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Ordered Weighted Averaging in Social Networks†

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Abstract

We study a stochastic model of influence where agents have yes-no inclinations on some issue, and opinions may change due to mutual influence among the agents. Each agent independently aggregates the opinions of the other agents and possibly herself. We study influence processes modelled by ordered weighted averaging operators. This allows to study situations where the influence process resembles a majority vote, which are not covered by the classical approach of weighted averaging aggregation. We provide an analysis of the speed of convergence and the probabilities of absorption by different terminal classes. We find a necessary and sufficient condition for convergence to consensus and characterize terminal states. Our results can also be used to understand more general situations, where ordered weighted averaging operators are only used to some extend. Furthermore, we apply our results to fuzzy linguistic quantifiers.

JEL classification: C7, D7, D85

Keywords: Social network, influence, convergence, speed of convergence, consensus, ordered weighted averaging operator, fuzzy linguistic quantifier

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1 Introduction

In the present work we study an important and widespread phenomenon which affects many aspects of human life – the phenomenon of *influence*. Being undoubtedly present, e.g., in economic, social and political behaviors, influence frequently appears as a dynamic process. Since social networks play a crucial role in the formation of opinions and the diffusion of information, it is not surprising that numerous scientific works investigate different dynamic models of influence in social networks. In what follows we mention only some works related to this paper, and for an overview of the vast literature on influence we refer, e.g., to Jackson (2008).

The seminal model of opinion and consensus formation is due to DeGroot (1974), where the opinion of an agent is a number in $[0, 1]$ and she aggregates the opinions (beliefs) of other agents through a weighted arithmetic mean. The interaction among agents is captured by the social influence matrix. Several scholars have analyzed the DeGroot framework and proposed different variations of it, in which the updating of opinions can vary in time and along circumstances. However, most of the influence models usually assume a convex combination as the way of aggregating opinions. Golub and Jackson (2010) examine convergence of the social influence matrix and reaching a consensus, and the speed of convergence of beliefs, among other things. DeMarzo et al. (2003) consider a model where an agent may place more or less weight on her own belief over time. Another framework related to the DeGroot model is presented in Asavathiratham (2000). Büchel et al. (2011) introduce a generalization of the DeGroot model by studying the transmission of cultural traits from one generation to the next one. Büchel et al. (2012) analyze an influence model in which agents may misrepresent their opinion in a conforming or counter-conforming way. Calvó-Armengol and Jackson (2009) study an overlapping-generations model in which agents, that represent some dynasties forming a community, take yes-no actions.

Also López-Pintado (2008, 2010), and López-Pintado and Watts (2008) investigate influence networks and the role of social influence in determining distinct collective outcomes. Related works can also be found in articles by van den Brink and his co-authors, see, e.g., van den Brink and Gilles (2000); Borm et al. (2002). A different approach to influence, i.e., a method based on simulations, is presented in Mäs (2010). Morris (2000) analyzes the phenomenon of contagion which occurs if an action can spread from a finite set of individuals to the whole population.

Another stream of related literature concerns models of Bayesian and observational learning where agents observe choices over time and update their beliefs accordingly, see, e.g., Banerjee (1992), Ellison (1993), Bala and Goyal (1998, 2001), Gale and Kariv (2003) and Banerjee and Fudenberg (2004). A model of strategic influence is studied in Galeotti and Goyal (2009). Mueller-Frank (2010) considers continuous aggregation functions with a special property called "constricting" and studies convergence applied to non-Bayesian learning in social networks.

Grabisch and Rusinowska (2010, 2011a) investigate a one-step deterministic model of influence, where agents have yes-no inclinations on a certain issue and their opinions may change due to mutual influence among the agents. Grabisch and Rusinowska (2011b) extend it to a dynamic stochastic model based on aggregation functions. Each agent independently aggregates the opinions of the other agents and possibly herself. Since any aggregation function is allowed when updating the opinions, the framework covers numerous existing models of opinion formation. The only restrictions come from the definition of an aggregation function, i.e., boundary conditions and nondecreasingness. The latter condition implies that only a kind of positive influence can be covered by the model. Grabisch and Rusinowska (2011b) provide a general analysis of convergence in the aggregation model and find all terminal classes and states.

In the present paper we study this influence model based on aggregation functions and continue the analysis done in Grabisch and Rusinowska (2011b). More precisely, we examine a particular way of aggregating the opinions and investigate influence processes modelled by *ordered weighted averaging operators (ordered weighted averages)*, commonly called *OWA operators* and introduced in Yager (1988). Roughly speaking, OWA operators are similar to the ordinary weighted averages, with the essential difference that weights are not attached to agents, but to the *ranks* of the agents in the input vector. As a consequence, OWA operators are in general nonlinear, and include as particular cases the median, the minimum and the maximum, as well as the (unweighted) arithmetic mean. More importantly, OWA operators can model *fuzzy linguistic quantifiers* introduced in Zadeh (1983), also called *soft quantifiers*. Typical examples of such quantifiers are expressions like "almost all", "most", "many" or "at least a few"; see Yager and Kacprzyk (1997). OWA operators have been widely applied, in particular, to multi-criteria decision-making. Jiang and Eastman (2000), for instance, apply OWA operators to geographical multi-criteria evaluation, and Malczewski and Rinner (2005) present a fuzzy linguistic quantifier extension of OWA in geographical multi-criteria evaluation.

Using ordered weighted averages in (social) networks is quite new, although some scholars have already initiated such an application; see Cornelis et al. (2010) who apply OWA operators to trust networks. To the best of our knowledge, ordered weighted averages have not been used in influence networks yet, although applying OWA operators to the analysis of influence has undoubtedly several advantages. In particular, it allows to study situations where the influence process resembles a *majority vote*, which are not covered by the classical (commonly used) approach of weighted averaging aggregation. By using OWA operators we can analyze situations in which only the number of agents that share the same opinion matters for the influence process. In other words, models in which all agents use OWA operators for aggregating opinions are *anonymous*. By applying our results on OWA operators to fuzzy linguistic quantifiers we can analyze *soft majority* and *soft minority* in voting. For instance, we can study situations in which an agent changes her opinion if "most" or "many" or "at least a few" of the agents say "yes".

In the face of such a high applicability of OWA operators, in the present paper we aim at providing a detailed analysis for aggregation models with these operators. Moreover, some new results on the general setup of Grabisch and Rusinowska (2011b) are delivered.

The main objectives of the paper and its theoretical results are the following:

- Analyzing the speed of convergence and the probabilities of absorption by different terminal classes in the general model (Propositions 1 and 2) and in anonymous models (Corollaries 1 and 2).

- Finding a necessary and sufficient condition for a coalition to be yes-influential or no-influential (Proposition 4) in the model with OWA operators.

- Characterizing terminal states in the model with OWA operators (Proposition 5).

- Finding a necessary and sufficient condition for convergence to consensus in the model with OWA operators (Theorem 2, Corollary 4).

- Finding a sufficient condition for convergence to consensus in the model with OWA-decomposable aggregation functions, i.e., in more general situations where ordered weighted averaging operators are only used to some extent (Corollary 5).

- Finding a sufficient condition for convergence to consensus in a model where agents use regular quantifiers and some are allowed to deviate from such a quantifier (Proposition 6, Remark 3).

- Characterizing terminal states in a model where agents use regular quantifiers (Proposition 7).

We illustrate our results by considering several examples, e.g., the model of Confucian society, majority voting, and mass psychology (also called herding behavior).

The remainder of the paper is organized as follows. In Section 2 we present the model and basic definitions. In Section 3 the speed of convergence and the absorption probabilities are studied. Section 4 concerns the convergence analysis in the aggregation model with OWA operators. In Section 5 we apply our results on ordered weighted averages to fuzzy linguistic quantifiers. Section 6 contains some concluding remarks. The longer proofs of some of our results are presented in the Appendix.

2 Model and Notation

Let $N := \{1, \ldots, n\}$, $n \geq 2$, be the set of agents that have to make a yes-no decision on some issue. Each agent $i \in N$ has an initial opinion $x_i \in \{0, 1\}$ (called *inclination*) on the issue, where "yes" is coded as 1.[1] During the influence process, agents may change their opinion due to mutual influence among the agents.

Definition 1 (**Aggregation function**)**.** An *n-place aggregation function* is any mapping $A : [0, 1]^n \to [0, 1]$ satisfying

(i) $A(0, \ldots, 0) = 0, A(1, \ldots, 1) = 1$ (boundary conditions) and

[1]Note that the model can be easily generalized to the case where agents have inclinations that are tendencies $x_i \in [0, 1]$ to say "yes". These tendencies generate a probability distribution of initial states. Our results still hold in this setting.

(ii) if $x \leq x'$ then $A(x) \leq A(x')$ (nondecreasingness).

To each agent i we assign an aggregation function A_i that determines the way she reacts to the opinions of the other agents and herself. Note that by using these functions we model positive influence only. Our aggregation model $\mathbf{A} = (A_1, \ldots, A_n)^t$ is stochastic, the outcome of one step of influence for each agent is her probability to say "yes". The aggregation functions our paper is mainly concerned with are *ordered weighted averaging operators* or simply *ordered weighted averages*. This class of aggregation functions was first introduced by Yager (1988).

Definition 2 (**Ordered weighted average**)**.** We say that an n-place aggregation function A is an *ordered weighted average* $A = \mathsf{OWA}_w$ with weight vector[2] w, if $A(x) = \sum_{k=1}^{n} w_k x_{(k)}$ for all $x \in [0,1]^n$, where $x_{(1)} \geq x_{(2)} \geq \ldots \geq x_{(n)}$ are the ordered components of x.

Let us denote by 1_S the characteristic vector of $S \subseteq N$, i.e., $(1_S)_j = 1$ if $j \in S$ and $(1_S)_j = 0$ otherwise. If 1_S represents the current opinions of the agents, then we say that the model is in *state* or *coalition* S. We sometimes denote a coalition $S = \{i, j, k\}$ simply by ijk and its cardinality or size by s. The definition of an aggregation function ensures that the two consensus states – the yes-consensus $\{N\}$ where all agents say "yes" and the no-consensus $\{\emptyset\}$ where all agents say "no" – are fixed points of the aggregation model $\mathbf{A} = (A_1, \ldots, A_n)^t$. We call them *trivial terminal classes*. In general, a terminal class is defined as follows:

Definition 3 (**Terminal class**)**.** A *terminal class* is a collection of states $\mathcal{C} \subseteq 2^N$ that forms a strongly connected and closed component, i.e., for all $S, T \in \mathcal{C}$, there exists a path[3] from S to T and there is no path from S to T if $S \in \mathcal{C}, T \notin \mathcal{C}$.

We can decompose the state space into disjoint terminal classes – also called absorbing classes – $\mathcal{C}_1, \ldots, \mathcal{C}_l \subseteq 2^N$, for some $l \geq 2$, and a set of transient states $\mathcal{T} = 2^N \backslash (\bigcup_{k=1}^{l} \mathcal{C}_k)$. For convenience, we denote by $\mathcal{C}^{\cup} := \bigcup_{k=1}^{l} \mathcal{C}_k$ the set of all states within terminal classes. Let us now define the notion of an influential agent (Grabisch and Rusinowska, 2011b).

[2] A vector $w \in [0,1]^n$ is a weight vector if $\sum_{i=1}^{n} w_i = 1$.

[3] We say that there is a path from S to T if there is $K \in \mathbb{N}$ and states $S = S_1, S_2, \ldots, S_{K-1}, S_K = T$ such that $A_i(S_k) > 0$ for all $i \in S_{k+1}$ and $A_i(S_k) < 1$ otherwise, for all $k = 1, \ldots, K-1$.

Definition 4 (**Influential agent**). (i) An agent $j \in N$ is *yes-influential* on $i \in N$ if $A_i(1_{\{j\}}) > 0$.

(ii) An agent $j \in N$ is *no-influential* on $i \in N$ if $A_i(1_{N \setminus \{j\}}) < 1$.

The idea is that j is yes-(or no-)influential on i if j's opinion to say "yes" (or "no") matters for i in the sense that there is a positive probability that i follows the opinion that is solely held by j. Analogously to influential agents, we can define influential coalitions (Grabisch and Rusinowska, 2011b).

Definition 5 (**Influential coalition**). (i) A nonempty coalition $S \subseteq N$ is *yes-influential* on $i \in N$ if $A_i(1_S) > 0$. It is *minimal yes-influential* if additionally $A_i(1_{S'}) = 0$ for all $S' \subsetneq S$.

(ii) A nonempty coalition $S \subseteq N$ is *no-influential* on $i \in N$ if $A_i(1_{N \setminus S}) < 1$. It is *minimal no-influential* if additionally $A_i(1_{N \setminus S'}) = 1$ for all $S' \subsetneq S$.

Making the assumption that the probabilities of saying "yes" are independent among agents[4] and only depend on the current state, we can represent our aggregation model by a time-homogeneous Markov chain with transition matrix $\mathbf{B} = (b_{S,T})_{S,T \subseteq N}$, where

$$b_{S,T} = \Pi_{i \in T} A_i(1_S) \Pi_{i \notin T}(1 - A_i(1_S)).$$

Note that for each coalition $S \subseteq N$, the transition probabilities to coalitions $T \subseteq N$ are represented by a certain row of $\mathbf{B}$. The m-th power of a matrix, e.g., $\mathbf{B} = (b_{S,T})_{S,T \subseteq N}$, is denoted by $\mathbf{B}^m = (b_{S,T}(m))_{S,T \subseteq N}$. Moreover, let $\{X_k\}_{k \in \mathbb{N}}$ be a homogeneous Markov chain and $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space corresponding to $\mathbf{B}$, i.e.,

$$\mathbb{P}(X_{k+1} = T | X_k = S) = b_{S,T} \text{ for all } k \in \mathbb{N}, S, T \subseteq N.$$

Note that this Markov chain is neither irreducible nor recurrent since it has at least two terminal classes – also called communication classes.

[4]This assumption is not limitative, and correlated opinions may be considered as well. In the latter case, only the next equation giving $b_{S,T}$ will differ.

3 Speed of Convergence and Absorption

We first study the *speed of convergence* – also called time before absorption – of the influence process to terminal classes. Secondly, we investigate the probabilities of convergence to each of the consensus states and possibly other terminal classes – we call them *absorption probabilities*. We provide an analysis for the general setup of Grabisch and Rusinowska (2011b) and for *anonymous models*, which particularly cover the case where all agents use ordered weighted averages. The concept of anonymity allows to reduce the computational demand a lot in many situations.

Before we study the convergence to terminal classes, we have a look at the possible types of these classes. Grabisch and Rusinowska show that there are three different types of terminal classes in the general model.

Theorem 1 (Grabisch and Rusinowska, 2011b). *In an aggregation model with aggregation functions $A_1, \dots, A_n$, terminal classes are*

(i) either singletons $\{S\}$, $S \subseteq N$,

(ii) cycles of nonempty sets $\{S_1, \dots, S_k\}$ of any length $2 \leq k \leq \binom{n}{\lfloor n/2 \rfloor}$ (and therefore they are periodic of period k) with the condition that all sets are pairwise incomparable (by inclusion) or

(iii) collections $\mathcal{R}$ of nonempty sets with the property that $\mathcal{R} = \mathcal{R}_1 \cup \cdots \cup \mathcal{R}_p$, where each subcollection $\mathcal{R}_j$ is an interval $\{S \in 2^N \mid S_j \subseteq S \subseteq S_j \cup K_j\}$, with $S_j \neq \emptyset, S_j \cup K_j \neq N$, and at least one K_j is nonempty.

To terminal classes of the first type we usually refer to as *terminal states*, to the second type as *cyclic terminal classes* and to the last type as *regular terminal classes*.

Suppose that $\mathbf{B}$ is obtained from an aggregation model $A_1, \dots, A_n$ and that there is at least one transient state, i.e., $\mathcal{T} \neq \emptyset$. We assume that the process starts from one of these states, that is, we take some $S \in \mathcal{T}$ as the *initial coalition*. Note that since the set of transient states is finite, we have convergence to the terminal classes almost surely. We say that the influence process $\mathbf{B}$ converges to the terminal classes after m steps of influence if $\{X_{m-1} \in \mathcal{T}, X_m \notin \mathcal{T}\}$. Thus, the speed of convergence is the time it takes for the process to leave the set of transient states.[5] To measure it, we use stopping times and rely on results provided in Brémaud (1999). Let τ_S be a stopping time such that $\{\tau_S = m\}$ if we have convergence to the terminal classes after m steps of influence when S is the initial coalition, i.e.,

$$\{\tau_S = m\} = \{X_m \notin \mathcal{T}, X_{m-1} \in \mathcal{T} | X_0 = S\}.$$

Our aim is to determine the distribution of the speed of convergence, given by the distribution of τ_S. It turns out that the latter is solely determined by the transition probabilities within the set of transient states.

Proposition 1. *Suppose* $\mathbf{B}$ *is obtained from an aggregation model with aggregation functions* $A_1, \ldots, A_n$*. If* $S \in \mathcal{T}$ *is the initial coalition, then*

$$\mathbb{P}(\tau_S > m) = \sum_{T \in \mathcal{T}} q_{S,T}(m),$$

where $Q = \mathbf{B}|_{\mathcal{T}}$*. Furthermore,*

$$\mathbb{E}[\tau_S] = \sum_{m=0}^{\infty} \sum_{T \in \mathcal{T}} q_{S,T}(m) < +\infty.$$

Proof. The first part follows from Brémaud (1999). For the expected value of τ_S, first note that it only takes nonnegative integer values. The first equality of the following computation follows from this fact, whereas the third equality and the inequality follow since $\mathcal{T}$ is finite and Q is strictly sub-stochastic, i.e., $\sum_{m=0}^{\infty} Q^m < +\infty$.[6]

$$\mathbb{E}[\tau_S] = \sum_{m=0}^{\infty} \mathbb{P}(\tau_S > m) = \sum_{m=0}^{\infty} \sum_{T \in \mathcal{T}} q_{S,T}(m) = \sum_{T \in \mathcal{T}} \sum_{m=0}^{\infty} q_{S,T}(m) < +\infty.$$

□

[5] Note that we do not consider the speed of convergence to certain terminal classes since its expected value will be infinite if there is a positive probability that this may not happen. Instead, we consider later on the absorption probabilities of certain terminal classes.

[6] cf. Brémaud (1999). Of course, it is understood that the right member is a matrix whose entries are all $+\infty$.

The next step is to look at the absorption probabilities of certain terminal classes. Define by

$$D = (d_{S,T})_{S\in\mathcal{T},T\in\mathcal{C}^{\cup}} := (b_{S,T})_{S\in\mathcal{T},T\in\mathcal{C}^{\cup}}$$

the matrix of transition probabilities from transient states to states within terminal classes. We can decompose D into matrices

$$D_k := (d_{S,T})_{S\in\mathcal{T},T\in\mathcal{C}_k}$$

of transition probabilities from transient states to states within a certain terminal class. For our analysis, it does not matter at which state the influence process enters a terminal class and hence we can reduce the matrix D by considering a terminal class $\mathcal{C}_k$ simply as a terminal state $\widetilde{\mathcal{C}_k}$. The transition probabilities from transient states to a terminal class $\mathcal{C}_k$ are then given by the vector

$$\widetilde{D_k} := \left(\sum_{T\in\mathcal{C}_k} d_{S,T}\right)_{S\in\mathcal{T}}.$$

Let us denote the matrix of transition probabilities from transient states to the terminal classes by $\widetilde{D} := (\widetilde{D_1} : \cdots : \widetilde{D_l})$ and define $F := (\mathbf{I} - Q)^{-1}$.[7] Furthermore, denote by τ_S^k a stopping time such that $\{\tau_S^k = m\}$ if we have absorption by the terminal class $\mathcal{C}_k$ after m steps of influence when starting in state S. The following result immediately follows from Brémaud (1999).

Proposition 2. *Suppose* $\mathbf{B}$ *is obtained from an aggregation model with aggregation functions* $A_1, \ldots, A_n$*. If* $S \in \mathcal{T}$ *is the initial coalition, then we get for the absorption probabilities:*

$$\mathbb{P}(\tau_S^k < \infty) = g_{S,\widetilde{\mathcal{C}_k}},\ \textit{for } k = 1,\ldots,l,\ \textit{where } (g_{S,\mathcal{C}})_{S\in\mathcal{T},\mathcal{C}\in\{\widetilde{\mathcal{C}}_1,\ldots,\widetilde{\mathcal{C}}_l\}} := F\widetilde{D}.$$

This completes the analysis of the general case. Let us now turn to anonymous models.

Definition 6 (Anonymity). (i) We say that an n-place aggregation function A is *anonymous* if for all $x \in [0,1]^n$ and any permutation $\sigma : N \to N$, $A(x_1, \ldots, x_n) = A(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$.

[7]Note that for absorbing Markov chains the matrix F always exists since $Q^m \to 0$ for $m \to \infty$.

(ii) Suppose $\mathbf{B}$ is obtained from an aggregation model with aggregation functions $A_1, \ldots, A_n$. We say that the model is *anonymous* if for all $s, t \in \{0, 1, \ldots, n\}$,

$$\sum_{\substack{T \subseteq N: \\ |T|=t}} b_{S,T} = \sum_{\substack{T \subseteq N: \\ |T|=t}} b_{S',T} \text{ for all } S, S' \subseteq N \text{ of size } s.$$

For an agent using an anonymous aggregation function, only the size of the current coalition matters. Similarly, in models that satisfy anonymity, only the size of the current coalition matters for the further influence process.

Proposition 3. *An aggregation model with anonymous aggregation functions* $A_1, \ldots, A_n$ *is anonymous.*

Note that the converse does not hold, a model can be anonymous although not all agents use anonymous aggregation functions. Anonymous models are particularly interesting since ordered weighted averages are anonymous and hence also models where agents use them.

Remark 1. Aggregation models with aggregation functions $A_i = \mathsf{OWA}_{w^i}, i \in N$, are anonymous.

Assume now that the aggregation model is anonymous. Then, we can reduce the $2^n \times 2^n$ transition matrix $\mathbf{B}$ to an $(n+1) \times (n+1)$ matrix $\mathbf{B}^a = (b^a_{s,t})_{s,t \in \{0,1,\ldots,n\}}$, where

$$b^a_{s,t} := \sum_{\substack{T \subseteq N: \\ |T|=t}} b_{S,T}, \text{ for any } S \subseteq N \text{ of size } s,$$

are the transition probabilities from coalitions of size s to coalitions of size t. However, note that the gain in computational tractability – the dimensions of the transition matrix grow only linearly instead of exponentially with the number of agents – comes at the cost of losing track of the transition probabilities to certain states. We define an *anonymous terminal class* as $\mathcal{C}^a := \{s \in \{0, 1, \ldots, n\} | \exists\ S \in \mathcal{C} \text{ such that } |S| = s\}$. Similarly, we define $\mathcal{T}^a := \{s \in \{0, 1, \ldots, n\} | S \in \mathcal{T} \text{ if } |S| = s\}$ and use a stopping time τ_s when the initial coalition is of size $s \in \mathcal{T}^a$. Otherwise the notation carries over straightforwardly from the general case. Note that anonymous terminal classes are extended by states of the same size as states within the original class. This comes without loss of generality regarding absorption probabilities: if the influence process is in a state that was not part of the original class, it will converge to that class immediately due to anonymity. This also justifies not considering such states as possible initial states.

However, the speed of convergence will be distorted in case it is possible that the process arrives at a state which is only part of the anonymous terminal class and not of the original class. We call such a model *distorted*. In this case, we need to use the original model to compute the speed of convergence. Models that only have singleton terminal classes are not distorted, though.

Corollary 1. *Suppose* $\mathbf{B}^a$ *is obtained from an anonymous aggregation model with aggregation functions* $A_1, \ldots, A_n$ *that is not distorted. If* $s \in \mathcal{T}^a$ *is the size of the initial coalition, then*

$$\mathbb{P}(\tau_s > m) = \sum_{t \in \mathcal{T}^a} q^a_{s,t}(m) \text{ and } \mathbb{E}[\tau_s] = \sum_{m=0}^{\infty} \sum_{t \in \mathcal{T}^a} q^a_{s,t}(m) < +\infty.$$

As already said, the result on absorption probabilities for anonymous models is straightforward.

Corollary 2. *Suppose* $\mathbf{B}^a$ *is obtained from an anonymous aggregation model with aggregation functions* $A_1, \ldots, A_n$*. If* $s \in \mathcal{T}^a$ *is the size of the initial coalition, then we get for the absorption probabilities:*

$$\mathbb{P}(\tau^k_s < \infty) = g^a_{s,\widetilde{\mathcal{C}^a_k}}, \text{ for } k = 1, \ldots, l.$$

The initial coalition $S \in \mathcal{T}$ (or its size s) in the results above can as well be seen as a coalition (or its size) at some stage of the influence process before entering a terminal class. This finishes our analysis of the speed of convergence and absorption probabilities.[8] To illustrate the results, let us consider the model of a Confucian society studied by Hu and Shapley (2003) and Grabisch and Rusinowska (2011a).

Example 1 (**Confucian society**)**.** Consider a four member Confucian society $N = \{1, 2, 3, 4\}$ that consists of the king (agent 1), the man (2), the

[8] Of course, we could also discuss the convergence after the process has entered a terminal class. This is obvious at least for singleton and cyclic terminal classes, though. For the latter, there is clearly no convergence to a stationary distribution. Furthermore, it holds that regular classes are convergent if and only if their corresponding transition matrix is aperiodic.

wife (3) and the child (4). There are three principles in the decision-making process:

(i) The man follows the king;

(ii) the wife and the child follow the man;

(iii) the king should respect his people.

From the principles (i) and (ii), we get the following aggregation model:

$$\mathsf{Conf}_2(x) = x_1, \mathsf{Conf}_3(x) = \mathsf{Conf}_4(x) = x_2.$$

We can interpret the principle (iii) in different ways. If the inclination of the king is "yes", then his decision could be "yes" if either

(a) at least one of his people has the inclination to say "yes",

(b) at least two of his people have the inclination to say "yes" (majority) or

(c) all of his people have the inclination to say "yes" (unanimity).

If his inclination is "no", then he could change his opinion and say "yes" if either

(a) at least two of his people have the inclination to say "yes" (majority),

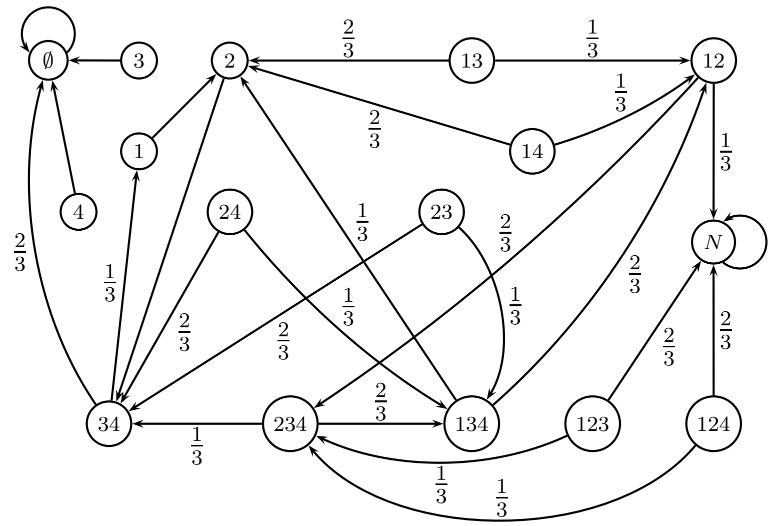
(b) all of his people have the inclination to say "yes" (unanimity) or he could

(c) stick to his inclination and say "no".

We consider a stochastic version where in each case the king chooses one of the three interpretations with equal probability. This leads to the following aggregation function of the king:

$$\mathsf{Conf}_1(x) = \frac{1}{3}(x_{(2)} + x_{(3)} + x_{(4)}).$$

The king is yes- and no-influential on the man, and so is the man on the wife and the child. Moreover, coalitions of size two or more are yes- and no-influential and all agents are no-influential on the king. This model corresponds to the following digraph of the Markov chain:



Note that there are only the trivial terminal classes and that the model does not satisfy anonymity.[9] The convergence to the terminal classes is immediate if initially only the wife or the child said "yes". We also see that the convergence is rather fast if initially only the king, only the man or both the wife and the child said "yes". It is slow if initially only the king or the man said "no". If initially only one agent or both the wife and the child said "yes", then there is convergence to the no-consensus for sure. For most coalitions, it is much more likely to reach the no-consensus. Only if initially at least the king and the man said "yes", there is a high probability to reach the yes-consensus. These findings reflect the intuition that the king and the man are somehow more powerful than the wife and the child in the Confucian society (see Table 1).

4 Convergence and Ordered Weighted Averages

In aggregation models where the influence process is determined by ordered weighted averages, it could be that the agents finally reach a consensus for sure. Or it could as well be possible to end in some other terminal class, e.g.,

[9]Both can be easily seen from the digraph. For anonymity, it is enough to observe that the probabilities to reach the no-consensus are different for coalitions $\{1\}$ and $\{3\}$.

$\mathbb{P}(\tau_S > m)$	1	3	5	10	15	25	$\mathbb{E}[\tau_S]$	$\mathbb{P}(\tau_S^N < \infty)$
1	1	.33	.33	.04	$< 10^{-2}$	$< 10^{-3}$	4.5	0
2	1	.33	.11	.04	$< 10^{-2}$	$< 10^{-3}$	3.5	0
3	0	0	0	0	0	0	1	0
4	0	0	0	0	0	0	1	0
12	.67	.52	.32	.08	.01	$< 10^{-3}$	4.53	.47
13	1	.44	.36	.06	.01	$< 10^{-3}$	4.84	.16
14	1	.44	.36	.06	.01	$< 10^{-3}$	4.84	.16
23	1	.48	.23	.06	.01	$< 10^{-3}$	4.39	.11
24	1	.48	.23	.06	.01	$< 10^{-3}$	4.39	.11
34	.33	.33	.11	.01	$< 10^{-2}$	$< 10^{-3}$	2.5	0
123	.33	.26	.16	.04	$< 10^{-2}$	$< 10^{-3}$	2.76	.74
124	.33	.26	.16	.04	$< 10^{-2}$	$< 10^{-3}$	2.76	.74
134	1	.56	.39	.07	.02	$< 10^{-3}$	5.18	.32
234	1	.63	.34	.08	.02	$< 10^{-3}$	5.29	.21

Table 1: Speed of convergence and absorption probabilities in the Confucian society (Example 1).

a state where the society is split up. Our aim is to investigate conditions for these outcomes. We also relax our setup and study the case where agents use ordered weighted averages only to some extend. As we already know, models where agents use ordered weighted averages are anonymous. In particular, this means that whether a yes- or no-consensus (or none of them) is finally reached does not depend on which agents initially said "yes", it just depends on how many did so. To give some intuition, let us consider a simple example presented in Grabisch and Rusinowska (2011b), where the influence process resembles a majority voting.

Example 2 (**Majority**)**.** A straightforward way of making a decision is based on majority voting. If the majority of the agents says "yes", then all agents agree to say "yes" after mutual influence and otherwise, they agree to say "no". We can model simple majorities as well as situations where far more than half of the agents are needed to reach the yes-consensus. Let $m \in \{\lfloor \frac{n}{2} \rfloor + 1, \ldots, n\}$. Then, the *majority* aggregation model is given by

$$\mathsf{Maj}_i^{[m]}(x) := x_{(m)} \text{ for all } i \in N.$$

All agents use an ordered weighted average where $w_m = 1$. Obviously, the convergence to consensus is immediate.

Not surprisingly, the influence of a coalition indeed solely depends on the number of individuals involved.

Proposition 4. *Consider an aggregation model with aggregation functions* $A_i = \mathsf{OWA}_{w^i}, i \in N$.

(i) A coalition of size $0 < s \leq n$ *is yes-influential on* $i \in N$ *if and only if* $\min\{k \in N | w_k^i > 0\} \leq s$. *It is minimal yes-influential on* i *if and only if equality holds.*

(ii) A coalition of size $0 < s \leq n$ *is no-influential on* $i \in N$ *if and only if* $\max\{k \in N | w_k^i > 0\} \geq n + 1 - s$. *It is minimal no-influential on* i *if and only if equality holds.*

Proof. Let $S \subseteq N$ have size $0 < s \leq n$ and be yes-influential on $i \in N$, i.e.,

$$A_i(1_S) = \sum_{k=1}^{s} w_k^i > 0 \Leftrightarrow \min\{k \in N | w_k^i > 0\} \leq s.$$

Suppose moreover that S is minimal yes-influential on i, i.e.,

$$A_i(1_S) = \sum_{k=1}^{s} w_k^i > 0 \text{ and } A_i(1_{S'}) = \sum_{k=1}^{s'} w_k^i = 0 \text{ for all } S' \subsetneq S$$
$$\Leftrightarrow \min\{k \in N | w_k^i > 0\} = s.$$

The second part is analogous.

□

The result on influential agents follows immediately.

Corollary 3. *Consider an aggregation model with aggregation functions* $A_i = \mathsf{OWA}_{w^i}, i \in N$.

(i) All agents $j \in N$ *are yes-influential on* $i \in N$ *if and only if* $w_1^i > 0$.

(ii) All agents $j \in N$ *are no-influential on* $i \in N$ *if and only if* $w_n^i > 0$.

Note that this means that either all agents are yes-(or no-)influential on some agent $i \in N$ or none. Next, we study non-trivial terminal classes. We characterize terminal states and show that there cannot be a cycle.

Proposition 5. *Consider an aggregation model with aggregation functions $A_i = \mathsf{OWA}_{w^i}, i \in N$.*

(i) A state $S \subseteq N$ of size s is a terminal state if and only if $\sum_{k=1}^{s} w_k^i = 1$ for all $i \in S$ and $\sum_{k=1}^{s} w_k^i = 0$ otherwise.

(ii) There does not exist any cycle.

Proof. The first part is obvious. For the second part, assume that there is a cycle $\{S_1, \ldots, S_k\}$ of length $2 \leq k \leq \binom{n}{\lfloor n/2 \rfloor}$. This implies that there exists $l \in \{1, \ldots, k\}$ such that $s_l \leq s_{l+1}$, where $S_{k+1} \equiv S_1$. Thus,

$$\sum_{j=1}^{s_l} w_j^i = 1 \text{ for all } i \in S_{l+1}$$

and hence $S_{l+1} \subseteq S_{l+2}$, which is a contradiction to pairwise incomparability by inclusion (see Theorem 1 (ii)).

$\square$

For regular terminal classes, note that an agent $i \in N$ such that $w_1^i = 1$ blocks a no-consensus and an agent $j \in N$ such that $w_n^j = 1$ blocks a yes-consensus – given that the process has not yet arrived at a consensus. Therefore, since there cannot be any cycle, these two conditions, while ensuring that there is no other terminal state, gives us a regular terminal class.

Example 3 (**Regular terminal class**)**.** Consider an aggregation model with aggregation functions $A_i = \mathsf{OWA}_{w^i}, i \in N = \{1, 2, 3\}$. Let agent 1 block a no-consensus and agent 3 block a yes-consensus, i.e., $w_1^1 = w_3^3 = 1$. Furthermore, choose $w_1^2 = w_3^2 = \frac{1}{2}$. Then $\{\{1\}, \{1, 2\}\}$ is a regular terminal class. We have $\mathbf{A}(1_{\{1\}}) = \mathbf{A}(1_{\{1,2\}}) = (1\ \frac{1}{2}\ 0)^t$.

It is left to find conditions that avoid both non-trivial terminal states and regular terminal classes and hence ensure that the society ends up in a consensus. The following result characterizes the non-existence of non-trivial terminal classes. The idea is that for reaching a consensus, there must be some threshold such that whenever the size of the coalition is at least equal to this threshold, there is some probability that after mutual influence, more people will say “yes”. And whenever the size is below this threshold, there is some probability that after mutual influence, more people will say “no”.

Theorem 2. *Consider an aggregation model with aggregation functions* $A_i = \mathsf{OWA}_{w^i}, i \in N$. *Then, there are no other terminal classes than the trivial terminal classes if and only if there exists* $\bar{k} \in \{1, \ldots, n\}$ *such that both:*

(i) For all $k = \bar{k}, \ldots, n-1$, *there are distinct agents* $i_1, \ldots, i_{k+1} \in N$ *such that*

$$\sum_{j=1}^{k} w_j^{i_l} > 0 \text{ for all } l = 1, \ldots, k+1.$$

(ii) For all $k = 1, \ldots, \bar{k}-1$, *there are distinct agents* $i_1, \ldots, i_{n-k+1} \in N$ *such that*

$$\sum_{j=1}^{k} w_j^{i_l} < 1 \text{ for all } l = 1, \ldots, n-k+1.$$

The proof is in the appendix. Note that Theorem 2 implies a straightforward – but very strict – sufficient condition:

Remark 2. Consider an aggregation model with aggregation functions $A_i = \mathsf{OWA}_{w^i}, i \in N$. Then, there are no other terminal classes than the trivial terminal classes if $w_1^i > 0$ for all $i \in N$ ($\bar{k} = 1$), or $w_n^i > 0$ for all $i \in N$ ($\bar{k} = n$).

We get a more intuitive formulation of Theorem 2 by using influential coalitions.

Corollary 4. *Consider an aggregation model with aggregation functions* $A_i = \mathsf{OWA}_{w^i}, i \in N$. *Then, there are no other terminal classes than the trivial terminal classes if and only if there exists* $\bar{k} \in \{1, \ldots, n\}$ *such that both:*

(i) For all $k = \bar{k}, \ldots, n-1$, *there are* $k+1$ *distinct agents such that coalitions of size* k *are yes-influential on each of them.*

(ii) For all $k = 1, \ldots, \bar{k}-1$, *there are* $n-k+1$ *distinct agents such that coalitions of size* $n-k$ *are no-influential on each of them.*

In more general situations, the agents' behavior might only partly be determined by ordered weighted averages.

Definition 7 (OWA**-decomposable aggregation function**)**.** We say that an n-place aggregation function A is OWA_w*-decomposable*, if there exists $\lambda \in (0,1]$ and an n-place aggregation function A' such that $A = \lambda \mathsf{OWA}_w + (1-\lambda)A'$.

Such aggregation functions do exist since convex combinations of aggregation functions are again aggregation functions. Note that a model where agents use these functions is not anonymous in general, though. However, the sufficiency part of Theorem 2 also holds if agents use such decomposable aggregation functions.[10]

Corollary 5. *Consider an aggregation model with* OWA_{w^i}*-decomposable aggregation functions* $A_i, i \in N$*. Then, there are no other terminal classes than the trivial terminal classes if there exists* $\bar{k} \in \{1, \ldots, n\}$ *such that both:*

(i) For all $k = \bar{k}, \ldots, n-1$*, there are distinct agents* $i_1, \ldots, i_{k+1} \in N$ *such that*

$$\sum_{j=1}^{k} w_j^{i_l} > 0 \text{ for all } l = 1, \ldots, k+1.$$

(ii) For all $k = 1, \ldots, \bar{k}-1$*, there are distinct agents* $i_1, \ldots, i_{n-k+1} \in N$ *such that*

$$\sum_{j=1}^{k} w_j^{i_l} < 1 \text{ for all } l = 1, \ldots, n-k+1.$$

To illustrate this result, we consider the phenomenon of mass psychology, also called herding behavior, given in Grabisch and Rusinowska (2011b).

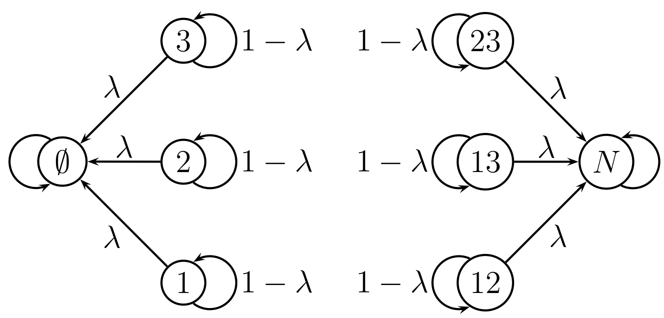
Example 4 (**Mass psychology**)**.** Mass psychology or herding behavior means that if at least a certain number $m \in \{\lfloor \frac{n}{2} \rfloor + 1, \ldots, n\}$ of agents share the same opinion, then these agents attract others, who had a different opinion before. We assume that an agent changes her opinion in this case with probability $\lambda \in (0,1]$. In particular, we consider $n = 3$ agents and a

[10]It is clear that in general, the necessity part does not hold since convergence to consensus may as well be (partly) ensured by the other component.

threshold of $m = 2$. This means whenever only two agents are of the same opinion, the third one might change her opinion. This corresponds to the following *mass psychology* aggregation model:

$$\mathsf{Mass}_i^{[2]}(x) = \lambda x_{(2)} + (1-\lambda)x_i \text{ for all } i \in N.$$

This aggregation function is OWA_w-decomposable, with $w_2 = 1$ and by Corollary 5, taking $\bar{k} = 2$, the group eventually reaches a consensus. Furthermore, agents are yes- and no-influential on themselves and coalitions of size two or more are yes- and no-influential on all agents. The model gives the following digraph of the Markov chain:



Although the aggregation functions are not anonymous, the model is so. We get for any initial coalition of size $1 \leq s \leq 2$:

$$\mathbb{P}(\tau_s > m) = (1-\lambda)^m \text{ and } \mathbb{E}[\tau_s] = \frac{1}{\lambda}.$$

So, the speed of convergence hinges on λ, the probability that an agent follows the herd. If it is small, the process can take a long time. If initially two agents said "yes", the process terminates in the yes-consensus and otherwise, it terminates in the no-consensus. This example shows that OWA-decomposable aggregation functions allow – differently from ordered weighted averages – to account for the own opinion of an agent.

5 Applications to fuzzy linguistic quantifiers

Instead of being sharp edged, the threshold of an agent initially saying "no" for changing her opinion might be rather "soft". For instance, she could change her opinion if "*most* of the agents say 'yes' ". This is called a *soft* *majority* and phrases like "most" or "many" are so-called *fuzzy linguistic quantifiers*. Furthermore, *soft minorities* are also possible, e.g., "*at least a few* of the agents say 'yes' ". Our aim is to apply our findings on ordered weighted averages to fuzzy linguistic quantifiers. Mathematically, we define them by a function which maps the agents' proportion that says "yes" to the degree to which the quantifier is satisfied.[11]

Definition 8 (Fuzzy linguistic quantifier). A *fuzzy linguistic quantifier* $\mathcal{Q}$ is defined by a nondecreasing function

$$\mu_{\mathcal{Q}} : [0,1] \to [0,1] \text{ such that } \mu_{\mathcal{Q}}(0) = 0 \text{ and } \mu_{\mathcal{Q}}(1) = 1.$$

Furthermore, we say that the quantifier is *regular* if the function is strictly increasing on some interval $(\underline{c}, \bar{c}) \subseteq [0,1]$ and otherwise constant.

Fuzzy linguistic quantifiers like "most" are ambiguous in the sense that it is not clear how to define them exactly mathematically. For example, one could well discuss which proportion of the agents should say "yes" for the quantifier "most" to be fully satisfied. Nevertheless, let us give some typical examples.[12]

Example 5 (Typical quantifiers). We define

(i) $\mathcal{Q}_{\text{aa}}$ = "almost all" by

$$\mu_{\mathcal{Q}_{\text{aa}}}(x) := \begin{cases} 1, & \text{if } x \geq \frac{9}{10} \\ \frac{5}{2}x - \frac{5}{4}, & \text{if } \frac{1}{2} < x < \frac{9}{10} \\ 0, & \text{otherwise} \end{cases},$$

(ii) $\mathcal{Q}_{\text{mo}}$ = "most" by

$$\mu_{\mathcal{Q}_{\text{mo}}}(x) := \begin{cases} 1, & \text{if } x \geq \frac{4}{5} \\ \frac{5}{2}x - 1, & \text{if } \frac{2}{5} < x < \frac{4}{5} \\ 0, & \text{otherwise} \end{cases},$$

(iii) $\mathcal{Q}_{\text{ma}}$ = "many" by

$$\mu_{\mathcal{Q}_{\text{ma}}}(x) := \begin{cases} 1, & \text{if } x \geq \frac{3}{5} \\ \frac{5}{2}x - \frac{1}{2}, & \text{if } \frac{1}{5} < x < \frac{3}{5} \\ 0, & \text{otherwise} \end{cases},$$

[11] cf. Zadeh (1983)
[12] cf. Yager and Kacprzyk (1997)

(iv) $\mathcal{Q}_{\text{af}}$ = "at least a few" by

$$\mu_{\mathcal{Q}_{\text{af}}}(x) := \begin{cases} 1, & \text{if } x \geq \frac{3}{10} \\ \frac{10}{3}x, & \text{otherwise} \end{cases} .$$

Note that these quantifiers are regular. For every quantifier, there exists a corresponding ordered weighted average in the sense that it represents the quantifier.[13] We can find its weights as follows.

Lemma 1 (Yager, 1988)**.** *Let $\mathcal{Q}$ be a fuzzy linguistic quantifier defined by $\mu_{\mathcal{Q}}$. Then, the weights of its corresponding ordered weighted average $\mathsf{OWA}_{\mathcal{Q}}$ are given by*

$$w_k = \mu_{\mathcal{Q}}\left(\frac{k}{n}\right) - \mu_{\mathcal{Q}}\left(\frac{k-1}{n}\right), \text{ for } k = 1, \ldots, n.$$

This allows us to apply our results to regular quantifiers. We find that if all agents use such a quantifier, then under some similarity condition, the group will finally reach a consensus. Moreover, we show that the result still holds if some agents deviate to a quantifier that is not similar in that sense. In the following, we denote the quantifier of an agent i by $\mathcal{Q}^i$.

Proposition 6. *Consider an aggregation model with aggregation functions $A_i = \mathsf{OWA}_{\mathcal{Q}^i}, i \in N$.*

(i) If $\mathcal{Q}^i$ is regular for all $i \in N$ and $\cap_{i\in N}(\underline{c}_i, \bar{c}_i) \neq \emptyset$, then there are no other terminal classes than the trivial terminal classes.

(ii) Suppose $\min_{i\in N} \underline{c}_i > 0$, then the result in (i) still holds if less than $\lceil \bar{c}_d n \rceil$ agents deviate to a regular quantifier $\mathcal{Q}_d$ such that $\bar{c}_d < \min_{i\in N} \underline{c}_i$.

(iii) Suppose $\max_{i\in N} \bar{c}_i < 1$, then the result in (i) still holds if less than $\lceil (1-\underline{c}_d)n \rceil$ agents deviate to a regular quantifier $\mathcal{Q}_d$ such that $\max_{i\in N} \bar{c}_i < \underline{c}_d$.

The proof is in the appendix. Note that the deviating agents can in fact also use different quantifiers, as follows.

[13] Note that this is due to our definition. The conditions in Definition 8 ensure that there exists such an ordered weighted average. In general, one can define quantifiers also by other functions, cf. Zadeh (1983).

Remark 3. (i) Suppose $\min_{i\in N} \underline{c}_i > 0$, then the result in part (i) of the Proposition still holds if $k < \lceil \min_d \bar{c}_d n \rceil$ agents $i_{d_1}, \ldots, i_{d_k}$ deviate to regular quantifiers $\mathcal{Q}_d$ such that $\bar{c}_d < \min_{i\in N} \underline{c}_i$ for all $d = d_1, \ldots, d_k$.

(ii) Suppose $\max_{i\in N} \bar{c}_i < 1$, then the result in part (i) of the Proposition still holds if $k < \lceil (1 - \max_d \underline{c}_d)n \rceil$ agents $i_{d_1}, \ldots, i_{d_k}$ deviate to regular quantifiers $\mathcal{Q}_d$ such that $\underline{c}_d > \max_{i\in N} \bar{c}_i$ for all $d = d_1, \ldots, d_k$.

(iii) Suppose $\min_{i\in N} \underline{c}_i > 0$ and $\max_{i\in N} \bar{c}_i < 1$, then the result in part (i) of the Proposition still holds if $k < \lceil \min_p \bar{c}_p n \rceil$ agents $i_{p_1}, \ldots, i_{p_k}$ deviate to regular quantifiers $\mathcal{Q}_p$ and $k' < \lceil (1 - \max_q \underline{c}_q)n \rceil$ agents $i_{q_1}, \ldots, i_{q_{k'}}$ deviate to regular quantifiers $\mathcal{Q}_q$ such that $\bar{c}_p < \min_{i\in N} \underline{c}_i$ and $\underline{c}_q > \max_{i\in N} \bar{c}_i$ for all $p = p_1, \ldots, p_k$ and $q = q_1, \ldots, q_{k'}$.

We can also characterize terminal states in a model where agents use regular quantifiers.

Proposition 7. *Consider an aggregation model with aggregation functions* $A_i = \mathsf{OWA}_{\mathcal{Q}^i}, i \in N$*. If* $\mathcal{Q}^i$ *is regular for all* $i \in N$*, then a state* $S \subseteq N$ *of size* s *is a terminal state if and only if*

$$\max_{i\in S} \bar{c}_i \leq \frac{s}{n} \leq \min_{i\in N\setminus S} \underline{c}_i.$$

Proof. Suppose $S \subseteq N$ of size s is a terminal state. By Proposition 5, we know that this is equivalent to

$$\begin{aligned}
&\sum_{k=1}^{s} w_k^i = 1 \text{ for all } i \in S \text{ and } \sum_{k=1}^{s} w_k^i = 0 \text{ otherwise} \\
\Leftrightarrow &\mu_{\mathcal{Q}^i}(s/n) = 1 \text{ for all } i \in S \text{ and } \mu_{\mathcal{Q}^i}(s/n) = 0 \text{ otherwise} \\
\Leftrightarrow &\max_{i\in S} \bar{c}_i \leq \frac{s}{n} \leq \min_{i\in N\setminus S} \underline{c}_i.
\end{aligned}$$

□

To provide some intuition, let us come back to Example 5 and look at the implications our findings have on the quantifiers defined therein.

Example 5 (Typical quantifiers, continued). Consider an aggregation model with aggregation functions $A_i = \mathsf{OWA}_{\mathcal{Q}^i}, i \in N$.

(i) If $\mathcal{Q}^i \in \{\mathcal{Q}_{\text{aa}}, \mathcal{Q}_{\text{mo}}, \mathcal{Q}_{\text{ma}}\}$ for all $i \in N$, then there are no other terminal classes than the trivial terminal classes. The result still holds if less than $\lceil \frac{3}{10} n \rceil$ agents deviate to $\mathcal{Q}_{\text{af}}$.

(ii) If $\mathcal{Q}^i \in \{\mathcal{Q}_{\text{ma}}, \mathcal{Q}_{\text{af}}\}$ for all $i \in N$, then there are no other terminal classes than the trivial terminal classes. The result still holds if less than $\lceil \frac{1}{2} n \rceil$ agents deviate, each of them either to $\mathcal{Q}_{\text{aa}}$ or $\mathcal{Q}_{\text{mo}}$.

(iii) A state $S \subseteq N$ of size s is a terminal state if $\mathcal{Q}^i = \mathcal{Q}_{\text{af}}$ for all $i \in S$, $\mathcal{Q}^i = \mathcal{Q}_{\text{aa}}$ ($\mathcal{Q}^i \in \{\mathcal{Q}_{\text{aa}}, \mathcal{Q}_{\text{mo}}\}$) otherwise and $\frac{3}{10} \leq \frac{s}{n} \leq \frac{1}{2}$ ($\leq \frac{2}{5}$).

It is left to provide concrete examples where agents use these quantifiers. We give an example where agents finally reach a consensus as well as one where this might not be the case.

Example 6 (**Typical quantifiers in a four-agents-society**)**.** Consider an aggregation model with aggregation functions $A_i = \mathsf{OWA}_{\mathcal{Q}^i}, i \in N = \{1, 2, 3, 4\}$.

(i) Let each quantifier that we introduced be used by one agent, i.e., $\mathcal{Q}^1 = \mathcal{Q}_{\text{aa}}$, $\mathcal{Q}^2 = \mathcal{Q}_{\text{mo}}$, $\mathcal{Q}^3 = \mathcal{Q}_{\text{ma}}$ and $\mathcal{Q}^4 = \mathcal{Q}_{\text{af}}$. By Example 5 (i), there are only the trivial terminal classes. If initially only one or two agents said "yes", the convergence can take quite long since the first two agents are likely to hold a different opinion than the fourth agent after mutual influence. However, we see that the group tends to converge to the yes-consensus for most initial coalitions. This is because the "at least a few" quantifier kind of blocks the no-consensus (see Table 2).

(ii) Let two agents use the "almost all" quantifier and the other two the "at least a few" quantifier, i.e., $\mathcal{Q}^1 = \mathcal{Q}^2 = \mathcal{Q}_{\text{aa}}$ and $\mathcal{Q}^3 = \mathcal{Q}^4 = \mathcal{Q}_{\text{af}}$. By Example 5 (iii), $S = \{3, 4\}$ is a terminal state, where the last two agents say "yes" and the others say "no". If initially only one agent said "yes", it is very likely that the society is split up eventually. If instead three agents said "yes", the group tends to converge to the yes-consensus. Overall, the convergence is fast (see Table 3). Note that for an initial coalition of size two other than $S = \{3, 4\}$, the convergence to S is immediate.

$\mathbb{P}(\tau_s > m)$	1	3	5	10	20	30	$\mathbb{E}[\tau_s]$	$\mathbb{P}(\tau_s^N < \infty)$
1	.85	.65	.48	.22	.04	$< 10^{-2}$	7.05	.26
2	1	.73	.5	.21	.04	$< 10^{-2}$	7.32	.61
3	.45	.13	.06	.02	$< 10^{-2}$	$< 10^{-3}$	2.26	.97

Table 2: Speed of convergence and absorption probabilities in Example 6 (i).

$\mathbb{P}(\tau_s > m)$	1	3	5	10	$\mathbb{E}[\tau_s]$	$\mathbb{P}(\tau_s^N < \infty)$	$\mathbb{P}(\tau_s^\emptyset < \infty)$
1	.28	.02	$< 10^{-2}$	$< 10^{-5}$	1.38	0	.04
3	.47	.10	.02	$< 10^{-3}$	1.88	.74	0

Table 3: Speed of convergence and absorption probabilities in Example 6 (ii).

We chose only the two extreme quantifiers in the second part because otherwise the group would reach a consensus although the condition in Example 5 (i) was violated. The reason is that the number of agents is small in the Example, the conditions on the deviating agents in Proposition 6 somehow get "closer to necessity" when n increases. In other words, reaching a consensus seems to be easier in our model for smaller groups.

6 Conclusion

We study a stochastic model of influence where agents aggregate opinions using *ordered weighted averaging operators*. These models are *anonymous* in the sense that only the number of agents sharing the same opinion matters. We analyse the speed of convergence to terminal classes as well as absorption probabilities of different classes. For anonymous models and in particular our setup, the computational demand is much lower than for general aggregation models. We show that cyclic terminal classes cannot exist and characterize terminal states. We provide a necessary and sufficient condition for convergence to consensus. It turns out that this is a condition on the influence that coalitions have on agents. We also extend our model to more general situations, where influence is determined by convex combinations of ordered weighted averages and general aggregation functions. Our previous condition is still sufficient for convergence to consensus in this generalized setting. Furthermore, we apply our results to *fuzzy linguistic quantifiers* and show that if agents use similar quantifiers and not too many agents deviate, the society will eventually reach a consensus.

References

Asavathiratham, C. (2000). The influence model: A tractable representation for the dynamics of networked Markov chains. *PhD thesis*. Massachusetts Institute of Technology.

Bala, V. and S. Goyal (1998). Learning from neighbours. *The Review of Economic Studies 65*(3), 595–621.

Bala, V. and S. Goyal (2001). Conformism and diversity under social learning. *Economic Theory 17*(1), 101–20.

Banerjee, A. (1992). A simple model of herd behavior. *The Quarterly Journal of Economics 107*(3), 797–817.

Banerjee, A. and D. Fudenberg (2004). Word-of-mouth learning. *Games and Economic Behavior 46*(1), 1–22.

Borm, P., R. van den Brink, and M. Slikker (2002). An iterative procedure for evaluating digraph competitions. *Annals of Operations Research 109*, 61–75.

Brémaud, P. (1999). *Markov Chains*. Springer.

Büchel, B., T. Hellmann, and S. Klößner (2012, July). Opinion dynamics under conformity. Institute of Mathematical Economics Working Papers 469, Bielefeld University.

Büchel, B., T. Hellmann, and M. Pichler (2011, January). The dynamics of continuous cultural traits in social networks. Institute of Mathematical Economics Working Papers 457, Bielefeld University.

Calvó-Armengol, A. and M. Jackson (2009). Like father, like son: social network externalities and parent-child correlation in behavior. *American Economic Journal: Microeconomics 1*, 124–150.

Cornelis, C., P. Victor, and E. Herrera-Viedma (2010). Ordered weighted averaging approaches for aggregating gradual trust and distrust. Proceedings of XV Congreso Español sobre Tecnologías y Lógica Fuzzy (ESTYLF 2010), pp. 555–60.

DeGroot, M. (1974). Reaching a consensus. *Journal of the American Statistical Association 69*(345), 118–21.

DeMarzo, P., D. Vayanos, and J. Zwiebel (2003). Persuasion bias, social influence, and unidimensional opinions. *Quarterly Journal of Economics 118*(3), 909–68.

Ellison, G. (1993). Learning, local interaction, and coordination. *Econometrica 61*(5), 1047–71.

Gale, D. and S. Kariv (2003). Bayesian learning in social networks. *Games and Economic Behavior 45*(2), 329–46.

Galeotti, A. and S. Goyal (2009). Influencing the influencers: a theory of strategic diffusion. *The RAND Journal of Economics 40*(3), 509–532.

Golub, B. and M. Jackson (2010). Naïve learning in social networks and the wisdom of crowds. *American Economic Journal: Microeconomics 2*(1), 112–49.

Grabisch, M. and A. Rusinowska (2010). A model of influence in a social network. *Theory and Decision 1*(69), 69–96.

Grabisch, M. and A. Rusinowska (2011a). Influence functions, followers and command games. *Games and Economic Behavior 1*(72), 123–38.

Grabisch, M. and A. Rusinowska (2011b, December). A model of influence based on aggregation functions. CES Working Paper 2011.58, Université Paris 1 Panthéon-Sorbonne.

Hu, X. and L. Shapley (2003). On authority distributions in organizations: equilibrium. *Games and Economic Behavior* (45), 132–52.

Jackson, M. (2008). *Social and Economic Networks*. Princeton University Press.

Jiang, H. and R. Eastman (2000). Application of fuzzy measures in multicriteria evaluation in GIS. *International Journal of Geographical Information Science 2*(14), 173–84.

López-Pintado, D. (2008). Diffusion in complex social networks. *Games and Economic Behavior 62*, 573–90.

López-Pintado, D. (2010, June). Influence networks. WP ECON 10.06, Universidad Pablo de Olavide.

López-Pintado, D. and D. Watts (2008). Social influence, binary decisions and collective dynamics. *Rationality and Society 20*(4), 399–443.

Malczewski, J. and C. Rinner (2005). Exploring multicriteria decision strategies in GIS with linguistic quantifiers: A case study of residential quality evaluation. *Journal of Geographical Systems 7*(2), 249–68.

Mäs, M. (2010). *The diversity puzzle*. ICS dissertation series. Ridderprint Offsetdrukkerij B.V., Ridderkerk.

Morris, S. (2000). Contagion. *The Review of Economic Studies 67*(1), 57–78.

Mueller-Frank, M. (2010, September). A fixed point convergence theorem in Euclidean spaces and its application to non-Bayesian learning in social networks. *Working paper*. University of Oxford.

van den Brink, R. and R. Gilles (2000). Measuring domination in directed networks. *Social Networks 22*, 141–157.

Yager, R. (1988). On ordered weighted averaging aggregation operators in multicriteria decisionmaking. *IEEE Transactions on Systems, Man and Cybernetics 18*(1), 183–90.

Yager, R. and J. Kacprzyk (1997). *The Ordered Weighted Averaging Operators*. Kluwer Academic Publishers.

Zadeh, L. (1983). A computational approach to fuzzy quantifiers in natural language. *Computers and Mathematics with Applications 9*(1), 149–84.

A Appendix

A.1 Proof of Theorem 2

First, suppose that there exists $\bar{k} \in \{1, \ldots, n\}$ such that (i) and (ii) hold. Let us take any coalition $S \subsetneq N$ of size $s \geq \bar{k}$ and show that it is possible to reach the yes-consensus, which implies that S is not part of a terminal class. By choice of S, it is sufficient to show that there is a positive probability that after mutual influence, the size of the coalition has strictly increased. That is, it is sufficient to show that there exists a coalition $S' \subseteq N$ of size $s' > s$, such that $A_i(1_S) > 0$ for all agents $i \in S'$. Set $k := s$, then by condition (i), there are distinct agents $i_1, \ldots, i_{k+1} \in N$ such that

$$A_{i_l}(1_S) = \sum_{j=1}^{k} w_j^{i_l} > 0 \text{ for all } l = 1, \ldots, k+1,$$

i.e., setting $S' := \{i_1, \ldots, i_{k+1}\}$ finishes this part. Analogously, we can show by condition (ii) that for any nonempty $S \subseteq N$ of size $s < \bar{k}$ it is possible to reach the no-consensus. Hence, there are only the trivial terminal classes.

Now, suppose to the contrary that for all $\bar{k} \in \{1, \ldots, n\}$ either (i) or (ii) does not hold. Note that in order to establish that there exists a non-trivial terminal class, it is sufficient to show that there are $k_*, k^* \in \{1, \ldots, n-1\}, k_* \leq k^*$, such that for all $S \subseteq N$ of size $s = k_*$,

$$A_i(1_S) < 1 \text{ for at most } n - k_* \text{ distinct agents } i \in N \qquad (C_*[k_*])$$

and for all $S \subseteq N$ of size $s = k^*$,

$$A_i(1_S) > 0 \text{ for at most } k^* \text{ distinct agents } i \in N. \qquad (C^*[k^*])$$

Indeed, condition $C_*[k_*]$ says that it is not possible to reach a coalition with less than k_* agents starting from a coalition with at least k_* agents. Similarly, condition $C^*[k^*]$ says that it is not possible to reach a coalition with more than k^* agents starting from a coalition with at most k^* agents.[14] Therefore, it is not possible to reach the trivial terminal states from any coalition S of size $k_* \leq s \leq k^*$, which proves the existence of a non-trivial terminal class.

[14]Note that monotonicity of the aggregation function implies that $(C_*[k_*])$ also holds if we replace S by a coalition $S' \subseteq N$ of size $s' > k_*$. Analogously for $(C^*[k^*])$.

Let now $\bar{k} = 1$. Then, clearly condition (ii) is satisfied and thus condition (i) cannot be satisfied by assumption. Hence, there exists $k^* \in \{1, \ldots, n-1\}$ such that there are at most k^* distinct agents $i_1, \ldots, i_{k^*}$ such that

$$\sum_{j=1}^{k^*} w_j^{i_l} > 0 \text{ for } l = 1, \ldots, k^*.$$

This implies that condition (i) is not satisfied for $\bar{k} = 1, \ldots, k^*$. If $k^* \geq 2$ and additionally condition (ii) was not satisfied for some $\bar{k} \in \{2, \ldots, k^*\}$, we were done since then there would exist $k_* \in \{1, \ldots, k^* - 1\}$ such that there are at most $n - k_*$ distinct agents $i_1, \ldots, i_{n-k_*}$ such that

$$\sum_{j=1}^{k_*} w_j^{i_l} < 1 \text{ for } l = 1, \ldots, n - k_*,$$

i.e., $(C_*[k_*])$ and $(C^*[k^*])$ were satisfied for $k_* \leq k^*$. Therefore, suppose w.l.o.g. that condition (ii) is satisfied for all $\bar{k} = 1, \ldots, k^*$. $(*)$ For $\bar{k} = n$, clearly condition (i) is satisfied and thus condition (ii) cannot be satisfied. Hence, using $(*)$, there exists $k_* \in \{k^*, \ldots, n-1\}$ such that there are at most $n - k_*$ distinct agents $i_1, \ldots, i_{n-k_*}$ such that

$$\sum_{j=1}^{k_*} w_j^{i_l} < 1 \text{ for } l = 1, \ldots, n - k_*,$$

i.e., $(C_*[k_*])$ and $(C^*[k^*])$ are satisfied. We now proceed by case distinction:

(1) If $k_* = k^*$, then we are done.

(2) If $k_* > k^*$, then let $\bar{k} = k_*$. By assumption, either (i) or (ii) does not hold.

   (2.1) If (i) does not hold, then there exists $k^{**} \in \{k_*, \ldots, n-1\}$ such that there are at most k^{**} distinct agents $i_1, \ldots, i_{k^{**}}$ such that

$$\sum_{j=1}^{k^{**}} w_j^{i_l} > 0 \text{ for } l = 1, \ldots, k^{**},$$

   i.e. $(C_*[k_*])$ and $(C^*[k^{**}])$ are satisfied for $k_* \leq k^{**}$ and hence we are done.

(2.2) If (ii) does not hold, then, using $(*)$, there exists $k_{**} \in \{k^*, \ldots, k_* - 1\}$ such that there are at most $n - k_{**}$ distinct agents $i_1, \ldots, i_{n-k_{**}}$ such that

$$\sum_{j=1}^{k_{**}} w_j^{i_l} < 1 \text{ for } l = 1, \ldots, n - k_{**},$$

i.e., $(C_*[k_{**}])$ is satisfied. If $k_{**} = k^*$, then we are done, otherwise we can repeat this procedure using k_{**} instead of k_*.

Since $k_{**} \lneq k_*$, we find $k_{**} = k^*$ after a finite number of repetitions, which finishes the proof.

A.2 Proof of Proposition 6

(i) By assumption, there exists $c \in \cap_{i \in N} (\underline{c}_i, \bar{c}_i)$. Let us define $\bar{k} := \min\{k \in \mathbb{N} | \frac{k}{n} > c\}$, then clearly $\frac{\bar{k}-1}{n} \leq c$. We show that conditions (i) and (ii) of Theorem 2 are satisfied for $\bar{k}$. Since for all $i \in N$, $\mu_{\mathcal{Q}^i}$ is nondecreasing and, in particular, strictly increasing on the open ball $B_\epsilon(c)$ around c for some $\epsilon > 0$, we get by Lemma 1 that

$$w_{\bar{k}}^i = \mu_{\mathcal{Q}}\left(\frac{\bar{k}}{n}\right) - \mu_{\mathcal{Q}}\left(\frac{\bar{k}-1}{n}\right) \geq \mu_{\mathcal{Q}}\left(\frac{\bar{k}}{n}\right) - \mu_{\mathcal{Q}}(c) > 0 \text{ for all } i \in N.$$

This implies that for all $k = \bar{k}, \ldots, n-1$,

$$\sum_{j=1}^{k} w_j^i \geq w_{\bar{k}}^i > 0 \text{ for all } i \in N$$

and for all $k = 1, \ldots, \bar{k} - 1$,

$$\sum_{j=1}^{k} w_j^i \leq \sum_{j \neq \bar{k}} w_j^i = 1 - w_{\bar{k}}^i < 1 \text{ for all } i \in N,$$

i.e., (i) and (ii) of Theorem 2 are satisfied for $\bar{k}$, which finishes the first part.

(ii) Suppose $\min_{i \in N} \underline{c}_i > 0$ and denote by $D \subseteq N$ the set of agents that deviate to the quantifier $\mathcal{Q}_d$. Similar to the first part, there exists

$c \in \cap_{i\in N\setminus D}(\underline{c}_i, \bar{c}_i)$ and we can define $\bar{k} := \min\{k \in \mathbb{N} | \frac{k}{n} > c\}$. This implies that for all $k = \bar{k}, \dots, n-1$,

$$\sum_{j=1}^{k} w_j^i > 0 \text{ for all } i \in N\setminus D \tag{$*$}$$

and for all $k = 1, \dots, \bar{k}-1$,

$$\sum_{j=1}^{k} w_j^i < 1 \text{ for all } i \in N\setminus D. \tag{$**$}$$

Furthermore, we have by assumption $\mu_{\mathcal{Q}_d}(\bar{k}/n) = 1$, which implies $w_j^i = 0$ for all $j = \bar{k}+1, \dots, n$ and $i \in D$. Thus, for all $k = \bar{k}, \dots, n-1$

$$\sum_{j=1}^{k} w_j^i = \sum_{j=1}^{\bar{k}} w_j^i = 1 > 0 \text{ for all } i \in D,$$

i.e., in combination with $(*)$, condition (i) of Theorem 2 is satisfied for $\bar{k}$. It is left to check condition (ii). Define for $i \in D$,

$$\tilde{k} := \max\{k \in \mathbb{N} | w_k^i > 0\} = \min\{k \in \mathbb{N} | k/n \geq \bar{c}_d\} \leq \bar{k}.$$

Hence, for $k = 1, \dots, \tilde{k}-1$,

$$\sum_{j=1}^{k} w_j^i < 1 \text{ for all } i \in D.$$

If $\tilde{k} = \bar{k}$, condition (ii) is – in combination with $(**)$ – satisfied for $\bar{k}$ and any $D \subseteq N$. Otherwise, we have $\tilde{k} < \bar{k}$ and then, for $k = \tilde{k}, \dots, \bar{k}-1$,

$$\sum_{j=1}^{k} w_j^i = 1 \text{ for all } i \in D.$$

This implies in combination with $(**)$ that condition (ii) is only satisfied if $\max_{k=\tilde{k},\dots,\bar{k}-1}(n-k+1) = n-\tilde{k}+1$ agents do not deviate, i.e.,

$$|D| \leq n - (n-\tilde{k}+1) = \tilde{k}-1 \;\Leftrightarrow\; |D| \lneq \tilde{k} \;\Leftrightarrow\; |D| \lneq \lceil \bar{c}_d n \rceil.$$

Thus, (i) and (ii) of Theorem 2 are satisfied for $\bar{k}$ if $|D| \lneq \lceil \bar{c}_d n \rceil$, which finishes the proof.

(iii) Analogous to the second part.