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Stéphane Gonzalez, Michel Grabisch

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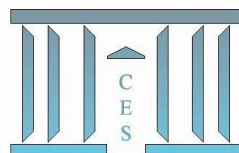


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Stéphane GONZALEZ, Michel GRABISCH

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Preserving coalitional rationality for non-balanced games

Stéphane GONZALEZ* and Michel GRABISCH†

University of Paris I

†Paris School of Economics

106-112, Bd de l'Hôpital, 75013 Paris, France

email Stephane.Gonzalez1@malix.univ-paris1.fr, michel.grabisch@univ-paris1.fr

Abstract

In cooperative games, the core is one of the most popular solution concept since it ensures coalitional rationality. For non-balanced games however, the core is empty, and other solution concepts have to be found. We propose the use of general solutions, that is, to distribute the total worth of the game among groups rather than among individuals. In particular, the k -additive core proposed by Grabisch and Miranda is a general solution preserving coalitional rationality which distributes among coalitions of size at most k , and is never empty for $k \geq 2$. The extended core of Bejan and Gomez can also be viewed as a general solution, since it implies to give an amount to the grand coalition. The k -additive core being an unbounded set and therefore difficult to use in practice, we propose a subset of it called the minimal bargaining set. The idea is to select elements of the k -additive core minimizing the total amount given to coalitions of size greater than 1. Thus the minimum bargaining set naturally reduces to the core for balanced games. We study this set, giving properties and axiomatizations, as well as its relation to the extended core of Bejan and Gomez. We introduce also the notion of unstable coalition, and show how to find them using the minimum bargaining set. Lastly, we give a method of computing the minimum bargaining set.

Keywords: cooperative game, core, balancedness, general solution

1 Introduction

One of the major challenges of the theory of cooperative games with transferable utility is to propose an allocation of gains obtained by a set N of players. Typically, this distribution (called a solution) is done among the individual players, and many concepts of solutions have been proposed so far, the core being one of the best known solution since it ensures coalitional rationality. However, in many real situations, the distributions of gains is not done to individuals but often to groups of individuals (associations, companies,

*Corresponding author.

families, etc.), which can further distribute among their members according to their own rules, which can differ from one group to another. Also, the drawbacks of the classical solutions like the core are well known. For example, the core is often empty, which obliges to take other solution concepts, often violating coalitional rationality.

A natural generalization of the classical view of solutions should be then to distribute among groups rather than individuals. We call this a *general solution*, and observe that we can already find in the literature two attempts in this direction. The first one is the k -additive core proposed by Grabisch and Miranda [6, 8], which was also proposed earlier by Vassil'ev [11] in a publication in russian. The extended core of Bejan and Gomez [1] can also be viewed as a general solution (although in its formulation it is not), since it implies to give an amount to the grand coalition. In the k -additive core, the distribution of $v(N)$ is done among coalitions of size at most k , and coalitional rationality is ensured (thus, the 1-additive core is the classical core). In the extended core, the distribution is done among individuals and (implicitly) the grand coalition. When the core is empty, these two general solutions allocate some negative amounts (debts) to some coalitions (the grand coalition in the case of the extended core).

The aim of this paper is to propose a subset of solutions of the k -additive core, and to relate it to the extended core. Indeed, if the k -additive core has some remarkable features (it is never empty as soon as $k \geq 2$, and it preserves coalitional rationality), its main drawback is that it is an unbounded set for any $k \geq 2$. We call the subset of the k -additive core we propose the *minimum bargaining set*, since it is based on the following idea: the less bargaining is necessary, the better the solution. Indeed, a general solution obliges individuals belonging to a group which has received an allocation to bargain among them in order to share the amount received by the group (often a negative amount, if the core is empty). Therefore, it is desirable to focus on solutions minimizing the number of groups and the allocation given to them. This is achieved by minimizing a norm of the vector of allocations to groups. We will show that the minimum bargaining set coincides with the core when the latter is nonempty, it is a singleton for all L_p norms with $p > 1$, and for the L_1 norm, it is a convex polyhedron, which among other properties, permits to find unstable coalitions (that is, which have incentive to leave the grand coalition).

The paper is organized as follows: Section 2 introduces the concept of general solution and the basic material which is needed. In particular, we present properties for general solutions, and we introduce the k -additive core and the G-extended core (a rewriting of the extended core as a general solution). Section 3 focusses on the main achievement of the paper, the minimum bargaining set. It studies its properties, and two particular cases of norms (L_1 and L_p for $p > 1$), and gives axiomatizations. Also, the notion of unstable coalition is introduced, as well as a method to find them. Lastly, Section 4 addresses the problem of computing the minimum bargaining set.

2 General payoff vectors and solutions

2.1 Basic definitions and notations

We start by fixing the notation and introducing basic definitions. Let $N = \{1, \dots, n\} \subset \mathbb{N}$ be a finite and nonempty set of agents or players. *Coalitions of players* are subsets of N , denoted by capital letters S , T , and so on. Whenever possible, we will omit braces for

singletons and pairs, denoting $\{i\}, \{i, j\}$ by i, ij respectively, in order to avoid a heavy notation. A *transferable utility* (TU) *game* on N is a pair (N, v) where v is a mapping $v : 2^N \rightarrow \mathbb{R}$ satisfying $v(\emptyset) = 0$. We will denote by $\mathcal{G}(N)$ the set of all games over N . For any coalition S , $v(S)$ represents the *worth* of S , i.e., what coalition S could earn regardless of other players.

To every game $v \in \mathcal{G}(N)$ we associate its *Möbius transform* [10] (also known as Harsanyi dividends [7]) $m^v : 2^N \rightarrow \mathbb{R}$, defined by

$$m^v(S) := \sum_{T \subseteq S} (-1)^{|S \setminus T|} v(T), \quad \forall S \subseteq N, \quad (1)$$

and $m^v(\emptyset) = 0$. Conversely, v can be recovered from m^v by the inverse transform

$$v(S) = \sum_{T \subseteq S} m^v(T), \quad \forall S \subseteq N. \quad (2)$$

Hence the Möbius transform is a linear bijection on $\mathcal{G}(N)$. Note in particular that $m^v(\{i\}) = v(\{i\})$ for all $i \in N$, implying that $m^v(S) = 0$ if $|S| > 1$ whenever v is additive. We recall also that the Möbius transform gives the coordinates of a game into the $(2^n - 1)$ -dimensional basis of unanimity games $\{u_S\}_{S \subseteq N, S \neq \emptyset}$, with $u_S(T) = 1$ if $T \supseteq S$, and 0 otherwise.

A *payoff vector* is a vector $x \in \mathbb{R}^n$ that assigns agent i the payoff x_i . Given $x \in \mathbb{R}^n$, and $S \subseteq N$, denote by $x(S)$ the sum $\sum_{i \in S} x_i$ with the convention that $x(\emptyset) = 0$. A payoff vector is *efficient* for game v if $x(N) = v(N)$. We will call *preimputation* of v each efficient payoff vector of v and we denote by $PI(v)$ the set of preimputations of v .

Coalition S is able to *improve upon* payoff x if $x(S) < v(S)$. The *core* $C(v)$ of a game v is the set of efficient payoffs that cannot be improved upon by any coalition, i.e.,

$$C(v) = \{x \in \mathbb{R}^n \mid \forall S \subsetneq N, x(S) \geq v(S), \text{ and } x(N) = v(N)\}.$$

A game with a nonempty core is said to be *balanced*. Whenever convenient, we consider the core as a mapping $C : \mathcal{G}(N) \rightarrow 2^{(\mathbb{R}^n)}$.

A *general payoff vector* is a vector $x \in \mathbb{R}^{2^N \setminus \emptyset}$ that assigns to a coalition $S \subseteq N$ a payoff x_S . Also, 1_S with $S \subseteq N$ indicates the vector with coordinate 1 for S and 0 otherwise.

A general payoff vector is said to be *efficient* for game v if $\sum_{\emptyset \neq S \subseteq N} x_S = v(N)$.

Example 1. A payoff vector $x \in \mathbb{R}^n$ defines a general payoff vector by considering $x(S)$ for all nonempty $S \subseteq N$. Hence, any element of the core induces a general efficient payoff vector.

Example 2. A less trivial example of general payoff vector, which will be central in our study, is the Möbius transform m^v of a game v . Note that by (2) m^v is efficient for v .

A *solution* on the set of games $\mathcal{G}(N)$ is a mapping $\sigma : \mathcal{G}(N) \rightarrow 2^{(\mathbb{R}^n)}$, i.e., it assigns to any game $v \in \mathcal{G}(N)$ a set of payoff vectors, with the additional property that they are all efficient for v . Analogously, a *general solution* assigns to every game v in $\mathcal{G}(N)$ a set of general payoff vectors efficient for v .

The core is a well-known solution for games. A trivial example of general solution is the Möbius transform m (see Example 2). Although the notion of general solution is not explicitly considered in the literature, there are two recent remarkable examples of general solutions. The first one is the k -additive core [8], where a payoff is given to each coalition of size at most k , while the second example is the extended core [1], which can be viewed as a general solution. In the extended core, apart giving a payoff to each player, a payoff is also (virtually) given to the grand coalition N . We will study this concept in Section 2.4. Our paper mainly focusses on the k -additive core, which is presented in Section 2.3. Before that, we introduce some properties for general solutions.

2.2 Properties of general solutions

We propose several properties for general solutions, some of them being direct generalization of classical properties of solutions. Let σ be a general solution on $\mathcal{G}(N)$. We say that σ

- is *covariant under strategic equivalence* (COV) if $\forall v \in \mathcal{G}(N), \forall \alpha > 0, \forall \beta \in \mathbb{R}^{2^n-1}$:

$$\sigma(\alpha v + \beta) = \alpha \sigma(v) + \beta.$$

- is an *idempotent solution* (IDEM) if

$$\sigma(m^{-1} \circ \sigma) = \sigma.$$

The property means that applying the same solution concept to the set of solutions found (after transforming them into games by m^{-1}) does not give more solutions.

- is *symmetric* (SYM) if $\forall v \in \mathcal{G}(N)$, for all permutations π such that $v(\pi(S)) = v(S)$ holds for all $S \subseteq N$:

$$\sigma(\pi(v)) = \sigma(v).$$

- is *coalitionally rational* (CR) if $\forall v \in \mathcal{G}(N), \forall S \subseteq N, \forall x \in \sigma(v)$,

$$\sum_{T \subseteq S} x_T \geq v(S).$$

- *gives preimputation* (PI) if $\forall v \in \mathcal{G}(N), \forall S \subseteq N$ such that $|S| \geq 2$ we have:

$$\forall x \in \sigma(v), x_S = 0.$$

Only classical solutions, like the core or the Shapley value, satisfy (PI).

- satisfies the *dummy player property* (DPP) if for any dummy player i , we have:

$$\forall x \in \sigma(v), x_i = v(i),$$

where as usual a player i is dummy if $v(S \cup i) = v(S) + v(i)$ for every $S \subseteq N \setminus i$.

- is a *minimization of the global debt* (MGD) if for all nonnegative games v , for all $x \in \sigma(v)$,

$$\sum_{\substack{S \subseteq N \\ |S| \geq 2}} \min(x_S, 0) = -\bar{t}(v),$$

where $\bar{t}(v) = \min\{t \geq 0 \mid C(v^t) \neq \emptyset\}$, with v^t the game defined by $v^t(N) = v(N) + t$, and $v^t(S) = v(S)$ otherwise (see Section 2.4). This axiom ensures that the grand coalition is not in debt more than necessary to ensure stability. We note that it is impossible to have $\sum_{\substack{S \subseteq N \\ |S| \geq 2}} \min(x_S, 0) > -\bar{t}(v)$ without losing (CR).

The set of general solutions is a partially ordered set with the relation \preceq defined by:

$$\sigma_1 \preceq \sigma_2 \iff \forall v \in \mathcal{G}(N), \sigma_1(v) \subseteq \sigma_2(v).$$

For any set A of general solutions, whenever it exists we denote by $\top(A)$ the top element of A for the relation \preceq , that is, the element satisfying $\sigma \preceq \top(A)$ for all $\sigma \in A$.

For example, if we denote by *PICR* the set of general solutions which satisfy (PI) and (CR), then

$$C = \top(\text{PICR}).$$

2.3 The k -additive core

A game $v \in \mathcal{G}(N)$ is said to be *k -additive* [4] if its Möbius transform m^v vanishes for subsets of more than k players: $m^v(S) = 0$ if $|S| > k$, and there exists $K \subseteq N$, $|K| = k$, such that $m^v(K) \neq 0$. Note that a 1-additive game is an additive game in the usual sense. A game v is said to be *at most k -additive* if it is q -additive for some $q \in \{1, \dots, k\}$. We denote by $\mathcal{G}^k(N)$ the set of all at most k -additive games.

The k -additive core, introduced by Grabisch and Miranda [5, 6, 8], and also by Vassil'ev [11], preserves the general spirit of coalitional rationality of the core.

Definition 1. Let v be a game. The *k -additive core* of v , denoted by $C^k(v)$, is the set of k -additive games ensuring coalitional rationality:

$$C^k(v) = \{\phi \in \mathcal{G}^k(N) \mid \phi(S) \geq v(S), \forall S \subseteq N, \phi(N) = v(N)\}.$$

As before we may consider the k -additive core as a mapping $C^k : \mathcal{G}(N) \rightarrow 2^{\mathcal{G}^k(N)}$. From any $\phi \in C^k(v)$, one determines an efficient general payoff vector m^ϕ for v through its Möbius transform given by (1). We also comment on coalitional rationality: by (2), the inequality $\phi(S) \geq v(S)$ means that the total amount received by coalition S , when summing all payoffs given to its members and to all its subcoalitions of size at most k , exceeds $v(S)$.

Summarizing, the mapping $m \circ C^k$ is a general solution. Put differently, any element of the k -additive core in the basis of unanimity games is an efficient general payoff vector. We may call the n -additive core the *general core*, since it gives (full dimensional) general payoff vectors.

Example 3. We consider a game v on $N = \{1, 2, 3\}$ defined by:

S	1	2	3	12	13	23	123
$v(S)$	1	1	1	1.5	1.5	1.5	2

Let ϕ be a game defined by

S	1	2	3	12	13	23	123
$\phi(S)$	2	2	3	3	3	3	2

We have a payoff for each player and for each pair of players given by the Möbius transform of ϕ .

S	1	2	3	12	13	23	123
$m^\phi(S)$	2	2	3	-1	-2	-2	0

We observe that $m^\phi(123) = 0$, $\phi(S) \geq v(S)$ and $\phi(N) = v(N)$, therefore $\phi \in \mathcal{C}^2(v)$.

Theorem 1. [8] The k -additive core is a convex polyhedron, nonempty for $k \geq 2$.

$$\mathcal{C}^k(v) \neq \emptyset, \quad \forall v \in \mathcal{G}(N), \forall k \geq 2.$$

Unlike the core, the k -additive core is always nonempty. However, it is an unbounded set, as shown by the next proposition.

Theorem 2. For all $k \geq 2$, the k -additive core is unbounded. Moreover, for $k = 2$, its extremal rays are of the form $1_i - 1_{ij}$, for all $i \neq j \in N$, in the basis of unanimity games (i.e., for $m \circ \mathcal{C}^2$).

Proof. Since for any game v , the inclusion $\mathcal{C}^2(v) \subseteq \mathcal{C}^k(v)$ holds for any $k \geq 2$, it suffices to show that the 2-additive core is unbounded.

From standard results in polyhedra, studying the unboundedness of the 2-additive core amounts to studying its recession cone, i.e., the set of inequalities where the right member is replaced by 0 (see, e.g., [3, Ch. 1]). If the recession cone is a pointed cone not reduced to $\{0\}$, then the corresponding polyhedron has vertices and rays. i.e., it is unbounded. We denote naturally by $\mathcal{C}^2(0)$ the recession cone of $\mathcal{C}^2(v)$.

Let us write the system of inequalities in the basis of unanimity games, i.e., for $m \circ \mathcal{C}^2(0)$. We have:

$$\begin{aligned} m^v(i) &\geq 0, & \forall i \in N \\ m^v(i) + m^v(j) + m^v(i, j) &\geq 0, & \forall i, j \in N, i \neq j \\ \sum_{i \in N} m^v(i) + \sum_{\{i, j\} \subseteq N} m^v(i, j) &= 0. \end{aligned}$$

We claim that any vector of the form $r = 1_i - 1_{ij}$, i.e., having value 1 for coordinate i , -1 for coordinate ij , and 0 otherwise, is an extremal ray. Indeed, the vector αr satisfies the above system for any $\alpha > 0$, and it cannot be expressed as the sum of other rays since any ray must have at least 2 nonzero coordinates. Note also that there is no other extremal ray.

As a conclusion, the 2-additive core is unbounded. □

Proposition 1. For all $k \in \{1, \dots, n\}$, $m \circ \mathbf{C}^k$ satisfies (COV), (IDEM), that is, $\mathbf{C}^k \circ \mathbf{C}^k = \mathbf{C}^k$, (SYM), and (CR). Moreover, if we denote by CR the set of general solutions which satisfies (CR), we have:

$$m \circ \mathbf{C}^n = \top(CR).$$

Proof. We prove only (IDEM), the rest being clear. Let v be a game on N . Take any $\phi \in \mathbf{C}^k(v)$. Then $\phi \in \mathbf{C}^k(\phi)$, therefore $\phi \in \mathbf{C}^k(\mathbf{C}^k(v))$

Conversely, if $\phi \in \mathbf{C}^k(\mathbf{C}^k(v))$, then $\exists \psi \in \mathbf{C}^k(v)$ such that $\phi \in \mathbf{C}^k(\psi)$. We have $\phi(S) \geq \psi(S) \geq v(S) \quad \forall S \subseteq N$ and $\phi(N) = \psi(N) = v(N)$. Since ϕ is k -additive we have $\phi \in \mathbf{C}^k(v)$. □

2.4 The G-extended core

In this section we introduce a new general solution based on the principle of the extended core $\mathbf{EC}(v)$ of Bejan and Gomez. Given a game v and $t \geq 0$, define its t -expansion v^t by $v^t(S) := v(S)$ for all $S \subset N$, and $v^t(N) := v(N) + t$. Now, to any game v we assign $\bar{t}(v) := \min\{t \geq 0 \mid \mathbf{C}(v^t) \neq \emptyset\}$, the minimum amount to be given to the grand coalition in order to ensure balancedness. Given v a game and x a preimputation, we introduce the real number $t(v, x) := \min\{l(N) \mid l \in \mathbb{R}_+^n \text{ and } (x + l)(S) \geq v(S), \forall S \subseteq N\}$.

Bejan and Gomez [1] have introduced the concept of *extended core* of v as follows:

$$\mathbf{EC}(v) := \{x \in PI(v) \mid t(v, x) = \bar{t}(v)\}.$$

The extended core of v coincides with the core whenever $\mathbf{C}(v) \neq \emptyset$. It is the set of preimputations x for which $\exists l^x \in \mathbb{R}_+^n$ such that $(x + l^x)(S) \geq v(S)$, $\forall S \subseteq N$ and $\sum_{i \in N} l_i^x = \bar{t}(v)$. In other words, the extended core is the set of those preimputations that require a minimal subsidy to the grand coalition.

Since the extended core is a set of preimputations, it is not a general solution as announced in the introduction. However, a general solution is inherent in the way Bejan and Gomez use the extended core: given a tax rule, take any vector in the core of $v^{\bar{t}(v)}$ (where precisely the amount $\bar{t}(v)$ has been given to the grand coalition) and tax every player according to the chosen rule, where the tax is $\bar{t}(v)$. In what follows, we propose a rewriting of the extended core as a general solution, which we call the *G-extended core*.

Consider $x \in \mathbf{EC}(v)$ and its associated vector $l^x \in \mathbb{R}_+^n$ as above. We build from x the game ϕ^x defined by $\phi^x(S) := x(S) + l^x(S)$, $\forall S \subset N$, and $\phi^x(N) = v(N)$. We observe that ϕ^x is additive on $2^N \setminus \{N\}$, therefore its Möbius transform reads:

$$m^{\phi^x}(S) = \begin{cases} x_i + l_i^x, & \text{if } S = \{i\} \\ -\bar{t}(v), & \text{if } S = N \\ 0, & \text{otherwise,} \end{cases}$$

as it can be easily checked from (2). Based on this, we define the *G-extended core* of v as follows.

Definition 2. Let v be a game. Its \mathbf{G} -extended core is the set defined by

$$\text{GEC}(v) := \{\phi^x \in \mathcal{G}(N) \mid x \in \text{EC}(v)\},$$

or equivalently

$$\begin{aligned} \text{GEC}(v) = \{\phi \in \mathcal{G}(N) \mid (\phi(i))_{i \in N} \in \mathbf{C}(v^{\bar{t}(v)}), \quad m^\phi(S) = 0, \forall S, 1 < |S| < n, \\ \text{and } m^\phi(N) = -\bar{t}(v)\}. \end{aligned}$$

As for the k -additive core, taking the Möbius transform of an element $\phi \in \text{GEC}(v)$ defines an efficient general payoff vector, therefore $m \circ \text{GEC}$ is a general solution, which can be directly expressed as:

$$m \circ \text{GEC}(v) = \{x \in \mathbb{R}^{2^N \setminus \emptyset} \mid (x_i)_{i \in N} \in \mathbf{C}(v^{\bar{t}(v)}), \quad x_N = -\bar{t}(v), \quad x_S = 0 \text{ otherwise}\}.$$

Proposition 2. The following properties hold for GEC .

- (i) $\text{GEC}(v)$ is a nonempty convex and compact polyhedral set for all $v \in \mathcal{G}(N)$. It is equal to the core if the latter is nonempty.
- (ii) GEC satisfies (CR). Hence, $\text{GEC}(v) \subseteq \mathbf{C}^n(v)$.
- (iii) GEC satisfies (IDEM) (i.e., $\text{GEC} \circ \text{GEC} = \text{GEC}$), (COV), (SYM) and (MGD).

Proof. Most of the properties are clear by definition. We detail only (IDEM). We suppose first that v is balanced. Therefore, $\text{GEC}(v) = \mathbf{C}(v)$ and the property holds since $\mathbf{C} \circ \mathbf{C} = \mathbf{C}$.

Suppose then that v is not balanced. Take any $\phi \in \text{GEC}(v)$ and let us prove that $\phi \in \text{GEC}(\phi)$. Since $\phi \geq v$ and $\phi(N) = v(N)$, we have $\bar{t}(\phi) \geq \bar{t}(v)$. We have $(\phi_i)_{i \in N} \in \mathbf{C}(\phi^{\bar{t}(v)})$, since for all $S \neq N$, $\sum_{i \in S} \phi_i \geq \phi(S)$, and $\sum_{i \in N} \phi_i = v(N) + \bar{t}(v) = \phi(N) + \bar{t}(v)$. This proves that $\bar{t}(\phi) = \bar{t}(v)$, and consequently $\phi \in \text{GEC}(\phi)$. In conclusion, $\text{GEC}(v) \subseteq \text{GEC} \circ \text{GEC}(v)$.

Conversely, let $\phi \in \text{GEC} \circ \text{GEC}(v)$. Then there exists $\psi \in \text{GEC}(v)$ such that $\phi \in \text{GEC}(\psi)$. Reasoning as above, we find that $\bar{t}(\psi) = \bar{t}(v)$. Then for any $S \neq N$ we have $\sum_{i \in S} \phi_i \geq \psi(S) \geq v(S)$, and $\sum_{i \in N} \phi_i = \psi(N) + \bar{t}(v) = v(N) + \bar{t}(v)$. Lastly, $m^\phi(N) = -\bar{t}(v)$, which proves that $\phi \in \text{GEC}(v)$. \square

We observe that although the global amount of debt is minimal, it is concentrated only on the grand coalition and therefore does not necessarily optimally express the balance of power of each coalition. By contrast, in the k -additive core, the smallest coalitions are in charge of distributing the debt.

3 The minimum bargaining set

The principle of a solution of the k -additive core is: if a payment is given to coalitions up to size k , that is, if each player accepts to pool a part of his gain with the other players, it is possible to preserve coalitional rationality. However, we can note two important problems of the k -additive core: First, for $k \geq 2$, the k -additive core is unbounded, secondly, there could be a conflict between the agents of a coalition on how the payment made by the k -additive core will be allocated to each player. Therefore, a solution of the

k -additive core implying few bargaining among players (that is, involving few coalitions and with small amounts) will be considered better than a solution far from the set of imputations implying a lot of bargaining.

We introduce a general solution which minimizes the amount of pooling or, equivalently, which maximizes individual payoffs. To achieve this, given $k \in \{2, \dots, n\}$ and a norm on \mathbb{R}^{2^n} , we consider that the payoff to each coalition of at least two players is an amount to be negotiated. We define here the bargaining level as the total amount of pooling for the selected norm. The bargaining level can also be defined equivalently as the distance between the k -additive core and the set of preimputations, or like a kind of measure of the deviation from additivity.

3.1 The set $I^k(v)$ of minimal bargaining

Let $2 \leq k \leq n$ be fixed and $\|\cdot\|$ be a norm on \mathbb{R}^{2^n} .

We consider the following nonlinear program:

$$\begin{aligned} & \text{Minimize } B(\phi) := \|(m^\phi(S))_{\substack{S \subseteq N \\ |S| \geq 2}}\| \\ & \text{subject to } \phi \in C^k(v). \end{aligned}$$

We call $B(\phi)$ the *bargaining level* of ϕ . For a game $v \in \mathcal{G}(N)$ and $\epsilon \geq 0$, we define the set $C_\epsilon^k(v)$ by

$$C_\epsilon^k(v) := \{\phi \in C^k(v), B(\phi) \leq \epsilon\}$$

and the set $I^k(v)$ by

$$I^k(v) := \bigcap_{\substack{\epsilon \geq 0 \\ C_\epsilon^k(v) \neq \emptyset}} C_\epsilon^k(v).$$

$I^k(v)$ exists for all games in $\mathcal{G}(N)$ because $C^k(v) \neq \emptyset$ implies $\exists \epsilon_0$ such that $C_{\epsilon_0}^k(v) \neq \emptyset$. Moreover, $I^k(v)$ is the set of optimal solutions of the above nonlinear program.

Theorem 3. Let $v \in \mathcal{G}(N)$, $2 \leq k \leq n$, and $\epsilon \geq 0$. The following statements hold:

- (i) $\forall \epsilon \geq 0$, $C_\epsilon^k(v)$ is a convex and compact set which is equal¹ to $C(v)$ if $\epsilon = 0$.
- (ii) $I^k(v)$ is a nonempty convex and compact set, equal to $C(v)$ if $C(v) \neq \emptyset$, otherwise $\exists \epsilon^* > 0$ such that $I^k(v) = C_{\epsilon^*}^k(v)$.

Proof. (i) It is obvious that $C_0^k(v) = C(v)$. Also, $C_\epsilon^k(v)$ is closed $\forall \epsilon \geq 0$, and if $0 \leq \epsilon_1 \leq \epsilon_2$, then $C_{\epsilon_1}^k(v) \subseteq C_{\epsilon_2}^k(v)$.

Let us prove convexity. If a and b are in $C_\epsilon^k(v)$ then $\forall t \in [0, 1]$, the game $ta + (1-t)b$ is in $C^k(v)$ because $C^k(v)$ is convex. Furthermore, by convexity of $x \mapsto \|x\|$,

$$\|(t(m^a(S)) + (1-t)(m^b(S)))_{\substack{S \subseteq N \\ |S| \geq 2}}\| \leq t \underbrace{\|(m^a(S))_{\substack{S \subseteq N \\ |S| \geq 2}}\|}_{\leq \epsilon} + (1-t) \underbrace{\|(m^b(S))_{\substack{S \subseteq N \\ |S| \geq 2}}\|}_{\leq \epsilon},$$

¹Up to the natural identification between an additive game and a payoff vector. The same remark applies to the next item.

from which we deduce that $\mathbf{C}_\epsilon^k(v)$ is convex.

It remains to prove compactness. Suppose $a \in \mathbf{C}_\epsilon^k(v)$. By definition,

$$\|(m^a(S))_{\substack{S \subseteq N \\ |S| \geq 2}}\| \leq \epsilon,$$

which implies that each component $m^a(S)$, $|S| \geq 2$, is bounded, and so is $\sum_{\substack{S \subseteq N \\ |S| \geq 2}} m^a(S)$.

Now we have:

$$v(N) = a(N) = \sum_{i \in N} m^a(i) + \sum_{\substack{S \subseteq N \\ |S| \geq 2}} (m^a(S)),$$

which implies that $\sum_{i \in N} m^a(i)$ is bounded. Since $a \in \mathbf{C}_\epsilon^k(v)$, we have $m^a(i) = a(i) \geq v(i)$ for all $i \in N$. Hence, each $m^a(i)$ is bounded. It follows that each coordinate $m^a(S)$ is bounded, therefore, $m \circ \mathbf{C}_\epsilon^k(v)$ is compact, and consequently, $\mathbf{C}_\epsilon^k(v)$ is compact too.

- (ii) $\mathbf{I}^k(v)$ is an intersection of compact, convex and nonempty nested sets, therefore $\mathbf{I}^k(v)$ is a compact, convex and nonempty set. It is easy to see that if $\mathbf{C}(v)$ is nonempty then $\mathbf{I}^k(v) = \mathbf{C}(v)$.

Let us define

$$\epsilon^* = \sup\{\epsilon \geq 0 \text{ such that } \mathbf{C}_\epsilon^k(v) = \emptyset\}$$

with the convention $\sup(\emptyset) = 0$. It is easily verified that $\mathbf{C}_{\epsilon^*}^k(v) \subseteq \mathbf{I}^k(v)$ and, $\forall n \in \mathbb{N}$, the set $\mathbf{I}^k(v)$ satisfies $\mathbf{I}^k(v) \subseteq \mathbf{C}_{\epsilon^* + \frac{1}{n}}^k(v)$. Therefore

$$\mathbf{I}^k(v) \subseteq \bigcap_{n \in \mathbb{N}^*} \mathbf{C}_{\epsilon^* + \frac{1}{n}}^k(v) = \mathbf{C}_{\epsilon^*}^k(v).$$

Hence $\exists \epsilon^* \geq 0$ such that $\mathbf{I}^k(v) = \mathbf{C}_{\epsilon^*}^k(v)$ and $\forall \phi \in \mathbf{I}^k(v)$, we have $B(\phi) = \epsilon^*$. □

The next proposition gathers properties of elements of $\mathbf{I}^k(v)$.

Proposition 3. Let $2 \leq k \leq n$ be fixed, $\|\cdot\|$ be a given norm, and $v \in \mathcal{G}(N)$. For any element $\phi \in \mathbf{I}^k(v)$, it holds:

- (i) $m^\phi(S) \leq 0$, $\forall S \subseteq N$ such that $|S| \geq 2$.
- (ii) ϕ is a concave game.
- (iii) The vector $(\phi(i))_{i \in N}$ belongs to $\mathbf{C}(v^{\|B(\phi)\|_1})$, where $\|\cdot\|_1$ is the L_1 norm.
- (iv) $\left| \sum_{\substack{S \subseteq N \\ |S| \geq 2}} m^\phi(S) \right| \geq \bar{t}(v)$.

Proof. (i) Let $\phi \in \mathbf{C}^k(v)$. Suppose that $m^\phi(S) > 0$ for some S with $|S| \geq 2$, then the game $\phi^{eq} \in \mathcal{G}(N)$ defined by :

$$\begin{cases} m^{\phi^{eq}}(i) &= \phi^{eq}(i) = \phi(i) + \frac{1}{|S|} m^\phi(S), \quad \forall i \in S \\ m^{\phi^{eq}}(S) &= 0 \\ m^{\phi^{eq}}(T) &= m^\phi(T) \end{cases} \quad \text{otherwise,}$$

belongs to $\mathbf{C}^k(v)$, satisfies $m^{\phi^{eq}}(S) = 0$ and $B(\phi^{eq}) < B(\phi)$.

(ii) It is known [2] that concavity of ϕ is equivalent to $\sum_{S|A \subseteq S \subseteq B} m^\phi(S) \leq 0$ for every A, B such that $|A| = 2$ and $B \supseteq A$, which holds by (i).

(iii) Let $\phi \in \mathcal{I}^k(v)$. By (i), $m^\phi(S) \leq 0$ for all S such that $|S| \geq 2$. Then, $\sum_{\substack{S \subseteq N \\ |S| \geq 2}} (m^\phi(S)) = -\|B(\phi)\|_1$ and the vector $(\phi(i))_{i \in N}$ satisfy

$$\begin{aligned} \sum_{i \in S} \phi(i) &\geq \sum_{i \in S} \phi(i) + \underbrace{\sum_{\substack{T \subseteq S \\ |T| \geq 2}} m^\phi(T)}_{\leq 0} = \phi(S) \geq v(S) = v^{\|B(\phi)\|_1}(S), \quad S \neq N, |S| \geq 2 \\ \sum_{i \in N} \phi(i) &= v(N) - \underbrace{\sum_{\substack{T \subseteq N \\ |T| \geq 2}} m^\phi(T)}_{=-\|B(\phi)\|_1} = v^{\|B(\phi)\|_1}(N). \end{aligned}$$

(iv) The result comes directly from (i) and (iii). □

3.2 The minimum bargaining set as a general solution

$\mathcal{I}^k(v)$ being a subset of the k -additive core $\mathcal{C}^k(v)$, it follows that $m \circ \mathcal{I}^k$ can be used as a general solution, preserving the desirable properties of the k -additive core without having its drawbacks. Indeed, it preserves coalitional rationality, it is never empty and coincides with the core when the latter is nonempty as shown in Theorem 3, and unlike the k -additive core, it is bounded, and can even be reduced to a singleton when the L_p norm with $p > 1$ is used, as it will be shown in Section 3.5.

In this section and the subsequent ones, we will investigate the properties of this general solution. The next proposition shows that the minimum bargaining set satisfies all properties of general solutions satisfied by the k -additive core.

Proposition 4. $m \circ \mathcal{I}^k$ satisfies (CR), (COV), (SYM) and (IDEM) for any norm.

Proof. (CR): Clear.

(COV): m being linear, $m \circ \mathcal{I}^k$ satisfies (COV) if and only if $\forall v \in \mathcal{G}(N), \forall \alpha > 0, \forall \beta \in \mathbb{R}^{2^n-1}$ we have:

$$\mathcal{I}^k(\alpha v + \beta) = \alpha \mathcal{I}^k(v) + \beta.$$

Since $m \circ \mathcal{C}^k(v)$ satisfies (COV),

$$\begin{aligned} &\text{Minimize } B(\phi) := \|(m^\phi(S))_{\substack{S \subseteq N \\ |S| \geq 2}}\| \\ &\text{subject to } \phi \in \mathcal{C}^k(\alpha v + \beta). \end{aligned}$$

is equivalent to

$$\begin{aligned} &\text{Minimize } B(\phi) := \|(m^\phi(S))_{\substack{S \subseteq N \\ |S| \geq 2}}\| \\ &\text{subject to } \phi \in \alpha \mathcal{C}^k(v) + \beta. \end{aligned}$$

(SYM): Similarly, since $m \circ \mathbb{C}^k(v)$ satisfies (SYM), we have that for all permutations π such that $v(\pi(S)) = v(S)$ for all $S \subseteq N$,

$$\begin{aligned} & \text{Minimize } B(\phi) := \|(m^\phi(S))_{\substack{S \subseteq N \\ |S| \geq 2}}\| \\ & \text{subject to } \phi \in \mathbb{C}^k(v). \end{aligned}$$

is equivalent to

$$\begin{aligned} & \text{Minimize } B(\phi) := \|(m^\phi(S))_{\substack{S \subseteq N \\ |S| \geq 2}}\| \\ & \text{subject to } \phi \in \mathbb{C}^k(\pi(v)). \end{aligned}$$

(IDEM): We have to show that $\mathbb{I}^k \circ \mathbb{I}^k = \mathbb{I}^k$. First, if $\mathbb{I}^k(v) = \mathbb{C}(v)$, then for each $x \in \mathbb{C}(v)$, considering x as an additive game, we have $\mathbb{I}^k(x) = \{x\}$. Therefore $\mathbb{I}^k(\mathbb{C}(v)) = \mathbb{C}(v)$.

Now, suppose $\mathbb{C}(v) = \emptyset$ and take any $\phi \in \mathbb{I}^k(v)$. Since $\phi \in \mathbb{C}^k(v)$, it follows that $\mathbb{C}^k(\phi) \subseteq \mathbb{C}^k(v)$. Therefore, since ϕ minimizes the bargaining level over $\mathbb{C}^k(v)$, ϕ minimizes the bargaining level over $\mathbb{C}^k(\phi)$, which implies $\phi \in \mathbb{I}^k(\phi)$ and

$$\mathbb{I}^k(v) \subseteq \mathbb{I}^k(\mathbb{I}^k(v)).$$

Let $\phi \in \mathbb{I}^k(\mathbb{I}^k(v))$. Then there exists $\psi \in \mathbb{I}^k(v)$ such that $\phi \in \mathbb{I}^k(\psi)$. Observe that $\psi \in \mathbb{C}^k(\psi) \subseteq \mathbb{C}^k(v)$. Since ψ minimizes the bargaining level over $\mathbb{C}^k(v)$ and ϕ minimizes the bargaining level over $\mathbb{C}^k(\psi)$, it follows that these bargaining levels are equal. Moreover, ϕ is a k -additive game which satisfies $\phi(S) \geq v(S)$ for all $S \subset N$ and $\phi(N) = v(N)$. Therefore $\phi \in \mathbb{I}^k(v)$ and

$$\mathbb{I}^k(\mathbb{I}^k(v)) \subseteq \mathbb{I}^k(v).$$

□

3.3 Continuity

We study in this section the continuity of the minimum bargaining set. For this, we recall the basic definitions and results on the continuity of set-valued functions (see, e.g., [9, Ch. 9]).

Let X and Y be two metric spaces, and consider a set-valued function φ from X to Y , i.e., $\emptyset \neq \varphi(x) \subseteq Y$ for all $x \in X$. The *graph* of φ is defined by

$$Gr(\varphi) := \{(x, y) \mid x \in X, \text{ and } y \in \varphi(x)\}.$$

We say that φ is *closed* if $Gr(\varphi)$ is a closed subset of $X \times Y$, and φ is *bounded* if for each compact subset B of X , the image of B , $\varphi(B) = \cup_{x \in B} \varphi(x)$, is a bounded subset of Y . A set-valued function φ is *upper hemicontinuous* at $x \in X$ if for every open subset U of Y such that $U \supseteq \varphi(x)$, there exists an open subset V of X such that $x \in V$ and $\varphi(z) \subseteq U$ for every $z \in V$. Moreover, φ is *upper hemicontinuous* (uhc) if it is upper hemicontinuous at each $x \in X$.

Similarly, φ is *lower hemicontinuous* at $x \in X$ if for every open subset U of Y such that $\varphi(x) \cap U \neq \emptyset$, there exists an open subset V of X such that $x \in V$ and $\varphi(z) \cap U \neq \emptyset$ for every $z \in V$. Moreover, φ is *lower hemicontinuous* (lhc) if it is lower hemicontinuous at each $x \in X$. If both upper and lower hemicontinuity hold, then φ is *continuous*.

Lemma 1. Let X be a metric space and let $Y = \mathbb{R}^M$, with M a finite and nonempty set. If φ is a closed and bounded set-valued function, then φ is uhc.

Lemma 2. Let X be a convex polyhedral set in \mathbb{R}^n , $X \neq \emptyset, \mathbb{R}^n$, and let $Y \subseteq \mathbb{R}^m$ be bounded ($m, n \in \mathbb{N}$). If φ is a bounded set-valued function with a convex graph, then φ is lhc.

Based on these lemmas, we can show the following.

Proposition 5. For any $k \in \{1, \dots, n\}$, for any norm, l^k is continuous.

Proof. We denote by ϵ_v^* the bargaining level of $l^k(v)$ for each game $v \in \mathcal{G}(N)$.

- l^k is uhc.

$$\begin{aligned} Gr(l^k) = \{ & (v, \phi) \in \mathcal{G}(N) \times \mathcal{G}^k(N), \phi(N) = v(N), \phi(S) \geq v(S), \forall S \subseteq N, \\ & \text{and } \|(m_S^\phi)_{\substack{S \subseteq N \\ |S| \geq 2}}\| = \epsilon_v^* \} \end{aligned}$$

is a closed subset of $\mathcal{G}(N) \times \mathcal{G}^k(N)$ and by linearity of the Möbius transform, efficiency of the k -additive core and (CR), $l^k(B)$ is bounded for all compact subsets B of $\mathcal{G}(N)$. We conclude by Lemma 1.

- l^k is lhc.

$\mathcal{G}(N)$ is a polyhedral convex set. Also l^k is bounded and its graph is convex. We conclude by Lemma 2.

□

3.4 Properties of l_1^k

In this section, $\|\cdot\|$ is the L_1 norm. We denote by l_1^k the solution l^k for the L_1 norm.

By Proposition 4, we know that $m \circ l_1^k$ satisfies (CR), (COV), (SYM) and (IDEM). In addition, we show that $m \circ l_1^n$ satisfies (DPP) and (MGD).

Proposition 6. $m \circ l_1^n$ satisfies (DPP) and (MGD).

Proof. (i) Let v be a game in $\mathcal{G}(N)$, and $x \in C(v^{\bar{t}(v)})$. It is clear that the game ψ defined by

$$\begin{cases} \psi(i) &= x_i, & \forall i \in N \\ m^\psi(S) &= 0, & \forall S \subseteq N, S \neq N \text{ and } |S| \geq 2 \\ m^\psi(N) &= -\bar{t}(v) \end{cases}$$

belongs to $C^n(v)$. We have

$$\|B(\psi)\|_1 = \sum_{\substack{S \subseteq N \\ |S| \geq 2}} |(m^\psi(S))| = \bar{t}(v).$$

Therefore $\forall \phi \in l_1^n(v)$, we have

$$\sum_{\substack{S \subseteq N \\ |S| \geq 2}} |(m^\phi(S))| = \|B(\phi)\|_1 \leq \|B(\psi)\|_1 = \bar{t}(v).$$

By Proposition 3 (i),

$$\sum_{\substack{S \subseteq N \\ |S| \geq 2}} |(m^\phi(S))| = \left| \sum_{\substack{S \subseteq N \\ |S| \geq 2}} (m^\phi(S)) \right|$$

and by Proposition 3 (iv),

$$\left| \sum_{\substack{S \subseteq N \\ |S| \geq 2}} m^\phi(S) \right| \geq \bar{t}(v).$$

Therefore, $m \circ l_1^n$ satisfies (MGD).

(ii) By (MGD) and Proposition 3 (i) the bargaining level is equal to $\bar{t}(v)$, and by Proposition 3 (iii), we deduce that for all $\phi \in l_1^n(v)$, we have $(\phi(i))_{i \in N} \in C(v^{\bar{t}(v)})$.

Suppose i is dummy for v and take $x \in C(v^{\bar{t}(v)})$. Then the vector $x^* \in \mathbb{R}^n$ defined by: $x_i^* = v(i)$ and $x_j^* = x_j$ for $j \in N \setminus i$ satisfies:

$$\begin{aligned} x^*(N) &\leq x(N) \\ x^*(S) &= x(S) \geq v(S), \quad \text{if } i \notin S \\ x^*(S) &= x(S \setminus i) + v(i) \geq v(S \setminus i) + v(i) = v(S), \quad \text{if } i \in S. \end{aligned}$$

Therefore $x^*(N) = x(N)$, which entails $x_i = x_i^* = v(i)$, proving that $m \circ l_1^n$ satisfies (DPP). \square

The next theorem is a characterization of $m \circ l_1^n$.

Theorem 4. Let $MGDCR$ be the set of general solutions satisfying (MGD) and (CR). Then:

$$m \circ l_1^n = \top(MGDCR).$$

In words, $m \circ l_1^n$ is the largest general solution satisfying coalitional rationality and minimizing the amount of the global debt.

Proof. We already know that $m \circ l_1^n$ satisfies (CR), and by Proposition 6, we know that $m \circ l_1^n$ satisfies (MGD).

Conversely, let σ be a general solution which satisfies (CR) and (MGD). Let $x \in \sigma(v)$ and ϕ the game defined by

$$\phi(S) = \sum_{T \subseteq S} x_T.$$

Since σ satisfies (CR), $\phi \in C^n(v)$ because $m \circ C^n(v) = \top(CR)$. Moreover, since σ satisfies (MGD), the bargaining level of ϕ is equal to the bargaining level of each element of $l_1^n(v)$. Therefore ϕ minimizes the bargaining level over $C^n(v)$.

We conclude that $\phi \in l_1^n(v)$ and $\forall v \in \mathcal{G}(N)$, $\sigma(v) \subseteq m \circ l_1^n(v)$. \square

Remark 1. We have seen that GEC satisfies (MGD) and (CR). Therefore we have the following inclusions:

$$C(v) \subseteq \text{GEC}(v) \subseteq l_1^n(v) \subseteq C^n(v),$$

for all games v . It follows that GEC inherits the properties of l_1^n .

From the above results, we notice the following important fact.

Corollary 1. Let $v \in \mathcal{G}(N)$. We have:

$$\{(\phi(i))_{i \in N} \mid \phi \in I_1^n(v)\} = C(v^{\bar{t}(v)}) = \{(\phi(i))_{i \in N} \mid \phi \in \text{GEC}(v)\}.$$

Proof. We prove the first equality. From the proof of Proposition 6, we see that for all $\phi \in I_1^n(v)$, it holds $(\phi(i))_{i \in N} \in C(v^{\bar{t}(v)})$. The reverse inclusion holds by Remark 1.

Now, the second equality holds by definition of GEC. \square

We introduce now the concept of unstable coalition. Consider a non-balanced game v and the core $C(v^{\bar{t}(v)})$ of its t -expansion. Suppose it exists a coalition $T \neq N$ such that, for any payoff $x \in C(v^{\bar{t}(v)})$, $x(T) = v(T)$. It means that the coalition can never receive more than its worth $v(T)$. Moreover, since the debt $\bar{t}(v)$ has to be paid anyway by the players, it is likely that the members of T will eventually receive in total strictly less than $v(T)$. Such a coalition is unstable, because it has incentive to leave the grand coalition. In what follows, first we show that there always exists an unstable coalition, and second, we show how to characterize them, and how it is possible to find them in a very easy way by $I_1^n(v)$.

Definition 3. We say that a coalition $T \subset N$ is *unstable* if for any payoff vector $x \in C(v^{\bar{t}(v)})$, it holds $x(T) = v(T)$.

For any coalition $T \subset N$, we denote by v_T the restriction of v to the coalition T .

Proposition 7. If T is unstable, then $C(v_T)$ is nonempty.

Proof. If T is unstable, then a solution $x \in C(v^{\bar{t}(v)})$ satisfies $x(S) \geq v(S) \forall S \subseteq T$ and $x(T) = v(T)$. Therefore $(x_i)_{i \in T} \in C(v_T)$. \square

Proposition 8. Any non-balanced game v contains an unstable coalition $T \subset N$.

Proof. Suppose that v is not balanced, and that $\forall T \subset N, \exists x \in C(v^{\bar{t}(v)})$ such that $x(T) > v(T)$. Then, the convexity of $C(v^{\bar{t}(v)})$ implies that there exists x' such that $\forall T \subset N, x'(T) > v(T)$. Let $T' \in \arg \min_{T \subset N} (x'(T) - v(T))$. The vector \tilde{x} defined by:

$$\tilde{x}_i = \begin{cases} x'_i - \frac{x(T') - v(T')}{|T'|}, & \text{if } i \in T' \\ x'_i, & \text{otherwise} \end{cases}$$

satisfies

$$\tilde{x}(S) \geq v(S), \forall S \subset N$$

and

$$\tilde{x}(N) < x(N) = v(N) + \bar{t}(v).$$

This is not possible by definition of $\bar{t}(v)$. \square

Proposition 9. Let v be a game on N . Suppose it exists $S \subset N$, $|S| \geq 2$, such that for all $\phi \in I_1^n(v)$, $m^\phi(S) = 0$. Then the following holds.

- (i) For all $\phi \in I_1^n(v)$, for all $T \subseteq S$, $|T| \geq 2$, we have $m^\phi(T) = 0$.
- (ii) There exists $T \supseteq S$, $T \neq N$, such that $C(v_T) \neq \emptyset$.

Proof. (i) Suppose there exists $\phi \in \mathcal{I}_1^n(v)$ and $T \subseteq S$ such that $m^\phi(S) = 0$ and $m^\phi(T) < 0$. Then the game ϕ' defined by

$$\begin{aligned} m^{\phi'}(T) &= 0 \\ m^{\phi'}(S) &= m^\phi(T) \\ m^{\phi'}(K) &= m^\phi(K), \quad \forall K \neq T, S \end{aligned}$$

belongs to $\mathcal{I}_1^n(v)$.

(ii) Take $S \subset N$, $|S| \geq 2$, such that for all $\phi \in \mathcal{I}_1^n(v)$, $m^\phi(S) = 0$. We prove that if $\forall T \supseteq S$, $T \neq N$, we have $\mathcal{C}(v_T) = \emptyset$, then there exists $\phi \in \mathcal{I}_1^n(v)$ such that $m^\phi(S) < 0$.

Let $\tilde{\phi} \in \text{GEC}(v) \subseteq \mathcal{I}_1^n(v)$. We put

$$t = \min_{T \supseteq S, T \neq N} \{\tilde{\phi}(T) - v(T)\}.$$

Observe that $t > 0$. Indeed, $t = 0$ would imply that for some $T \supseteq S$, $T \neq N$, $\tilde{\phi}(T) = v(T)$. Since $\tilde{\phi} \in \text{GEC}$, its Möbius transform is 0 for all nonsingleton subsets of T . Therefore, $\sum_{i \in T'} \tilde{\phi}(i) = \tilde{\phi}(T') \geq v(T')$ for all $T' \subset T$, and $\sum_{i \in T} \tilde{\phi}(i) = \tilde{\phi}(T) = v(T)$. It follows that the vector $(\tilde{\phi}(i))_{i \in N}$ is an element of the core $\mathcal{C}(v_T)$, a contradiction.

Then the game ϕ defined by

$$\begin{aligned} m^\phi(K) &= m^{\tilde{\phi}}(K), \quad \forall K \subset N, \quad K \neq S \\ m^\phi(S) &= -t \\ m^\phi(N) &= -\bar{t}(v) + t \end{aligned}$$

belongs to $\mathcal{I}_1^n(v)$. □

Theorem 5. Let v be a game on N , and S be a coalition of N such that $|S| \geq 2$. The following properties are equivalent:

- (i) $\exists T \supseteq S$ such that T is unstable.
- (ii) $\forall \phi \in \mathcal{I}_1^n(v), m^\phi(S) = 0$.

Proof. (i) \Rightarrow (ii). Let $\phi \in \mathcal{I}_1^n(v)$. By Corollary 1, we have $(\phi(1), \dots, \phi(n)) \in \mathcal{C}(v^{\bar{t}(v)})$, which yields $\sum_{i \in T} \phi(i) = v(T)$. Therefore if $m^\phi(S) < 0$, we would have $\phi(T) < v(T)$, which contradicts $\phi \in \mathcal{I}_1^n(v)$.

(ii) \Rightarrow (i). Define T as a maximal element (in the sense of inclusion) of $\{L \supseteq S, m^\phi(L) = 0, \forall \phi \in \mathcal{I}_1^n(v)\}$. Suppose $T = N$. Then for all $\phi \in \mathcal{I}_1^n(v)$, $m^\phi(N) = 0$, which implies that $\bar{t}(v) = 0$ since $\text{GEC}(v) \subseteq \mathcal{I}_1^n(v)$. Therefore $\mathcal{C}(v) \neq \emptyset$, and (i) holds for $T = N$.

We assume now that $T \neq N$. Due to Proposition 9 (i) and the definition of T , for any set $L \subseteq T$, $|L| \geq 2$, we have $m^\phi(L) = 0$ for all $\phi \in \mathcal{I}_1^n(v)$, and for all $L \supset T$, $m^\phi(L) \neq 0$ for at least one $\phi \in \mathcal{I}_1^n(v)$. It follows that

$$\sum_{i \in T} \phi(i) = \phi(T), \quad \forall \phi \in \mathcal{I}_1^n(v).$$

From the above equation and Corollary 1, it suffices to show that $\phi(T) = v(T)$ for all $\phi \in I_1^n(v)$ in order to prove that $x(T) = v(T)$ for all $x \in C(v^{\bar{i}(v)})$. Suppose then that there exists $\phi_0 \in I_1^n(v)$ such that $\phi_0(T) > v(T)$. We distinguish two cases.

Suppose first that $T = N \setminus i$. Suppose there exists $\phi \in I_1^n(v)$ such that $\phi(N \setminus i) > v(N \setminus i)$. Take $\phi' \in I_1^n(v)$ such that $m^{\phi'}(N) < 0$ (necessarily exists otherwise $T = N$), and build the game $\phi'' = \frac{1}{2}(\phi + \phi')$. Define $\tilde{\phi}$ by

$$m^{\tilde{\phi}}(K) = \begin{cases} \max(v(N \setminus i) - \phi''(N \setminus i), m^{\phi''}(N)) < 0, & \text{if } K = N \setminus i \\ m^{\phi''}(N) - m^{\tilde{\phi}}(N \setminus i), & \text{if } K = N \\ m^{\phi''}(K), & \text{otherwise.} \end{cases}$$

Then $\tilde{\phi} \in I_1^n(v)$, although $m^{\tilde{\phi}}(N \setminus i) < 0$, a contradiction.

Suppose next that T is not of the form $N \setminus i$. We know that for all $T' \supset T$, $T' \neq N$, there exists $\phi^{T'} \in I_1^n(v)$ such that $m^{\phi^{T'}}(T') < 0$. Define $\tilde{\phi} = \sum_{T' \supset T, T' \neq N} \alpha_{T'} \phi^{T'}$, with $\alpha_{T'} \in (0, 1)$, $\sum_{T' \supset T, T' \neq N} \alpha_{T'} = 1$. Then by convexity of $I_1^n(v)$, it follows that $\tilde{\phi} \in I_1^n(v)$, and we have $m^{\tilde{\phi}}(T') < 0$, for all $T' \supset T$, $T' \neq N$. Consider the game $\phi' = \frac{1}{2}(\phi_0 + \tilde{\phi})$, which belongs to $I_1^n(v)$ by convexity, and put

$$\epsilon = \min(\phi'(T) - v(T), \min_{T \subset K \subset N} (-m^{\phi'}(K))) > 0,$$

and choose \tilde{K} which realizes the minimum of $\min_{T \subset K \subset N} (-m^{\phi'}(K))$. Build ϕ^* as follows:

$$m^{\phi^*}(K) = \begin{cases} -\epsilon, & \text{if } K = T \\ m^{\phi'}(\tilde{K}) + \epsilon, & \text{if } K = \tilde{K} \\ m^{\phi'}(K), & \text{otherwise.} \end{cases}$$

We claim that $\phi^* \in I_1^n(v)$, which contradicts the hypothesis $m^{\phi}(T) = 0$ for all $\phi \in I_1^n(v)$. Indeed, it suffices to verify coalitional rationality:

- (i) If $K \supseteq \tilde{K}$, we have $\phi^*(K) = \phi'(K) \geq v(K)$.
- (ii) If $K \supseteq T$ and $K \not\supseteq \tilde{K}$, then

$$\phi^*(K) = \phi'(K) - \epsilon \geq \phi'(K) - (\phi'(K) - v(K)) = v(K).$$

- (iii) If otherwise $K \subseteq T$, then $\phi^*(K) = \phi'(K) \geq v(K)$.

□

It is easy to find all unstable coalitions. It suffices to compute the coordinates $(\bar{\phi}(i))_{i \in N}$ of the barycenter of $\text{GEC}(v)$ (or equivalently of the barycenter of $I_1^n(v)$). By Corollary 1, it is plain that $T \neq N$ is unstable if and only if $\sum_{i \in T} \bar{\phi}(i) = v(T)$.

3.5 Properties of \mathbf{l}_p^k with $p > 1$

In this section, $\|\cdot\|$ is the L_p norm, with $p > 1$:

$$\|x\|_p = \left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}}.$$

We denote by \mathbf{l}_p^k the general solution \mathbf{l}^k for the norm L_p .

Theorem 6. Let $p > 1$. Then $\mathbf{l}_p^k(v) = \mathbf{C}(v)$ if $\mathbf{C}(v) \neq \emptyset$, otherwise $\mathbf{l}_p^k(v)$ reduces to the unique element of $\mathbf{C}^k(v)$ which minimizes the bargaining level.

Proof. Let a and b be two different games in $\mathbf{l}_p^k(v)$. From Theorem 3 (ii), there exists $\epsilon^* \geq 0$ such that $\mathbf{l}_p^k(v) = \mathbf{C}_{\epsilon^*}^k(v)$, and, for $t \in]0, 1[$, the game $ta + (1-t)b$ belongs to $\mathbf{l}_p^k(v)$ by convexity, so in particular for $t = \frac{1}{2}$. If $\epsilon^* = 0$ then $\mathbf{l}_p^k = \mathbf{C}(v)$, otherwise we have, by Proposition 3 (i) :

$$\begin{aligned} (B(a))^p &= \sum_{\substack{S \subseteq N \\ |S| \geq 2}} (-m^a(S))^p = (\epsilon^*)^p \\ (B(b))^p &= \sum_{\substack{S \subseteq N \\ |S| \geq 2}} (-m^b(S))^p = (\epsilon^*)^p \\ \left(B\left(\frac{a+b}{2}\right) \right)^p &= \sum_{\substack{S \subseteq N \\ |S| \geq 2}} \left(-\frac{1}{2}m^a(S) - \frac{1}{2}m^b(S) \right)^p = (\epsilon^*)^p. \end{aligned}$$

Now, if $\epsilon^* \neq 0$ and $a \neq b$, we have by strict convexity of $x \mapsto x^p$ on $[0, +\infty)$:

$$(\epsilon^*)^p = \sum_{\substack{S \subseteq N \\ |S| \geq 2}} \left(-\frac{1}{2}m^a(S) - \frac{1}{2}m^b(S) \right)^p < \frac{1}{2} \sum_{\substack{S \subseteq N \\ |S| \geq 2}} (-m^a(S))^p + \frac{1}{2} \sum_{\substack{S \subseteq N \\ |S| \geq 2}} (-m^b(S))^p = (\epsilon^*)^p,$$

a contradiction. Therefore $\epsilon^* = 0$ or $a = b$.

To conclude, if $\epsilon^* = 0$ then $\mathbf{l}_p^k(v) = \mathbf{C}(v)$, otherwise $a = b$ and $\mathbf{l}_p^k(v) = \{a\}$. □

Unlike $m \circ \mathbf{l}_1^n$, the general solution $m \circ \mathbf{l}_p^k$ does not satisfy (MGD), as shown by the following example with $p = 2$.

Example 4. We suppose that $k = 2$, and consider a game v on $N = \{1, 2, 3\}$ defined by:

S	1	2	3	12	13	23	123
$v(S)$	0	0	0	1	1	0	$1 - \epsilon$

Clearly, $\bar{t}(v) = \epsilon$ and $\mathbf{l}_2^2(v)$ reduces to a singleton. If (MGD) holds, the minimum bargaining level is ϵ , hence by Proposition 3 (iii), $(m \circ \mathbf{l}_2^2(v)(i))_{i \in N}$ belongs to $\mathbf{C}(v^\epsilon) = \{(1, 0, 0)\}$. This implies in turn

$$\begin{cases} 1 + m \circ \mathbf{l}_2^2(v)(12) \geq 1 & \Rightarrow & m \circ \mathbf{l}_2^2(v)(12) \geq 0 \\ 1 + m \circ \mathbf{l}_2^2(v)(13) \geq 1 & \Rightarrow & m \circ \mathbf{l}_2^2(v)(13) \geq 0 \\ m \circ \mathbf{l}_2^2(v)(23) \geq 0. \end{cases}$$

This is impossible because by Proposition 3 (i), $m \circ \mathbf{l}_2^2(v)(ij) \geq 0$ for all $\{i, j\} \subseteq N$, and by MGD, $m \circ \mathbf{l}_2^2(v)(12) + m \circ \mathbf{l}_2^2(v)(13) + m \circ \mathbf{l}_2^2(v)(23) = -\epsilon$.

4 Computation of the minimum bargaining set

In general the minimum bargaining set $l^k(v)$ is difficult to compute because it is a general nonlinear optimization problem. However, in the case of the L_1 norm, the optimization problem reduces to a linear program. We have also some partial result for the case of the L_2 norm.

4.1 Computation of l_1^k

Proposition 10. The set $m \circ l_1^k(v)$ is the solution of the linear program:

$$\text{maximize } \sum_{\substack{S \subseteq N \\ 2 \leq |S| \leq k}} x_S$$

under the conditions :

$$\begin{aligned} i) \quad & \sum_{S \subseteq N} x_S = v(N) \\ ii) \quad & \sum_{S \subseteq T} x_S \geq v(T) \quad \forall T \subseteq N \\ iii) \quad & x_S \leq 0 \quad \forall S \subseteq N \quad \text{such that } |S| \geq 2. \\ iv) \quad & x_S = 0 \quad \forall S \subseteq N \quad \text{such that } |S| > k. \end{aligned}$$

Proof. Each $\phi \in l_1^k(v) \subseteq C^k(v)$ is such that m^ϕ satisfies *i*), *ii*), and *iv*). Furthermore Proposition 3 (i) implies that $m^\phi(S) \leq 0 \quad \forall |S| \geq 2$, which is *iii*). Lastly, minimizing $\sum_{\substack{S \subseteq N \\ 2 \leq |S| \leq k}} |m^\tau(S)|$ for $\tau \in C^k(v)$ is equivalent to maximize $\sum_{\substack{S \subseteq N \\ 2 \leq |S| \leq k}} m^\tau(S)$ under the conditions $\tau \in C^k(v)$ and $m^\tau(S) \leq 0, \quad \forall |S| \geq 2$. □

Example 5. We consider a game v on $N = \{1, 2, 3\}$ defined by:

S	1	2	3	12	13	23	123
$v(S)$	20	40	30	40	40	40	50

We have $C(v) = \emptyset$, and $\forall t \in [0, 1]$, the game ϕ_t defined by:

S	1	2	3	12	13	23	123
ϕ_t	20	40	30	$60 - 20t$	$50 - 10t$	$70 - 10t$	50

belongs to $l_1^n(v)$. Its Möbius transform is

S	1	2	3	12	13	23	123
$m^{\phi_t}(S)$	20	40	30	$-20t$	$-10t$	$-10t$	$-40(1 - t)$

and we observe that the optimal value of the objective function is $-\bar{t}(v) = -40$.

4.2 Computation of l_2^k

Giving an exact expression of l_2^k in the general case is quite difficult because it is necessary to solve a quadratic program. However, we can obtain an exact expression of l_2^2 in the case of a symmetric² and subadditive game.

Proposition 11. If v is a nonnegative, symmetric and subadditive game on N then :

$$l_2^2(v)(i) = \max_{k \in \{1, \dots, n-1\}} \left\{ \frac{n-1}{k(n-k)} v_k - \frac{k-1}{n(n-k)} v_n \right\}, \forall i \in N$$

$$l_2^2(v)(ij) = \frac{2}{n(n-1)} (v(N) + n(n-2)l_2(v)(1)), \forall i, j \in N$$

where $v_k := v(K)$, $|K| = k$.

Proof. Subadditivity of v implies $C(v) = \emptyset$ or reduced to a singleton. Therefore $l_2^2(v)$ is a singleton.

Now, symmetry of v implies by definition of l_2^2 that $\forall i \in N$, $l_2^2(v)(i) = l_2^2(v)(1)$ and $\forall i, j \in N$, $l_2^2(v)(ij) = l_2^2(v)(12)$.

Denote by α the value of $l_2^2(v)(1)$ and by β the value of $l_2^2(v)(12)$. $l_2^2(v) \in C^2(v)$ therefore, $\forall k \in \{1, \dots, n-1\}$,

$$\frac{k(k-1)}{2} \beta + (2-k)k\alpha \geq v_k$$

and

$$\frac{n(n-1)}{2} \beta + (2-n)n\alpha = v_n.$$

These two expressions are equivalent to:

$$\beta = 2 \left(\frac{v_n + n(n-2)\alpha}{n(n-1)} \right) (*)$$

and $\forall k \neq n$ and $n \neq 1$

$$\alpha \geq \frac{n-1}{k(n-k)} v_k - \frac{k-1}{n(n-k)} v_n (**)$$

Furthermore, v is symmetric, therefore, the minimization of $B(\phi)$ under the condition $\phi \in C^2(v)$ is equivalent to the minimization of $(\beta - 2\alpha)^2$ under the conditions (*) and (**).

Since v is subadditive, $\beta - 2\alpha < 0$. Therefore, minimizing $(\beta - 2\alpha)^2$ is equivalent to maximize $\beta - 2\alpha$, and this maximization can be reduced to the maximization of $v_n - n\alpha$.

Consequently, we must have α as small as possible. The minimum is reached for $\alpha = \max_{k \in \{1, \dots, n-1\}} \left\{ \frac{n-1}{k(n-k)} v_k - \frac{k-1}{n(n-k)} v_n \right\}$ and $\beta = 2 \left(\frac{v_n + n(n-2)\alpha}{n(n-1)} \right)$. \square

Example 6. We consider a symmetric subadditive game v on $N = \{1, 2, 3\}$ defined by:

S	1	2	3	12	13	23	123
$v(S)$	1	1	1	1.5	1.5	1.5	2

²A game v is symmetric if $v(S)$ depends only on the cardinality of S for each S .

We have $C(v) = \emptyset$. Using the above result, we get:

S	1	2	3	12	13	23	123
$l_2^2(v)$	1	1	1	$\frac{5}{3}$	$\frac{5}{3}$	$\frac{5}{3}$	2

S	1	2	3	12	13	23	123
$m \circ l_2^2(v)(S)$	1	1	1	$-\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	0

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