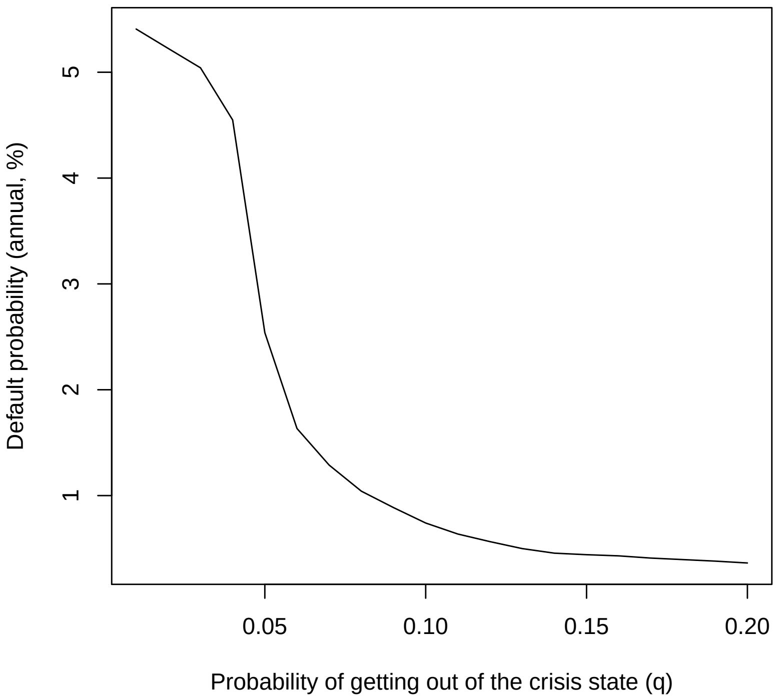


Figure 3 shows the sensitivity of the mean debt-to-GDP ratio to the recovery value V of the debt. The mean debt-to-GDP ratio is an increasing function of the recovery value: this was to be expected since a higher recovery value means that default is more costly, and therefore a higher level of debt can be sustained by the sovereign country. On this graph we also plot the line corresponding to a fixed 40% haircut: one can see that the recovery level consistent with this observed historical haircut is close to the 25% that we have chosen for our calibration.

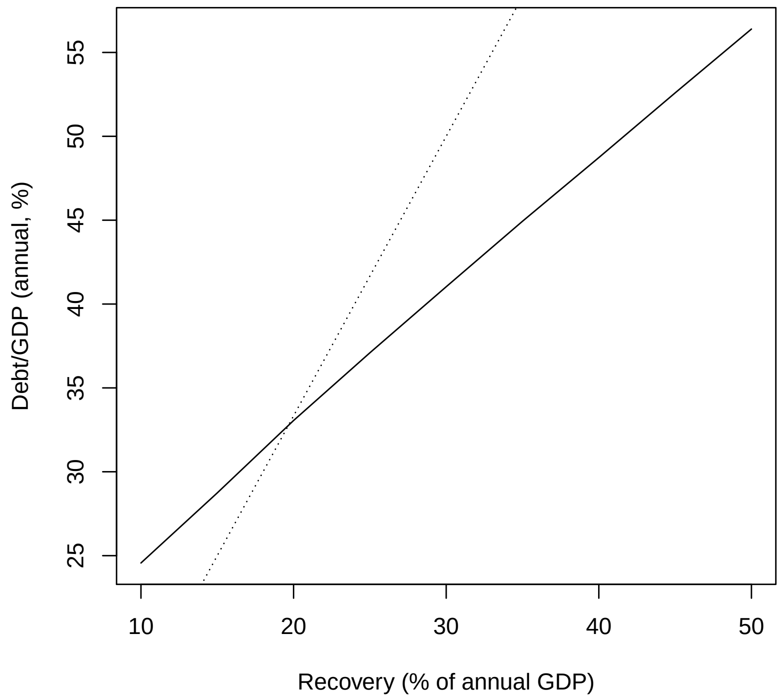
4.4.3 A self-fulfilling re-interpretation

We show in this section that, when q is low enough, it is possible to reinterpret our model in the spirit of self-fulfilling models pioneered by Cole and Kehoe (1996, 2000). Taking into account the possibility of a self-fullfilling effect is important because, as shown by Cohen and Villemot (2011), this effect plays a role in a significant minority of crises (around 20%).

In the cases when q is low enough, the trembling shock always triggers a default: this is so because "the default is prepaid" through the crisis. A self-fulfilling reinterpretation becomes possible, as outlined below.

Assume that markets fear a default anytime they see a sunspot. When markets starts anticipating a default on their own, assume that they create a negative wave which is expected to be long lasting, corresponding to low values of q. The country then defaults with probability one. The shock is self-fulfilling. The difference with the model that we presented above is that, in the self-fulfilling interpretation, the shock is triggered because of the fear of a default

Figure 3: Mean debt-to-GDP as a function of recovery value



The dotted line indicates the debt-to-GDP value corresponding to a 40% haircut.

rather than for reasons independent of the fear of default. But for low values of q the two are observationally equivalent.

5 Eurozone policies

5.1 Analysis at business cycle frequencies

As we already mentionned, our modelling strategy for the output process is close to that of Hamilton (1989). Let us now see what the consequences would be of plugging the parameter values estimated by the Markov-switching literature into our model.

The original model of Hamilton (1989) estimated on US data for the period 1952–1984 gives $p = 9.5\%$ and $q = 24.5\%$. Later, Goodwin (1993) has estimated a similar model on 8 advanced economies from the late 1950s/early 1960s to the late 1990s, and came up with values for p ranging from 1% to 9%, and for q ranging from 21% to 49%.[8] Table 6 reports default probabilities and mean debt-to-GDP ratios obtained with the model of section 4 for values of p and q lying in the estimated range for business cycles of advanced countries.

As we can see from this table, the most prominent fact exhibited by this exercise is that the risk of default at business cycle frequencies of advanced economies is negligible. The trembling times that we are talking about are therefore events which are less frequent and more severe

[8]We disregard the result for Italy, since the author labels it a pathological case and explains that a 3-state specification would probably better fit the data for that country.

Table 6: Model simulations using business cycle frequencies in advanced economies

Probability of entering "trembling times" (p, per quarter)	1%	1%	10%	10%
Probability of exiting "trembling times" (q, per quarter)	20%	50%	20%	50%
Rate of default (per year)	0.38%	0.27%	0.32%	0.29%
Mean debt output ratio (annualized)	45%	47%	43%	46%

The simulations are done using the model presented in section 4. The values tested for p and q correspond approximately to the extreme values estimated by Hamilton (1989) and Goodwin (1993) on 7 advanced economies over the postwar period. Other parameters of the model are set to their benchmark values as in Table 4.

downturns than business cycles downturns (such as a banking crises or a very severe recession). This is why we chose parameter values such that crises are less frequent (lower p) and longer lasting (lower q) than mere recessions.

5.2 Credit ceilings

Following the Greek crisis, the eurozone imposed a new set of stringent rules to avoid future crises. The idea of policymakers is to impose a tougher credit ceiling on eurozone countries in order to protect the zone from any risk of default. The following questions are therefore of major importance from a policy perspective: how low should the debt ceilings be to avert any risk of crisis? What are the welfare implications of these constraints?

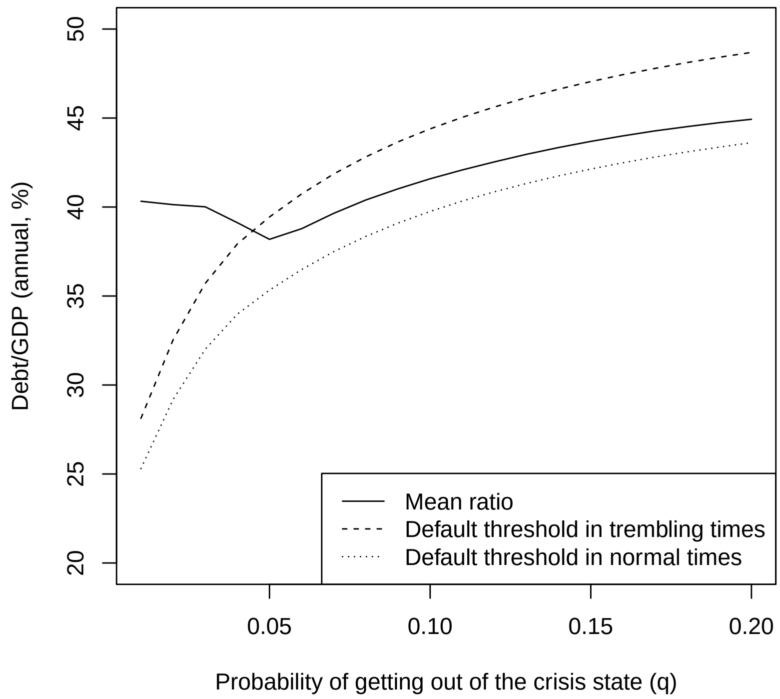
In order to address these questions, we have computed the levels of debt that are consistent with no default in either "normal" or "trembling" times in our model. Figure 4 summarizes the results as a function of the key parameter q. The solid line represents the mean debt-to-GDP ratio, the dotted line represents the maximum debt-to-GDP level under which there is no default in "normal times," and the dashed line represents the maximum debt-to-GDP level under which there is no default in "trembling times."

It is interesting to note that, for values of q less than 5%, the mean debt-to-GDP ratio is a decreasing function of q; the country becomes less prudent and is willing to take on more debt as the risk of default becomes higher because it knows that it will not repay its debt in bad states of nature. This is the "Panglossian effect" described in Cohen and Villemot (2011).

One should also note that the proper way to introduce a relevant credit ceiling is to allow for two different levels: one pertaining to "normal times" and one pertaining to "trembling times." Imposing one only ceiling for both states of nature would be quite an inefficient way to avoid default. Extraordinary times call at extraordinary debt ceilings. This is a key distinction that tends to be lost in current policy debates.

Figure 4 shows, as expected, that debt ceilings to avoid default are an increasing function of the parameter q. For a short-lived crisis episode (q around 20%), countries take care of themselves as they do not default and are able to stabilize their debt. Thus, no debt ceiling is needed in this situation. This is in line with the business cycle properties that we examined in section 5.1.

Figure 4: Mean debt-to-GDP and credit ceilings as a function of q



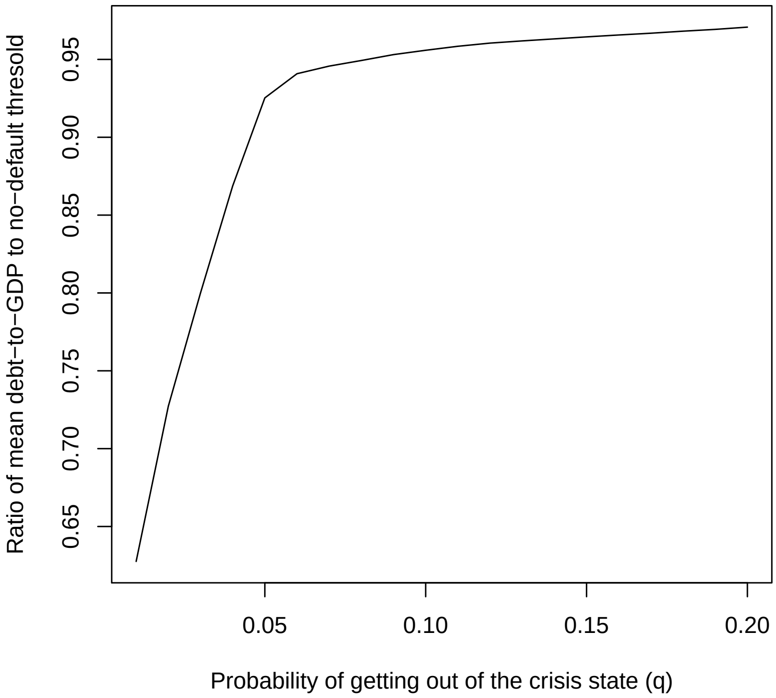
However, in the range when q is around or below 5%, the risk of default rises and stringent debt ceilings are needed if default is to be avoided in all circumstances. In the cases when q becomes very low, the constraint of a debt ceiling becomes quite strong; the ratio of the credit ceiling in normal times to the natural level of debt drops below two-thirds. For the critical value of 5%, the ratio is closer to one, slightly above 90%. These results are shown in Figure 5 which plots the ratio of mean debt-to-GDP to no-default threshold, as a function of q.

We also measure, in welfare terms, the impact of enforcing the credit ceiling. From a modelling perspective, we compute alternative solutions of the model where the credit ceiling is enforced,[9] and we compare the welfare obtained within the constrained model with the welfare obtained without constraints. Results are reported in Table 7.

As expected, the welfare cost is insignificant in the region of large q, and becomes quite significant for the low q zone. In particular, the welfare cost of debt ceilings is much larger than the cost of fluctuations. As a mean of comparison, Lucas (2003) estimates that the welfare cost of fluctuations in the US is about 0.1% of GDP (assuming the same value we do for risk aversion). In the median range for values of q (between 5% and 10%), the welfare cost appears to be moderate and commensurate with Lucas's estimate. If avoiding a sovereign default is of systemic importance, this may be worth a try. In the lower end of q's values however, the cost is much higher (about 15 times Lucas's numbers) and it would be clearly inefficient to

[9]Technically we restrict the state space for the debt level by setting a lower bound equal to the ceiling that we want to impose. The ceiling that we use corresponds to the lower line of Figure 3. As expected, the constrained models exhibit a zero probability of default.

Figure 5: Credit ceilings as a fraction of equilibrium levels in normal times



This graph shows the ratio of the credit ceiling to avoid default (in "normal times") to the mean debt-to-GDP ratio, as presented on Figure 4.

Table 7: Welfare cost of imposing credit ceilings

Probability of exiting "trembling times" (q, per quarter)	1%	5%	10%	20%
Unconstrained welfare	−18.273	−18.510	−18.524	−18.570
Constrained welfare	−18.573	−18.581	−18.578	−18.573
Cost of ceiling (permanent loss of GDP)	1.64%	0.39%	0.30%	0.02%

Welfare is computed in both models for a level of debt equal to the ceiling, and at the mean productivity level ($y = \mu_y$, $z = 0$) and in "normal times" ($\theta = N$). The cost of imposing the ceiling as a percentage of GDP is computed using a Lucas (1987) type calculation; we first consider an economy with the same preferences but no fluctuations $\left(i.e.\right.$ an economy where the welfare is $W = \frac{u((1-\Delta)\bar{Q})}{1-\beta}\Big)$, and then we report the value of Δ that generates the same drop in welfare as the one observed between the unconstrained and constrained economies reported in the table.

target a zero default equilibrium. In our simulations, we picked a 5% value for q, for which the constrained equilibrium remains reasonably close to the unconstrained one.

5.3 Further insights

When investigating the policy implications of our model for the eurozone, we did not modify the value of the parameter governing the magnitude of the trembling shock (μ_z) from the value set for the emerging country benchmark; we left it at 1 percentage point of the GDP growth rate. Quantitatively, this means that following a trembling shock, the GDP level will be permanently lowered by 3.8% relative to the pre-shock trend.[10] This is a sizable shock, but not so big compared to what Greece is currently undergoing.[11] When it comes to the eurozone, it could therefore make sense to use greater values for the magnitude of the trembling shock, and for the two other parameters governing the cost of default (the direct penalty on output λ and the recovery value V).[12] This would reflect the fact that in a monetary union with a highly integrated banking system, eurozone countries face a higher cost for default. For example, if we increase these three parameters by a factor of 1.5 (*i.e.* $\mu_z = 0.015$, $\lambda = 3\%$ and $V = 37.5\%$ of GDP), and with other parameters being held at the benchmark values given in Table 4, the mean debt-to-GDP ratio jumps to 58.8% (which is an increase by a factor of 1.5 compared to the benchmark case given in Table 5). Only the default frequency remains relatively unchanged at 2.4%. Our model is therefore capable of delivering an arbitrarily high level of debt-to-GDP through a homothetic re-scaling of the three parameters μ_z, λ and V, while keeping the default probabilities at a constant level.

Another point worth mentioning is that in the framework under which the eurozone operates, up to 50% of public debt is held by foreigners. In fact the three countries which have been most vulnerable to the recent crisis (Greece, Portugal and Ireland) all had more than 70% of their public debt held by foreigners. A straightforward policy lesson to avoid default risk on sovereign issuers could be to make sure that sovereign debt is primarily held by domestic institutions or individuals, as is the case in Japan for instance. Of course, for a given path of current accounts, this would imply that other (private) debt would be held outside the country. This is irrelevant in our model, since we don't distinguish between public and private external debt.[13] However,

[10] If g_t is the growth rate without trembling shock, and $\tilde{g}_t$ is the growth rate following a trembling shock at $t = 0$, then the long-term ratio of the two corresponding GDP levels is:

$$\frac{\prod_{t=0}^{+\infty} g_t}{\prod_{t=0}^{+\infty} \tilde{g}_t} = \prod_{t=0}^{+\infty} \left(1 + \frac{\rho_z^t \mu_z}{e^{\mu_y}}\right) \simeq 0.962$$

[11] According to some estimates, Greece GDP level could be more than 10% below its pre-crisis trend.

[12] Note that the size of the trembling shock μ_z influences the cost of default, because upon default the country loses the possibility of having its output restored to the pre-crisis level; since the size of the output restoration depends on μ_z as shown in (3), so does the cost of default. If we were to distinguish the size of the trembling shock in (2) from the size of the restoration in (3), then only the latter would have an impact on the default cost.

[13] The rationale for not distinguishing between public and private debt is that private external loans generally come with an explicit or implicit public guarantee, as documented by Reinhart and Rogoff (2011). This is confirmed by a positive correlation between sovereign default and access of the private sector to foreign credit, as documented by Arteta and Hale (2008). These considerations have led Mendoza and Yue (2011) to present a model where private and public external debt bear the same interest rate spread and the same credit risk.

in a more thoughtful model where the distinction is introduced, this could have interesting policy implications.

6 Conclusion

We have analyzed a model in which countries usually do not like to default, but rather are forced into default when the economy turns sour, arguing that this modelling choice better fits the historical reality. Based on this postulate, the key message of the paper is simply that in order to avoid default, the critical parameter to analyze is the speed at which economies can move out of these "trembling times" (the parameter q in our model). Clearly, this is a lesson that European policymakers should understand, as the more protracted the economic crisis (and, hence, the perception that countries entering into "trembling times" will stay there for a while), the higher the risk of default. In the worst case scenario when a crisis is expected to be very long lasting, the debt ceiling needed to avoid default may become very low. Building institutions that avoid default risk should not only rely on debt ceilings, but also on mechanisms that limit the duration of the "trembling times." One key distinction between advanced and poor countries is the supposedly superior ability of advanced economies to recover from crisis (rather than sheer recessions), as documented by Hausmann et al. (2006). The mess created in Europe by the management of the sovereign crisis has certainly shifted the perception of the European ability to exit "trembling times," making the risk of a default much higher for all sovereigns within the eurozone. This is where the debate on the macro-management of the crisis would certainly need to be addressed. Reassuring investors of the policymakers' ability to address trembling episodes is perhaps more important than imposing credit ceilings that are too stringent.

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A Extra results and proofs

A.1 General case

Proposition 6 (Eaton and Gersovitz (1981)) *Default incentives are stronger the higher the debt*

Proposition 7 *Default occurs if and only if debt-to-GDP ratio is higher than a given threshold d^*.*

Proof. This is a straightforward implication of the isoelasticity of preferences and of the Markovian nature of the stochastic process driving output and recovery. ■

A.2 Brownian or Poisson case

In this section we establish a lemma valid for both the Brownian case (section 3.1.2) and the Poisson case (section 3.1.3).

Lemma 8 *For both the Brownian and the Poisson cases, the country never chooses an indebtment level such that default is sure tomorrow (i.e. a level for which $\tilde{L}'(s, \tilde{D}'(s)) = 0$).*

Proof. By contradiction.

We denote by g^+ the growth rate in the good case and g^- in the bad case,[14] and p is the probability of a being in the bad case.

Suppose that for some state s, the optimal choice is to repay and have $\tilde{L}'(s, \tilde{D}'(s)) = 0$. Since it means that default is sure tomorrow, we have:

$$d^* < \frac{\tilde{D}'(s)}{g^+ Q}$$

We also have:

$$J^r(s) = u(Q - D) + (1-p)e^{-\rho h} J^d(g^+ Q) + p\, e^{-\rho h} J^d(g^- Q)$$

Now let $D'_2 = d^* g^- Q$. It is clear that this level of indebtment is in the safe zone, so that $\tilde{L}(s, D'_2) = e^{-rh} D'_2$. If the country was choosing that level of indebtment, it would get:

$$J^r_2(s) = u(Q - D + e^{-rh} D'_2) + (1-p)e^{-\rho h} J^r(D'_2, g^+ Q) + p\, e^{-\rho h} J^r(D'_2, g^- Q)$$

But since $J^r(D'_2, g^+ Q) \geq J^d(g^+ Q)$ and $J^r(D'_2, g^- Q) \geq J^d(g^- Q)$ by construction of D'_2, we therefore have:

$$J^r_2(s) > J^r(s)$$

[14] In the Poisson case, g^- can actually be a random variable (see section 3.1.3). The demonstration still applies in this case, with the following modifications:

- in the definition of D'_2, replace g^- by the minimum value that growth can take in the case of a bad shock;
- add conditional expectation operators in the last terms defining $J^r(s)$ and $J^r_2(s)$.

This is in contradiction with the optimality of $\tilde{D}'(s)$. ■

A.3 Brownian case

In this section, we denote by p the probability that output is low tomorrow, *i.e.* $p = \frac{1}{2} - \frac{\mu}{2\sigma}\sqrt{h}$. In order to demonstrate proposition 2, we begin by establishing the following lemma:

Lemma 9 *In the Brownian case, if $h \leq \frac{1}{\left(4\sigma+\frac{\mu}{\sigma}\right)^2}$, the risky interest rate ($r+p$ in first approximation) does not happen in equilibrium.*

Proof. By contraposition. Assume that for some state s, the optimal choice is to repay and $\tilde{L}(s, \tilde{D}'(s))$ is equal to $e^{-rh}(1-p)\tilde{D}'(s)$: this is the risky case where the country will repay next period in the good state of nature, but default in the bad state (the other two possible values for $\tilde{L}(s, \tilde{D}'(s))$ are $e^{-rh}\tilde{D}'(s)$ and 0, since output can take only two values). This implies that:

$$\frac{\tilde{D}'(s)}{e^{\sigma\sqrt{h}}Q} \leq d^* < \frac{\tilde{D}'(s)}{e^{-\sigma\sqrt{h}}Q}$$

We also have:

$$J^r(s) = u(Q - D + e^{-rh}(1-p)\tilde{D}'(s)) + (1-p)e^{-\rho h}J^r(\tilde{D}'(s), e^{\sigma\sqrt{h}}Q) + p\, e^{-\rho h}J^d(e^{-\sigma\sqrt{h}}Q)$$

Let $D_2' = e^{-2\sigma\sqrt{h}}\tilde{D}'(s)$. We therefore have $\frac{D_2'}{e^{-\sigma\sqrt{h}}Q} \leq d^*$, which means that this level of indebtment is in the safe zone, and that $\tilde{L}(s, D_2') = e^{-rh}D_2'$. If the country was choosing that level of indebtment, it would get:

$$J_2^r(s) = u(Q - D + e^{-rh}D_2') + (1-p)e^{-\rho h}J^r(D_2', e^{\sigma\sqrt{h}}Q) + p\, e^{-\rho h}J^r(D_2', e^{-\sigma\sqrt{h}}Q)$$

By optimality of $\tilde{D}'(s)$, we have $J^r(s) > J_2^r(s)$. And since we have $J^r(D_2', e^{\sigma\sqrt{h}}Q) > J^r(\tilde{D}'(s), e^{\sigma\sqrt{h}}Q)$ (by proposition 6) and $J^r(D_2', e^{-\sigma\sqrt{h}}Q) > J^d(e^{-\sigma\sqrt{h}}Q)$ (by construction of D_2'), this implies that $u(Q-D+e^{-rh}(1-p)\tilde{D}'(s)) > u(Q-D+e^{-rh}D_2')$. In turn, this implies that $(1-p) > e^{-2\sigma\sqrt{h}}$, or $\ln(1-p) > -2\sigma\sqrt{h}$. Using the concavity of the logarithm, this implies $p < 2\sigma\sqrt{h}$, which is equivalent to $h > \frac{1}{\left(4\sigma+\frac{\mu}{\sigma}\right)^2}$. ■

Intuitively, when h is small, then the variance of next period output conditionally to today's output is small. The country will prefer to borrow a little less, in order to be in the safe zone, since the effort to be done is small and tends towards zero, while the cost of a default remains high.

Proof of proposition 2. Assume that the country decides to repay. Given that next period output can take only two values, three cases are possible for tomorrow:

1. the country will repay in both states,

2. the country will repay in the good state of nature and default in the bad state,

3. the country will default in both states.

The second and third cases are excluded by lemmas 8 and 9.

By forward recursion, it is clear that the country will always repay in the future. ■