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From Kant to Hilbert: French philosophy of concepts in the beginning of the XXth century

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The legacy of the Kantian theory of knowledge was very much alive among French philosophers of science at the beginning of the twentieth century. To adopt, not the letter of the Kantian system, but its spirit of critique, seemed a good perspective from which to undertake a rational study of science in general, and of mathematics in particular. The critical attitude does indeed invite us to turn away from things in themselves, inaccessible to human nature, and to keep at arm's length metaphysical questions, examining instead the defining conditions of knowledge.

There is no better way to conduct this examination than in the exercise of a scientific activity. Several brilliant philosophers were to engage in this exercise as they completed their philosophical education through the study of mathematics. They were greatly encouraged by the teaching of Leon Brunschvicg (1869–1944) at L'Ecole Normale Supérieure and at the Sorbonne; they were tempted by the example of Gaston Bachelard (1884–1962). A whole generation set about bringing to bear the power of critique on the new disciplines that were transforming traditional concepts of the nature and object of mathematics: non-Euclidean geometries, axiomatic methods, algebraic and topological structures, etc.

On the one hand, Brunschvicg praised Kant for having brought scientific objectivity to bear on human reason, so destroying the idea of an absolute rationality prior to rational activity. On the other hand, he deplored the fact that Kant had established immutable a priori forms, contradicted as these were by the *evidence* of the *indefinite progress* of mathematics. The decisive discoveries did not obey the schema of categories. It was therefore necessary to rethink the Kantian epistemology taking into account the historical character of mathematical results. If the search for truth is an essential task of philosophy, so an examination of the history of mathematics is indispensable, mathematics being, according to Brunschvicg, the discipline that had brought 'the most scruple and subtlety' to this search.

Brunschvicg thus proposed to separate out the critical attitude from the Kantian canvas of a priori forms of intuition and categories of reason.¹ Furthermore, he presented this attitude in a positivist light, insisting, like Auguste Comte (1798–1857), on the necessity of starting from the actual fact of science, and drawing philosophical lessons from its history. In place of the determination of the conditions of the possibility of knowledge in general, Brunschvicg substituted a kind of half-historical, half-philosophical, enquiry into the development of particular sciences. Reason being intimately linked to scientific activity, the inquiry had as its aim to discover in this activity that which Gaston Bachelard emphasizing the intimate link between rationality and historicity, called ‘the events of reason’.²

Those who followed Brunschvicg held onto several of his points:

- To consider the advent of the critical idea as a ‘decisive date in the history of humanity’³, because it sets out to focus on the power of intellectual and scientific creativity;
- therefore to take the Kantian epistemology as a starting point, but to amend it, modify it, or go beyond it, since it is necessary to
- take account of the indefinite progress of mathematics,
- and underline the unpredictable and complex nature of its results,
- and so give priority to the *development* of mathematical knowledge over the consideration of fixed and timeless frameworks of knowledge,
- to arrange the history of mathematical theories from the point of view of critical examination, that is to submit the chronological succession of results to a ‘reflective analysis’, which would bring out the internal rationality of their connections,
- to show that these dynamic connections cannot be reduced to static logical relationships.

Brunschvicg made the dynamism of mathematics correspond to the inexhaustible dynamism of the mind and attributed to consciousness the generative power of creation and progress. It is this fundamental choice that brings Brunschvicg’s philosophy into line with the western tradition of the philosophy of subject, and it was a choice firmly disavowed by his successors.

A pupil of Brunschvicg, Jean Cavailles (1903–1944) started by accepting the parallels between mathematical progress and the progress of consciousness, and, more generally, between the enriching of experience and the expansion of consciousness. This is particularly apparent in the closing pages of *Méthode axiomatique et formalisme*⁴, the book that formed Cavailles’ thesis under the

¹ Brunschvicg, [1924].

² Bachelard, [1945], no 1–2, reprinted in *Léon Brunschvicg. L’œuvre. L’homme*, Paris, A. Colin, 1945, p. 77.

³ Brunschvicg, [1924], p. 229.

⁴ Cavailles, [1938], in *Euvres complètes de philosophie des sciences*, Paris, Hermann, 1994, pp. 177, 179.

supervision of Brunschvicg. But in his last work, *Sur la logique et la théorie de la science*, written in prison and published posthumously⁵ three years after he was put to death by the Nazis, Cavaillès proposed to substitute for the philosophy of consciousness a ‘philosophy of concept’. This programme, which the author did not have the chance to expound fully, nonetheless left a deep and enduring mark on the landscape of the philosophy of mathematics in France. It to some extent overshadowed Brunschvicg’s legacy and reverberated throughout French philosophy of science. Figures with starting points as diverse as Georges Canguilhem’s (1904–1995), Gilles Gaston Granger’s (1920–) and Jean Toussaint Desanti’s (1914–2002) forged the essential part of their arguments from the perspective that Cavaillès had begun to open. Thinkers who ranged widely across philosophical ideas without focusing exclusively on this or that science, such as Michel Foucault (1926–84) or Gilles Deleuze (1925–95), also hailed the virtues of the concept in the construction of a *structural* theory of knowledge. But if so many people pondered the programme floated by Cavaillès like a bottle in the sea, none of them was to accept all the consequences of its author’s conceptual idealism.

Here I propose a kind of ‘tableau’ of the French philosophy of concept, centred on the dominant figure of Jean Cavaillès. The work of Cavaillès⁶ remained, for at least thirty years (1940–1970), a source of inspiration all the more diffuse for being dense and difficult. I want to show how the results of axiomatics and mathematical logic, as developed by David Hilbert’s (1862–43) proof theory (Beweistheorie), refined by the objections raised by L.E.J. Brouwer (1881–1966), and extended by A. Tarski’s (1901–83) formal semantics, served as a reformation of Kantian epistemology, and led to the placing of concept above consciousness. Naturally, I will attempt to make clear what should be understood by the term ‘philosophy of concept’. But we can note right away that while Anglo-Saxon analytical philosophy turned away from consciousness to invest in the objective

⁵ Cavaillès, [1947b], reprinted in *Œuvres* . . . pp. 473–560.

⁶ In addition to the works cited in the preceding notes, it is necessary to add:

— ‘Réflexions sur le fondement des mathématiques’, *Travaux du IXe Congrès international de philosophie*, t. VI, Paris, Hermann, 1937. Reprint in *Œuvres* . . . pp. 577–580.

— *Remarques sur la formation de la théorie abstraite des ensembles, étude historique et critique*, Paris, Hermann, 1938. Reprint in *Œuvres* . . . pp. 221–374.

— ‘La pensée mathématique’ (en collaboration avec Albert Lautman), *Bulletin de la Société française de philosophie*, **40**, n° 1, 1946, pp. 1–39. Reprint in *Œuvres* . . . pp. 593–630.

— ‘Transfini et continu’, Paris, Hermann, 1947. Reprint in *Œuvres* . . . pp. 451–472.

See also Hourya Benis Sinaceur:

— ‘Structure et concept dans l’épistémologie de Jean Cavaillès’, *Revue d’Histoire des Sciences*, XL-I, 1987, p. 5–30.

— *Jean Cavaillès. Philosophie mathématique*, Paris, Presses Universitaires de France, 1994. This work contains the most complete bibliography of the writings of Cavaillès and a list of works about him.

factuality of language, it was a concept that French philosophy made the source of scientific objectivity.

My tableau will be far from complete. It will nonetheless bear the principle features that characterize the radical change of perspective achieved by Cavailles. These features are the collapsing of the transcendental onto the factual, the substitution of that which unfolds for that which is, of a developmental 'logic' for the usual static one, the taking into account of the symbolic material of mathematics, the elimination of the subject in favour of the object, and the objectification of the concept, the internal dialectic of concepts. It is in so far as they contribute towards underlining one or more of these features that I will also bring in the works of G.G. Granger and J.T. Desanti.⁷ These two authors expressly lay claim to Cavailles and to the philosophy of concept in their works on the epistemology of mathematics.

I Kant appropriated: the rejection of the transcendental and of the a priori

Until the 1960s, and occasionally to this day, the Kantian doctrine remained a backdrop to epistemological discussions; it was at once a support and a foil. Taking Brunschvicg's example, Cavailles and Granger engaged in the determined project of reforming of Kantianism, Foucault in a collapsing of the transcendental onto the empirical. Reflection was focused on the notion of the 'synthetic a priori' and, to a lesser extent, on notions of experience, of object and subject, of concept and reason. To be sure, these notions no longer necessarily held the specific meaning Kant had given to them; they were reinterpreted as they were filtered through the bundle of notions brought forward to sketch out the framework of the epistemological project: notions of the act, of activity, of work, of effectiveness, of event or 'moment', of development, of dynamism, of dialectic.

I.1 The philosophy of mathematical practice

Mathematics has traditionally played a paradigmatic role in the development of theories of objective knowledge. Cavailles embraced tradition in affirming

⁷ Of J. T. Desanti's writings, I have essentially used *Les idéalités mathématiques*, Paris, Le Seuil, 1968. Among G. G. Granger's works, I have restricted myself to:

— *Pensée formelle et sciences de l'homme*, Paris, Aubier, 1960.

— 'Pour une épistémologie du travail scientifique', *La philosophie des sciences aujourd'hui*, sous la direction de Jean Hamburger, Paris, Gauthier-Villars, 1986.

— *Pour la connaissance philosophique*, Paris, Odile Jacob, 1988.

— *Formes, opérations, objets*, Paris, Vrin, 1994.

See also Hourya Benis Sinaceur, 'Formes et concepts', in *La connaissance philosophique. Essais sur l'œuvre de G. G. Granger*, Paris, Presses Universitaires de France, 1995, pp. 95–119.

that 'mathematical knowledge is central to knowing what knowledge is'.⁸ Nonetheless, he specifies that 'critical reflection on the very essence of mathematical work' . . . leads to 'digging beyond what can strictly be called mathematics, in the ground common to all rational activities'.⁹ And there, he sets himself apart from the classical understanding through his reference to an activity and to work. Cavaillès has in mind a theory of knowledge that acknowledges the *practice of mathematics*, the work of the mathematician. Similarly, Granger underlines that 'the epistemological attitude looks to the *practice* of science, in its process of creation and bringing about'.¹⁰ Again, Desanti is interested in '*productive practices*' in the field of mathematics.¹¹ So it is not a case of determining a priori categories of thought, but of explaining, from the inside, thought in action, of grasping the actual process of production, by tracing the stages and the tangled and many-pathed routes that lead to a novel result. One does not occupy oneself with the empty forms that reason, according to Kant, is bound to impose on empirical facts in order to transform them into objects of knowledge, but rather with the substance itself, with the contents of knowledge, which are at once the object and the product mathematical practice. Mathematical knowledge is something original, which constitutes a positive reality, existing of itself, irreducible to anything other than itself. It is necessary to put oneself into this specific reality without bringing in any pre-conceived philosophical idea. The question one is looking to answer is no longer 'How can mathematical knowledge be possible a priori?', but simply this: 'How does mathematical knowledge come about?'. It is to bring back to the level of *phenomenon* the question that Kant asked at the 'transcendental' level (that is to say at the level of first principles, universally applicable to *all* knowledge in general).

1.2 Demonstration

To be sure, a certain dose of pragmatism enters this interest in the actual and material aspect of mathematical activity. But for Cavaillès, as for Granger and Desanti, it is not a case of clarifying the psychological, sociological, cultural or anthropological context of the activity. The mathematical activity is considered in itself, in abstracto; that is to say, its effects and results are thought of independently of real circumstances, contingent and possibly resistant to a totally rational explanation. Pragmatism serves to forbid an external and general discourse *about* mathematics, to try to show mathematics in the process of happening, and to expose in detail situations and problems.

⁸ 'La pensée mathématique', p. 625. In a similar way, Granger thought that 'mathematics remains a paradigm on which can be modelled every objective thought, however distant.', *Pour la connaissance philosophique*, p. 113.

⁹ *Méthode axiomatique et formalisme*, p. 29.

¹⁰ *Pensée formelle et sciences de l'homme*, p. 9.

¹¹ *Les idéalités mathématiques*, Avant-propos, p. VII.

Across the firm, known facts of mathematical situations, Cavaillès has his eye on what it is that constitutes the *validity* of results. In abstracto, a result is unconditionally affirmed if it is demonstrated. And, in this case, it is ‘absolutely intelligible’. Cavaillès’ insistence on the role of demonstration is correlative with putting into question intuitive evidence as a reliable way of accessing the truth. Cavaillès takes on board the attitude of the mathematical movement which, since the nineteenth century, had been emphasizing the objective dependencies between definitions and theorems. The first chapter of *Méthode axiomatique et formalisme* surveys the contributions of numerous mathematicians, among others Gauss, Cauchy, Grassmann, Bolzano, Frege, Dedekind, Riemann, Pasch, and Hilbert. Cavaillès notes, for example, that the definition of quantities by Grassmann and the discovery of non-Euclidean geometries mark a break from intuition and tangible experience. He recalls the exhortation of Moritz Pasch in a famous passage of Lectures on the new geometry¹²: ‘The process of demonstration must be entirely independent of the *meaning* of the concepts, just as it must be of the figures: Only the *relationships* established between the principles or definitions should be taken into consideration’. He lingers longer over Hilbert’s Foundations of geometry, in which points, lines and planes no longer have any intuitive meaning. Indeed, application of the axiomatic method consists in revealing the different possible geometrical architectures as a function of the statements chosen as axioms. A theorem is not an absolutely true proposition, it is one proposition demonstrated from others, taken as axioms. Cavaillès emphasizes that Hilbert’s merit is to have made geometry the equal of the axiomatic arithmetic of Grassmann, Frege, Dedekind and Peano. Which is to say, in brief, that geometric space is not *the representation* of real space. It is a mathematical concept, tied to mathematical experience and not to real experience. A representational philosophy, such as classical philosophy, is therefore no longer relevant.

1.3 Truth from the perspective of structural mathematics

Thus, demonstration is at the same time the norm of exploration and the norm of the production of truth. The only place truth is to be found is in a system defined by a set of relationships that the mathematician proposes as, or accepts as, fundamental. Varying the axioms opens perspectives that act retrospectively on our understanding of known truths, or even of a whole branch of mathematics. Hilbert had illustrated this in Foundations of geometry, in specifying, for example, the conditions and the limits of validity of Desargues’ theorem, or of Pappus–Pascal’s. The same proposition can result from different proofs, or might not be demonstrable in certain systems. Truth, therefore, is not monolithic, it has multiple aspects; and above all, it is not bound by the dogmatism of the obvious.

Modern mathematics in fact plays down the role of obviousness in the multiplicity of possible geometries: non-Euclidean, non-Archimedean, projective,

¹² *Méthode axiomatique et formalisme*, p. 64, p. 70.

Hermitian, metric, etc. It dissolves the ideal of a universal truth in the construction of non-standard models of arithmetic, of real analysis, of set theory. It puts structures in place of objects, and builds concepts that bring together varied structures: arithmetical, algebraic, topological, etc. Cavaillès, Granger and Desanti are strongly influenced by this architectonic work that shows mathematics as a stratified network of concepts, ‘an indivisible whole, an organism whose vital force depends on the connections of its parts’.¹³ In 1936, long before Bourbaki popularized the word in an article now become emblematic¹⁴, Cavaillès described to his friend Albert Lautman the effect on him of the texts that he was studying. He said he was literally submerged by ‘architectural images of mathematical development’.¹⁵

Architecture eclipsed the object; or, more exactly, it itself became the object. In French historical epistemology, the object does not have the characteristics of permanence, separate identity and indivisible unity of a *substance*. Conceived in line with the lessons of axiomatics and the ‘*begriffliche Mathematik*’ developed at Göttingen, an object is some element of a domain associated with a system, in which the relationships define a structure, a concept. The mathematician constructs concepts, of a group, for example, or a ring, a field, vector space, metric space, etc. The concept brings together in a complex *functional unit* a collection, which can vary, of schemas of operations prescribed by axioms. It is an anchor-point for reasoning, but, as Desanti wrote, nothing in it is fundamental or foundational.¹⁶ It is not a given fact, but a construct. And when, for example, one talks about *representing* a group, one means that the concept of a group brings about a *figurative effect* in which the operative possibilities indicated by the concept can be realized. From the *anti-psychological* perspective of the French philosophy of mathematics, representation is not essential, but only secondary.

2 Mathematical progression

2.1 Progressive reason

Mathematics shows an ‘original dynamism’ that escapes ‘all prior order’.¹⁷ Cavaillès broadly adopted this idea of Brunschvicg’s. He brought it closer to some ideas of Brouwer on the ‘auto-deployment’ of mathematics, notably in the article

¹³ Hilbert, 2^e Congrès international des mathématiciens, Paris, 1900.

¹⁴ N. Bourbaki, ‘L’architecture des mathématiques’, *Les grands courants de la pensée mathématique*, Paris, Cahiers du Sud, 1948, pp. 35–47.

¹⁵ Letter of 13 June 1936, in Sinaceur 1987, p. 120.

¹⁶ Desanti, [1968], p. 230.

¹⁷ Brunschvicg, [1912], pp. 562–577.

‘Mathematik, Wissenschaft und Sprache’, published in 1929.¹⁸ He himself said several times that mathematics constituted a ‘real progression’, both uninterrupted and unpredictable. On the one hand, there is the continuity between the past and the future: each stage of the progression results from earlier stages. On the other hand, each novelty is complete, in so far as ‘one cannot, by simple analysis of notions already used, find within them the new notions’.¹⁹ Mathematical truths are not analytical truths. A real progression stretches into the future.

‘Progression belongs to the essence’ wrote Cavaillès²⁰, meaning that nothing comes at a single stroke but by successive steps. It is therefore useless to look for conditions other than the particular conditions of a given mathematical situation. These conditions are not formal. They prevail because they are elements or aspects of an earlier mathematical situation. They are conditions *internal* to the practice and always given to modification. And above all, the conditions and the system they pertain to are *not separable but reciprocally correlative*.

2.2 The mathematical experiment

The biological metaphor of a burgeoning and organic growth in mathematics favoured an ‘experimental’ conception of mathematical knowledge. Intuitionist ideas also played a role. ‘There is no truth which has not been the object of an experiment, and logic is not an entirely safe instrument for discovering truths’, wrote Brouwer.²¹ Cavaillès thought, on this score, that the activity of mathematicians was an experimental activity. He intended to write a book on *L’expérience mathématique*, in which he would probably have shown at the same time the absence of any break from real-world experiment and the specific difference between the two. Unlike empirical experiment, mathematical experiment is knowledge. That is why it is an experiment of truth. But like any experiment, it is to venture, to test, to try, to risk, and also a thing of custom, acquisition, practice, and expertise. Knowledge proceeds by tentative advances and by reorganizations ‘based on experience’.²² It finds *strategies* (not *principles*) for its action in that action itself. The theory of mathematical knowledge is a theory of action, more precisely a *theory of the rules of the action*.

Here, it is necessary to point out a harmony noted by Cavaillès between Brouwer’s intuitionism and certain themes from phenomenology. Cavaillès, Granger and Desanti had read the work of Husserl, notably the *Logische Untersuchungen* and *Formale und transzendente Logik*. For all that they vigorously

¹⁸ Cavaillès alluded to this article in *Sur la logique et la théorie de la science*, p. 497.

¹⁹ ‘La Pensée mathématique’, *Œuvres* . . . , p. 601.

²⁰ *Sur la logique et la théorie de la science*, p. 552.

²¹ ‘Consciousness, Philosophy, and Mathematics’, *Proceedings of the tenth International Congress of Philosophy*, Amsterdam, 1948, I, Fasc. 2, North-Holland. Reprint in *Collected Works I*, A. Heyting ed. North-Holland, 1975.

²² The expression is due to Gonsseth.

argued over certain aspects, the phenomenological direction left its mark in their works. Here are the principal traces:

- First of all there is the primacy of *Sachverhalt* [situation], which determines the decision, as I pointed out at the start, to turn one's attention towards the practice of the mathematician and the contents of mathematical knowledge.
- There is also the importance given to meaning. On the one hand, epistemology consists (according to Granger and Desanti):
 - of digging in the archaeological subsoil of those structures now in use to find their ancient roots (which is where history comes in)
 - of making explicit the latent meanings (which is where the hermeneutic method comes in), which the mathematician (as a mathematician) cannot explore without losing the thread of what he is doing
 - And to 'shed light on the relationships a posteriori necessary to the organization of concepts'.
- Above all there is the analysis of mathematical thought as essentially constituted from procedures of idealization and thematization. I will come back to these procedures later, which Husserl himself described in terms of the contributions of the axiomatic method.
- There is an attention to language, paid particularly by Granger, for whom 'it is without a doubt due to Husserl that epistemology has been reoriented down the difficult path of research on two levels: that of language and that of the object'.²³ Granger brought phenomenology closer to the Anglo-saxon²⁴ analytical current and was clear that the most significant contribution to epistemology consists of the linguistic analysis of knowledge.²⁵ He thus took an original path in the field of French epistemology.
- Finally there is a whole vocabulary that would be barely intelligible without reference to the intentionality that Husserl forges from premises that he found in Brentano. In Cavaillès, the words 'act', 'moves' and 'target' betray the impact of what seemed to him an authentic discovery. Granger made providential use of the notion of the 'categorical outline of the object' to understand the multiple polarity of concepts. Desanti, more directly than he would admit, made broad use of intentional structures and the phenomenon of the horizon.

Despite all these points, faithfulness to Husserl is fundamentally contradicted by the rejection, more radical for Cavaillès than for Granger and Desanti, of the subjective perspective that attributes to consciousness the initiative in the formation

²³ *Pensée formelle et sciences de l'homme*, p. 14.

²⁴ Granger is one of the first French authors to have explained and commented upon the works of Wittgenstein. The notion of 'a language game' naturally drew his attention. He compared it with the Husserlian 'eidetic variation' and made precise the difference: Husserl sought to determine the essentials in creating acts of thought; Wittgenstein sought to determine the function of language in creating the universe of linguistic behaviour.

²⁵ *Pensée formelle et sciences de l'homme*, p. 12–13.

of mathematical processes. The rejection of consciousness dismisses *de facto* a representational philosophy, which we have already seen does not correspond to axiomatic practice. But it leads to a major difficulty. How are we to sustain a philosophy of practice, of experiment and of action while excluding the subject that experiments and acts?

For Cavaillès, this problem is eliminated by his definition, entirely desubjectivized, and surprising, of what an experiment is. In fact Cavaillès says:

'By experiment, I understand a system, governed by a rule and under conditions independent of the actions . . . I mean that each mathematical process is defined in relation to an earlier mathematical situation on which it partially depends, and in relation to which it maintains an independence such that the result of the action must be observed in its accomplishment . . . That is to say . . . the act having been accomplished, by the very fact of it appearing, takes its place in a mathematical system extending the earlier system'.²⁶

Cavaillès emphasizes that a mathematical result is always situated in an historical context, which it extends and modifies. He suggests that, as a consequence, a mathematician's act is to be explained in terms of mathematics and not the mathematician's psychology. Finally, for him, an experiment obeys a system of rules dictated by a *state* of the problem to be resolved. To say that mathematicians' activity is an experimental activity, is simply to say that it is *subject to objective conditions*. These conditions are *internal* to mathematics and are concretely incarnated each time in a package of results, methods and problems. For example, for set theory or set topology in the last twenty years of the nineteenth century, as Granger notes, 'the attentive study of the state of analysis after Cauchy is manifestly indispensable; but one can reasonably doubt that the examination of the situation of the means of production in France and in Germany, and of the development of the battle of classes and of ideologies are of any great help.'²⁷

2.3 The historical method

Mathematics is a progression and the mathematical experiment is the continuation of a specific history, which is not to be confused with empirical history. The philosopher who seeks access to this experiment cannot get there except by way of this history, which he shall explore. The historicity of the experiment is not the cause, but the consequence of the moving, successive, winding, at times abrupt, character of the progression, the course of which never ceases to modify and deepen its own traces. There is a history because reason is (of its essence) progressive; and not the other way round.

²⁶ 'La pensée mathématique', pp. 601–602.

²⁷ 'Are there internal dialectics in the development of science?', *Formes, opérations, objets*, chapitre 18, p. 348.

It has been said and said often that history is to philosophy what a laboratory is to the scientist: a place where practical observations can be made and where the tools of analysis can be crafted. In a polemical spirit against the logicism of Bertrand Russell, promoted in France by the work of Louis Couturat (1868–1913), Brunschvicg insisted on the importance of the historical study of mathematics. He presented history as a *method* for revealing the *multiple*, albeit structured, character of mathematics, for explaining its progress, and, on this basis, for elaborating a supple and open theory of rationality. Bachelard, Cavailles, Gonthier, Granger, and Desanti systematically applied this method and spent a long time in the laboratory of reason that is the history of mathematics. But the history that these philosophers wrote is far from the landscapes familiar to mathematicians. None claimed to be rivalling the ‘Historical notes’ of Bourkabi’s *Éléments de mathématiques*. Instead it was a case of approaching mathematics as an experiment, that is to say not just by its successes but also by its speculative attempts, its difficulties, its setbacks, its errors. To try to capture its branching, generative evolution. And above all to discern in the succession of events connections that give them significance and make them intelligible. In this way the philosophers defended a structuralist concept of history.

Cavaillès, for example, held that ‘one can, by studying the contingent historical development of mathematics as it presents itself to us, perceive necessary facts beneath the string of notions and processes’.²⁸ It was from this perspective that he undertook the history of abstract set theory. On the one hand, as scrupulous historian, he reviewed an impressive number of mathematical works and memoirs. On the other hand, as philosopher, he summed up in a few words the originality of the Cantorian creation: it is not, according to him, in the *objects*, but in the *methods*, not in the consideration of sets of points, already entailed in the analysis of the representability of function in trigonometric series, but in the processes of the diagonal and transfinite iteration. All these recounted historical details have, therefore, to converge to show what it was that made the invention of these processes necessary. By the same token, in the history of axiomatics, it is necessary to emphasize the constraints that lead to crossing the boundaries of more restrained theories, to establish the new more general procedures of reasoning. Taking inspiration from the reflections of Dedekind²⁹ and Hilbert³⁰, Cavaillès exposed the working of these procedures, which he called paradigm and thematization. Given the repercussions of these notions for French philosophers, I will come back to them in section 4.2.

History is thus an essential instrument of philosophy. It allows us to stand at a distance from the present, the better to home in on originality. It offers

²⁸ ‘La pensée mathématique’, p. 600.

²⁹ Notably in the Habilitation lecture given in 1854 before Gauss and published only in 1932 by Emmy Noether in volume III of his *Gesammelte Werke*, O. Ore, R. Fricke and E. Noether eds, Braunschweig.

³⁰ Notably ‘Axiomatisches Denken’, *Mathematische Annalen*, L. 18, 1918 and ‘Über das Unendliche’, *Mathematische Annalen*, t. 95, 1925.

reason the chance to test its own critical power and to undo the old prejudices of traditional philosophy. It shows flux rather than things, processes rather than entities, multiplicity rather than singularity, singularities rather than the universal. History is the refracting prism of philosophical notions: being, subject, object, concept, intuition. And above all, history sets philosophers (of mathematics) a major problem: how to conceive of the link between the contingency of events and the internal necessity of the development of notions? The solution to this problem, if it exists at all, is not easy.

3 Form and content

3.1 The autonomy of content

Despite its external links with the physical, social, economic or political real world, mathematical activity produces a reality whose relative autonomy has been highly promoted by French philosophers. This had been the view of Brunschvicg. Cavailles and Granger had probably noticed its relationship with the Vienna Circle's neo-positive thesis of the autonomy of science. This is straightforwardly suggested by the fact that Cavailles wrote in 1935 a brief account of the activities of the Circle, and that Granger had good knowledge of the work of Wittgenstein.

Mathematical reality is made up of 'objective contents' endowed with an autonomous dynamism. For Cavailles, these contents 'are, in their progression, themselves the essential'.³¹ They have a 'creative autonomy'.³² This progress is driven less by the relationship between object and subject than in a *relationship between the object and it itself*.³³ The result is a complexity and an increasing unification. As Hilbert underlined in his lecture to International Congress of Mathematicians in Paris (1900), complexity and unification go hand-in-hand. Cavailles takes advantage of this lesson from modern mathematics to criticise Kant. 'The synthesis which Kant finds in thought requires nothing more or different to be provided but just itself, made multiple by its moments and its progress: that which is unified is not first of all present as varied', he writes.³⁴ The rhapsody of variety is internal to the development of mathematics. It is not given by the external, it is constructed according to a specific rhythm with unexpected bifurcations. The unpredictable character of mathematics may be caught by the kantian notion of synthetic judgement. But Cavailles focuses on the *content of judgement*, and disregards the *faculty of judgement*. He is looking for a theory of content, not

³¹ *Sur la logique et la théorie de la science*, p. 486.

³² *Sur la logique et la théorie de la science*, p. 501.

³³ Desanti thought, moreover, that the object occurs in a mobile and self-regulated relationship with itself.

³⁴ *Sur la logique et la théorie de la science*, p. 510.

a theory of judgement. Here he distinguishes himself notably from his teacher Brunschvicg and adopts a perspective that owes much to Gottlob Frege.

3.2 Formal content

In evoking in Kantian terms the genesis of the notion of group, Brunschvicg remarked that ‘this notion does not have, as the notions of negative numbers or imaginary numbers did retain, the external appearance of a concept to which an object might correspond; . . . it presents itself as an intellectual relationship, establishing . . . the most significant trait of modern intellectualism, . . . the primacy of judgement over concept’.³⁵ Cavallès, Granger and Desanti understood that structural mathematics leads to nullifying the exteriority between concept and object (in Kant’s sense), that is to say between form and matter. But they drew a different conclusion from Brunschvicg. For them, content is primary. But in fact, mathematical contents are always already formal, and the forms are called upon to become contents for the construction of more abstract forms. The very wide and differentiated application of group theory showed, for example, the incarnation of structure in many varied aspects of mathematics: in algebra (the group of permutations of the roots of an equation), geometry (the group of transformations), matrices, topology, etc. The notion of group is *generic*: it talks not about a determined structure, but a type of structure. This is what explains its usefulness in the search for solutions in transversal problems. Put another way, to make axiomatic is not simply to give form; it gives rise to content.³⁶

The border between form and content was crossed another way by Hilbert’s metamathematics (*Beweistheorie*), which took the formal tool *par excellence* for forming mathematical content and made it a mathematical object (mathematical content): ‘a formal demonstration is just as much a visualisable and concrete object as is a number’ wrote Hilbert.³⁷ Further, Tarski’s formal semantics had perceived various layered levels in the formal. It had accentuated and made a systematic method of the back-and-forth movement between forms: considered structure, and contents: the various models, known or possible, of this structure.

The entire second part of Cavallès’ *Sur la logique et la théorie de la science* is a – difficult – commentary on the philosophical consequences of Hilbert’s theory of demonstration, and Carnap and Tarski’s introduction of the semantic point of view. The conceptual framework and the language are borrowed from Husserl.³⁸ Contents and forms are described in terms of objective meanings, as introduced

³⁵ *Les Étapes de la philosophie mathématique*, §349, p. 550.

³⁶ See notably Granger, *Pensée formelle et sciences de l’homme*, p. 170.

³⁷ ‘Über das Unendliche’, p. 179.

³⁸ This can seem surprising, but it is necessary to recall that Tarski’s formal semantics developed an affinity with the semantics of traditional philosophy fashionable with Bolzano and Husserl. Husserl thought that ‘everything in the domain of logic is contained in the categories that, correlatively, present the signification and the object’. *Logische Untersuchungen*, I, §29.

in the *Logische Untersuchungen*. Like Husserl, Cavaillès establishes a correlation between meaning and the act by which it exists. This constitutes a serious difficulty for a philosophy of content, which wants to expel the idea of consciousness. Desanti and Granger faced this difficulty. The first deliberately associated with structural analysis an archaeological analysis, developed along the characteristic lines of phenomenology: the relationship between the explicit and the implicit, the link back from operations to acts, the position of objects, thematization, etc. He noticed that these themes imply the pre-supposition of a 'mathematical activity', correlative to the domain where actions, properties and objects would link up. But he suggested that one can understand this mathematical activity without necessarily referring it to a transcendental subject. I will shed more light on this point of view in Sections 5 and 6 of this chapter, as well as in the conclusion.

For his part, Granger grasped the *internal* link between form and content through the original notion of 'formal content'.³⁹ He intended to substitute this notion for that of the synthetic a priori, the key notion around which revolved all attempts to realign Kantism. His argument asks us to consider that the opposition between form and content had been understood on an ontological level (by Aristotle) or on an epistemological level (by Kant), whereas it consists of an *opposition of meaning*, that is a functional distinction between two elements correlative in a symbolic universe. The revelation of a form always coincides with the use of symbols, and the opposition of form and content fundamentally accompanies every act of meaning.

4 The theory of sign and schematics

4.1 The being of the sign

Formal content in fact arise in a universe of signs governed by explicit rules. The signs relate back to content, to varied meanings, and they are themselves the object, the material of formal transformations. Transformations have laws (which define structures and structures of structures) and limits (impossibility theorems). Laws and limits determine the generic specifications of possible content.

The principal element of Kantian epistemology comes from schematization which is the function of presenting of a concept to an intuition. For Kant, the concept needs to be made schematic: each concept has its schema, that is to say a procedure, a rule for the application of formal conditions of intuition to the empirical material of intuition, so as to give the concept a relationship with objects, that is to say, to give meaning to the concept.

Hilbert had noted that it was necessary to have symbolic mediation and pointed out the primacy of the sign in mathematics. Mathematical objects are the signs

³⁹ The notion of formal content, in *Formes, Opérations, Objets*, Chapter 2.

themselves: digits, geometric figures, indeterminate equations, formulae, schema, matrix tables, homomorphical diagrams of morphisms, diagrams of proofs, etc. 'In the beginning was the sign'.⁴⁰ Called upon by the critiques of Poincaré, Kronecker and Brouwer to make room for intuition in the formal procedures of his structural and metamathematical methods, Hilbert made the sign the intuitive basis, the content of mathematical thought, and made a plea for an intuitive metamathematics. Signs permitted a concrete handling of the finite sequence of formulae that constituted the diagram of a proof. Signs are external objects; they constitute the irreducible given facts, prior to all thought, of operating intuition. Hilbert thus collapsed the sign, the carrier of meaning, onto the material prior to thought. To rescue intuition, Hilbert sacrificed the meaning of signs to perception, and came back to a type of philosophy where matter is considered as external and prior to thought.

Cavaillès grasped the strategic importance of what he called Hilbert's 'theory of the sign'. Unlike the majority of historians, who followed the example of Bernays, Cavaillès did not argue for Hilbert's declared adherence, from the 1920s, to a position close to Kronecker's. It was to the theory of signs in and of itself to which he turned his attention, and not to the thesis that the integer number is a datum. He emphasized that this theory was not just a psychological description, but picks out an essential characteristic of mathematical terms: the expression of a mathematical situation is itself a mathematical situation. Moreover, it is this characteristic that gives rise to the reflexive disposition of modern mathematics. It is the signs, expressions and formulae that make up mathematical reality. The work of the mathematician is to experiment on its formulae. The sign is the substance of thought, but is not prior to thought. Unlike Hilbert, Cavaillès has no polemical reason to place so much emphasis on just the material, tangible, visualizable aspect of the sign. He therefore emphasized the function of the sign as a meeting place for form and matter. The sign has two faces – perceivable, because it really is the mathematician's material in his work, and formal, because it is defined by the rules of its use and its possible transformations. In the realm of perception, the sign is a sign just because it points to one (or more) *meanings*. The sign is the *symbolic matter* and not empirical matter; sign and sense are indivisible. In the realm of the intelligible, the sign is operative content, it acts. The intelligible content is not inert or isolated or fixed. It is a sign for the operations it points to and that act back on its meaning. The sign is what is permanent in that which is changeable in it and by it: the meaning.

So, Cavaillès adjusted Hilbert's theory. He reinstated the sign in the sphere where it belongs, that of meaning, and gave it a philosophical status more adequate to symbolic mathematics. But, at the same time, he had found what was needed to amend Kant's schematization theory. The sign is a mixed, tangible-intelligible, concrete-abstract, intuitive-formal thing. To it, therefore, belongs the

⁴⁰ 'Neue Begründung der Mathematik. Erste Mitteilung', *Gesam. Abh.* III, p. 163.

schematic function; it itself is the rule for applying form to content. There is therefore no need for a scheme in the Kantian sense. And intuition has no need to be split into the a priori and the tangible content. The sign condenses the matter: the sign's received meaning; and reveals the form : the new meaning.⁴¹

4.2 Constitutive properties of mathematical thought

The autonomy of the sphere of meanings was brought to light by the semantic tradition that was partially conveyed by Husserl phenomenology. In taking the semantic point of view of mathematical content, French epistemology never intended to deal with the question of the *ontological* status of meaning. From mathematical thought it did not describe the thing in itself, but its attributes. Moreover, the hypothesis of a mathematics of itself, that is to say of a region of ideal objects to which mathematics could refer, seemed superfluous. Cavailles is very clear in his refusal of Platonic realism, that is to say, of the philosophical option of the prime supporters of semantics, Bolzano and Frege. 'I believe', he wrote, 'that a concept of systems of mathematical objects is in no way necessary to guarantee mathematical reasoning'.⁴² When we consider this system of objects, all that we think of them as, are rules for reasoning demanded by the problems that arise; and it is these unresolved problems that push us to propose new objects or to change the meaning of the previous objects. This point of view is also that of Granger and Desanti (despite the metaphor of the expressions 'ideal objects' and 'idealities' constantly used by Desanti).

The progress of meaning acts to multiply it; the lines of generalization and of abstraction of modern mathematics appear like lines of meaning. Cavailles sees two principle axes of the ordered proliferation of meaning. Horizontally, idealization consists in the adjunction of ideals: imaginary numbers, Kummer's ideal numbers, points at infinity in the plane.⁴³ Idealization frees meaning from particular constraints: operations are dissociated from the elements of the field on which they operate. Vertically, thematization⁴⁴ superposes a different levels: autonomous operations are transformed into objects of a higher operating field. These two constitutive modalities of thought intersect again and again in the non-uniform, non-linear timeline of the mathematical experiment. Tangled layers of meaning give the sign its semantic substance, which grows more substantial all the time.

We can illustrate this by taking Granger's examples of conics. The idealization, which Cavailles also calls 'the paradigm moment', came into play when instead

⁴¹ The relationship, called dialectic, of matter and form, of the concrete and the abstract, is presented alike in the commentaries by P. Bernays on the metamathematical writings of Hilbert, in the work of Gonsseth, and in that of Granger.

⁴² 'La pensée mathématique', p. 603.

⁴³ Cavailles was inspired here by Hilbert's article 'Über das Unendliche'.

⁴⁴ This term is taken from Husserl, who created it in contact with mathematics. In his *Formale und transzendente Logik* (Halle, Niemeyer, 1929), he remarked that the '*thematische Einstellung*' is a solid mathematical tradition.

of making conics from different cones, with acute, obtuse and right angles at their point, Apollonius got them from the *same* cone, by varying the plane of the section. The production of content, the conics, comes from an *internal principle of variation*, and not from an *external principle of unification*. In thematization, we consider the form itself as variable: for example, Desargues defining, by the operation of projection, *the* conic itself as an invariant. The specification of curves then depends on the choice of type of transformation; for example, the class of parabola is correlative with the invariance of a line chosen to represent the points at infinity.

Granger accentuates the *semiotic* reform, sketched by Cavaillès, of the transcendental aesthetic. He makes explicit the status of formal contents as correlates of acts of meaning, clearly taking on the inheritance of phenomenology. But at the same time, he is less concerned with the acts themselves than with their structure. He thinks that mathematical axiomatization is a way of determining the objective categories of thought. Therefore he translates the action–meaning correlation into the operation–object duality, underlying the ambiguity of the sign and the back-and-forth movement of meaning between idealization and thematization. This ‘duality principle’ seems to him to be the essential feature of the act of cognition in general. It consists, according to him, of a ‘primitive and radical functional category of knowledge’, which constitutes an ultimate condition. This category therefore *takes the place of the transcendental subject*. Granger thus completes Cavaillès’ programme of abandoning the philosophy of the subject in favour of the philosophy of concept.

5 The subject displaced, the concept objectivized

5.1 The rational chain-sequence of contents

Formal content forms the *objective* network of thought. But the chain *sequence* that links the contents together has the same objective status. Content and chain sequence are homogeneous terms: an item of content is the result of earlier links, which can be updated from a new perspective. ‘Doing mathematics’ is to extend it, and to understand mathematics is to redo it. There is a global continuity to mathematics, despite local discontinuities. There is a coherence that surmounts or covers the aleatory ingredients of history. The author can change, the moment can differ: in the long run, the sequence is always taken up again and continued in an objective interdependence with what came before. There is an *internal necessity* to the chain sequence, whatever the historical moment at which it is being extended, and in whatever way someone is extending it. As Desanti expressed it through a paradigmatic image, on one side there is Archimedes, Leibniz and Riemann as like so many ‘contingent apparitions’, and on the other side we have the

quadratures, the definite integrals and the Riemann sums as so many ‘necessary chain-sequences’.⁴⁵

As I pointed out earlier, the link between the two series is problematic.

One solution, which is the one adopted by Cavailles⁴⁶ and Granger, is to consider that necessity is not a priori, as was the case in classical rationalism, but a posteriori. That does not mean to say that the necessity of the chain-sequence is chronologically posterior to its actualization, or that it is injected retrospectively thanks to some ‘rational reconstruction’. It rather means that necessity is there, from the start, but that it only *appears* after the event. It was hidden and only comes to light bit by bit, by the measure of successes. But it is in this that the distinctive mark of mathematics can be seen. Cavailles and Granger think that what characterizes mathematical progression is not the contingent aspect, which it shares with all other products of culture, but the rational structure of its chain-sequences. Besides, the rational interdependence of mathematical moments goes beyond purely deductive logic. Mathematical demonstration is more than a simple logical deduction. Mathematics cannot be reduced to logic, as was Frege and Russell had held. The French philosophers and mathematicians sided with Kant on this point and adopted the anti-logicism of Bruschvicg, Poincarée or Brouwer, who emphasized so strongly the originality of the mathematical experiment. And they took the semantic turn all the more quickly as they saw an opportunity to support, against Kant, the idea of an unbreakable intricacy linking content and form.

5.2 The motor of progress: constraints of mathematical problems environment

The interiority of the chain links to the contents that it produces destroys the idea of a creating subject.

Hilbert, all the while professing Kantianism, was firmly opposed to the subjective idealism of those who would contradict him: Kronecker, Poincarée and Brouwer. The philosophical idea underpinning his theory of demonstration was, he said, to draw up an inventory of the rules that our thoughts follow in order to function effectively, with a view to freeing us from the arbitrary, from sentiment and from custom, by protecting us from subjectivity.⁴⁷ Hilbert therefore searched for rules where others looked to custom, to chance inspiration, arbitrary convention, or, like Brouwer, ‘the exodus of consciousness from its deepest home’⁴⁸, oscillating between rest and feeling, towards a limitless introspective deployment of innate intuition.

⁴⁵ *Les idéalités mathématiques*, p. 32.

⁴⁶ For the details, see Sinaceur, *Jean Cavailles. Philosophie mathématique*, pp. 30–33.

⁴⁷ ‘Die Grundlagen der Mathematik’, *Abhandlungen aus dem mathematischen Seminar der Hamburgischen Universität*, VI. Band, Leipzig, Teubner, p. 79–80.

⁴⁸ ‘Consciousness, Philosophy, and Mathematics’, French translation in Largeault, p. 423 et seq.

Like Hilbert, Cavaillès, Granger and Desanti picked out as ways of thinking not empty forms, but effective, repeatable, combinable procedures that create content. Earlier I located the principal axes (idealization and thematization). For Cavaillès, the chain sequence did not depend on the initiative of a consciousness, be it empirical or transcendental. It is useless to postulate the Kantian 'I think' as an agent of rational *unification*. Because a content is already the bringing together of a polymorphism, in part actual and in part potential, of properties or methods; and the various items of content hold the one to the other via internal links: related through problems, structural identities, methodological or functional analogies, inversion or duality of perspective. As for Brunschvicg's *generative* consciousness, it is threatened by psychologism. According to Cavaillès, progress comes about from the *endogenous* development of content. Items of content, he writes, 'literally are the essential in their development, and the primordial pseudo-experiment of consciousness disappears before the autonomous dynamism which they reveal and which leaves no place for anything other than them'.⁴⁹

In this passage, Cavaillès accomplishes the preliminary step to his insistence on the philosophy of concept: he clearly eliminates the subject as an actor in the emergence of content and turns his attention towards the functional and symbolic unity offered by content. The linking of the sequence does not maintain its authority because of a legislative or creative consciousness; in its historicity, it possesses an authority of its own. Its accomplishment brings it up to date and legitimizes it at the same time. Because it is problems that need to be solved and difficulties to be overcome that engage solutions. It is the power of a method that surpasses the original field of its application and moves towards new territory and new problems. There is a reciprocal and objective conditioning between the methods and the extensions of the domains provoked by their application. The rational grows and branches of itself, according to the local constraints.⁵⁰ It has a self-organizing, expansive force.

As J.-T. Desanti explained metaphorically, 'the consciousness of the object lives on the life of the object itself'.⁵¹ The subject, 'reduced to the status of an anonymous spectator, [is] nothing other than the manner, different each time, by which the object is manifested'.⁵² Here the roles have been reversed: it is the subject that's the instrument of the object, the subject that acts as the medium for the object to express itself.

⁴⁹ *Sur la logique et la théorie de la science*, p. 486.

⁵⁰ A good example of a local constraint is recalled by Granger, who remarks that in algebra nothing in the primitive properties that define a field let one predict that a finite field must be commutative. A certain amount of work is needed to establish this property of finite fields.

⁵¹ *Les idéalités mathématiques*, p. 91–92.

⁵² *Les idéalités mathématiques*, p. 290.

5.3 Concept without subject

The paradox of the Kantian epistemology had been effectively revealed by Cavailles and Desanti. It consisted in abstracting from all content of knowledge to free up a formal framework, which, while being all the while inaccessible to consciousness, nonetheless posits consciousness as subjective structure of objectivity.

In Kant's philosophy, pure concepts of understanding determine the rules (the schemas) of subjective unification of representations related in their diversity. Nonetheless, without intuition, they are empty; just as intuition, without concept, is blind. Concepts serve, through the mediation of their respective schemas, to work out the substance of intuition as objective experience. The synthetic unity of the varied, in intuition, and the analytic unity of concept come together in the 'I think', a transcendental act of unification by consciousness. The 'I think' is neither the intuition nor the concept of an object, but the form of consciousness that comes with these two types of representation in the guise of a subjective condition of knowledge. Therefore concepts are functions of the spontaneity of the understanding of the subject.

In French philosophy of science following Cavailles, the link between concept and subject is broken. Concept moves to the side of object and content, reversing the normal paradigm of knowledge. In determining the objective structures of objectivity, the examination was directed at the objects themselves: the mathematical concepts. A truly Ptolemaic revolution.

The objectivity of the concept had been underlined by the work of Bolzano and Frege, who relieved it of all reference to activity of the soul. For his part, in his *Logical Investigations*, Husserl insisted upon the autonomy of the sphere of meaning; here belonged concepts as 'units of ideal meaning' representing 'constituent moments' of the construction of systematic theories similar to mathematical theories.⁵³ The mathematical theories that Husserl speaks of, and that Cavailles had studied in his two theses, are the structures of abstract axiomatics. In German, mathematicians generally called them concepts (*Begriffe*) and spoke of an architecture of concepts that organized the flourishing field of new inventions: the concepts of group, field, ring, ideal, algebra, etc.

I have shown the reverberations of this *begriffliche Mathematik* on the philosophical programme proposed by Cavailles.⁵⁴ The roles were redistributed: instead of setting consciousness and concept on the one hand in opposition to intuition and object on the other, Cavailles ignored the constituent function of consciousness and brought over to the side of the object (to the side of content) concept and intuition. The mathematical experiment is knowledge, and, in that respect, it is not an experiment of consciousness, but an experiment of concept. Thus concept,

⁵³ *Logische Untersuchungen*, I.

⁵⁴ Hourya Benis Sinaceur, 'Structure et concept dans l'épistémologie de Jean Cavailles', *Revue d'Histoire des Sciences*, XL-I, 1987, p. 5-30.

considered as an evolutionary crystallization of meaning, becomes a driving force. From this perspective, what becomes of intuition?

5.4 The parallelism of intuition and concept

Cavaillès was of the opinion that it is difficult to go further than Kant in the analysis of the role of intuition, which is 'not the contemplation of a completed event, but the apprehension in the performance of the act of the very conditions which make it possible'. This is how we can escape, he says, 'from the irrationality of that which is pointed up by the internal necessity of construction'.⁵⁵ But he deplores the influence of poorly understood Kantian epistemology: mathematicians (Kronecker, Poincaré, Brouwer and Hilbert) believe themselves faithful to Kant when they look for a zone of irreducible intuition, a kind of minimal reserve of guaranteed first entities. We have already seen that there are no such absolute objects at the start; the point, the continuum or the number are nothing but elements in the series 'from where they take their meaning and which goes beyond them'.⁵⁶ An object only has meaning by its function in a system of relationships and by the processes that these relationships put into motion. The idea of an irreducible intuition is contrary to thought, which is by nature systematic and progressive. According to Cavaillès, the intuition of Poincaré's pure number is nothing but the substantiation of reasoning by complete induction, which could be analogously applied to other 'numbers', for example Cantor's transfinite ordinals. Also, the mathematical continuum differed from the intuitive continuum. Hilbert had shown this in *foundations of geometry*, by constructing an algebraic model of the geometric continuum, such that the numbers in this model were sufficient for all Euclidean constructions, without needing recourse to what Hermann Weyl called the 'spatial sauce spread between them'.⁵⁷

For French epistemologists, mathematical intuition, which is distinct from perceptual intuition, is not something that can be isolated as itself, and so does not offer a permanent ground on which to build objects; because it itself shifts and lines itself up in parallel with the construction of the objects, which are not *given facts* for thought, but are rather *produced* by earlier conceptual chain-sequences. Bachelard notes that an intuition reveals itself progressively in a *discursive* manner, by variation of the examples where the associated notions act.⁵⁸ Desanti recommends⁵⁹ giving up a theory where intuition serves as the *foundation* to the constitution of content and its chain-sequence.

⁵⁵ *Méthode axiomatique et formalisme*, in *Œuvres*. . . , p. 35.

⁵⁶ 'Transfini et continu', in *Œuvres*. . . , p. 469, 472.

⁵⁷ The expression is due to Hermann Weyl, *Das Kontinuum*, Leipzig, 1918. Reprint New York, Chelsea Publishing company, 1960, p. 70–71.

⁵⁸ *Le nouvel esprit scientifique*, Paris, Félix Alcan, 1934. 9^e édition, Paris, Presses Universitaires de France, 1966, p. 145.

⁵⁹ *Les idéalités mathématiques*, p. 230.

It is important to understand that what one must give up is not intuition itself, but the foundational perspective. One of the most characteristic features of French epistemology is its anti-foundationalism. Cavailles, for example, does not think that intuition plays no role in conceptual mathematics. Quite the contrary, it is to be found at the highest levels of abstraction. Rather, he thinks that intuition is not a founding force and that it is neither internal nor external to concept. For him, intuition organizes itself in a systematic manner and by reference to an associated conceptual system. These two systems, intuitive and conceptual, are correlative and transform in parallel. The separation between the intuitive and the formal cannot be done in one step, nor is the relationship one-to-one. Intuition bears witness, at every stage, to the supposed 'naturalness' of a system of sequences conforming to specific rules. It is, in short, the mark of independence acquired by theories and methods, the sign of the objectivity of the concept. It is not the radiance from a consciousness of thought (the Cartesian *cogito*, the Kantian 'I think', the pure creative act of Brouwer, the intentionality of Husserl), but the effect of an effort of thought, which renders the formal intuitive.

So, intuition is not a subjective disposition. Or at least, what is of interest to French epistemology are intuition's objective traces, just as it only wants to examine the objective traces of experience and the objective traces of conceptualization. One is bound to recognize in that an original attempt to destroy the *myth of interiority*.

6 The dialectic of concept

6.1 The false problem of foundation

Anglo-Saxon pragmatism is often given credit for having eradicated the false problem that consists of wanting to make consciousness rational. Wittgenstein observed that all the reasons that one could choose as the foundation for whatever it be, are in general less certain than those that we want to build upon them.⁶⁰ It must be recognized that French epistemology should be given credit for having repudiated the perspective of foundationalism and having understood that it is only a piece of knowledge that can justify another. There are no justifications for mathematics that are not themselves mathematics, teaches Cavailles, closely followed by his successors. Put another way, mathematics contains its reason within itself. This is in any case what is correctly meant by its 'autonomy'. To say that mathematics is autonomous is not to say that it is connected to nothing, not to a specific society, nor history, nor culture. It means that none of these links are useful in explaining the contents, which can only be explained between themselves, in

⁶⁰ *Über Gewissheit*, §307.

relation to each other. 'The structure speaks of itself.'⁶¹ To understand mathematics, you have to do some mathematics, either directly, or, as a philosopher, by way of its history. The historian goes over the mathematician's *way of working*, which constitutes a real remedy against the temptation of foundationalist prejudice. For French epistemology, cultivating historical research means the elimination of the problem of foundations in favour of an understanding of the progress of knowledge. And in the arena of history, the ground was readied for the putting in place of a new interpretative tool: the dialectic.

6.2 The internal development of concept

Concept is therefore not a free creation of the human spirit (as certain mathematicians have affirmed, Cantor and Dedekind in particular). Concept, or expression, or meaning, is *the mathematical fact*. Mathematical theories develop from the formation of concepts. As we saw earlier, a concept lays down forms by the operation of idealization and 'lays itself down' by the operation of thematization. In the effective and objective operation of mathematical thought, concept is at once form and substance. Concept is born of concept and engenders new concepts.⁶² Development is an internal dimension of the concept.

There is a strong analogy, often underlined by authors who knew Cavailles, with the philosophy of Spinoza. Cavailles himself, in the last part of his posthumous *La logique et la théorie de la science*, recalls that the true idea, in Spinoza's sense, leads to nothing else that is not a true idea. Nevertheless, in mathematics, the links are made across a complex network scattered with concepts connected to each other by organic links of different kinds. This 'organism' is not stable. It evolves constantly under the influence of local changes, which have repercussions on the configuration of the whole. The development of the concept is more important than the concept itself. With mathematics we are dealing with a 'conceptual progression'. The concept lives, and develops.

6.1 Explaining the primacy of progress: The dialectic of concepts

Cavaillès was looking for a theory of rational chain-sequences that would justify content, that is to say erases from it chance and arbitrariness, and explains progression, by which we do not mean an increase in volume but a perpetual revision of content through deepening. His answer was more or less this: it is in the perpetual development that the justification of contents is realized. *The very pursuit of development*, which integrates what came before into what comes next while modifying it, is a guarantee against the arbitrary and the contingent.

This development is 'material' in as much as, stimulated by the problems that crop up in the mathematician's work, it arises from the contents. But at the same

⁶¹ Cavailles, *Sur la logique et la théorie de la science*, *Œuvres*. . . , p. 506.

⁶² This expression is literally due to Granger, *Formes, opérations, objets*, p. 345.

time it is conceptual, since the contents are methods and concepts, always 'intelligibles', like Spinoza's ideas. To understand the development, it is not necessary to presuppose a driving act. On the other hand, it is useful to find the trajectories, the different 'moments' of the shifting of one concept into another: 'The idea of the idea manifests its generative power on the plane of limitless superposition which it defines without suffering harm'.⁶³ Development is a source of surprises without being a factor of contingency. Development is 'necessary' because it is inscribed in the 'internal bonds' between concepts, in their organization and systems. The progressive structure neutralizes contingency at source. The accidental and the accessory fade away in favour of rational connections.

Cavaillès called this material and rational progression a 'dialectic', employing a term much used at the time, notably by G. Bachelard, A. Lautman and F. Gonseth⁶⁴, and preserved, at least to begin with, in the works of Granger and Desanti. The dialectic of concepts, which was to replace the philosophy of consciousness, 'materializes' autonomous and, so to speak, spontaneous, progression between contents. The dialectic is a logic, but it is not a formal logic. It is a logic of content and development; we can speak of an 'internal dialectic'. This is required to reflect the development of knowledge in its two *inseparable* aspects, material and formal. It expresses the, so to speak, substantial link between the necessity and the unpredictability of mathematical development. It is a 'creative dialectic': it, and not the subject, brings about the creation of concepts. The last sentence Cavaillès wrote in *Sur la logique et la théorie de la science* is this: 'The generative necessity is not that of an activity, but of a dialectic'.

The dialectic seemed at the beginning of the twentieth century to be the most adequate method to the sciences, because every scientific principle is called upon to be specified, differentiated, revised. As the physician J. L. Destouches wrote, the dialectic was understood as a strategy to '*mouvoir dans le mouvant*', to drive forward with the development. Although these authors did not much recognize themselves much in Hegel's or Marx's dialectic, they could not entirely avoid the vocabulary and the new ideas these brought; G. Bachelard's *Philosophie du non* is witness to this. As for Cavaillès, he acknowledged that his own ideas were compatible with dialectical materialism, even if they were not a priori guided by it.⁶⁵

But two things in particular were brought to the fore. 1) The dialectic was an improved replacement for logic, found much too rigid to fit the dynamism of thought. Thus Bachelard advised us to be wary of a concept that 'no-one has yet

⁶³ *Sur la logique et la théorie de la science*, *Cœuvres* . . . , p. 514.

⁶⁴ In 1947, Gonseth and P. Bernays founded the revue *Dialectica*. The first two volumes of this revue contained a semantic explanation of the term 'dialectic'. One also finds here Gonseth's reaction to the work of Cavaillès, *Sur la logique et la théorie de la science*, which was about to be published.

⁶⁵ Letter of Cavaillès to the marxist mathematician Paul Labérenne, cited in Henri Mouglin, Jean Cavaillès, *La Pensée*, n° 4, Juillet-Août-Septembre 1945, p. 79: 'Although philosophically I am not oriented by the materialistic dialectic [. . .] I have said to you that I find myself led to results which are not exactly excluded by your attitude.'

managed to dialectalize'.⁶⁶ 2) The dialectic introduced a dimension in which it was easy for the *possibility of progress* to find a place. In a way, the dialectic was seen less in the light of Marxism than as the shadow cast by Comte's 'positive spirit', which Léon Brunschvicg had injected into mathematics.

7 Conclusion

The reflective analysis of knowledge had shown the inevitable circularity that brings *the position* of the subject back to *the position* of the object, and vice versa. The circularity results from the replacement of the exteriority between these *positions* occupied, respectively, by the subject and object, concept and intuition, the abstract and the concrete, with an *internal* linking. It therefore is not useful to break this circularity, but instead we should look for a method to fit it. For a while, the intentional structure that Husserl aspired to had seemed a good candidate. Cavailles, Granger and Desanti examined it. Without totally relinquishing it, they distanced themselves from it because it was rooted in consciousness. They chose the internal dialectic of concept because it avoided the difficulties of a subjective theory at the same time as it avoided the problems of a deductive-logical theory of knowledge. The dialectic installed an 'other' logic, or perhaps another meaning of the word 'logic', considering the circularity of thought not as a source of aporia, but as a normal, inevitable, fertile characteristic. Above all it allows the historical dimension to be put at the heart of this circularity. Thus is liberated all the dynamic potential of the continual and multiple back-and-forth between the opposing and correlative poles of the production of knowledge.

We can boil down all the correlations into just one of their number, such that it can be thought of as fundamental, that of subject and object. We can then attribute a *constituent* role to the subject, or a *constituting* role to the object. Bachelard and Gonthier chose the first path. For his part, Cavailles resolutely privileged the mathematical object, which he saw as a functional unit, capable of evolving in meaning, and as a structure (or concept). The dialectic of concept that he proposed avoids the need to resort to a transcendental or empirical subjectivity. But if the subject has been thrown out, its role persists: the *reflexive position* of the subject is taken on by the concept, as it appeared in expressions like 'the concept reveals itself', 'the concept transforms itself', 'mathematics occurs', 'it reflects on itself', 'it organizes itself into structures', etc. So it is legitimate to ask if this is not just a straightforward displacement, a transferral to object and concept of characteristics commonly recognized as belonging to the subject: autonomy, spontaneity, and dynamism.

⁶⁶ *Philosophie du non. Essai d'une philosophie du nouvel esprit scientifique*, Paris, Presses Universitaires de France, 1940, p. 134.

It is an altogether good question, and a pertinent one, that Granger and Desanti recognized, even if they did not tackle it head on. If they were categorical in choosing philosophy of concept, they nonetheless tried to find by what means, or within which limits, it is possible to eradicate all reference to subject.

By nuance, and to some extent by playing word games, Granger 'finally' settled on a form of transcendentalism that oscillated between Kant's doctrine and Husserl's. We might say of him the same thing *he* thinks of the Wittgenstein of the first part of philosophical *Grammaire*: he transposes the idea of the transcendental to an examination of the usage of language.⁶⁷ Let us quickly say how. Granger opposes consciousness and concept as two modes of experience, the first centred on the subject, the second decentralized, organized on and open to a hierarchy of possible obviousness.⁶⁸ On the one hand, concept takes on the Kantian transcendental function: it allows us 'to establish the conditions of possibility for considering as objects the entities to which it relates'.⁶⁹ On the other hand, concept can itself be thematized (in the sense of the process of thematization discussed earlier) as a higher level object. Thus, the different actualizations of concept are 'categorical outlines' of the object. Therefore the progression of the transformation of the concept is projected onto the transcendental plane, of which the vanishing point is intuition itself; but here, it is Husserl's phenomenology that is entailed, that is to say a perspective that, despite its leaning towards the object, reintroduces the transcendental subject, in so far as it sets out the rules of knowledge and turns them into objects. In fact, Granger dismisses only the intimist aspect of phenomenology, which, bogged down in perception and affection, knows nothing of science. Conscious experience is simply mistaken if it presents itself as a prototype of *all* experience and creates an illusion of stability and centrality, when scientific experience shows on the contrary that the subject is not the centre of the world. In the end, however decentralized objective thought might be, the transcendental subject reappears as a founder. Only, it is the subject of language, and not the subject of consciousness. An objectivized subject, so to speak. In fact language is, according to Granger, a 'store of forms, and the only thing responsible for their organization into a system'.⁷⁰ That is why it is necessary to root the forms of objectivity not in perception but in expression and to substitute for the aesthetic the semiotic, which sets up the 'general conditions of symbolic thought' and institutes the constituent categories of the object.

Desanti went another way, that of 'deconstructionalist philosophy'. If concepts were undeniably 'science's gift of a network', subject, history and work considered as entities were illusory, chimeras. Philosophy should undo them as units, break

⁶⁷ *Pour la connaissance philosophique*, pp. 235–239.

⁶⁸ *Pensée formelle et sciences de l'homme*, chapitre 6.

⁶⁹ *Le transcendantal et le formel en mathématiques*, in *Formes, Opérations, Objets*, chapitre 9, p. 150.

⁷⁰ On the idea of the mathematical concept 'natural', *Formes, Opérations, Objets*, chapitre 10.

the chimeras in order to approach mathematics as a cultural phenomenon.⁷¹ In particular, it is necessary to decode the language and try to discover what expressions like ‘the development of ideas’, ‘chain-sequence’, ‘conceptual necessity’, ‘the field of consciousness’, and so on really mean. This decoding, which Desanti initiated without developing the work in a systematic way, led him, in principle, to the destruction of the subject and all phenomenological reasoning. In a way entirely original among supporters of historical epistemology, he further suggested the likely impossibility of comprehending the connection between historical contingency and necessary sequences. With Archimedes, Leibniz and Riemann on one side, quadratures, definite integrals and Riemann sums on the other, can we hope to do any better than *acknowledge* (empirically) the fact of the coexistence of these two distinct orders of reality? Desanti accepted what neither Cavaillès nor Granger was willing to do: to give up the principle of necessary reason.

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⁷¹ *Les idéalisés mathématiques*, p. 11.

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