Critical Review of: ”Pourquoi les mathématiques sont-elles difficiles”

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Critical Review of *Pourquoi les mathématiques sont-elles difficiles?* by Lény Oumraou, preface by Jacques Dubucs, (Vuibert, « Philosophie des sciences » series, Paris, 2009, viii pages, 216 pages)

Mathematics is a notoriously difficult discipline. There are pedagogical and psychological aspects to the difficulty, psychological explanations of why some students fare poorly even when taught by outstanding teachers while others cope with intricacies of higher order and sometimes become creative in the field. The difficulty Oumraou is concerned with in this excellent and demanding book is the *objective* difficulty of mathematics, the kind that students of mathematics face when asked to prove Euclid's theorem that the sum of the three interior angles of a triangle is equal to two right angles, or Desargues' theorem that two triangles are in perspective axially just in case they are in perspective centrally. The book is also concerned with the more complex difficulties with which first rate mathematicians struggle when looking for a proof of Fermat's last theorem (found, at last), of Riemann's hypothesis, or for a general proof of Goldbach's conjecture (still struggling after partial proofs for small exponents by Euler, Fermat, Dirichlet, Legendre, Lamé and Lebesgue).

Now that the word "objective" has been thrown in, a philosophical problem arises: how do we account for the objective difficulty of mathematical problems? It isn't an easy task and Oumraou does a terrific job at probing a number of hypotheses and frameworks that have been proposed on which to base arguments for particular construals of the nature of the difficulty, both of the time-honored problems we have brought under our control and of the higher level problems for outstandingly smart grown-ups.

The book is divided into four parts. The first one clears the ground, analyzing the paralogism of the internalist position according to which the mathematical mind only deals with what it establishes. The second and third parts explore, respectively, the thesis that mathematics is analytic and the thesis that mathematics is synthetic. The fourth and longest part is devoted to heuristics. Finally, there is a useful appendix listing Hilbert's five sets of axioms for geometry (for belonging, order, congruence, parallels and continuity).

It seems at first blush that internalists are unable to explain why mathematics is *so* difficult. If the mind deals only with what it establishes, it is mysterious why it has to work so hard to reach the desired mathematical results. Worse: why does it sometimes utterly fail? The first part of the book is devoted to a refinement and defense of the internalist position, as opposed to the externalist (or platonist) view, according to which the business of mathemacians is to discover a suprasensible external reality. In a nutshell, Oumraou argues that one may be an internalist *and* hold that mathematical objects are externally related to the knowing subject, i.e., that, although the human mind creates them, the *bona fide* mathematical products possess their own autonomous productivity, at least in the sense that one may not construe such objects as merely resulting from the 'rules' or 'laws' we have chosen for them, e.g., the axioms.

The author rejects the analytic conception of mathematics and favours a synthetic conception. This naturally leads him to consider the case of heuristics. Heuristics matters in this instance if only because of a particular construal of the rejected analyticity thesis according to which a correct analysis of the *formulation* of mathematical problems necessary leads to their *resolution*—something an internalist relying blindly on the principle of necessary internality (we may not know with perfection what we haven't produced ourselves) might find appealing. Mathematics may not be merely conventional either (and therefore analytic) in the syntactic sense, as Hahn and Schlick thought it was. Even though Carnap showed in *The Logical Syntax of Language* how the syntactic viewpoint could be vindicated for axioms, it remains that, by Gödel's second incompleteness theorem, the consistency of a system of axioms may not be proved by finitary means. Consequently, the *philosophical dividends* of the syntactic viewpoint may *not* be obtained, the general philosophical result being that, by merely positing the mathematical axioms to be true by convention, one may not eliminate either mathematical intuition or empirical induction. Oumraou gives a very clear presentation of the issue, defending Gödel against recent criticisms by Awodey and Carus. It is therefore all the more unfortunate that an account of Gödel's own kind of platonist position is left out of the picture.

The criticism of the semantic conception of the analyticity of mathematics is generously detailed and offers a precise and useful overview of various philosophical theses that have shaped much of contemporary philosophy of mathematics. Oumraou distinguishes the formal understanding of a mathematical statement (the ability to recognize its logical variants) from its contentual understanding (the ability to identify its contentual variants via the substitutivity of a *definiedum* for a *definiens*), but also from its justificational understanding (the possession of a proof) and from its pragmatic understanding (grasping the conceptual role of a proof, knowing the contexts in which the statement may usefully be applied). This last aspect is crucial, especially in view of the fact that Oumraou, against both Bolzano and Frege, insists that a proof may bring with it a new heretofore unknown context of justification. Neither purity in Bolzano's sense (i.e., the requirement that proofs appeal only to notions that appear in the conclusion), nor full Fregean decomposition (no gaps allowed in the deductive chain) are required for the full justificational function of proofs. One, then, would have liked a more detailed account of the relation between justificational and pragmatic understanding than the one proposed by Oumraou.

There is a useful exposition and discussion of Bolzano's requirement that proofs be pure and of Bolzano's claim that proofs must provide foundations (*Begründungen*) rather than certificates (*Bewissmachungen*). Oumraou favours the view that proofs must not only provide a verification that what they are a proof of is true, but also yield an explanation of *why* this is the case. This is of course a very strong requirement, the strengh of which depends here on what one intends the explanation to be. For Bolzano, only pure proofs are explanatory, but, as Oumraou appropriately remarks, the logical relations indicated by impure proofs are also objective.

There is another distinction to which Oumraou devotes much attention, due to Pòlya. Pòlya distinguishes problems which must be solved from problems which require a proof: in the first case, we are looking for some unknown related to some previously fixed set of data; in the second case we must deduce a conclusion from a set of hypotheses. Oumraou insists, with Pòlya, on the value of the distinction between the way in which a mathematical problem has, as a matter of historical fact, been solved, and the way in which *it could have been*. He claims that Pòlya's distinction is akin to the distinction between the analysis of the constitution of mathematical knowledge (which pertains to psychology) and the verification that some proof is indeed a proof (which pertains to logic). Such a "verification" must certainly be related to what Oumraou calls "justificational understanding" and one would have liked him to expand upon this, especially in view of the fact that he usefully distinguishes three forms of heuristics: the heuristics of facts, which deals with the content of conjectures *per se* ; the heuristics of proofs, which deals with strategies for proof-searching ; and the heuristics of resolutions, which deals with the values of some unkown satisfying a particular condition. Heuristics is usually conceived as leading to a dilemma: either it is an *ars inveniendi*, or it reduces to a mere historical catalogue of past mathematical discoveries. If one follows Pòlya's third way, one must conceive mathematical problems as being probed by a somewhat idealized mathematician. Just how much idealization is warranted here?

The question is relevant especially in the context of Brouwer's intuitionism (perhaps even more so than in the context of Pòlya's own analysis in terms of patterns of plausible inference). Of course, in the perspective of Brouwer's intuitionism the distinction between heuristics and logic is even stronger (see in particular pp. 148-154 on this point). Since finding proofs of mathematical propositions consists in finding means of constructing objects — the Brouwerian creative subject being thus in a privileged epistemic situation since such objects are a creation of his mental acitivity —, it is hard to see how the verification that a proof is indeed a proof may also count as a heuristics of resolutions. May the "verification" that a proof is one indeed consist in the (re-)effectuation of the very same act (i.e., of the very same proof)? An unsatisfying regress is involved here, or at any rate something that is deeply *un*informative for the creative subject. Finally, this seems to be a difficulty for someone like Oumraou, who also defends the discursivity principle (a statement may only be grounded by another statement), which clearly flies in the face of any philosophy of mathematics founded on strong intuition (on the idea that mathematical objects have the properties that they have *necessarily*, i.e. *by the fact that they are the objects of our intuition*). But it would be churlish to complain here that this question is left unresolved because that part of the book devoted to the relation between mathematical logic and philosophies of mathematics based on intuition is one of the more fruitful of the book.

A subject index and a name index would have been useful, but this is indeed a minor complaint in the light of the excellent philosophical and mathematical discussions the book has to offer. It should be recommended reading to anyone interested in mathematics, the teaching of mathematics at any level, the philosophy of mathematics and, of course, philosophy *tout court*.

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