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# On the Stochastic Properties of Carbon Futures Prices\*

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## Abstract

In this paper, we are interested in deriving the stochastic properties of CO<sub>2</sub> emission allowance futures prices. First, we review the extant mathematical and environmental economics literatures, and explain the main findings from both continuous- and jump-diffusion models. Second, we derive our own empirical application by means of the activity signature function (Todorov and Tauchen (2010,2011)). The central result is that the evolution of the carbon futures price can be described in terms of a pure jump-diffusion process. For instance, important cases of information shocks leading to allowance price jump can be addressed when modelled as an appropriately sampled, centered Lévy or Poisson process.

*JEL Classification:* C14; C32; G1; Q4

*Keywords:* Carbon Price; Stochastic Modeling; Activity Signature Function.

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# 1 Introduction

The theory behind tradable permits markets certainly found its roots in the writing of environmental economists (Dales (1968), Montgomery (1972)). Dynamic properties were then progressively established in the very same literature (Cronshaw and Kruse (1996), Ellerman (2005), Kling and Rubin (1997), Leiby and Rubin (2001), Maeda (2004), Rubin (1996), Schennach (2000), Stevens and Rose (2002)).

Recently, a parallel mathematical literature has developed to establish the properties of tradable permits programs, in the wider context of the European Union Emissions Trading Scheme (EU ETS) and the price formation of European Union Allowances (EUAs). These papers are typically interested in recovering the stochastic properties of allowance prices, either to model the dynamic price equilibrium (Carmona et al. (2009,2010)), or to address the question of derivatives valuation in emissions markets (Chesney and Taschini (2008), Çetin and Verschuere (2009), Borovkov et al. (2011), Carmona et al. (2011), Hinz and Novikov (2010)).

With the notable exception of Seifert et al. (2008), there seems to be a gap in the environmental economics literature to document these stochastic properties of carbon prices. Some papers have contributed to the econometric modeling of carbon assets in the ARCH framework (Paolella and Taschini (2008), Benz and Trück (2009)), but only Daskalakis et al. (2009) have derived some stochastic properties. In particular, the authors focus on the issue of optimally choosing the stochastic process among a series of models.

Reasons for stochastic modeling of carbon prices are amongst other expected allowance demand and fluctuating fuel prices, which potentially lead to a change in the merit order of power production, fuel-switch costs, weather changes, economic growth, etc. (Seifert et al. (2008), Carmona et al. (2009)). The Brownian motion assumption is intended to capture these uncertainties in a simplified way. This process finds its roots in option pricing since at least Black-Scholes, its main advantage being to allow for continuous dynamic hedging (which is good news for option writers). Other popular stochastic processes include mean-reversion (Vasicek, Ornstein-Uhlenbeck) and/or jumps (Poisson, Lévy, Bernoulli), with finite or infinite activity<sup>1</sup>.

For modeling emission allowances, it appears necessary to identify a stochastic process which can map the expected properties of the price as a proxy for marginal abatement costs, as well as the time series properties of returns. Uncovering the stochastic properties of carbon futures prices is also a necessary step for building an equilibrium pricing model for futures, choosing the appropriate option pricing

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<sup>1</sup>If the assumption of infinite vs. finite activity jump models has little impact on long-term decisions (for instance in a real-option model), it is of primary importance for option traders. Since options are increasingly traded in the carbon market, identifying the type of jumps in the stochastic process therefore stands out as a relevant task.

model, and evaluating investment decisions - for example through a real-option valuation approach (Daskalakis et al. (2009), Zhu et al. (2009)).

Here, we can draw a parallel with the economic literature on oil. Papers by Miller and Zhang (1996) and Postali and Picchetti (2006) have investigated the Geometric Brownian Motion hypothesis in order to identify the stochastic properties of the crude oil price process for real-option valuation. Drawing conclusions about the optimal stochastic process for carbon price appears relevant as well.

The goals pursued by this paper are twofold: (i) review the main results on the stochastic properties of carbon prices derived in previous literature, and (ii) apply an original empirical method to establish the stochastic properties of carbon futures prices by means of the activity signature function (Todorov and Tauchen (2010,2011)).

Our approach makes use of intraday transactions to investigate the presence of jumps of infinite activity. This is not the case for the nonparametric approach using recent bipower variations measures proposed in Barndorff-Nielsen and Shephard (2004, 2006) that are only robust to the presence of large but rare jumps<sup>2</sup>.

If we are interested in derivatives pricing, this limitation strongly advocates the use of a methodology able to consider both finite and infinite activity jumps and, possibly, to distinguish between them. This is precisely the objective of the activity signature function. Hence, we shall be able to distinguish between what should be considered as pure diffusion, jump-diffusion or pure-jump processes.

To give a sketch of the procedure, we first compute for a given interval of time the realized power variation<sup>3</sup> over two different time scales (Zhang et al. (2005), Aït-Sahalia and Jacod (2009a,b)). Second, we compute a nonlinear transformation of the ratio of realized power variations (over the two time scales) and plot it as a function of the exponent over the range  $p \in (0, 4]$ .

The activity signature function identifies the activity of the observed process: it will either increase linearly or stay flat depending on whether the process contains jumps or not. Its overall behavior reveals the appropriate modeling for both the ‘small’ and ‘big’ moves in the discretely observed process. This estimation strategy is useful for discriminating across classes of models of stochastic processes, as contradictory conclusions have been reached in previous literature dealing with carbon data (as it will

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<sup>2</sup>Barndorff-Nielsen and Shephard (2004, 2006) propose a methodology to investigate the presence of jumps, but this methodology is not robust to the presence of jumps of infinite activity. As emphasized in Andersen et al. (2007) using such measures: “*The main limitation is that we exclude Lévy jump processes with infinite jump intensity, see Carr et al. (2002), as we only allow for ‘rare’ jumps occurring at a finite rate. Hence, key issues are to what extent the jump-diffusion representation is consistent with empirical data and what specific features of the specification are necessary in order to adequately describe the observed return processes.*”

<sup>3</sup>The realized power variation is the sum of the absolute values of the increments of the process within the interval raised to some power  $p$ .

be shown in the paper).

Our empirical application based on the activity signature function reveals that a pure-jump model is appropriate for the modeling of carbon futures. We find that the process is very vibrant. Hence, a good stochastic model could be one of the pure-jump type, whose driving jumps come from a very active Lévy process<sup>4</sup>.

These findings lead to several economically important conclusions. First, jumps (coming from institutional news releases, energy or macroeconomic markets shocks, etc.) need to be explicitly taken into account when modelling spot and futures carbon price series. Second, the pure-jump dynamics of the process would imply that hedging is quite complicated on the carbon market, since volatility risk cannot be fully spanned with derivatives instruments.

The rest of the paper is organized as follows. Section 2 details the previous studies. Section 3 contains the empirical work. Section 4 concludes.

## 2 Previous studies

Several studies have attempted to identify the stochastic processes of carbon prices (both spot and futures). We recall briefly the main findings from this literature for continuous- and jump-diffusion models.

### 2.1 Continuous-diffusion models

#### 2.1.1 Spot price

Seifert et al. (2008) first derived the characteristic partial differential equation that fully describes the solution of a theoretical emissions trading model for CO<sub>2</sub> emissions certificates under the EU ETS:

$$V^{(t)} = -\frac{1}{2} (G(t))^2 V^{(xx)} - \frac{1}{2c} e^{rt} \left( V^{(x)} \right)^2 \quad (1)$$

with  $V^{(t)}$ ,  $V^{(x)}$  and  $V^{(xx)}$  the partial derivatives of the expected value  $V(t, x_t)$  of an optimal policy given the total expected emissions<sup>5</sup>  $x_t$  over the trading period  $[0, T]$ .  $G(t)$  denotes the volatility of the uncontrolled stochastic part of the process  $x_t$ , and is determined by drift and volatility of the underlying

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<sup>4</sup>No need to say that the pure-jump hypothesis has a very limited impact at longer horizons. Clearly, our methodology will have much more to say about the very short-term behavior of carbon prices, and will find many more applications in derivatives trading than in real asset valuation.

<sup>5</sup> $x_t$  is a controlled stochastic process, which describes the expectation of the cumulative emissions over the whole trading period at time  $t$ , taking into account both known emissions and abatement actions in the past up to time  $t$ , as well as expected future emissions from  $t$  onwards.

emissions rate<sup>6</sup>.  $c$  reflects the constant cost coefficient of abatement, and  $r$  constant interest rates.

Eq.(1) can be translated into the spot price function  $S(t, x_t)$  by using the fact that the spot price equals the marginal abatement costs. In a risk-neutral special case<sup>7</sup>, the analytically derived properties of the equilibrium CO<sub>2</sub> spot price yield to:

$$S(t, x_t) = -V^{(x)} = \frac{p}{1 - \frac{e^{(-p(p(T-t)+2c(e_0-x_t))/2c^2\sigma^2)} (-2 + \text{Erfc}(e_0 - x_t/\sqrt{2}\sigma\sqrt{T-t}))}{\text{Erfc}(p(T-t) + c(e_0 - x_t)/\sqrt{2}c\sigma\sqrt{T-t})}} \quad (2)$$

with  $p$  the penalty,  $e_0$  the initial endowment with certificates, and  $\sigma$  the volatility of the emissions rate.  $\text{Erfc}$  is the complementary error function. Eq.(2) states that:

- the spot price may not rise above the penalty (because CO<sub>2</sub>-regulated companies would no longer increase abatement but would rather pay the cheaper penalty);
- the spot price never actually reaches zero (which reflects the option value of emissions certificates);
- companies are always willing to abate emissions in order to mitigate some of the expected penalty costs (resulting in a positive spot price).

Note these results can be further generalized to the case of banking behavior on the emissions market<sup>8</sup>.

In terms of volatility dynamics, the resulting CO<sub>2</sub> spot prices follow:

$$dS = \left( -u_t S^{(x)} + S^{(t)} + \frac{1}{2} (G(t))^2 S^{(xx)} \right) dt + G(t) S^{(x)} dW_t \quad (3)$$

with  $u_t$  the optimal rate of abatement, and  $dW_t$  the increment of a standard Wiener process. Eq.(3) reveals that the volatility of  $S(t, x_t)$  depends on both  $t$  and the total expected emissions term  $x_t$ . Besides, the authors are able to show that the volatility of the spot price:

- approaches zero when  $x_t$  is far away from the initial endowment  $e_0$ ;
- increases with positive changes in the penalty cost, abatement costs, and emissions rate volatility;
- should be bounded between zero and the penalty.

<sup>6</sup> $G(t)$  can be exemplified for three processes where the emissions rate follows (i) an arithmetic Brownian motion without drift, (ii) a white noise process, or (iii) an Ornstein-Uhlenbeck process.

<sup>7</sup>Note that these results can be extended to the potential effect of risk-aversion on CO<sub>2</sub> prices.

<sup>8</sup>Banking consists in saving certificates for future use, by emitting less or abating more than would be required to cover the emissions of the current year.

Seifert et al. (2008) suggest to calibrate this CO<sub>2</sub> spot price process to market data, as an input to pricing CO<sub>2</sub> derivatives. For instance, these volatility properties are of interest to potential buyers or sellers of option contracts on CO<sub>2</sub> allowances, which are usually linked to some underlying spot price process to value the contract. In case of misspecified volatility properties, significant mispricing errors can occur. Besides, the spot price process can be used as an additional component in a power price model with emissions trading.

### 2.1.2 Futures price

Zhu et al. (2009) have proposed a stochastic model to represent the underlying stochastic properties of carbon futures prices:

$$d \ln S = [\theta(t) - a \ln S]dt + \sigma(t)dZ \quad (4)$$

with  $S$  the spot price movement at time  $t$ ,  $a$  the mean-reversion rate,  $\theta(t)$  the value of the mean-reversion level multiplied by the reversion rate, and  $\sigma$  the volatility of the logarithmic value of the spot price.  $dZ$  is a Wiener process of zero mean value and of variance  $dt$ .

This model can be seen as representing the carbon process in the risk-neutral world. When futures prices and implied volatilities of vanilla options are available (and liquid enough), the market data can be used to calibrate the parameters in Eq.(4). Once the volatility term structure, the mean-reversion level and the mean-reversion rate are determined, Eq.(4) can indeed be employed to simulate futures carbon prices<sup>9</sup>.

Note that this is a mean-reverting model inspired by Vasicek to capture the major stochastic features exhibited in carbon futures prices. In Section 2.2.2, we discuss whether it is indeed appropriate to retain this characteristic, which is common to model commodities price behavior<sup>10</sup>, also in the context of carbon prices.

Carmona and Hinz (2011) have developed another approach to model carbon futures prices. Consider historical observations of a futures contract on an allowance  $(A_t)_{t \in [0, T]}$  with compliance date  $T$  for a penalty  $\pi \in (0, \infty)$  recorded at times  $t_1 < t_2 < \dots < t_n$ . Denote  $a_t = \frac{1}{\pi} A_t$  the normalized futures

<sup>9</sup>The authors typically use Monte-Carlo simulations or trinomial tree methods to study or estimate costs or risks directly dependent on the futures carbon prices.

<sup>10</sup>The rationale for mean-reversion is that the price of a commodity tends to be pulled back to its production cost plus a margin (Schwartz (1997), Schwartz and Smith (2000)). For commodities such as energy, the mean is determined by the marginal cost of production and the extent of demand. In the short-run, there can be deviations from this arithmetic mean, but in the long-run the price converges towards the marginal costs of production as a result of competition among the producers. That is why the modeling of energy prices using the mean-reversion process is quite common.

price process, and  $\Phi$  the cumulative distributive function of the standard normal distribution. Their stochastic differential equation writes:

$$da_t = \Phi'(\Phi^{-1}(a_t)) \sqrt{z_t} dW_t \quad (5)$$

with  $(W_t)_{t \in [0, T]}$  the process of Brownian motion. The positive-valued function  $z_t$  is given by:

$$z_t = \frac{\sigma_t^2}{\int_t^T \sigma_u^2 du}, t \in (0, T) \quad (6)$$

with the variance  $\sigma^2$ . By calibrating this model to historical daily prices of a futures contract written on the EUA with maturity in December 2012, and by applying it to the valuation of European call options, the authors demonstrate that their model is capable of matching the observed historical or implied volatility of the underlying allowance futures price  $(A_t)_{t \in [0, T]}$ . As such, it possesses desirable properties from a hedging perspective and for efficient option valuation schemes.

Next, we review papers modeling explicitly the occurrence of jumps.

## 2.2 Jump-diffusion models

Identifying jumps in a stochastic process is important because it has implications for risk management, option pricing, portfolio selection and for optimal hedging strategies. Cont and Tankov (2004) discuss the issue of option pricing with jump processes as the underlying stochastic process. They show that the option prices differ significantly with or without jumps. Similarly, portfolio selection can be dramatically modified when some assets in the investment universe are potentially jumping (Liu et al. (2003), Cvitanic et al. (2008)).

### 2.2.1 Spot price

The first theoretical pricing of emissions allowances spot prices with jumps is given by Dannenberg and Ehrenfeld (2011) under a discretized form of the standard Vasicek process:

$$\Delta S_{t+1} = \alpha(S_{t+1}^* - S_t) + u_{t+1} \quad (7)$$

with  $S_t$  the spot market price in  $t$ ,  $\Delta S_{t+1} = S_{t+1} - S_t$  the expected price change,  $S^*$  the equilibrium price level of the process. After a stochastic shock  $u_{t+1}$  occurring between  $t$  and  $t + 1$ , the spot market price is ‘pulled back’ towards the mean level  $S^*$  by the mean reversion formula. The mean reversion rate



$\alpha$  stands for the strength with which the price is drawn back to the mean. This modelling is therefore close in its spirit to Zhu et al. (2009), as seen in Section 2.1.2.

Jumps can affect the allowance price due to the arrival of new information which fundamentally changes the expectations of the marginal abatement costs. The change of expectations is by definition very abrupt in this case. To account for jumps, the mean of the process  $S^*$  is shifted by this new information:

$$S_{t+1}^* = \begin{cases} S_t^* & \text{if } J(p) = 0, \\ H & \text{if } J(p) = 1. \end{cases} \quad (8)$$

with  $J$  the jump variable which acts like a switch: either a shift of the mean-reversion level occurs from  $t$  to  $t + 1$  ( $J = 1$ ) or not ( $J = 0$ ). For instance, the occurrence of jumps can be linked to institutional events on the carbon market. Every year, the European Commission aggregates submitted emissions data, and compares this to the quantity of allowances surrendered. The processing of emissions data for the entire zone takes a couple of months, and announcements on the market's net position are not released until mid-April (Ellerman and Buchner (2008)).

$J$  follows a Bernoulli process, with  $p$  the probability of a shift causing information to emerge. If the mean-reversion level is shifted to the new state  $H$ , this level is bounded above ( $S_{\max}$ ) and below ( $S_{\min}$ ). For the modeling of the new mean-reversion level  $H$ , a PERT distribution is chosen<sup>11</sup>. This modeling of the jump height is chosen to achieve an approximately unbiased price modelling. Overall, Dannenberg and Ehrenfeld (2011) suggest to adopt a Mean-Reversion Jump Diffusion (MRJD) process for allowance spot prices.

Çetin and Verschuere (2009) note that Seifert et al. (2008) avoid the natural complication of an equilibrium analysis by assuming every market participant risk-neutral or the existence of a representative agent with a logarithmic utility, thereby reducing the problem to a central planner who aims to maximize the total profit of all agents. By contrast, Çetin and Verschuere (2009) avoid the equilibrium approach to spot prices, but use the no-arbitrage principle between the spot and forward prices of carbon allowances<sup>12</sup>.

Without jumps, the EUA spot price is a continuous process  $S$  satisfying:

$$dS_t = S_t \mu(t, S_t, \theta_t) dt + S_t \sigma(t, S_t, \theta_t) dW_t \quad (9)$$

<sup>11</sup>We leave to the interested reader the explanation of the PERT distribution in the original article.

<sup>12</sup>By assuming an exogenous price process for the forward contract.

with  $\mu, \sigma$  some constants for all  $t$ ,  $S_0 = s$ , and  $W$  a Brownian motion.  $\theta$  is a Markov chain modelling the net position of the allowance market:

$$\theta_t = \theta_0 - 2 \int_0^t \theta_s \lambda(\theta_s) ds + N_t \quad (10)$$

with  $N$  a martingale. The Markov chain  $\theta$  stays in state  $i$  for an exponential amount of time with parameter  $\lambda(i)$ .  $\theta$  takes values in  $E = \{-1, 1\}$ :  $\theta_t = 1$  ( $\theta_t = -1$ ) corresponds to the market being long (short) at time  $t$ . To sum up, the price process  $S$  of carbon allowances depends on two sources of uncertainty:  $W$  and  $\theta$ .

Jumps are introduced under the form of intermediate announcements regarding the market's net position. Suppose at some  $t_0 < T$  the true position of the allowance market is revealed. Typically, there will be a jump at time  $t_0$  and the jump size equals:

$$\Delta P_{t_0} = h(t_0, S_{t_0}, \theta_{t_0}) - h(t_0, S_{t_0}, \tilde{\theta}_{t_0}) - Z_{t_0}^h \quad (11)$$

with  $h$  a function solution to a boundary value problem<sup>13</sup>,  $\tilde{\theta}_{t_0}$  the true state of  $\theta$  revealed at time  $t_0$ , and  $Z$  the Lévy's characterization of Brownian motion.

By numerical implementation<sup>14</sup>, the authors illustrate how intermediate announcements of a long position can cause a sharp decline in EUA prices. This situation mimics the information release in April 2006, which revealed that National Allocation Plans were so generous that there was almost no possibility for the market to end up short at the end of the period (Ellerman and Buchner (2008)). Hence, the model is satisfactory in capturing the main features observed for carbon spot prices during the first phase (2005-2007) of the EU ETS without banking and borrowing.

Borovkov et al. (2011) also propose a model of allowance spot pricing in the presence of jumps. According to the current practice of the EU ETS, revised decision on the amount of allocated certificates may yield to a jump of allowance market prices. Similarly, a sudden change in demand and/or price for fuel can result in the pollution levels changing dramatically, which impacts the allowance price.

Suppose that, at any time  $t = 0, \dots, T$ , credits can be traded at the spot price  $A_t$ :

$$A_t = \alpha(t, X_t), \quad t \in [0, T] \quad (12)$$

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<sup>13</sup>Sufficient conditions for the existence of a unique classical solution to stochastic differential equations are often given in terms of a boundary function.

<sup>14</sup>The calculation of the carbon spot price can be calculated in a Monte Carlo routine.

with  $\alpha$  a deterministic function on  $[0, T]$ .  $X_t$  is a state process given by:

$$X_t = -r(\alpha(t, X_t))dt + \sigma(t, X_t)dW_t + \int_{\mathbb{R}} a(t-, X_{t-}, y)(p_v - q_v)(dy \times dt) \quad (13)$$

with  $r$  the abatement function,  $\sigma$  the volatility level,  $W_t$  a Brownian motion adapted to a general jump-diffusion process,  $p_v$  an adapted random Poisson measure with intensity  $q_v(dy \times dt) = \lambda v(dy)dt$ ,  $v$  a probability distribution on  $\mathbb{R}$ , and  $\lambda \in (0, \infty)$  a positive constant. Eq.(13) allows to model jumps with great flexibility, as the distribution of the jump is state- and time-dependent. The authors establish the existence of a unique solution for the stochastic differential equation.

Then, by means of simulation, they conclude that in presence of jumps the allowance spot price tends to converge slower than expected to its boundary value. The main reason is that, following jumps, agents can revise their anticipations towards a sudden increase or decrease in pollution emissions (which impacts directly the allowance price). Similar modeling results can be found in Hinz and Novikov (2010).

### 2.2.2 Futures price

To capture the dynamics of EUAs, Daskalakis et al. (2009) reject the standard Brownian motion process in favor of jump-diffusion processes under the form:

$$dS_t = \mu(S_t, t)dt + \sigma(S_t, t)dW_t + y(S_t, t)dq_t \quad (14)$$

with  $S_t$  the EUA spot price at time  $t$ ,  $W_t$  a standard Wiener process,  $\mu(S_t, t)$  the drift,  $\sigma(S_t, t)$  the diffusion coefficient, and  $y(S_t, t)$  the jump amplitude. The drift, the diffusion and the jump coefficients are assumed to be general functions of time and the EUA spot price. The jump component is controlled by a Poisson process  $q_t$  with constant arrival parameter  $\lambda$ , i.e.  $Pr\{dq_t = 1\} = \lambda dt$  and  $Pr\{dq_t = 0\} = 1 - \lambda dt$ .  $dq_t$  and  $y$  are assumed to be mutually independent.

After estimating various configurations<sup>15</sup>, the authors find that Eq.(14) is better approximated by a Geometric Brownian motion process augmented by Jumps (GBMPJ):

$$dS_t = (\mu - \lambda\mu_j)S_tdt + \sigma S_t dW_t + (y - 1)S_t dq_t \quad (15)$$

<sup>15</sup>Namely, Geometric Brownian motion process, Mean-reverting square-root process, Mean-reverting logarithmic process, Constant-elasticity of variance, Geometric Brownian motion process augmented by Jumps, and Mean-reverting square-root process augmented by Jumps.

with  $\mu$  the expected return of the asset per unit of time, and  $\sigma$  its volatility. In Eq.(15),  $\lambda$  is the average number of jumps per year, and  $y$  the jump size distributed normally according to the following density:

$$f(y) = \frac{1}{\sqrt{2\pi}\sigma_j} \exp\left(-\frac{(y - \mu_j)^2}{2\sigma_j^2}\right) \quad (16)$$

with  $\mu_j$  and  $\sigma_j^2$  the mean and variance, respectively.

Their results indicate that Eq.(15) performs better against competing models. The addition of jumps improves significantly the performance of diffusion processes. Overall, their findings indicate that EUA spot prices have a proportional, non-mean reverting structure with jumps, i.e., they are subject to large movements that cannot be explained by standard diffusion processes. Note, however, that the authors do not investigate the presence of infinite activity jumps (as in our setting).

Besides, it is worth noting that the addition of mean-reversion appears to decrease the goodness-of-fit, especially in the case of jump-diffusion models. This finding stands therefore in sharp contrast to the approaches developed by Dannenberg and Ehrenfeld (2011), Lin and Lin (2007) and Zhu et al. (2009)<sup>16</sup>.

When the EUA spot prices are given by the jump-diffusion process detailed Eq.(15), the corresponding pricing formula for inter-Phase emission allowance futures<sup>17</sup> writes:

$$dS_t = (r - \delta - \lambda^* \mu_j^*) S_t dt + \sigma S_t dW_t + (y^* - 1) S_t dq_t \quad (17)$$

with  $r$  the risk-free rate, and  $\lambda^*, \mu_j^*, \sigma, y^*$  the equivalent risk adjusted parameters of Eq.(15). Assuming a non-zero mean-reverting convenience yield  $\delta$ , the inter-Phase EUA futures price is finally given by:

$$F_t(S_t, T) = \frac{S_t}{P(t, T)} A(t, T) e^{-H_c(t, T) \delta_t} \quad (18)$$

with  $\delta_t$  the current level of convenience yield,  $P(t, T)$  the price of a zero-coupon bond with maturity at time  $T$ ,  $\kappa_c$  the speed of mean-reversion of the convenience yield,  $\theta_c$  the long-run mean yield,  $\lambda_c$  the convenience yield market price of risk, and  $\sigma_c^2$  the variance of the change in the marginal convenience yield,  $H_c(t, T) = \frac{1 - e^{-\kappa_c(T-t)}}{\kappa_c}$ , and  $A(t, T) = \exp\left[\frac{(H_c(t, T) - (T-t))(\kappa_c^2 \theta_c - \kappa_c \lambda_c \sigma_c - \sigma_c^2/2 + \rho \sigma \sigma_c \kappa_c)}{\kappa_c^2} - \frac{\sigma_c^2 H_c^2}{4\kappa_c}\right]$ .

According to Daskalakis et al. (2009), Eq.(18) provides a satisfactory formula for the pricing of inter-

<sup>16</sup>While the modeling of two former papers has been detailed, we only mention here the findings of Lin and Lin (2007). They model carbon dioxide spot prices as a result of mean-reversion with varying trends, combined with state-dependent price jumps and volatility structure. Globally, their results show that mean-reversion with state-dependent price jumps performs the best in forecasting the EUA futures prices.

<sup>17</sup>i.e. with banking. Results with no banking are also provided.

Phase EUA futures as a two-factor model that assumes a jump diffusion process for the underlying and a stochastic, mean-reverting convenience yield<sup>18</sup>.

## 2.3 Summary

Table 1 provides a useful summary of the findings from previous studies:

**Table 1**  
Stochastic properties of carbon prices in previous literature

Diffusion Process	Brownian motion	Continuous Vasicek	Ornstein- Uhlenbeck	Poisson	Jump Lévy	Bernoulli
Seifert et al. (2008)			X			
Zhu et al. (2009)		X				
Carmona and Hinz (2011)	X					
Çetin and Verschuere (2009)	X				X	
Borovkov et al. (2011)	X			X		
Hinz and Novikov (2010)	X			X		
Carmona et al. (2009)			X			
Daskalakis et al. (2009)	X			X		
Lin and Lin (2007)		X		X		
Dannenberg and Ehrenfeld (2011)		X				X

As discussed in Section 2.1.2 and Section 2.2.2, controversy remains on the usefulness of modeling the carbon price process with mean-reversion. Besides, the inclusion of jumps (either under the form of Poisson- or Lévy-like processes) appears as a salient characteristic of most models developed.

Next, we develop our own empirical investigation through the use of high-frequency data.

## 3 Recovering the stochastic properties of the carbon price via the activity signature function

The rest of the paper is focused on determining the activity signature function of the carbon futures price. As emphasized in the Introduction, the previous literature does not make use of intraday data to infer the optimal stochastic process of carbon prices. In addition, the issue of infinite activity jumps, which is of central importance for option traders and investors has not been addressed. This Section fills this gap, and draws some conclusions on the fine structure of carbon prices. We explain first the methodology, second the data used, and third the empirical results.

<sup>18</sup>For a model with a stochastic convenience yield, see Hilliard and Reis (1998).

### 3.1 Methodology

Todorov and Tauchen (2010) define a new concept termed activity signature function, which is constructed from high-frequency observations of a continuous-time process. This function can be seen as a useful device for estimating the activity level of the underlying process, and in particular to determine its stochastic properties.

It is based on the Blumenthal-Gettoor (1961) index, which allows to classify the different processes used in continuous-time modeling in the following order (from low to high activity): finite activity jumps, finite variation (but infinite activity) jumps, absolutely continuous processes, jumps of infinite variation and continuous martingales<sup>19</sup>. Hence, the different models considered differ in whether the stochastic process contains continuous martingale and/or jumps:

$$\begin{aligned} Y_t &= \int_0^t b_{1s} ds + \int_0^t \sigma_{1s} dW_s \\ X_t &= \int_0^t b_{2s} ds + \int_0^t \int_{\mathbb{R}} \sigma_{2s-\kappa}(\chi) \tilde{\mu}(ds, dx) + \int_0^t \int_{\mathbb{R}} \sigma_{2s-\kappa'}(\chi) \mu(ds, dx) \\ Z_t &= X_t + Y_t \end{aligned} \tag{19}$$

with  $\int_0^t b_{1s} ds$  and  $\int_0^t b_{2s} ds$  the drift terms in respectively  $Y$  and  $X$ ,  $W_t$  a standard Brownian motion,  $\mu$  a jump measure,  $\kappa(\chi)$  a continuous truncation function. Stochastic volatility is generated through time-varying jump size  $\sigma_{2t}$  and time-varying intensity of the jumps.

In Eq.(19),  $Y_t$  represents a continuous model,  $X_t$  a pure-jump model, and  $Z_t$  a continuous plus jumps model. For instance, pure-jump models have recently been proposed as an alternative to standard jump-diffusions in finance<sup>20</sup>.

As mentioned in the Introduction, the idea to measure activity via the activity signature function is to compute the realized power variation at two different frequencies, and use the fact that the scaling factors at the two different frequencies will differ. Formally, let  $x_i$ ,  $i = 1, 2, \dots, n$ , denote the  $i$ th increment of the process over the period, and define the realized power variation:

$$V(p) = \sum_{i=1}^n |x_i|^p \tag{20}$$

In practice, the domain  $p \in (0, 4]$  exhausts the relevant information in the observed  $x_i$ . Next, define

<sup>19</sup>Note that if the underlying process is the superposition of several different processes from this list, then the measure will be equal to that of the most active component.

<sup>20</sup>A good example is the CGMY model by Carr et al. (2002).

the realized power variation using a sampling frequency half that of the original data:

$$\tilde{V}(p) = \sum_{i=1}^{\lfloor n/2 \rfloor} |x_{2i-1} + x_{2i}|^p \quad (21)$$

The activity signature function viewed as a function  $p$  on the domain  $(0, 4]$  is given by:

$$b(p) = \frac{\ln(2)p}{\ln(2) + \ln[\tilde{V}(p)] - \ln[V(p)]} \quad (22)$$

For high-frequency data with  $N$  intervals, it should be possible to compute the signature function for each of the  $N$  intervals over a fine grid of  $p$ , yielding  $\{\{b_s(p)\}_{p \in (0,4]}\}_{s=1}^N$  for  $s$  indexes intervals. Instead of reporting all  $N$  signature functions, Todorov and Tauchen (2010) suggest to look only for measures of central tendency and dispersion of the signature functions. For this purpose, the Quantile Activity Signature Function (QASF) is given by:

$$b_\alpha(p) = \text{Quantile}_\alpha [\{b_s(p)\}_{s=1}^N], \quad \text{for each } p \in (0, 4] \quad (23)$$

with  $\text{Quantile}_\alpha$  the  $\alpha$ th quantile of a sequence of numbers. The most informative plots are obtained from the lower  $b_{0.25}(p)$  and upper  $b_{0.75}(p)$  quartiles, and the median  $b_{0.50}(p)$  over the range  $0 \leq p \leq 4$ . As Todorov and Tauchen (2011) put it, robust methods such as using quantiles are crucial because the data sets can contain extreme observations.

In addition to this graphical approach, Todorov and Tauchen (2010) have proposed a point estimator based on the activity signature function. They note that there is information about the activity level embedded in the entire signature function, especially for ordinates (heights) of  $b(p)$  evaluated at the lower values of  $p$ . This observation suggests generating a consistent estimator of activity  $\beta$ :

$$\hat{\beta} = \frac{1}{b(\tau) - \tau} \int_{p=\tau}^{p=b(\tau)} b(p) dp \quad (24)$$

with  $\tau = 0.10$  the tuning parameter.

The activity index is always in the interval  $[0, 2]$ . For an absolutely continuous process, the index is always 1. If the estimated activity index is in the interval  $(0, 1)$ , that means that the observed process is from a pure-jump model with no drift and with jumps having finite variation. If the estimated index is in the interval  $(1, 2)$ , then the appropriate model for the observed process is again a pure-jump model, but now with jumps exhibiting infinite variation. If the estimated activity is 2, then a continuous

martingale is indicated.

Todorov and Tauchen (2010) have constructed a test for the presence of a continuous component against the alternative of a pure-jump process by using the activity signature function evaluated at a fixed power<sup>21</sup>. We will follow thoroughly their approach in our empirical application to carbon futures.

### 3.2 Data

We use high-frequency data on the EUA Futures contract traded on the European Climate Exchange (ECX) for the period going from January 2, 2009 to June 15, 2010 (369 trading days). Our data set includes transactions prices (as opposed to quotes) with time stamps. Each trading day is 11-hour and sampling every five-minutes gives 132 log-returns (increments).

We first investigate the quality of the data by using filter rules as in Barndorff-Nielsen and Shephard (2004). After cleaning the data for outliers<sup>22</sup>, the tick-by-tick data results in 437,801 observations, thus demonstrating the high liquidity of the carbon market in recent periods<sup>23</sup>.

The middle panel of Figure 1 shows that the liquidity of the market is continuously increasing, with the augmentation over time of the number of transactions and ticks available in intraday data. We record an average of 1,187 transactions/day, i.e. one transaction every 33.38 seconds.

The EUA ECX futures price series for the period under investigation is plotted in the top panel of Figure 1. Large moves can be observed in the allowance futures price, in particular during January-March 2009 which corresponds to a period of transmission of the financial crisis to the carbon market.

However, this graphical inspection does not inform us of the number of jumps in CO<sub>2</sub> intraday data, which are displayed in the bottom panel of Figure 1 following Barndorff-Nielsen and Shephard (2006)'s bipower variation measure. We notice the similarity with the previous plot, as the highest intensity of jumps is recorded during the period of large allowance price movements in January-March 2009.

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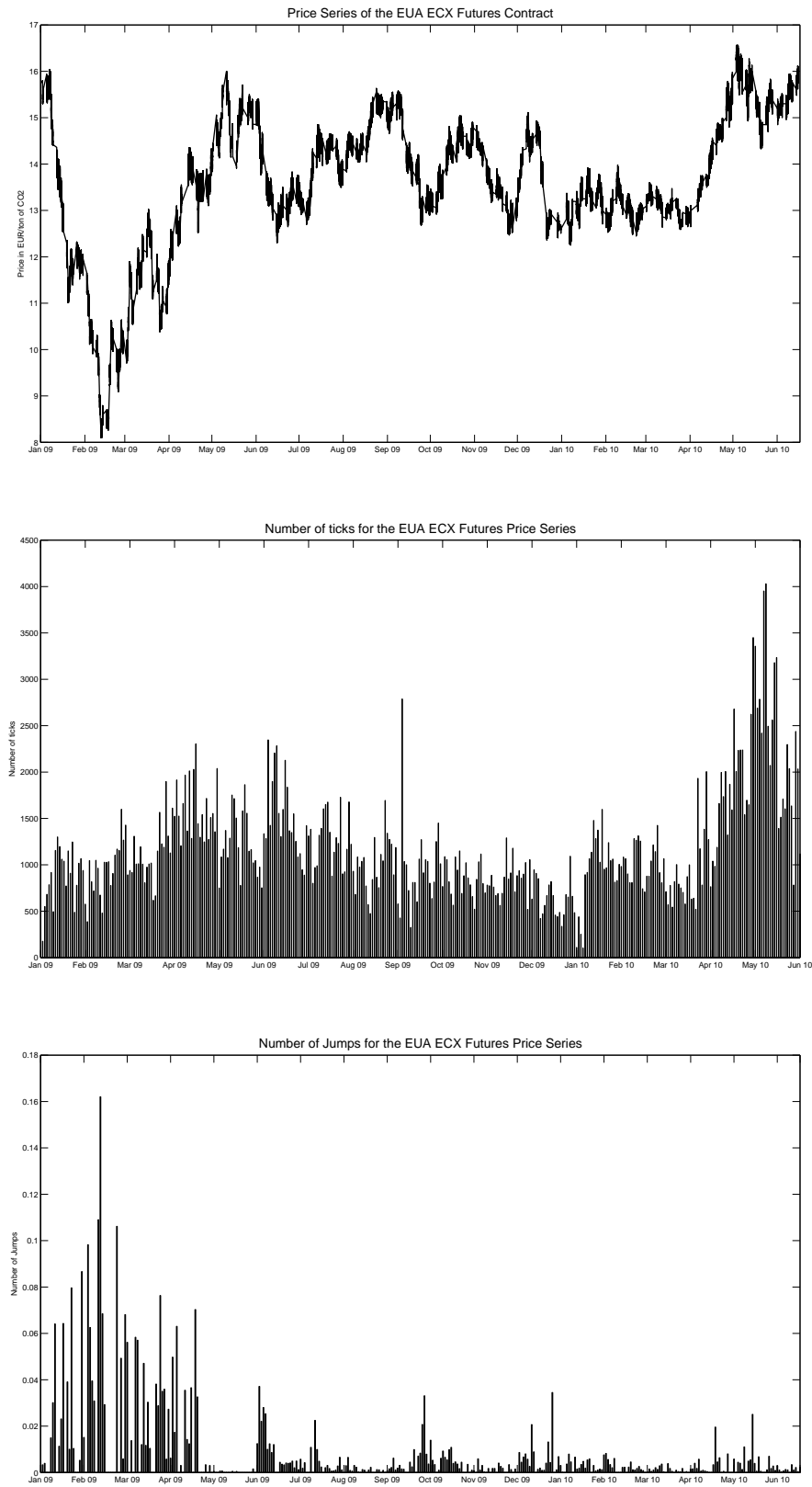
<sup>21</sup>Details on the setting of the test and the calculation of standard errors are not detailed here to conserve space, and can be found in Todorov and Tauchen (2010).

<sup>22</sup>Note that the cleaning of the data is a mandatory step when dealing with intraday data as, in our case, 22,177 observations (4.8% of the original data) have been removed after applying standard checks.

<sup>23</sup>Due to data limitations, note that the analysis is restricted to futures trading, since the frequency of intraday transactions available on the BlueNext spot market for CO<sub>2</sub> allowances is insufficient to yield to reliable estimates (i.e. only 38,924 ticks are available during our sample period).



**Figure 1**  
EUA ECX Futures Price Series (top), Number of Ticks (middle) and Number of Jumps (bottom)



Source: European Climate Exchange

### 3.3 Results

For each day, we calculate the activity signature function  $b_s(p)$  where  $s = 1, \dots, 369$  and then evaluate the quantile activity function QASF as  $b_\alpha(p)$  defined in Eq.(23). We enhance Todorov and Tauchen (2010)'s methodology by relying on a two-scale methodology as in Zhang et al. (2005)<sup>24</sup>.

Figure 2 show the QASFs for  $\alpha = 0.25, 0.50, 0.75$  computed on the observed data with 1-day and 5-day blocks. Inspecting this graph can be helpful to characterize the stochastic properties of the carbon futures price process. The abscissa represents the values of  $p$ . The ordinates represent the heights of  $b(p)$ . In the absence of jumps, one can expect the median QASF to appear horizontally flat around 2 over the entire domain  $p \in (0, 4]$ . For abscissa above 2, only jumps matter asymptotically. Hence, for  $p > 2$ , the behavior of the activity signature function can reveal us whether there are jumps, large or small, in the process. The upper  $b_{0.75}(p)$  and lower  $b_{0.25}(p)$  QASFs provide an indication of sampling dispersion.

In the top panel of Figure 2, the QASF suggests that the process generating the carbon futures price data lacks a continuous component, as it deviates strongly from the reference horizontal line set at 2. The QASF also provides evidence on the intensity of the jumps present in the discretely-observed process. The 50th and the 75th quantile of QASF computed on observed data clearly increase for  $p > 2$ , reflecting the high intensity of the jumps. This indicates a pure-jump model, which we need to verify with Todorov and Tauchen (2010)'s statistical test.

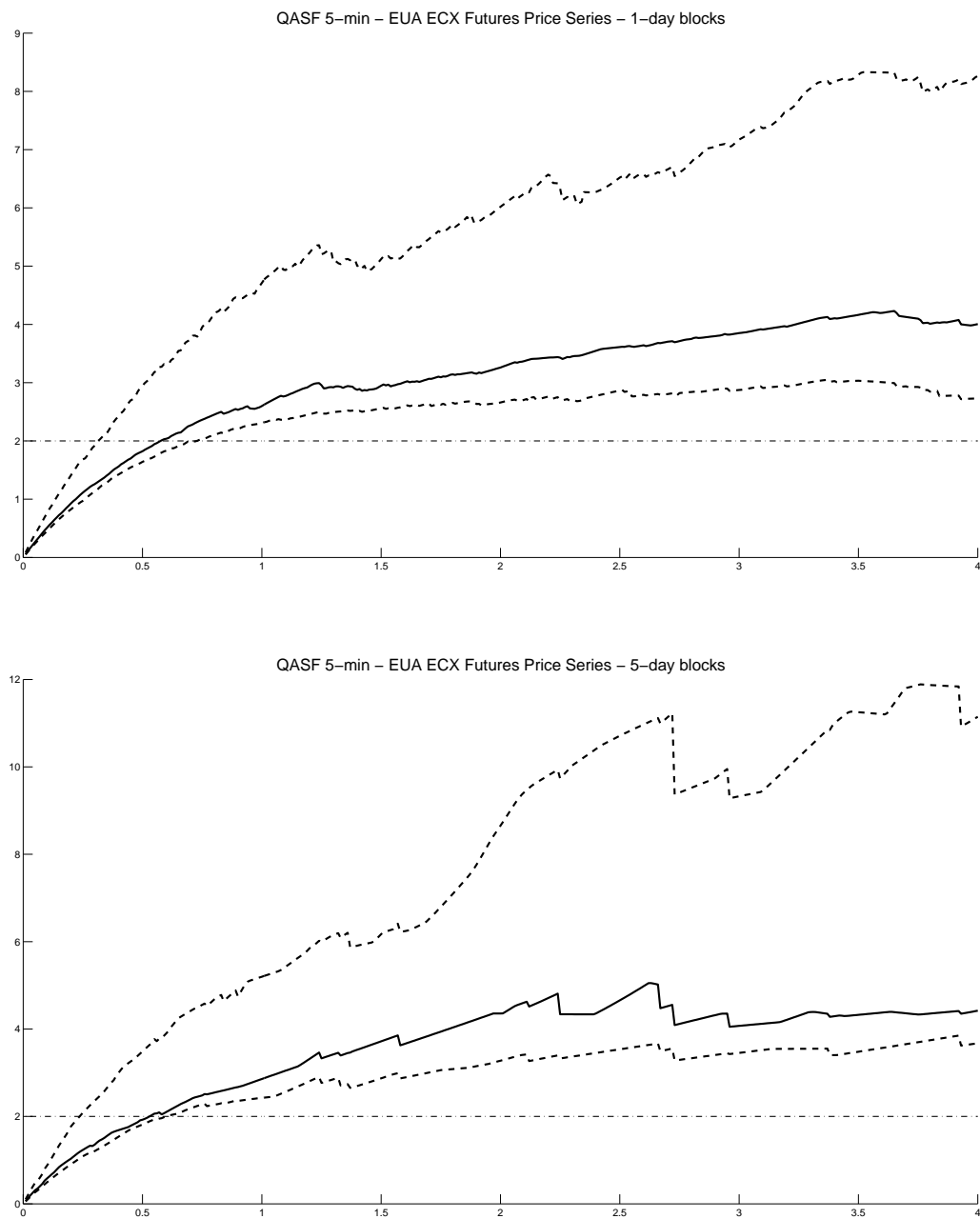
Note that similar comments arise from the bottom panel of Figure 2 (with 5-day blocks), which can be seen as a sensitivity test to the choice of length of the block.

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<sup>24</sup>In short, we keep a sampling interval for intraday returns of 5 minutes as in in the bulk of the literature using this kind of data, but we sample with starting points at each minute. Thus, we have 5 more estimates of the ratio at each period and an average of these ratios provides much more robust results.

**Figure 2**

QASF for the EUA ECX Futures Price Series for 1-day blocks (top), and 5-day blocks (bottom)



Note: QASF is computed using 5-minute sampling intervals. Lower and upper-quartiles are represented in dashed lines, along with the reference horizontal line fixed at 2.

**Table 2**

Test for presence of Brownian component in EUA ECX Futures Price Series

$p$	$\text{med}(\hat{\beta})$	MAD	Rejections (5%)
0.10	0.5099	0.1702	98.04
0.20	0.9258	0.2907	89.10
0.30	1.2588	0.3884	78.60
0.40	1.5611	0.4668	62.80
0.50	1.8234	0.5611	45.30

Note: MAD stands for the median absolute deviation about the median.

As stated in Section 3.1, an activity level of 2 has a special meaning since it separates pure-jump models from models containing continuous martingales. In Table 2, we conduct a test whether  $\hat{\beta}$  is statistically less than 2. The null hypothesis is that a continuous martingale component is present in the process (i.e. that its activity index equals 2). The rejection rates are for 5 values of  $p$  between 0.45 and 0.98 at the 5% level. Hence, we find strong statistical evidence against a model with a continuous component<sup>25</sup>.

Besides, column (2) in Table 2 shows the median of the point estimates based on Eq.(24), along with the median absolute deviation about the median ( $\text{MAD}=\text{med}|\hat{\beta}-\text{med}(\hat{\beta})|$ ) in column (3). The estimates indicate that the activity index for carbon futures is in the range of 1.82. This result is higher than the findings by Todorov and Tauchen concerning the NASA internet traffic (1.10-1.50), and close to the value found for the VIX (1.73-1.83) by Todorov and Tauchen (2011). The conclusion is that the carbon futures allow for a pure-jump model without a continuous component and of infinite variation.

### 3.4 Comparison with previous literature

**Table 3**

Comparison of the stochastic properties of carbon prices with previous literature

Diffusion Process	Brownian motion	Continuous Vasicek	Ornstein- Uhlenbeck	Poisson	Jump Lévy	Bernoulli
Seifert et al. (2008)			X			
Zhu et al. (2009)		X				
Carmona and Hinz (2011)	X					
Çetin and Verschuere (2009)	X				X	
Borovkov et al. (2011)	X			X		
Hinz and Novikov (2010)	X			X		
Carmona et al. (2009)			X			
Daskalakis et al. (2009)	X			X		
Lin and Lin (2007)		X		X		
Dannenberg and Ehrenfeld (2011)		X				X
<b>Our study</b>				X	X	X

With reference to the literature survey in Section 2, our study tends to invalidate the mean-reversion

<sup>25</sup>As discussed by Todorov and Tauchen (2011), the findings regarding the volatility activity seem reasonably insensitive to the choice of the tuning parameter.

processes identified in Danneberg and Ehrenfeld (2011), Lin and Lin (2007), Zhu et al. (2009) for carbon futures data<sup>26</sup>. Our results are the closest in spirit to the findings by Çetin and Verschuere (2010). Most notably, compared to previous literature, we find evidence against using Brownian motion.

## 4 Conclusion

In this paper, we are interested in documenting empirically the stochastic properties of carbon futures prices. Following an exhaustive literature review in mathematical finance, we apply Todorov and Tauchen (2010,2011)’s recent methodology termed activity signature function.

In short, Todorov and Tauchen (2010) propose to compute the logarithm of the sum of power variations for different values of the power argument, and show that the ratio of this sum for two different sampling frequencies has interesting properties.

This estimation strategy is based on inferring from the data the value of a generalized Blumenthal-Gettoor (1961) index. It lies in the interval  $[0,2]$  and measures the vibrancy of the process. The Brownian motion (and any diffusion process) has an activity level of 2. Values of the index in  $(0,1)$  correspond to jump processes of finite variations. Values of the index in  $(1,2)$  correspond to jump processes of infinite variation.

We estimate the activity index of high-frequency data on carbon futures from the ECX. Our estimation of the activity is based on constructing from the high-frequency data a diagnostic tool proposed in Todorov and Tauchen: Quantile Activity Signature Functions. Using the QASFs allow to limit the impact of extreme realizations, while keeping a reliable idea of the behavior of  $b(p)$  over the different periods of observations.

The visual inspection of QASFs for carbon futures data reveals that the process should be modelled as a pure-jump process, as it crosses the horizontal axis at a value below 2. Both the median and the upper quartile illustrate this result.

Moreover, we test formally for the presence of a Brownian component. The null hypothesis of the test is that the underlying process contains continuous martingale plus possibly jumps (i.e. the activity index equals 2). The alternative is that the underlying process lacks a continuous martingale, and the index is thereby less than 2, so the test is one-sided. We find statistical evidence against using the Brownian motion component.

To summarize, the evidence suggests that carbon futures are a pure-jump process without a continuous

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<sup>26</sup>Note, nevertheless, that mean-reversion is difficult to infer from a limited number of daily data.

component and a relatively high activity index. The occurrence of jumps on the carbon market may be related to information disclosure about allocation, or changes in perimeter of the scheme (Çetin and Verschuere (2009), Zhu (2009), Dannenberg and Ehrenfeld (2011)).

These findings have important implications for modeling of the spot and future processes of the carbon price. Derivatives models à la Cont and Tankov (2004), the CGMY model (Carr et al. (2002)) and Exponential-Ornstein-Uhlenbeck-Lévy type stochastic models (Andersen et al. (2002), Barndorff-Nielsen and Shephard (2001), Chernov et al. (2003)) would be plausible candidates, since these models accommodate a pure-jump model with activity indices above unity, as found in the carbon futures data.

Finally, another implication of our results relate to our understanding of the fine structure of asset prices. Indeed, microfounded theories (as in Geman et al. (2000) or Pakkanen (2010) among others) describe the price of assets as being pure jump processes. This result arises from market clearing conditions, the arrival of news which is discrete by nature and the trading behavior of investors who trade on news. Our results validate their hypothesis using high-frequency data.

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