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Abstract: *We consider a two-sector two-good two-periods overlapping generations model with inelastic labor, consumption in both period of life and homothetic CES preferences. There are two consumption goods, one pure (non-durable) consumption and one consumable (durable) capital good which can be either consumed or invested. Assuming gross substitutability and a capital intensive pure consumption good, we prove the existence of efficient endogenous fluctuations through a Hopf bifurcation if the share of the consumption of young in the composite good is low enough. We also show that some fiscal policy rules can improve welfare and prevent the existence of business-cycle fluctuations in the economy by driving it to the optimal steady state as soon as it is announced.*

Keywords: *Two-sector OLG model, multiple consumption goods, dynamic efficiency, endogenous fluctuations.*

Journal of Economic Literature Classification Numbers: C62, E32, O41.

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1 Introduction

The existence of endogenous fluctuations under gross substitutability is a well established fact in overlapping generations (OLG) models. In particular, a large proportion of the literature considers that endogenous fluctuations occur through sunspot equilibria and are driven by changes in expectations about fundamentals. As shown by Woodford [19], the existence of sunspot equilibria is a consequence of local indeterminacy of the equilibrium under perfect foresight, i.e. the existence of a continuum of equilibrium paths converging towards one steady state from the same initial value of the state variable.

In a seminal contribution, Reichlin [15] shows how the co-existence of dynamic efficiency, *i.e.*, Pareto-optimal equilibrium paths, and local indeterminacy in OLG models is an important question in terms of stabilization policies. If sunspot fluctuations occur under dynamic *efficiency*, a public policy can simultaneously stabilize the economy and reach the Pareto optimal steady state. On the contrary, when local indeterminacy occurs under dynamic *inefficiency*, stabilization policies targeting the steady state leave room for welfare losses. As shown by Reichlin [15], locally indeterminate dynamically efficient equilibria occurs in aggregate OLG models, but only throughout flip bifurcations and period-two cycles. This feature generates fluctuations that are far to what is generally observed. Indeed, in presence of period-two cycles, the main variables are characterized by unrealistic negative auto-correlations. Dufourt *et al.* [8] show on the contrary that when a Hopf bifurcation instead occurs, persistent and non-monotonous convergence to the steady-state with positive auto-correlations provide empirically plausible endogenous fluctuations.

In Galor [10] type two-sector OLG models,[1] with one pure consumption good and one pure investment good, the occurrence of endogenous fluctuations and local indeterminacy is based upon a capital intensive consumption good. The factor allocation across sectors generates oscillations of the capital accumulation path which can propagate in the economy through the savings behavior of the agents. Nourry and Venditti [13] have proved that local indeterminacy is likely to occur under dynamic efficiency, but again throughout a flip bifurcation.[2]

[1] See also Venditti [18].

[2] The sectoral technologies need to be close enough to Leontief functions. See also Drugeon *et*

However, when heterogeneous sectors are introduced, the assumption of a unique consumption good is a strong simplification likely to generate singular properties. In a multisector framework is introduced, it is quite common to consider many consumption goods as Benhabib and Nishimura [2] for an optimal growth infinite-horizon model. In such a case, the existence of additional substitution mechanisms between the different consumption goods suggests that new rooms for the occurrence of endogenous business cycles could be obtained. Kalra [12] and Nourry and Venditti [14] consider a two-sector model in which the first sector produces a pure consumption good (non-durable), and the second sector produces a mixed good which can be either consumed or used as capital (durable). They show that local indeterminacy can now occur through the occurrence of a Hopf bifurcation.

Nevertheless this last approach has two major flaws.
First it considers a structure of the life cycle consumption which is symmetric, meaning that agents' tastes concerning their consumption bundle are invariant. As a result they basically consume the same amount of durable goods throughout their life. This is in contradiction with the recent study of Fernandez-Villaverde and Krueger [9]. Using a Consumer Expenditure Survey data to estimate life cycle profile, they show that consumption expenditures differ qualitatively among age, retired agents consuming relatively less durable goods. Such patterns highlight that consumption expenditures evolve over time and thus should have an impact on the model dynamics through saving behaviors.
Secondly, Nourry and Venditti [14] consider that the bifurcation parameter is also the parameter allowing the normalization of the steady state. As a consequence, it is almost impossible to show the existence of a non-empty set of parameters leading to an indeterminate efficient steady state characterized by a Hopf bifurcation.

In this paper, we extend the formulation of Nourry and Venditti [14] by considering heterogenous intratemporal preferences among the period of life. This gives us an additional degree of freedom which allows to show the existence of efficient endogenous fluctuations through a Hopf bifurcation if the share of the pure consumption (non-durable) good in the utility of young agents is low enough, or equivalently if the share of the mixed (durable) good in the utility of young is large enough. Since this condition is now compatible with a large share of the

al. [7].

durable good in the utility of old agents, our results are in line with the empirical evidence provided by Fernandez-Villaverde and Krueger [9]. We also prove that a fiscal policy based on transfers and taxes could simultaneously stabilize the economy, and reach the efficient steady state. Finally, a numerical illustration is provided and shows that there exists a non empty set of parameters for which local indeterminacy under dynamic efficiency occurs through a Hopf bifurcation.

The paper is organized as follows: Section 2 presents the model and proves the existence of an efficient steady state. Section 3 contains our main results on the occurrence of efficient endogenous fluctuations through Hopf bifurcation. Section 4 depicts a stabilization policy. Section 5 illustrates the theoretical result of the previous section by a numerical example. Concluding remarks are in Section 6 and the proofs are gathered in a final Appendix.

2 The model

2.1 Technology

Consider a competitive economy in which there are two sectors, one representative firm for each sector and each firm producing one final good. In this economy there exist two goods: one consumption good $Y_{0,t}$ and one consumable capital good Y_t which can be either consumed or invested. The consumption good is taken as a numeraire. Each sector uses two factors, capital K_t and labor L_t, both factors are mobile between sectors. Depreciation of capital is complete within one period[3]: $K_{t+1} = Y_t - Z_t$ where K_{t+1} is the total amount of capital in period $t+1$ and Z_t is the consumption share of the consumable capital good in period t. A constant returns to scale technology is used for each sector: $Y_{0,t} = F^0\left(K_t^0, L_t^0\right)$, $Y_t = F^1\left((K_t^1, L_t^1\right)$ with $K_t^0 + K_t^1 \leq K_t$, and $L_t^0 + L_t^1 \leq L_t$, and satisfy the following properties:

Assumption 1. *The production function $F^j : \mathbb{R}_+ \to \mathbb{R}_+$, is C^2, increasing, concave, homogeneous of degree one and satisfy the Inada conditions such that for*

[3] In a two-periods OLG model, full depreciation of capital is justified by the fact that one period is about thirty years.

any $\chi > 0$, $F_1^j(0,\chi) = F_2^j(\chi, 0) = \infty$, $F_1^j(\infty,\chi) = F_2^j(\chi,\infty) = 0$, *where* $j \in \{0, 1\}$.

The optimal allocation of factors between sectors is defined by the social production function $T(K_t, Y_t, L_t)$:

$$T(K_t, Y_t, L_t) = \max_{K_t^j, L_t^j} \left\{ Y_{0,t} \mid Y_t \leq F^1\left(K_t^1, L_t^1\right),\ K_t^0 + K_t^1 \leq K_t,\ L_t^0 + L_t^1 \leq L_t \right\} \quad (1)$$

Under Assumption 1, the function $T(K_t, Y_t, L_t)$ is homogeneous of degree one, concave and twice continuously differentiable[4]. Denoting r_t the rental rate of capital, p_t the price of the consumable capital good and w_t the wage rate, all in terms of the price of the consumption good, using the envelope theorem the following relationships hold:

$$\begin{aligned} r(K_t, Y_t, L_t) &= T_1(K_t, Y_t, L_t) \\ p(K_t, Y_t, L_t) &= -T_2(K_t, Y_t, L_t) \\ w(K_t, Y_t, L_t) &= T_3(K_t, Y_t, L_t) \end{aligned} \quad (2)$$

The relative capital intensity difference is derived from the factor-price frontier

$$b = \tfrac{L^1}{Y}\left(\tfrac{K^1}{L^1} - \tfrac{K^0}{L^0}\right) \quad (3)$$

The sign of b is positive (resp. negative) if and only if the consumption good is labor (resp. capital) intensive. The Stolper-Samuelson effect $\frac{dr}{dp}$ and the Rybczynski effect $\frac{dY}{dK}$ are determined respectively by the factor-price frontier and the full employment condition.

$$\tfrac{dr}{dp} = \tfrac{dY}{dK} = b^{-1} \quad (4)$$

Under a consumption good labor intensive, i.e. $b > 0$, the Stolper-Samuelson effect states that an increase of the relative price decreases the rental rate of capital and raises the wage rate whereas the Rybczynski effect specifies that an increase of the capital-labor ratio decreases the production of the consumption good and increases the production of the consumable capital good. Furthermore, from the GDP function $T(K_t, Y_t, L_t) + p_t Y_t = w_t L_t + r_t K_t$, we get the share of capital in the economy:

$$s(K_t, Y_t, L_t) = \tfrac{r_t K_t}{T(K_t, Y_t, L_t) + p_t Y_t} \quad (5)$$

[4] See Benhabib and Nishimura [2].

2.2 Preferences

Consider an infinite-horizon discrete time economy that is populated by overlapping generations who live for two periods: young and old. At each period, a new generation N_t is born. The population is constant over time and normalized to one. In the first period, young agents inelastically supply one unit of labor and receive an income. They assign this income between saving ϕ_t and the consumption of the composite good C_t. The composite good C_t is defined by a Cobb-Douglas composition of a consumption good 0, $c_{0,t}$, and a consumption good 1, $c_{1,t}$:

$$C_t = c_{0,t}^{\theta_c} c_{1,t}^{1-\theta_c} \tag{6}$$

where $0 < \theta_c \leq 1$ is the share of the consumption good 0 in the composite good C_t. In the second period, old agents are retired. The return on saving, $R_{t+1}\phi_t$, give their income which they spend entirely in the consumption of the composite good D_{t+1}. D_{t+1} is determined by a Cobb-Douglas composition of a consumption good 0, $d_{0,t+1}$, and a consumption good 1, $d_{1,t+1}$:

$$D_{t+1} = d_{0,t+1}^{\theta_d} d_{1,t+1}^{1-\theta_d} \tag{7}$$

where $0 < \theta_d \leq 1$ is the share of the consumption good 0 in the composite good D_{t+1}. An agent born in period t has preferences defined over consumption of the composite good C_t and D_{t+1}. Intertemporal preferences of agents are described by the following CES utility function[5]

$$U\left(C_t, \frac{D_{t+1}}{\Gamma}\right) = \left[C_t^{\frac{\gamma-1}{\gamma}} + \delta\left(\frac{D_{t+1}}{\Gamma}\right)^{\frac{\gamma-1}{\gamma}}\right]^{\frac{\gamma}{\gamma-1}} \tag{8}$$

where δ is the discount factor, γ is the elasticity of intertemporal substitution in consumption and Γ is a scaling constant parameter. Let us define $\pi_{c,t}$ (resp. $\pi_{d,t+1}$) the price of the composite good C_t (resp. D_{t+1}). An agent born in period t has to solve the optimal composition of the composite goods C_t and D_{t+1} which are derived from the following static optimizations:

$$\max_{c_{0,t}, c_{1,t}} \left\{c_{0,t}^{\theta_c} c_{1,t}^{1-\theta_c} \mid \pi_{c,t} C_t = c_{0,t} + p_t c_{1,t}\right\} \tag{9}$$

[5] All the conclusion of this paper can be obtained with a general concave and homothetic utility function $U\left(C_t, \frac{D_{t+1}}{\Gamma}\right)$.

$$\max_{d_{0,t+1},d_{1,t+1}} \left\{ d_{0,t+1}^{\theta_d} d_{1,t+1}^{1-\theta_d} \mid \pi_{d,t+1} D_{t+1} = d_{0,t+1} + p_{t+1} d_{1,t+1} \right\} \tag{10}$$

The first order conditions gives:

$$c_{0,t} = \theta_c \pi_{c,t} C_t, \ c_{1,t} = \frac{(1-\theta_c)\pi_{c,t} C_t}{p_t}, \ \pi_{c,t} = \left(\frac{p_t}{1-\theta_c}\right)^{1-\theta_c} \theta_c^{-\theta_c}. \tag{11}$$

$$d_{0,t+1} = \theta_d \pi_{d,t+1} D_{t+1}, \ d_{1,t+1} = \frac{(1-\theta_d)\pi_{d,t+1} D_{t+1}}{p_{t+1}}, \ \pi_{d,t+1} = \left(\frac{p_{t+1}}{1-\theta_d}\right)^{1-\theta_d} \theta_d^{-\theta_d}. \tag{12}$$

Contrary to Nourry and Venditti [14], we assume that young and old agents allocate their consumption differently between the non-durable and durable consumption goods, i.e. $\theta_c \neq \theta_d$. As shown by Fernandez-Villaverde and Krueger [9], the consumption of durable goods fall when agent are retired. In order to satisfy this empirical evidence, and since θ_c and θ_d are the shares of the non-durable good in the consumption of young and old agents, we assume in the rest of the paper that $\theta_c < \theta_d$. Under perfect foresight w_t and R_{t+1} are considered as given. A young agent born at period t solves the following dynamic program:

$$\max_{C_t, D_{t+1}} \left\{ U\left(C_t, \frac{D_{t+1}}{\Gamma}\right) \mid w_t = \pi_{c,t} C_t + \phi_t, \pi_{d,t+1} D_{t+1} = R_{t+1}\phi_t \right\} \tag{13}$$

Solving the first order condition gives:

$$\pi_{c,t} C_t = \alpha\left(\frac{R_{t+1}\pi_{c,t}}{\Gamma \pi_{d,t+1}}\right) w_t \equiv \frac{w_t}{1+\delta^\gamma \left(\frac{R_{t+1}\pi_{c,t}}{\Gamma \pi_{d,t+1}}\right)^{\gamma-1}} \in (0,1) \tag{14}$$

where $\alpha\left(\frac{R_{t+1}\pi_{c,t}}{\Gamma \pi_{d,t+1}}\right)$ is the propensity to consume of young agent. Similarly using (13), the saving function is $\phi_t = 1 - \alpha\left(\frac{R_{t+1}\pi_{c,t}}{\Gamma \pi_{d,t+1}}\right)$. In the following, we assume that the saving function is increasing with respect to the gross rate of return R_{t+1}.

Assumption 2. $\gamma > 1$.

This assumption states that the substitution effect following an increase in the gross rate of return R_{t+1} is greater than the income effect.

2.3 Dynamic equilibrium

Let us introduce $k_t = K_t/L_t$ the capital-labor ratio at time t and $y_t = Y_t/L_t$ the output-labor ratio at time t. The dynamics of the economy is described by the evolution of the capital stock and the market clearing condition for the consumable capital good:

$$k_{t+1} - \frac{w(k_t,y_t)}{p(k_t,y_t)} \left\{ 1 - \alpha \left[\frac{\Theta r(k_{t+1},y_{t+1})}{\Gamma p(k_t,y_t)^{\theta_c} p(k_{t+1},y_{t+1})^{1-\theta_d}} \right] \right\} = 0 \tag{15}$$

$$\theta_c k_{t+1} - y_t + (1-\theta_d)\, k_t \frac{r(k_t,y_t)}{p(k_t,y_t)} + (1-\theta_c) \frac{w(k_t,y_t)}{p(k_t,y_t)} = 0 \tag{16}$$

with a constant term $\Theta = \frac{(1-\theta_d)^{1-\theta_d} \theta_d^{\theta_d}}{(1-\theta_c)^{1-\theta_c} \theta_c^{\theta_c}}$. The set of admissible paths is defined as follows:

$$\omega = \left\{ (k_t, y_t) \in \mathbb{R}_+^2 \mid k_t \leq \overline{k},\ y_t \leq F^1(k_t, 1) \right\} \tag{17}$$

where the maximum admissible value of capital $\overline{k}$ is solution of $k - F^1(k, 1) = 0$. A perfect-foresight competitive equilibrium, defined as a sequence of allocations $\{k_t, y_t\}_{t=0}^{\infty}$, satisfies the two differences equations (15)-(16) for every period t, with the pair (k_0, y_0) given.

2.4 An efficient normalized steady state

A steady state $(k_t, y_t) = (k^*, y^*)$ is defined by:

$$k^* - \frac{w(k^*,y^*)}{p(k^*,y^*)} \left\{ 1 - \alpha \left[\frac{\Theta r(k^*,y^*)}{\Gamma p(k^*,y^*)^{1+\theta_c-\theta_d}} \right] \right\} = 0 \tag{18}$$

$$\theta_c k^* - y^* + (1-\theta_d)\, k^* \frac{r(k^*,y^*)}{p(k^*,y^*)} + (1-\theta_c) \frac{w(k^*,y^*)}{p(k^*,y^*)} = 0 \tag{19}$$

In two-sector OLG model, any steady state depends on both consumption and production characteristics. Any variation of the elasticity of intertemporal substitution in consumption γ induces a change in the stationary capital-labor ratio and thus involves a modification of all the previously defined shares and elasticities evaluated at the steady state. In the following, we consider a set of economies parametrized by γ. In order to guarantee that the steady state remains unaltered when γ varies, the steady state (k^*, y^*) is normalized by using two scaling parameters (Γ, θ_d).

Remark 1. *Nourry and Venditti [13] assume that $\theta_c = \theta_d = \theta$ in (19). As a result the existence of a normalized steady state (NSS) is guaranted by two scaling* *parameters* (Γ, θ)*. However,* θ *is also used as a bifurcation parameter. Such a dual role significantly complicates the analysis of the occurrence of endogenous fluctuations. In our model, we introduce a heterogeneity in the intratemporal preferences* $(\theta_c \neq \theta_d)$ *which allows to disentangle the dual role of* θ*:* θ_d *is taken as a scaling parameter while* θ_c *is a bifurcation parameter.*

Let us define $\nu = \frac{y}{k}$, the output-capital ratio, the inverse of ν represents the share of the consumable capital good which is invested. From (19), the scaling parameter θ_d lies between 0 and 1 if ν lies between $\underline{\nu}$ and $\overline{\nu}$, with $\underline{\nu} < \overline{\nu}$:

$$\underline{\nu} = \frac{1-\alpha\theta_c}{1-\alpha},\ \overline{\nu} = \frac{1-(1-s)\alpha\theta_c}{(1-\alpha)(1-s)} \tag{20}$$

Then, the following Proposition holds:

Proposition 1. *Under Assumptions 1-2,* (k^*, y^*) *is a normalized steady state if and only if* $\Gamma = \Gamma(k^*, y^*, \gamma) > 0$ *and* $\theta_d = \theta_d(k^*, y^*) \in (0, 1)$*.*

Proof: See Appendix 7.1. ■

In the rest of the paper we made the following Assumption in order to guarantee the existence of NSS.

Assumption 3. $\Gamma = \Gamma(k^*, y^*, \gamma)$ *and* $\theta_d = \theta_d(k^*, y^*)$.

In our model, the Golden-Rule level of capital, denoted $\hat{k}$, is characterized from the total stationary consumption $C + D = T(k, y) + p[y - k]$. Denoting $R(k, y) = -T_1(k, y)/T_2(k, y)$, $\hat{k}$ satisfies as usual $R(\hat{k}, \hat{y}) \equiv \hat{R} = 1$. Let us derive R the stationary gross rate of return at the NSS:

$$R = \frac{s}{(1-\alpha)(1-s)} \tag{21}$$

If $R > 1$, the steady state (k^*, y^*) is lower than the Golden-Rule level, i.e. the equilibrium is dynamically efficient. From Drugeon et al. [7], the following Lemma holds:

Lemma 1. *Under Assumptions 1-3, let* $\underline{\alpha} = \frac{1-2s}{1-s}$*. An intertemporal competitive equilibrium converging towards a normalized steady state is dynamically efficient if* $\alpha \in (\underline{\alpha}, 1)$ *and dynamically inefficient if* $\alpha \in (0, \underline{\alpha})$*.*

Lemma 1 asserts that under-accumulation of capital occurs when labor income of agents is less than capital income, i.e. $s \geqslant \frac{1}{2}$. A young agent does not have enough labor income to save a sufficient amount. On the contrary, over-accumulation of capital occurs if the labor income of agents is higher than the capital income, i.e. $s < \frac{1}{2}$. A young agent have enough labor income to save a sufficient amount. However, over-accumulation of capital can be avoided provided the share of consumption of young agents is high enough, i.e. $\alpha > \underline{\alpha}$. In the rest of the paper, the following Assumption is made:

Assumption 4. $\alpha \in \left(\underline{\alpha}, 1\right)$, $s \in \left(\frac{1}{3}, \frac{1}{2}\right)$ *and* $b < 0$.

Assumption 4 states that we consider dynamically efficient paths, that we restrain the share of capital in the economy s in order to get positive value for $\underline{\alpha}$ and that we concentrate on a capital intensive consumption good case. Cecchi and Garcia-Peñalosa [5] show that over the period 1960-2003, OECD countries were characterized by a share of capital between 0.35 and 0.5 with an average level of 0.36. Using national accounting data on the main developed countries, Takahashi et al. [17] show that over the last three decades the aggregate consumption good sector is more capital intensive than the capital good sector[6].

3 Efficient endogenous fluctuations

This section discusses the existence of efficient endogenous fluctuations derived from changes in expectations about fundamentals. In our setting, the existence of local indeterminacy occurs if the two characteristic roots associated with the linearization of the dynamical system (15)-(16) around the normalized steady state have a modulus less than one. Moreover, a Hopf bifurcation occurs if there exists some value of γ for which these roots are complex conjugate and cross the unit circle in the complex plane. To pursue the analysis, the elasticity of the rental rate of capital is introduced

$$\varepsilon_{rk} = -\frac{T_{11}(k^*, y^*)k^*}{T_1(k^*, y^*)} \tag{22}$$

[6] See also Baxter [1].

Drugeon [6] points out that the elasticity of the rental rate of capital is negatively linked to the elasticities of capital-labor substitution.

$$\Sigma = \frac{(y_0+py)\left(pyk^0l^0\sigma_0+y_0k^1l^1\sigma_1\right)}{pyky_0}, \ \varepsilon_{rk} = \left(\frac{l^0}{y_0}\right)^2 \frac{w(y_0+py)}{\Sigma} \tag{23}$$

with σ_0, σ_1 the sectoral elasticities of input substitution. Let us denote two critical bounds on ν which appears to be important for the stability properties of the normalized steady state (k^*, y^*):

$$\nu_0 = \frac{1}{1-\alpha}, \ \nu_1 = \frac{1-\alpha\theta_c}{(1-\alpha)(1-s)}. \tag{24}$$

Then the following Lemma applies:

Lemma 2. *Under Assumptions 1-3, the characteristic polynomial is defined by* $\mathscr{P}(\lambda) = \lambda^2 - \lambda\mathscr{T} + \mathscr{D}$ *where:*

$$\mathscr{T} = \frac{1+\alpha(\gamma-1)\varepsilon_{rk}\left\{\left[1-b\left(\nu-\underline{\nu}\right)\right]^2+\frac{s\theta_c^2b^2}{(1-\alpha)(1-s)}\right\}+\varepsilon_{rk}b\left(\nu_1-\nu+\frac{sb\alpha\theta_c\nu}{(1-\alpha)(1-s)}\right)}{\left[1-b\left(\nu-\underline{\nu}\right)\right]\alpha(\gamma-1)b\theta_c\varepsilon_{rk}} \tag{25}$$

$$\mathscr{D} = \frac{s\left\{1-b\left[\nu-\nu_0-\frac{\theta_c\alpha(\gamma-1)}{1-\alpha}\right]\right\}}{(1-s)b\theta_c\alpha(\gamma-1)} \tag{26}$$

Proof: See Appendix 7.2. ■

Nourry and Venditti [14] assume symmetric preferences over the life cycle with $\theta_c = \theta_d = \theta$. They show the possibility of efficient endogenous fluctuations through Hopf bifurcation when θ is low enough.[7] But the structure of their model appears in contradiction with the empirical evidence of Fernandez-Villaverde and Krueger [9]. Although these authors show that old agents consume less durable goods than young agents, the assumption $\theta_c = \theta_d = \theta$ implies a symmetric consumption behavior across the two periods of life. In our extended framework, we then focus on the case where the share of the pure consumption (non-durable) good is larger for old agents than for young agents, i.e. $\theta_c < \theta_d$. Put differently, we consider that young agents consume more durable goods than old agents. We then obtain the following Proposition:

[7] Indeed, when θ is close to 1, only efficient endogenous fluctuations through a flip bifurcation can occur. The Hopf bifurcation is ruled out. See also Nourry and Venditti [13].

Proposition 2. *Under Assumptions 1-4, there exists* $\overline{\theta}_c < \theta_d(k^*, y^*) < 1$, $\nu_0 > \tilde{\nu} > 1$, $\underline{b} < \overline{b} < 0$, $\overline{\varepsilon}_{rk} > \underline{\varepsilon}_{rk} > 0$ *and* $\gamma^{\mathcal{F}} > \gamma^{\mathcal{H}} > \gamma^{\mathcal{T}} > 1$ *such that if* $\alpha \in (\frac{1}{2}, \frac{s}{1-s})$, $\theta_c < \overline{\theta}_c$, $\nu \in (\tilde{\nu}, \nu_0)$, $b \in (\underline{b}, \overline{b})$ *and* $\varepsilon_{rk} \in (\underline{\varepsilon}_{rk}, \overline{\varepsilon}_{rk})$, *the efficient steady state is locally indeterminate when* $\gamma \in (\gamma^{\mathcal{T}}, \gamma^{\mathcal{H}})$, *undergoes a transcritical bifurcation when* $\gamma = \gamma^{\mathcal{T}}$[8], *and undergoes a Hopf bifurcation when* $\gamma = \gamma^{\mathcal{H}}$. *Moreover, there generically exist locally indeterminate (resp. saddle-point stable) period-2 cycles when* $\gamma \in (\gamma^{\mathcal{F}} - \epsilon, \gamma^{\mathcal{F}})$ *(resp.* $\gamma \in (\gamma^{\mathcal{F}}, \gamma^{\mathcal{F}} + \epsilon)$*) with* $\epsilon > 0$, *i.e., when the bifurcation is super- (resp. sub-) critical, and locally indeterminate (resp. locally unstable) quasi-periodic cycles when* $\gamma \in (\gamma^{\mathcal{H}}, \gamma^{\mathcal{H}} + \varepsilon)$ *(resp.* $\gamma \in (\gamma^{\mathcal{H}} - \varepsilon, \gamma^{\mathcal{H}})$*) with* $\varepsilon > 0$, *i.e., when the bifurcation is super- (resp. sub-) critical.*

Proof: See Appendix 7.3. ■

Remark 2. *Whether the bifurcation is super- or sub-critical is driven by the sign of some coefficient computed from the second- and third-order approximations to two difference equations (15)-(16). This property determines whether the bifurcation leads to the occurrence of locally indeterminate or unstable (resp. saddle point stable) quasi-periodic (resp. period-2) cycles near the bifurcation value.*

Contrary to the formulation of Nourry and Venditti [14] with symmetric preferences over the life cycle, under dynamic efficiency a large share of the consumption of young agent, does not ruled out self-fulfilling fluctuations. The intuition of

[8] When the bifurcation parameter γ crosses a $\gamma^{\mathcal{T}}$, one characteristic roots crosses 1. We cannot a priori differentiate between the transcritical, the pitchfork or the saddle-node bifurcation from the linearized dynamic system (15)-(16). Under Assumption 3, the existence of the NSS is always ensured and a saddle-node bifurcation cannot occur. Moreover, the pitchfork bifurcation requires non-generic conditions, see Ruelle [16]. In order to simplify the exposition we focus on the generic case and we relate the existence of one characteristic root going through 1 to a transcritical bifurcation.

this result is quite simple. Starting from an arbitrary equilibrium, consider another one in which the agents expect a higher rate of investment at time t leading to some higher capital stock at time $t + 1$. This expectation will be self-fulfilling provided the amount of saving at date t is large enough to support the increase of the investment good output which directly provides the capital stock of the next period. When the equilibrium is dynamically efficient, the share of first period consumption is large enough to generate a stationary capital stock lower than the Golden Rule and thus prevents the agent from saving enough. At the same time, when the share of the pure consumption good in the composite good θ_c is low enough, the production of the mixed good is consequently increased. This increase may then compensate the lack of savings and lead to an increase of capital compatible with the expectations. As a result, the initial expectation can be realized as an equilibrium, and, under dynamic efficiency, local indeterminacy together with fluctuations based on local sunspots are possible. Such sunspot fluctuations arise through a Hopf bifurcation. That generates quasi-periodic cycles which coincide to various period of expansion followed by various periods of boom. Therefore, the associated macroeconomic volatility corresponds to empirically plausible endogenous fluctuations.

4 Stabilization policy

The result of the previous section emphasizes that the economy may exhibits efficient endogenous fluctuations. It raises the question of a stabilization policy which may simultaneously stabilize the economy and move the equilibrium to the optimal steady state which provide an equal level of utility to all generations[9]. In the present section, it is proven that a fiscal policy exists under the assumption that agents and policy-maker do not make forecasting mistakes. Consider that the policy-maker buys goods, levies taxes and makes transfers under balanced budget rule. Let G_t be the flows of consumption goods which is bought, $\beta_{g,t} > 0 (< 0)$ the taxes (transfers) on the income of period of life $g \in \{c, d\}$, for generation t. The

[9] For an example of stabilization policy in aggregate OLG model see Reichlin [15], and in Galor [10] type two-sector OLG model see Nourry and Venditti [13].

intertemporal maximization problem (13) turns to:

$$\max_{C_t, D_{t+1}} \left\{ U\left(C_t, \frac{D_{t+1}}{\Gamma}\right) \mid w_t + \beta_{c,t} = \pi_{c,t}C_t + \phi_t, \pi_{d,t+1}D_{t+1} = R_{t+1}\phi_t + \beta_{d,t+1} \right\} \tag{27}$$

The optimal saving function becomes:

$$\phi_t = \frac{\Gamma\left(w_t + \beta_{c,t}\right)\left(\frac{\delta R_{t+1}\Pi_{c,d}}{\Gamma}\right)^{\gamma} - \beta_{d,t+1}\Pi_{c,d}}{R_{t+1}\Pi_{c,d} + \Gamma\left(\frac{\delta R_{t+1}\Pi_{c,d}}{\Gamma}\right)^{\gamma}} \tag{28}$$

where $\Pi_{c,d} = \frac{\pi_{c,t}}{\pi_{d,t+1}}$ Consider Proposition 1, the scaling parameter Γ is written as:

$$\Gamma(k,y) = R(k,y)\,\Pi_{c,d}\left(\frac{kp(k,y)}{\delta^{\gamma}\left[w(k,y) - k(k,y)p(k,y)\right]}\right)^{\frac{1}{1-\gamma}} \tag{29}$$

where $r(k,y) = T_1(k,y),\ p(k,y) = -T_2(k,y),\ w(k,y) = T_3(k,y)$. Let,

$$\begin{aligned} \beta^{o}_{t+1} = kp(k,y)\left[R(k,y)\,\Pi_{c,d} + \Gamma(k,y)\left(\frac{\delta R(k,y)\Pi_{c,d}}{\Gamma}\right)^{\gamma}\right]\left(\frac{R_{t+1}}{\Pi_{c,d}R(k,y)}\right)^{\gamma} \\ -k_{t+1}p_t\left[R_{t+1}\Pi_{c,d} + \Gamma(k,y)\left(\frac{\delta R_{t+1}\Pi_{c,d}}{\Gamma}\right)^{\gamma}\right] \end{aligned} \tag{30}$$

$$\beta_{c,t} = w(k,y) - w_t \tag{31}$$

where $R(k,y) = \frac{r(k,y)}{p(k,y)}$ Plugging $\beta_{c,t} = \widehat{\beta}_{c,t},\ \beta_{d,t+1} = \widehat{\beta}_{d,t+1},\ \beta_{c,t+1} = \beta_{d,t+2} = 0$ into (28) gives

$$\phi_t = \frac{\Gamma w(k,y)\left(\frac{\delta R(k,y)\Pi_{c,d}}{\Gamma}\right)^{\gamma}}{R(k,y)\Pi_{c,d} + \Gamma\left(\frac{\delta R(k,y)\Pi_{c,d}}{\Gamma}\right)^{\gamma}} \tag{32}$$

It follows that if agents believe the announced policy rule, they will expect the optimal steady state to hold in the future. This expectation in turn drives the system to the steady state and keeps it there forever.

5 Numerical illustration

In order to show that when dynamic efficiency holds, local indeterminacy and endogenous fluctuations through Hopf bifurcation also holds, we provide a numerical illustration. Our aim is to prove the non-emptiness of the set of parameters' values for which the conditions of Proposition 2 holds.

5.1 A CES-Leontief Economy

Let us consider the following sectoral technologies:

$$\begin{aligned} F^0\left(K^0, L^0\right) &= \left[\mu\left(K^0\right)^{-\rho_0} + (1-\mu)\left(L^0\right)^{-\rho_0}\right]^{-\frac{1}{\rho_0}} \\ F^1\left(K^1, L^1\right) &= \min\left\{\frac{K^1}{\eta}, L^1\right\} \end{aligned} \quad (33)$$

with $\eta > 0$, $\mu \in (0,1)$ and $\rho_0 > -1$. The sectoral elasticity of capital-labor substitution in the consumption good sector is $\sigma_0 = \frac{1}{1+\rho_0}$ while the sectoral elasticity of capital-labor substitution in the consumable capital good sector is $\sigma_1 = 0$. Then the social production function is defined as:

$$T(K, Y, L) = \left[\mu(K - \eta Y)^{-\rho_0} + (1-\mu)(L - Y)^{-\rho_0}\right]^{-\frac{1}{\rho_0}} \quad (34)$$

Let $L = 1$ so that $K = k^*$ and $Y = y^*$. We derive the stationary propensity to consume of young agents α, the share of capital in the economy s, the relative capital intensity difference b and the output-capital ratio ν^*:

$$b = \frac{\eta - k^*}{1-y^*}, \quad \nu^* = \frac{y^*}{k^*} \quad (35)$$

$$\alpha = \frac{(1-\mu)(1-k^*)k^{*\rho_0}(1-\eta\nu^*)^{\rho_0+1} - \eta\mu(1-y)^{\rho_0+1}}{(1-\mu)k^{*\rho_0}(1-\eta\nu^*)^{\rho_0+1}} \quad s = \frac{\mu(1-y^*)^{\rho_0+1}}{(1-\mu)k^{*\rho_0}(1-\eta\nu^*)^{\rho_0+1} + \mu(1-y^*)^{\rho_0+1}} \quad (36)$$

The numerical example is computed in line with the following approach: we choose θ_c, η, μ, ρ_0, γ and ν^* in their admissible ranges. Substituting these parameters' values into (35)-(36), we then compute the values of α, b, s, ε_{rk} and $\theta_d(k^*, y^*)$ and the critical bounds on b, ε_{rk}, ν^* and γ.

5.2 Efficient endogenous fluctuations

Let us consider the following set of parameters: $\mu = 0.99865, \eta = 0.022, \theta_c = 0.125, \rho_0 = 3, k^* = 0.45, y^* = 0.9, \nu^* = 2, \gamma = 2$. We get an efficient steady state with $\alpha \simeq 0.53$ and $\underline{\alpha} \simeq 0.03$. The corresponding share of capital is $s \approx 0.49$. We find the share of the consumption good 0 in the composite good D_{t+1}, $\theta_d(k^*, y^*) \simeq 0.99$, and the relative capital intensity difference $b \simeq -4.28$. We obtain the elasticity of the rental rate of capital $\varepsilon_{rk} \simeq 1.68$. Then the efficient normalized steady state is locally indeterminate for any $\gamma \in \left(\gamma^{\mathcal{T}}, \gamma^{\mathcal{H}}\right)$, where $\gamma^{\mathcal{T}} = 1.69$

is a Transcritical bifurcation value inducing the existence of a second steady state and $\gamma^{\mathcal{H}} = 2.55$ corresponds to a Hopf bifurcation value leading to quasi-periodic cycle. The associated macroeconomic volatility is compatible with empirically relevant properties of fluctuations, in the sense that the Hopf bifurcation causes quasi-periodic cycles where several periods of expansion are followed by several periods of recession. Proposition 2 is satisfied for any $b \in (-6.99, -4.18)$, $\nu^* \in (1.98, 2.12)$, $\varepsilon_{rk} \in (0.38, 1.68)$, $\theta_c < 0.98$ and $\gamma \in (1.69, 2.55)$.

Our findings $(c_0, c_1, d_0, d_1) = (0.02, 0.3, 0.41, 0.01)$ are qualitatively in line with the empirical evidence provided in Fernandez-Villaverde and Krueger [9] along which young agents consume more durable goods than old agents while old agents consume more non durable goods.

6 Concluding Remarks

The main objective of this paper is to study the influence of heterogeneous consumption goods on the occurrence of efficient endogenous fluctuations in two-sector OLG model. We show that efficient endogenous fluctuations occur through Hopf bifurcation if the share of the pure consumption in the composite good is low enough for young agent. This condition is in line with the empirical evidence provided in Fernandez-Villaverde and Krueger [9] along which old agents consume less durable goods than young agents. We also prove that a fiscal policy based on transfers and taxes could simultaneously stabilize the economy and give an equal level of utility to each generation. Finally, we provide a numerical illustration showing that there exists a non-empty set of parameters' values for which efficient endogenous fluctuations arise through Hopf bifurcation. The associated macroeconomic volatility generates several periods of expansion, followed by several periods of recession.

7 Appendix

7.1 Proof of Proposition 1

Let $(k^*, y^*) \in \left(0, \overline{k}\right) \times \left(0, \overline{k}\right)$ and $\Pi_{c,d} = \Theta p\,(k^*, y^*)^{\theta_d - \theta_c}$. Solving equation (18) with respect to Γ yields to:

$$\Gamma\,(k^*, y^*, \gamma) = \Pi_{c,d}\,(k^*, y^*)\,R\,(k^*, y^*)\left\{\frac{\delta^\gamma\left[w(k^*,y^*) - k^* p(k^*,y^*)\right]}{k^* p(k^*,y^*)}\right\}^{\frac{1}{\gamma-1}} > 0 \tag{37}$$

Solving equation (19) with respect to θ_d yields to:

$$\theta_d\,(k^*, y^*) = 1 + \frac{\underline{\nu}(k^*,y^*) - \nu(k^*,y^*)}{R(k^*,y^*)} \tag{38}$$

(k^*, y^*) is a normalized steady state if and only if $\Gamma = \Gamma\,(k^*, y^*, \gamma)$ and $\theta_d = \theta_d\,(k^*, y^*)$. ∎

7.2 Proof of Lemma 2

From (14), one gets:

$$\alpha' = -\frac{\alpha(\gamma-1)(1-\alpha)\Gamma}{R\Pi_{c,d}} \tag{39}$$

Under Assumption 1, the first order conditions of firm's profit maxmization problem[10] (1) yield to $T_{12} = -T_{11}b,\ T_{22} = T_{11}b^2,\ T_{31} = -T_{11}a$ and $T_{32} = T_{11}ab$ with $a \equiv \frac{K^0}{L^0} > 0$, b as defined by (3) and $T_{11} < 0$. Consider ε_{rk} as given by (22) and the fact that the homogeneity of $T\,(K, Y, L)$ implies $a = (1-b)\,k^*$. $\mathscr{T}$ and $\mathscr{D}$ are obtained from the linearization of the two difference equations (15)-(16) in the neighborhood of the normalized steady state. ∎

7.3 Proof of Proposition 2

Following Grandmont et al. [11], we study the local dynamic properties by analysing the trace $\mathscr{T}$ and the determinant $\mathscr{D}$. This methodology allows to analyse the variation of the trace $\mathscr{T}$ and the determinant $\mathscr{D}$ by choosing a bifurcation

[10] See Benhabib and Nishimura [3] and Bosi et al. [4]

parameter. In this setting, the bifurcation parameter chosen is the elasticity of intertemporal substitution in consumption γ. Then the variation of the trace $\mathscr{T}$ and the determinant $\mathscr{D}$ in the $(\mathscr{T}, \mathscr{D})$ plane will be studied as γ evolves continuously within $(1, +\infty)$. The relationship between $\mathscr{T}$ and $\mathscr{D}$ is given by a half-line $\Delta(\mathscr{T})$ which is characterized from the consideration of its extremities (Figure 1). The starting point is the pair $(\lim_{\gamma \to +\infty} \mathscr{T} \equiv \mathscr{T}_\infty, \lim_{\gamma \to +\infty} \mathscr{D} \equiv \mathscr{D}_\infty)$, while the end point is the pair $(\lim_{\gamma \to 1} \mathscr{T} \equiv \mathscr{T}_1, \lim_{\gamma \to 1} \mathscr{D} \equiv \mathscr{D}_1)$. $\Delta(\mathscr{T})$ is obtained from the two difference equations (15)-(16), by solving $\mathscr{T}$ and $\mathscr{D}$ with respect to $\alpha(\gamma - 1)$:

$$\Delta(\mathscr{T}) = \mathscr{D}_\infty + \mathscr{S}(\mathscr{T} - \mathscr{T}_\infty) \tag{40}$$

where the slope $\mathscr{S}$, $\mathscr{D}_\infty$ and $\mathscr{T}_\infty$ are:

$$\mathscr{S} = \frac{s\varepsilon_{rk}[1-b(\nu-\nu_0)]\left[1-b\left(\nu-\underline{\nu}\right)\right]}{(1-s)\{1+\varepsilon_{rk}b(\nu_1-\nu+Rb\alpha\theta_c\nu)\}}, \ \mathscr{D}_\infty = \frac{s}{(1-\alpha)(1-s)}, \ \mathscr{T}_\infty = \frac{\left[1-b\left(\nu-\underline{\nu}\right)\right]^2+\theta_c^2 b^2 R}{\left[1-b\left(\nu-\underline{\nu}\right)\right]b\theta_c} \tag{41}$$

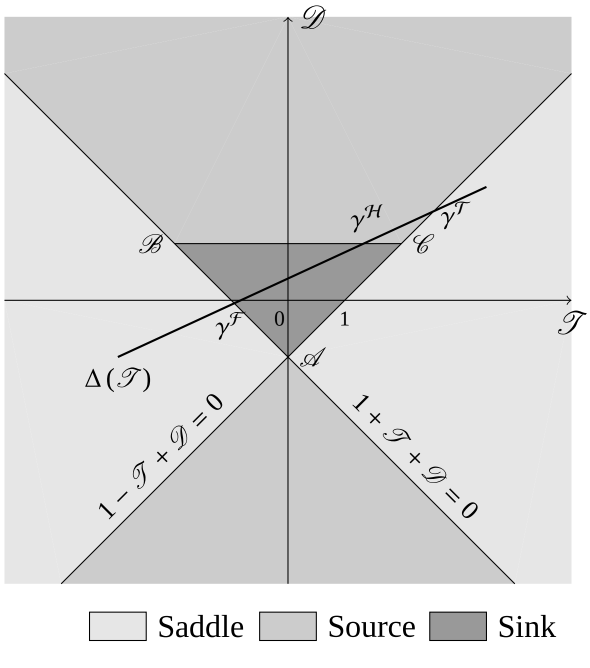

Figure 1: Stability triangle and $\Delta(\mathscr{T})$ segment.

Consider that $b < 0$ and $\alpha \in \left(\frac{1}{2}, \frac{s}{1-s}\right)$ so that $\mathscr{D}_\infty > 1$, $\mathscr{T}_\infty < 0$ and $\mathscr{P}_\infty(1) > 0$. Let assume that $\theta_c < \overline{\theta}_c$ implying that $\nu_0 < \nu_1$, defined in (24), and $b_0 < \underline{b}$, where:

$$\overline{\theta}_c = \frac{s}{\alpha+(1-2\alpha)(1-s)},\ \underline{b} = -\frac{\nu_1-\nu}{R\alpha\nu\theta_c},\ b_0 = \frac{1}{\nu-\nu_0} \tag{42}$$

Note that $\overline{\theta}_c < \theta_d(k^*, y^*)$ if $\nu \in \left(\underline{\nu}, \nu_0\right)$ and $\alpha > 0$. Under $\nu \in \left(\underline{\nu}, \nu_0\right)$ and $b \in \left(b_0, \overline{b}\right)$ we get $\mathscr{D}_1 = -\infty$ and $\mathscr{P}_\infty(-1) = 1 + \mathscr{T}_\infty + \mathscr{D}_\infty < 0$, where

$$\overline{b} = -\frac{1-\alpha}{\alpha+\left(\alpha+\frac{s}{1-s}\right)\theta_c} \tag{43}$$

Finally, $\mathscr{T}_1 = +\infty$ if and only if $\nu < \nu_1$, $b > \underline{b}$ and $\varepsilon_{rk1} > \varepsilon_{rk1}$ with

$$\varepsilon_{rk1} = -\frac{1}{Rb\alpha\theta_c\nu(b-b_1)} \tag{44}$$

Using the expression of $\mathscr{T}$ and $\mathscr{D}$ allows to show that when $\mathscr{D} = 1$, $\mathscr{T} > -2$ if and only if:

$$1 + \varepsilon_{rk} \underbrace{\left\{\left[\alpha(\gamma-1)\right]|_{\mathscr{D}=1}\ \mathscr{P}_{-2}(b) + b\left(\nu_1 - \nu + Rb\alpha\theta_c\nu\right)\right\}}_{=\Xi} \leq 0 \tag{45}$$

where:

$$\mathscr{P}_{-2}(b) = \theta_c^2 R b^2 + 2\theta_c b\left[1 - b\left(\nu - \underline{\nu}\right)\right] + \left[1 - b\left(\nu - \underline{\nu}\right)\right]^2 > 0 \tag{46}$$

and

$$\left[\alpha(\gamma-1)\right]|_{\mathscr{D}=1} = \frac{s(\nu-\nu_0)(b_0-b)}{b\theta_c\left(\underline{\alpha}-\alpha\right)} \tag{47}$$

Assume that $\nu = \nu_0 - d\nu$ with $d\nu > 0$ small, it follows that Ξ in (45) is negative. As a result $\mathscr{T} \geq -2$ if and only if:

$$\varepsilon_{rk} > \underline{\varepsilon}_{rk} = -\frac{1}{\left[\alpha(\gamma-1)\right]|_{\mathscr{D}=1}\mathscr{P}_{-2}(b) + Rb\alpha\theta_c\nu(b-b_1)} \tag{48}$$

Moreover there exist $\overline{\varepsilon}_{rk} > \underline{\varepsilon}_{rk}$ such that $\mathscr{T} = 2$ when $\varepsilon_{rk} = \overline{\varepsilon}_{rk}$. Therefore, $\mathscr{T} \in (-2, 2)$ as long as $\varepsilon_{rk}\left(\underline{\varepsilon}_{rk}, \overline{\varepsilon}_{rk}\right)$. Denote $d\tilde{\nu}$ the value of $d\nu$ such that the denominator of (48) is equal to zero. The maximal admissible value of $d\nu$ is such that $d\overline{\nu} = \min\left\{d\tilde{\nu}, \nu_0 - \underline{\nu}\right\}$. It follows that when $\nu \in (\tilde{\nu}, \nu_0)$ with $\tilde{\nu} = \nu_0 - d\nu$, $\theta_c < \overline{\theta}$, $b \in \left(\underline{b}, \overline{b}\right)$ and $\epsilon_{rk} \in \left(\underline{\varepsilon}_{rk}, \overline{\varepsilon}_{rk}\right)$, $\Delta(\mathscr{T})$ is between $\mathscr{T} \in (-2, 2)$ when $\mathscr{D} = 1$. Result follows. ■

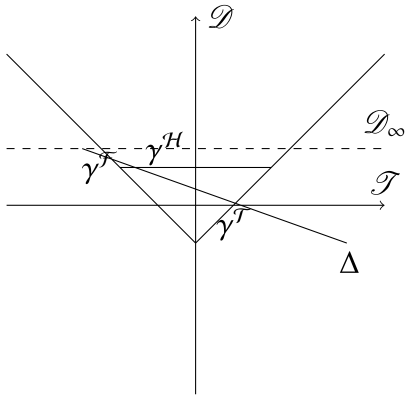

Figure 2: Hopf and transcritical bifurcation.

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