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## Network Games under Strategic Complementarities

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# Network Games under Strategic Complementarities

Mohamed Belhaj, Yann Bramoullé and Frédéric Deroïan\*

October 2012

*Abstract:* We study network games with linear best-replies and strategic complementarities. We assume that actions are continuous but bounded from above. We show that there is always a unique equilibrium. We find that two key features of these games under small network effects may not hold when network effects are large. Action may not be aligned with network centrality and the interdependence between agents' actions may be broken.

*Keywords:* Network Games, Strategic Complementarities, Supermodular Games, Bonacich Centrality.

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## I. Introduction

In this paper, we study network games with strategic complementarities. Agents are embedded in a fixed network and interact with their network neighbors. They play a game characterized by positive interactions and linear best-replies, so an agent's action is increasing in her neighbors' actions. We assume that actions are continuous *but bounded from above*, which means that the game is supermodular. Our main result, Theorem 1, establishes that this game always possesses a unique equilibrium. We then build on this result to further characterize the equilibrium. Overall, we find that the presence of an upper bound on actions strongly affects the outcomes of the game.

Our paper contributes to the analysis of games played on fixed networks. Ballester, Calvó-Armengol & Zenou (2006) study network games under linear best-replies and small network effects. They find that the equilibrium is necessarily unique and that action is related to network centrality. However, no equilibrium exists under strategic complements when network effects are large. Agents' actions feed back into each other in an explosive way and essentially diverge to infinity. This divergence seems unrealistic in many contexts where actions possess natural limits. Indeed, in their empirical implementation of that model, Calvó-Armengol, Patacchini & Zenou (2009) discuss the possibility of introducing such bounds. They state:

“Let us bound the strategy space in such a game rather naturally by simply acknowledging the fact that students have a time constraint and allocate their time between leisure and school work. In that case, multiple equilibria will certainly emerge, which is a plausible outcome in the school setting.”, Calvó-Armengol, Patacchini & Zenou (2009, p.1254)

We show that this conjecture does *not* hold under positive interactions, which is the usual empirical case. We extend Ballester, Calvó-Armengol & Zenou (2006)'s analysis to situations with large positive network effects and bounded actions. We find that uniqueness is guaranteed and we study how the equilibrium depends on the network structure. We develop our analysis in three steps. First, we apply results of monotone comparative statics for supermodular games to our setup. Second, we study how network position is related to action. We show that more central agents may end up playing a lower action. This confirms Bramoullé, Kranton & D'amours (2011)'s finding that the tight link between action and centrality only holds for small network

effects. We show that this link is preserved for interesting families of networks, including nested split graphs, the line and related hierarchical graphs, and regular graphs. And third, we study the extent of interdependence in the game. We find that the interdependence between agents' actions may be broken under large network effects, especially in situations where bridging agents are central.

At first glance, uniqueness in this context may indeed seem surprising. On one hand, supermodular games typically admit multiple equilibria. On the other hand, equilibrium multiplicity may be quite drastic in network games with linear best replies, under strategic substitutes (see Bramoullé & Kranton (2007) and Bramoullé, Kranton & D'amours (2011)).<sup>1</sup> Our proof makes clear that uniqueness relies on the combination of linearity and strategic complementarities. Introducing enough non-linearity in the best-replies, or enough substitutabilities in the interactions, would lead to multiple equilibria. In a way, linearity and strategic complementarities discipline each other.

Our study provides one of the first crossover between the theories of supermodular games and of network games. Galeotti, Goyal, Jackson, Vega-Redondo & Yariv (2008) analyze network games under strategic complementarities when agents have incomplete information on the network. We analyze a game of complete information here.<sup>2</sup> Belhaj & Deroïan (2009) analyze communication efforts under strategic complements and indirect network interactions. They look at the line and related networks. In contrast, we study direct network interactions and obtain results valid for arbitrary networks. Finally, the proof of our main result relies on a principle of partial contraction that, to our knowledge, had not yet been identified in the literature on supermodular games. This principle could potentially be useful in other setups.

The rest of the paper is organized as follows. We present the model in section 2. We prove uniqueness and derive some general properties of the equilibrium in Section 3. We study the relation between network position and action and the extent of interdependence in Section 4 and conclude in Section 5.

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<sup>1</sup>At the extreme under perfect substitutes, the number of equilibria may increase exponentially with the number of nodes in the network, see Bramoullé & Kranton (2007).

<sup>2</sup>However, we note that an agent only needs to know his neighbors' actions to be able to play a best-reply.

## II. The model

Consider  $n$  agents embedded in a fixed network, represented by a  $n \times n$  matrix  $\mathbf{G}$ . The  $(i, j)^{\text{th}}$  entry  $g_{ij}$  is a non-negative real number representing the link between  $i$  and  $j$ . We consider an arbitrary weighted network with no self-loop. Formally,  $g_{ij} \in [0, \infty[$  and  $g_{ii} = 0$ . Agent  $i$  is directly affected by agent  $j$  if  $g_{ij} > 0$  and  $g_{ij}$  then measures the strength of their relation. We do not impose symmetry so  $g_{ij}$  could differ from  $g_{ji}$ . In many of our examples, we will look at binary networks where  $g_{ij} \in \{0, 1\}$  and links do not differ in strength.

Agents choose an action  $x_i \in [0, L]$  and play a game with best-response

$$f_i(\mathbf{x}_{-i}) = \min(a_i + \delta \sum_{j=1}^n g_{ij} x_j, L) \quad (1)$$

where  $a_i$  denotes the optimal action of agent  $i$  absent social interactions with  $0 < a_i < L$  and  $\delta > 0$  denotes a global interaction parameter. A Nash equilibrium of the game is a profile of actions  $\mathbf{x}$  such that  $\forall i, x_i = f_i(\mathbf{x}_{-i})$ .

We take best-replies as primitives, as in Bramoullé, Kranton & D'amours (2011). So our results apply to any game characterized by these best-replies. In particular, the game  $\Gamma$  with quadratic utilities  $u_i(x_i, \mathbf{x}_{-i}) = -\frac{1}{2}x_i^2 + a_i x_i + \delta \sum_{j=1}^n g_{ij} x_i x_j$  belongs to this class. More generally, consider any functions  $f_i$  defined over  $\mathbb{R}$ , increasing over  $] -\infty, 0]$  and decreasing over  $[0, \infty[$  and any real-valued function  $v_i$  defined over  $[0, L]^{n-1}$ . Then the game with payoffs  $\pi_i(x_i, \mathbf{x}_{-i}) = f_i(x_i - a_i - \delta \sum_{j=1}^n g_{ij} x_j) + v_i(\mathbf{x}_{-i})$  yields best-replies (1).

In the absence of an upper bound ( $L = \infty$ ), these games have been analyzed in Ballester, Calvó-Armengol & Zenou (2006) and Ballester & Calvó-Armengol (2010). To see what happens, observe that if  $\mathbf{x}$  is an equilibrium we have  $\mathbf{x} = \mathbf{a} + \delta \mathbf{G} \mathbf{x}$  and hence, through repeated substitutions,

$$\mathbf{x} = \mathbf{a} + \delta \mathbf{G} \mathbf{a} + \delta^2 \mathbf{G}^2 \mathbf{a} + \dots + \delta^t \mathbf{G}^t \mathbf{a} + \delta^{t+1} \mathbf{G}^{t+1} \mathbf{x}$$

for any  $t \in \mathbb{N}$ . Denote by  $\lambda_{\max}(\mathbf{G})$  the largest eigenvalue of matrix  $\mathbf{G}$ . There are two cases. If  $\delta \lambda_{\max}(\mathbf{G}) < 1$ , then there is a unique equilibrium given by  $\mathbf{x} = (\mathbf{I} - \delta \mathbf{G})^{-1} \mathbf{a}$ . Under homogeneity ( $\forall i, a_i = 1$ ), actions are related to Bonacich centralities in the network.<sup>3</sup> In contrast if  $\delta \lambda_{\max}(\mathbf{G}) \geq$

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<sup>3</sup>The profile of Bonacich centralities is defined as  $\mathbf{c} = (\mathbf{I} - \delta \mathbf{G}) \mathbf{G} \mathbf{1}$  (see Bonacich (1987)), which yields  $\mathbf{x} = \mathbf{1} + \delta \mathbf{c}$ .

1, there is no equilibrium since the previous series diverges to infinity.

Our main new assumption is that actions are bounded from above:  $\forall i, x_i \leq L$ . Under this assumption, the strategy space  $[0, L]$  is now a complete lattice and since  $\partial^2 u_i / \partial x_i \partial x_j = \delta g_{ij} \geq 0$ , the game with quadratic utilities  $\Gamma$  is now a supermodular game (see e.g. Definitions and Theorem 4 in Milgrom & Roberts (1990)). Therefore we can apply classical results from the theory of supermodular games. In particular, a smallest and a largest Nash equilibrium always exist (see e.g. Theorem 5 in Milgrom & Roberts (1990)). Note that equilibrium conditions only depend on best-reply functions, so this property holds for *any* game in our class and not only  $\Gamma$ .

### III. Uniqueness

We now derive our main result.

**Theorem 1.** *Let  $\mathbf{a} \in \mathbb{R}^n$  and  $\mathbf{G} \in \mathbb{R}^{n^2}$  such that  $a_i > 0$  and  $g_{ij} \geq 0, \forall i, j$ . Any game with bounded actions,  $x_i \in [0, L]$ , and best-replies  $f_i(x_{-i}) = \min(a_i + \delta \sum_{j=1}^n g_{ij} x_j, L)$  has a unique Nash equilibrium.*

**Proof:** Denote by  $\mathbf{x}^*$  and  $\bar{\mathbf{x}}$  the smallest and largest Nash equilibrium such that for any equilibrium  $\mathbf{x}$  and any  $i$ ,  $x_i^* \leq x_i \leq \bar{x}_i$ . Denote by  $I$  the set of agents who play an interior action in the smallest equilibrium:  $I = \{i : x_i^* < L\}$ . Then,  $x_i = L$  for any  $i \notin I$  and any equilibrium  $\mathbf{x}$ . If  $I$  is empty, the equilibrium is unique. Next, assume that  $I \neq \emptyset$ . Given some arbitrary profile of actions for agents in  $I$ ,  $\mathbf{y} \in [0, L]^I$ , define  $\hat{\mathbf{y}}$  on  $[0, L]^N$  by  $\hat{\mathbf{y}}_I = \mathbf{y}$  and  $\hat{\mathbf{y}}_i = L$  if  $i \notin I$ . Consider  $\varphi$  the restriction of the best-reply function  $\mathbf{f}$  to  $[0, L]^I$ :  $\varphi(\mathbf{y}) = \mathbf{f}(\hat{\mathbf{y}})_I$ . Holding actions for agents in  $N \setminus I$  at the upper bound,  $\varphi$  describes the best-reply among agents in  $I$ .

Observe, first, that a profile  $\mathbf{x}$  is an equilibrium iff  $\mathbf{x} = \hat{\mathbf{y}}$  where  $\mathbf{y}$  is a fixed point of  $\varphi$ . Agents in  $N \setminus I$  play the upper bound  $L$  in all equilibria. And the equilibrium conditions for agents in  $I$  correspond to the fixed point equations of  $\varphi$ . Next, we show the following Lemma.

Lemma 1. Consider a system of linear equations of the form  $y_i = b_i + \sum_j h_{ij} y_j$  for all  $i \in I$ , with  $b_i > 0$  and  $h_{ij} \geq 0$ . If this system admits a non-negative solution  $\mathbf{y} \neq \mathbf{0}$ , then  $\lambda_{\max}(\mathbf{H}) < 1$ .

Proof of Lemma 1: Consider a non-negative solution  $\mathbf{y} \neq \mathbf{0}$ . For any  $t$ , we have  $\mathbf{y} = \sum_{s=0}^t \mathbf{H}^s \mathbf{b} + \mathbf{H}^{t+1} \mathbf{y}$ . If  $\lambda_{\max}(\mathbf{H}) \geq 1$ , the sequence on the right increases without bounds which is a contradiction.  $\square$

Finally, we use Lemma 1 to show that the restricted best-reply  $\varphi$  is contracting on  $[0, L]^I$ . To see why, observe that  $\mathbf{y}_I$  is a positive solution of the following system of equations:  $\forall i \in I, y_i = a_i + \delta L(\sum_{j \notin I} g_{ij}) + \delta \sum_{j \in I} g_{ij} y_j$ . Apply Lemma 1 to  $b_i = a_i + \delta L(\sum_{j \notin I} g_{ij})$  and  $\mathbf{H} = \delta \mathbf{G}_I$ . This means that  $\lambda_{\max}(\delta \mathbf{G}_I) < 1$ . Denote by  $\psi_i(\mathbf{y}) = b_i + \delta \sum_{j \in I} g_{ij} y_j$  and observe that  $\varphi_i(\mathbf{y}) = f_i(\hat{\mathbf{y}}) = \min(\psi_i(\mathbf{y}), L)$ . A linear function is contracting iff the largest eigenvalue of its associated matrix is lower than 1, and hence  $\psi$  is contracting. Then,  $\forall i, |\varphi_i(y) - \varphi_i(z)| \leq |\psi_i(y) - \psi_i(z)|$  and hence  $\varphi$  is contracting as well. Thus,  $\varphi$  has a unique fixed point, and hence the equilibrium is unique. QED.

Our proof relies on a principle of *partial contraction* that could, potentially, have bite in many other contexts. Note that here when network effects are large, the best-reply function is not contracting. Small differences in actions may be strongly amplified after a few rounds of best-replies. However, we show that the best-reply function is contracting on a critical subset of the original strategy space; namely, the set of actions lying between the smallest and largest equilibrium. Since any equilibrium belongs to that set, this property of partial contraction is sufficient to guarantee uniqueness. More generally, any supermodular game with such a partially contracting best-reply has a unique equilibrium.

Therefore, uniqueness prevails even in the presence of large positive network effects. The structure imposed by linearity somehow disciplines the natural tendency of strategic complementarities to generate multiple equilibria. Or, viewed from a complementary perspective, the structure imposed by the strategic complementarities somehow disciplines the tendency of linear network games to yield multiple equilibria. In short, we could say that linearity and complementarities discipline each other.<sup>4</sup>

Theorem 1 allows us to apply and adapt standard results and techniques from the theory of supermodular games. We begin with comparative statics.

**Corollary 1.** (*Monotone Comparative Statics*) *For any agent, action in the unique equilibrium is weakly increasing in  $\mathbf{a}$ ,  $\delta$ ,  $\mathbf{G}$  and  $L$ .*

Proof: Note that in the game with quadratic utilities  $\Gamma$ , all the following second-order cross

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<sup>4</sup>Uniqueness also hinges on the assumption that idiosyncratic actions  $a_i$  are all strictly positive. Uniqueness holds if some, but not all,  $a_i$  are equal to zero as long as the network is connected. In contrast, in the degenerate case where  $\forall i, a_i = 0$ , multiple equilibria may emerge.



derivatives are greater than or equal to zero:  $\partial^2 u_i / \partial x_i \partial a_i = 1$ ;  $\partial^2 u_i / \partial x_i \partial a_j = 0$  if  $j \neq i$ ;  $\partial^2 u_i / \partial x_i \partial \delta = \sum_j g_{ij} x_j$ ;  $\partial^2 u_i / \partial x_i \partial g_{ij} = \delta x_j$ ;  $\partial^2 u_i / \partial x_i \partial g_{kl} = 0$  if  $kl \neq ij$ . By Theorem 6 in Milgrom & Roberts (1990), each individual action  $x_i$  in the unique equilibrium is then weakly increasing in all these parameters. Next, let  $\mathbf{x}^*(L)$  be the equilibrium at  $L$  and consider a change in upper bound to  $L' > L$ . A first round of simultaneous best-replies from  $\mathbf{x}^*(L)$  necessarily leads to a profile  $\mathbf{x} \geq \mathbf{x}^*(L)$ . Then, a process of repeated best-replies converges monotonically to the unique equilibrium at  $L'$  and  $\mathbf{x}^*(L') \geq \mathbf{x}^*(L)$ . QED.

In this context of complementarities, direct and indirect network effects are fully aligned. Consider, for instance, the impact of connecting two agents. The direct effect of the new link is to induce both agents to increase their actions if they can. As a consequence, their neighbors may also increase their actions and hence their neighbors' neighbors may increase theirs. The impact of the new link propagates in the network and all the indirect effects are greater than or equal to zero. So the action of *every* other agent increases weakly following the addition of a new link.<sup>5</sup> This stands in sharp contrast to the case of strategic substitutes, where direct and indirect network effects are generally not aligned and comparative statics are more complicated, see Bramoullé, Kranton & D'amours (2011).

We are especially interested in the effect of an increase in  $\delta$ , which allows to vary the strength of interactions holding the network structure fixed. Our comparative statics result shows that once an agent reaches the upper bound, he necessarily stays there. This leads to a separation of the parameter space in three regions. Introduce  $\delta_1 = \inf\{\delta : \exists i : [(\mathbf{I} - \delta \mathbf{G})^{-1} \mathbf{a}]_i \geq L\}$  and  $\delta_2 = \max_i (L - a_i) / (\sum_{j=1}^n g_{ij} L)$ . Under homogeneity ( $\forall i, a_i = 1$ ), observe that  $\delta_2 = \frac{1}{k_{\min}} (1 - \frac{1}{L})$  where  $k_{\min}$  is the lowest degree of the network.

**Corollary 2.** *If  $\mathbf{G}$  has no isolated agent, the unique equilibrium  $\mathbf{x}^*$  is such that  $\forall i, x_i^* < L \Leftrightarrow \delta < \delta_1$  and  $\forall i, x_i^* = L \Leftrightarrow \delta \geq \delta_2$ .*

Proof. If  $\delta$  is low, the unique equilibrium is given by  $\mathbf{x}^* = (\mathbf{I} - \delta \mathbf{G})^{-1} \mathbf{a}$ . In particular,  $\forall i, x_i^* \geq a_i + \delta \sum_j g_{ij} a_j$ . So as  $\delta$  increases, a non-isolated agent necessarily reaches his upper bound and this first happens at  $\delta_1$ . Next, the profile where all agents play  $L$  is an equilibrium iff  $\forall i, a_i + \delta \sum_{j=1}^n g_{ij} L \geq L$ . This is equivalent to  $\delta \geq \delta_2$ . QED.

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<sup>5</sup>The increase is strict in a connected network under small network effects. The increase may not be strict here, due to the presence of an upper bound, see our discussion of broken interdependence below.

Thus, we see three domains emerging as a function of  $\delta$ . When  $\delta < \delta_1$ , the equilibrium is interior and action is proportional to Bonacich centrality in the network. When  $\delta_1 \leq \delta < \delta_2$ , some agents have reached the upper bound but others have not. When  $\delta \geq \delta_2$ , all agents have reached the upper bound and action does not depend on the network position. These two thresholds depend on the upper bound  $L$  and on the structure of the network. We emphasize that  $\delta_2$  is often larger than  $1/\lambda_{\max}(\mathbf{G})$ , the critical value above which actions diverge in the absence of bound. When some agents reach the upper bound, this dampens the explosive feedbacks and further postpones the levels for which the remaining agents will also reach it. In addition, the order in which agents reach the upper bound defines a network-specific ranking. We study below how this ranking depends on the network structure.

Finally, we note that we can import standard algorithms to our setup. In particular, we know that in supermodular games, repeated myopic best-replies converge rapidly and monotonically towards the smallest equilibrium when starting from the lowest profile ( $\forall i, x_i = 0$ ) and towards the largest equilibrium when starting from the highest profile ( $\forall i, x_i = L$ ), see e.g. Vives (1990).<sup>6</sup> Therefore, both processes will converge towards the unique equilibrium here. We make use of these algorithms in our examples below.

## IV. Structural properties of the equilibrium

### A. Network position and action

In this section, we study how the position of an agent in the network affects his action in equilibrium. To better identify the effect of the structure, we assume throughout that agents' actions in isolation are homogenous ( $\forall i, a_i = 1$ ) and that links are binary ( $\forall i, j, g_{ij} \in \{0, 1\}$ ). So individuals only differ in their network characteristics. Recall that when network effects are small, action is aligned with Bonacich centrality. We wish to understand how this is modified under large network effects.

We first provide an example where a more central agent ends up playing a lower action. This shows that action and centrality are generally not aligned under large network effects. In a second stage, we identify a number of cases where the alignment between action and centrality

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<sup>6</sup>In addition, a “round-robin” implementation, when agents take turn in best-replying, converges faster than a simultaneous implementation, when all agents best-reply at the same time.

is preserved.

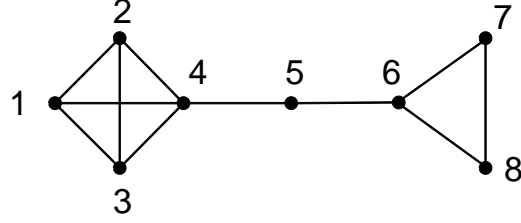


Figure 1: An example where action and centrality are not aligned.

Consider the graph depicted in Figure 1. It has eight nodes and two cliques: One composed of agents 1 to 4 and the triangle 6-7-8. In addition, agent 5 in the middle is connected to agents 4 and 6. When  $\delta = 0.3$  and actions are not bounded, the equilibrium is such that  $x_5 \approx 7.9 > x_6 \approx 5.7$ .<sup>7</sup> Even though agent 6 has one more neighbor than agent 5, his neighbors are not very central. In contrast, agent 5 is connected to agent 4 who is the most central agent in the graph. When  $\delta$  is not too low, the effect of indirect paths dominate and agent 5 is more central than agent 6 and play a higher action. Suppose next that actions are bounded from above by  $L = 5$ . The equilibrium is now such that  $x_5 \approx 3.7 < x_6 \approx 4.0$ .<sup>8</sup> Agents 1-4 reach the upper bound and this reduces the action premium that agent 5 gets from his link with agent 4. Agent 6, who is less central, now plays a higher action.

We next identify interesting cases where the alignment between action and centrality is preserved. We first look at nested neighborhoods. The following result shows that agents who have less neighbors in the sense of set inclusion always play a weakly lower action.

**Proposition 1.** *Consider two agents  $i$  and  $j$ . If every neighbor  $k \neq j$  of agent  $i$  is also a neighbor of agent  $j$ , then  $x_i^* \leq x_j^*$  in the unique equilibrium.*

Proof. Suppose first that  $g_{ij} = 0$ . Then  $N_i \subset N_j \Rightarrow \sum_{k \in N_i} x_k^* \leq \sum_{k \in N_j} x_k^*$  and hence  $x_i^* = \min(1 + \delta \sum_{k \in N_i} x_k^*, L) \leq \min(1 + \delta \sum_{k \in N_j} x_k^*, L) = x_j^*$ . Suppose next that  $g_{ij} = 1$ . Then,  $\{i\} \cup N_i \subset \{j\} \cup N_j$ . We can assume that  $x_j^* < L$  (otherwise  $x_i^* \leq x_j^* = L$ ). Then  $(1 + \delta)x_j^* = 1 + \delta \sum_{k \in \{j\} \cup N_j} x_k^* \geq 1 + \delta \sum_{k \in \{i\} \cup N_i} x_k^* \geq f_{-i}(\mathbf{x}_{-i}^*) + \delta x_i^* = (1 + \delta)x_i^*$ . QED.

<sup>7</sup>The whole equilibrium is  $x \approx (15.46, 15.46, 15.46, 17.28, 7.89, 5.69, 3.87, 3.87)$ .

<sup>8</sup>The whole equilibrium is  $x \approx (5, 5, 5, 5, 3.70, 3.99, 3.14, 3.14)$ .

This property of Bonacich centrality therefore extends to large network effects. Thus, an agent with additional neighbors will reach the upper bound first as  $\delta$  increases. This allows us to further characterize the equilibrium for graphs where agents are ordered in a way consistent with nested neighborhoods. This is, in particular, the case for *nested split* graphs. In these graphs, agents can be ordered so that  $g_{kl} = 1 \Rightarrow g_{ij} = 1$  whenever  $i \leq k$  and  $j \leq l$ . They appear, for instance, as outcomes of network formation processes based on centrality, see König, Tessone & Zenou (2011). On nested split graphs,  $k_i < k_j \Rightarrow N_i \setminus \{j\} \subset N_j$  and centrality and degree are aligned. A direct implication of Proposition 1 is that on nested split graphs, agents reach the upper bound precisely in the order of their degrees. A more central agent thus never plays a lower action.

Nested neighborhoods are not necessary, however, to preserve the alignment between action and centrality. In particular, we can apply the analysis of Belhaj & Deroïan (2010) to our setup. They study supermodular games played on the line and on related hierarchical graphs. These graphs are defined by the following three features. (1) They are structured around a geometric center and increasingly peripheral layers. (2) Degrees decrease weakly as nodes get further away from the center. (3) All agents in a given geometric layer have symmetric positions. (See Belhaj & Deroïan (2010) for a formal definition). Their main result then states that more central agents cannot play a lower action in the lowest and in the highest equilibrium in these graphs. In our context, the equilibrium is unique and this yields:

**Corollary 3.** *(Belhaj & Deroïan (2010)) On the line and on related hierarchical graphs, an agent who is more central plays a weakly higher action in equilibrium.*

Finally, the alignment between action and centrality can also be preserved in circumstances where some agents play the same action. A first observation, here, is that two individuals who have symmetric positions in the network must play the same action.<sup>9</sup> Next, consider regular networks. A network is regular of degree  $k$  if every agent has  $k$  links:  $\forall i, \sum_j g_{ij} = k$ . On regular networks, agents have the same degree but may have structurally different positions. Introduce  $\delta^* = \frac{1}{k}(1 - \frac{1}{L})$ . We can easily check that in a regular graph of degree  $k$ , the unique equilibrium  $\mathbf{x}^*$  is such that  $x_i^* = 1/(1 - \delta k)$  if  $\delta \leq \delta^*$  and  $x_i^* = L$  if  $\delta \geq \delta^*$ .

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<sup>9</sup>Otherwise, we could build another equilibrium by simply permutating the actions of the players.

In regular graphs all agents play the same action even under large network effects. All agents reach their upper bound for the same level of network effects and this threshold level does not depend on the specific structure of the graph. This stands in sharp contrast to what happens under strategic substitutes, where agents play the same action in a stable equilibrium only for small network effects and where the structure of the regular graph strongly affects the level of interactions above which asymmetric actions emerge, see Bramoullé, Kranton & D’amours (2011).

## B. Broken interdependence

In this section, we look at the pattern and extent of interdependencies. An important lesson of the previous literature is that interdependence is very high under small network effects. Even though agents interact directly with their network neighbors only, the interplay of strategic interactions implies that every agent is eventually affected by every other agent in the population when the network is connected. In contrast, we show here that this interdependence may be broken under large network effects. The reason is that when an agent reaches his upper bound, he stops being a transmitter of influences in the network.

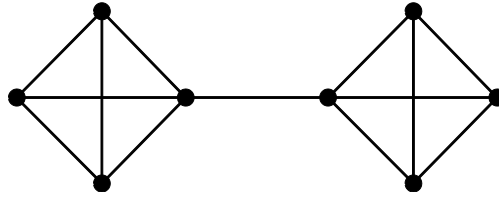


Figure 3: Two communities connected by a bridge.

To illustrate this effect consider the following stylized bridge example, depicted in Figure 3. Society is composed of two communities of equal size. In each community, every agent is connected to every other agent. In addition, one agent in the first community is connected to one agent in the second. So there is a unique link bridging the two communities. Then, if idiosyncratic actions  $a_i$  are not too different, we can show that there exists two threshold level  $\delta_1^*$  and  $\delta_2^*$  such that the following conditions hold. If  $\delta < \delta_1^*$ ,  $\partial x_i^*/\partial a_j > 0$  for any two agents  $i$  and  $j$  in the population. In particular, agents in one community are affected by shocks on agents in the other community. Every agent is affected by every other agent. If  $\delta_1^* \leq \delta < \delta_2^*$ ,  $x_i^* = L$  for both bridge agents;  $\partial x_i^*/\partial a_j > 0$  for any two non-bridge agents in the same community while  $\partial x_i^*/\partial a_j = 0$  if  $i$  lies in one community and  $j$  in the other. Finally, if  $\delta \geq \delta_2^*$ , every agent plays the upper bound.

Under small network effects, the bridging link plays a crucial role: It transmits shocks from one community to the other. When network effects increase, bridge agents tend to reach the maximal action first, because of their more central position. They then become unresponsive to shocks and this turns off the transmission channel.

More generally, denote by  $P_i(\delta) = \{j : (\frac{\partial x_i^*}{\partial a_j})^+ > 0\}$  the set of agents  $j$  who indirectly affect  $i$  in the sense that a positive shock on  $j$ 's action leads to an increase in  $i$ 's action.<sup>10</sup> Note that  $P_i = N \setminus \{i\}$  if the graph is connected and  $\delta < \delta_1$  and that  $P_i = \emptyset$  if  $x_i^* = L$ . We obtain the following result:

**Proposition 2.** *As  $\delta$  increases, the set of agents who indirectly affect  $i$ ,  $P_i(\delta)$ , shrinks monotonically towards  $\emptyset$ .*

Proof: Suppose that  $x_i^* < L$  and denote by  $I$  the set of agents playing an interior action. Denote by  $b_i = \delta L \sum_{j \notin I} g_{ij}$ . We have:  $\mathbf{x}^* = (I - \delta \mathbf{G}_I)^{-1}(\mathbf{a} + \mathbf{b})$  and hence  $\partial x_i^* / \partial a_j = (I - \delta \mathbf{G}_I)^{-1}_{ij} = \sum_{t=0}^{\infty} \delta^t (\mathbf{G}_I^t)_{ij}$  if  $j \in I$ . Therefore, an agent  $j$  belongs to  $P_i$  iff there is a positive integer  $t$  such that  $(\mathbf{G}_I^t)_{ij} > 0$ . This means that there exists a path from  $i$  to  $j$  in which all agents play an interior action. As  $\delta$  increases, more agents reach the upper bound and this set shrinks. When  $\delta$  is large enough, everyone plays  $L$  and actions are insensitive to marginal changes in  $a_j$ . QED.

Therefore, the extent of interdependencies is always smaller under higher network effects. Clearly, the way interdependence is broken depends on the shape of the network and on the number and the locations of the bridges. This, in turn, depends on the prominence of bridging agents within their communities. If bridging agents are equally central, or more, within their own community, as in the example above, we can expect interdependence to be reduced quite quickly because bridging agents will reach the upper bound first. But there are contexts where agents who are better connected externally are also less connected internally. In these situations, interdependence may be more robust as bridging agents within communities may reach the upper bound last.

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<sup>10</sup>Note that the left- and right- derivatives of individual action  $x_i^*$  with respect to  $a_j$  may differ and  $(\frac{\partial x_i^*}{\partial a_j})^+$  represents the right-derivative here. The analysis carries over to the set of agents  $j$  such that a negative shock on  $j$ 's action leads to a decrease in  $i$ 's action.

## V. Conclusion

In this paper, we analyze linear network games under strategic complementarities and with an upper bound on actions. We show that there is always a unique equilibrium. We apply standard results from the literature on supermodular games and characterize structural features of the equilibrium. In particular, we show that action may not be aligned with Bonacich centrality and that large network effects tend to break the interdependence.

Our results could be useful for empirical studies of peer effects in networks, see Calvó-Armengol, Patacchini & Zenou (2009) and Bramoullé, Djebbari & Fortin (2009). A typical econometric model aimed at studying whether some variable of interest  $x$  is subject to peer effects can be written:

$$x_i = a_i + \delta \sum_j g_{ij} x_j + \varepsilon_i$$

where  $\delta$  is the main parameter of interest to be estimated and  $\varepsilon_i$  is an error term.<sup>11</sup> Most choices and socio-economic outcomes are naturally bounded from above but existing empirical studies have neglected these bounds. This likely generates biases in existing estimates. While equilibrium multiplicity complicates the econometric analysis of games (see e.g. Tamer (2003)), Theorem 1 shows that this is not an issue here. Since the equilibrium  $\mathbf{x}^*$  is a function of  $\mathbf{a}$ ,  $\delta$ ,  $\mathbf{G}$  and  $\boldsymbol{\varepsilon}$  we can, in principle and given some assumption on the error terms, compute the likelihood  $L(\mathbf{x}^*|\mathbf{a}, \delta, \mathbf{G})$  and estimate  $\delta$  through maximum likelihood in a straightforward manner. Thus our analysis provides a stepping stone for an empirical study of peer effects in networks with continuous but bounded outcomes.

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<sup>11</sup>There are various possible empirical specifications for the  $a$ 's (including individual covariates and, possibly, contextual peer effects), the  $g$ 's (linear-in-sum or linear-in-means), and the error terms  $\varepsilon$ .

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