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► **To cite this version:**

Francoise Forges, Vincent Iehlé. Afriat's theorem for indivisible goods. *Journal of Mathematical Economics*, 2014, 54, pp.1-6. 10.1016/j.jmateco.2014.08.001 . halshs-00870052v2

HAL Id: halshs-00870052

<https://shs.hal.science/halshs-00870052v2>

Submitted on 26 Aug 2014

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AFRIAT'S THEOREM FOR INDIVISIBLE GOODS*

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May 16, 2014

Abstract

We identify a natural counterpart of the standard GARP for demand data in which goods are all indivisible. We show that the new axiom (DARP, for “discrete axiom of revealed preference”) is necessary and sufficient for the rationalization of the data by a well-behaved utility function. Our results complement the main finding of Polisson and Quah, *Am. Econ. J.: Micro.* **5**(1) p.28-34 (2013), who rather minimally modify the original consumer problem with indivisible goods so that the standard GARP still applies.

JEL classification numbers: D11,C81.

Keywords: Afriat's theorem, GARP, indivisible goods, rationalization, revealed preference.

*We are grateful to an anonymous referee for suggesting the form of the basic data that are considered in this version of the paper, which greatly contributed to the clarification of the results. We wish to thank Don Brown, Sam Cosaert, Thomas Demuyneck, John Quah and seminar audiences at Paris School of Economics and Institut Henri Poincaré for helpful comments.

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1. INTRODUCTION

When goods are perfectly divisible, Afriat (1967)'s theorem tells us that the general axiom of revealed preference (GARP) is a necessary and sufficient condition for consumption data to be consistent with utility maximization (see, e.g., Diewert, 1973; Varian, 1982). The proof of the result is fully constructive, namely yields an explicit well-behaved utility function when GARP is satisfied. In the standard formulation of GARP, it is understood that rational preferences are locally nonsatiated. However, in practice, goods are often indivisible and traded in discrete quantities in the field or in the laboratory. In this case, as recently acknowledged by Polisson and Quah (2013) and Fujishige and Yang (2012), local nonsatiation becomes meaningless so that GARP, in its usual form, is no longer a necessary condition of rationalization.

“Does this mean that we should drop or modify GARP when studying consumer choice over a discrete consumption space?” ask Polisson and Quah (2013, p.31). They answer “no” by considering a broader notion of rationality. We take another direction and do modify GARP by elaborating a new axiom that accounts fully for the discrete structure of the consumption space.

This note proposes an analog of Afriat's theorem in the case where all goods are indivisible. The data consist of finitely many observed prices and consumption bundles that the consumer could afford given his budget. In particular, we assume that the analyst observes the consumer's revenue at each date. The reason for this assumption is that, if the consumption space is discrete, a rational consumer with monotonic preferences does not necessarily exhaust his revenue. Consumption of a bundle x at price p is compatible with any budget above $p \cdot x$ and below $p \cdot (x + \mathbf{1})$.¹ The data considered in this paper

¹Echenique *et al.* (2011b) already assume that the consumer's income is part of the data, both in the case of indivisible goods and infinitely divisible goods. In the latter framework, Forges and Minelli (2009) even assume that the analyst has access to a full description of general, possibly complex, budget sets, while Forges and Iehlé (2013) explore a minimalist approach in which the analyst only observes the consumers' budget constraints insofar as they are revealed by his choices.

are not only appropriate in the understood idealized, theoretical framework, in which all consumption goods are indivisible, but also in experimental designs in which the designer controls the budget of the subjects. For instance, our setting is adequate to describe the economic environment in the experiments conducted by Février and Visser (2004), Hammond and Traub (2012) and Mattei (2000) where subjects, endowed with a given budget (e.g., tokens), face discrete choices.

We identify a discrete axiom of revealed preference (DARP) and show that it is necessary and sufficient for rationalization of our data. We start by defining a relation of direct preference, exactly as for the standard GARP. More precisely, let x and x' be bundles of indivisible goods that have been purchased at price p and p' respectively; the bundle x is directly revealed preferred to the bundle x' if x' was feasible at price p given the consumer's budget. We go on by defining an indirect revealed preference relation as the transitive closure of the previous relation. A set of observations satisfies DARP if, whenever a bundle x is indirectly revealed preferred to a bundle x' the bundle $x + \mathbf{1}$, which is obtained by adding one unit of every good to x , is not feasible at price p' . DARP can thus be described as a discrete analog of GARP, in which the interior of a budget set consists of those bundles that remain in the budget set when all their components increase by one unit. As expected, DARP implies GARP but the reverse is not true.

Our main result (proposition 1) states that DARP, as described above, is a necessary and sufficient condition for the rationalization of a finite set of discrete data by a discrete quasi-concave and monotonic utility function. Surprisingly, the proof can still make use of Afriat's methodology. The key difference is that, extended over a continuous consumption space, our utility function would be flat on a small domain. However, its restriction to integer bundles turns out to be well-behaved, in particular monotonic.

So far, the utility function that we have proposed to rationalize the data when DARP is satisfied is monotonic but not strictly monotonic, while the latter property is specially desirable in the context of indivisible goods. We show (in proposition 2) that a strengthening of DARP, which we denote as DARP*, is necessary and sufficient for rationalization

by a strictly monotonic utility function. As in the continuous case, a basic tool to establish propositions 1 and 2 is that DARP and DARP* are equivalent to the cyclical consistency of matrices that are associated to the data. This property is formalized in proposition 3.

1.1. Related literature

Polisson and Quah (2013) also investigate the problem of revealed preferences in the context of indivisible goods but adopt a different approach. To solve the problem that is generated by the lack of meaningful local nonsatiation, Polisson and Quah (2013) allow for an implicit additional consumption good, which can be purchased in continuous quantities. Formally their model of rationality comes close to the standard consumer problem with quasilinear utility where the additional good plays the role of money. They show that this is enough to guarantee that the standard GARP be a necessary and sufficient condition for the existence of a strictly increasing utility function on the discrete consumption space that rationalizes price and demand observations. In Fujishige and Yang (2012), an identical conclusion is obtained without any additional good but the problem related to local nonsatiation is evicted right away by assuming cost efficiency.

Models with continuous goods and money are central in Brown and Calsamiglia (2007) and Sákovics (2013). They introduce an axiom that is stronger than GARP, which they call respectively cyclical monotonicity condition and axiom of revealed valuation. They show that such an axiom is relevant for rationalization by a quasilinear utility when the data consist of finitely many observed prices and bundles of continuous goods. A similar conclusion is reached as a by-product in Echenique *et al.* (2011a, p.1211). In any case, as soon as the presence of (continuous) money is explicitly acknowledged, imposing discrete quantities for the consumption goods becomes innocuous from a revealed preference perspective and an identical test applies for rationalizing the data.

Recently, independently of our work, Cosaert and Demuyne (2014) have considered the rationalization of observations from finite consumption sets. In this general framework, they identify two variants of GARP, called WMARP and SMARP, that reduce to

DARP and DARP* in the standard consumer problem with indivisible goods. It follows that a by-product of their conclusions comes close to our main findings. However, with respect to the problem addressed here, the utility functions we construct behave better than the ones obtained by Cosaert and Demuynck (2014) under WMARP and SMARP and satisfy especially (discrete) quasi-concavity. The reason for this is that we can make full use of the “linear” structure of budget sets and the uniform incrementation by the vector $\mathbf{1}$ to define the interior of the budgets sets and follow Afriat’s methodology. This contrasts with the approach of Cosaert and Demuynck (2014) who posit no assumption, except finiteness, on the structure of the choice sets and use therefore a different strategy.²

We will make a more precise comparison between our results and the ones of Polisson and Quah (2013) and of Cosaert and Demuynck (2014) once we are equipped with precise definitions, in section 2.

Starting with Richter (1966, 1971), the relationship between the existence of a rationalization and the strong axiom of revealed preference (SARP) has also been investigated in abstract environment of choices. As recalled by Mas-Colell *et al.* (1995, chap.3, p.92), this approach can be applied to competitive budget sets. Echenique *et al.* (2011b) proceed in this way to deal with indivisibilities (see also Chambers and Echenique (2009) in the framework of finite lattices). In this paper we rather follow Afriat’s constructive approach, which enables us to check whether the possible rationalization is well-behaved (monotone, concave, etc.).

1.2. Notations and terminology.

The vector $\mathbf{1}$ is the characteristic vector of \mathbb{R}^K whose components are all equal to 1; and for any $\ell = 1, \dots, K$, \mathbf{e}^ℓ is the vector of \mathbb{R}^K whose component ℓ is 1 and remaining components are all 0. The mapping $u : \mathbb{N}^K \rightarrow \mathbb{R}$ is monotonic if for every $x, x' \in \mathbb{N}^K$ such that $x \gg x'$, $u(x) > u(x')$; u is strictly monotonic if for every $x, x' \in \mathbb{N}^K$ such

²Interestingly, they apply their results to reappraise the number of inconsistent subjects in past experimental studies that use GARP instead of its discrete/finite variants.

that $x \succ x'$, $u(x) > u(x')$. Given a set $A \subset \mathbb{N}^K$, $\mathbb{1}_A$ is the indicator function of A , that is, $\mathbb{1}_A(x) = 1$ if $x \in A$ and $\mathbb{1}_A(x) = 0$ otherwise; and A^c is the complement of A in \mathbb{N}^K . A set $A \subset \mathbb{N}^K$ is discrete convex if for every $x_1, \dots, x_m \in A$, with $m \in \mathbb{N}$, and every $\lambda_1, \dots, \lambda_m \geq 0$ with $\sum_{i=1}^m \lambda_i = 1$ and $\sum_i \lambda_i x_i \in \mathbb{N}^K$ it holds that $\sum_i \lambda_i x_i \in A$. A mapping $u : \mathbb{N}^K \rightarrow \mathbb{R}$ is discrete quasi-concave if $\{z \in \mathbb{N}^K : u(z) \geq k\}$ is a discrete convex set for any $k \in \mathbb{R}$. Observe that if $u : \mathbb{R}_+^K \rightarrow \mathbb{R}$ is quasi-concave then its restriction to \mathbb{N}^K is discrete quasi-concave.

2. RATIONALIZATION AND DARP

Consider an analyst observing at each date $t = 1, \dots, n$ the bundle $x_t \in \mathbb{R}_+^K$ purchased by a single consumer, the positive prices $p_t \in \mathbb{R}_{++}^K$ and the available budget $r_t \in \mathbb{R}_+$. The consumption set is \mathbb{N}^K , and the budget set at any date t is

$$B_t := \{x \in \mathbb{N}^K : p_t \cdot x \leq r_t\}.$$

Throughout the analysis we assume that the observations $(x_t, p_t, r_t)_{t=1, \dots, n}$ satisfy $x_t \in B_t$ at each date $t = 1, \dots, n$.

To assess whether the consumer behaves rationally, the analyst can use this basic data to check whether the observed choices match the solutions of the standard consumer problem.

Definition 1 *A utility function $u : \mathbb{N}^K \rightarrow \mathbb{R}$ is called a rationalization of the observations $(x_t, p_t, r_t)_{t=1, \dots, n}$ if, at each date t , x_t solves*

$$\max u(x) \text{ subject to } x \in B_t. \tag{1}$$

To formulate our axiom we proceed as usual by defining first the direct revealed preference relation, denoted by R . The bundle x_i is said to be *directly revealed preferred* to x_j , $x_i R x_j$, if $x_j \in B_i$. The transitive closure of R is denoted by H , and x_i is said to be

revealed preferred to x_s if $x_i H x_s$.³

Definition 2 The observations $(x_t, p_t, r_t)_{t=1, \dots, n}$ satisfy the discrete axiom of revealed preference (DARP) if for any $i, j = 1, \dots, n$

$$x_i H x_j \Rightarrow x_i + \mathbf{1} \notin B_j.$$

Remark 1 Observe that the previous definition makes sense if $i = j$, in which case it implies that $x_i + \mathbf{1} \notin B_i$, namely, that x_i is maximal in B_i . This is clearly a *sine qua non* condition for the existence of an increasing rationalization.

2.1. Comparison with other formulations of GARP

Now that DARP is formally defined, we can enter the details of the comparison between our approach and the other recent ones that were mentioned in the introduction, namely, Polisson and Quah (2013) and Cosaert and Demuynck (2014). Polisson and Quah (2013) study to which extent the standard GARP, which is tailored for perfectly divisible goods (and linear budget sets), is still appropriate when goods are indivisible.

Let us first recall the definition of GARP that is used in Polisson and Quah (2013):

The observations $(p_t, x_t)_{t=1, \dots, n}$ satisfy GARP if for every ordered subset $\{i, j, k, \dots, r, s\} \subset \{1, \dots, n\}$, the inequalities $p_i \cdot x_j \leq p_i \cdot x_i, p_j \cdot x_k \leq p_j \cdot x_j, \dots, p_r \cdot x_s \leq p_r \cdot x_r$ cannot hold unless they are all equalities.

Written in this way, GARP can be applied to discrete bundles x_t and is a stronger requirement than DARP. The observations in figure 1 are consistent with DARP but not with GARP, while those in figure 2 violate both DARP and GARP.

The relationship between DARP and GARP can be better understood by recalling an equivalent formulation of the latter axiom in the case of perfectly divisible goods. To this

³That is, $x_i H x_s$ if there exists an ordered subset $\{i, j, k, \dots, r, s\} \subset \{1, \dots, n\}$ such that $x_i R x_j, x_j R x_k, \dots, x_r R x_s$.

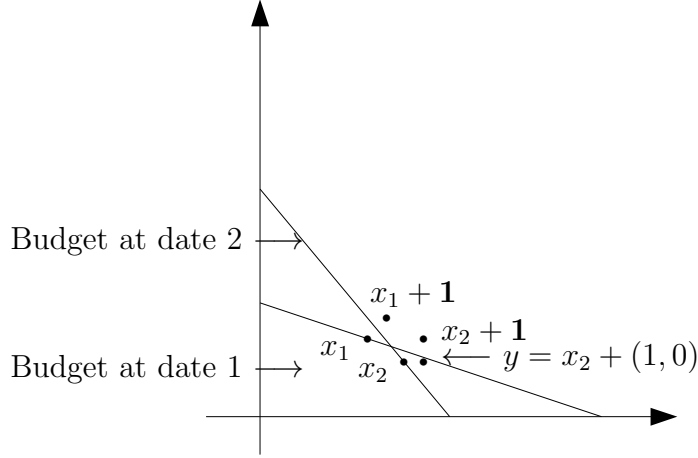


Figure 1: Passing DARP - Violation of GARP

aim, let us introduce the continuous version of our budget sets, namely,

$$C_t = \{x_t \in \mathbb{R}_+^K : p_t \cdot x \leq r_t\}$$

Then DARP says that for every $i, j = 1, \dots, n$, $x_i H x_j \Rightarrow (\{x_i\} + \mathbb{N}_{++}^K) \cap C_j = \emptyset$. On the other hand, if there exists a well-behaved (say, increasing) utility function defined all over \mathbb{R}_+^K (i.e., $u : \mathbb{R}_+^K \rightarrow \mathbb{R}$) such that x_t solves $\max u(x)$ subject to $x \in C_t$, we must have $r_t = p_t \cdot x_t$ for every $t = 1, \dots, n$. By defining the direct revealed preference relation R_c by $(x_i R_c x_j \text{ if } x_j \in C_i)$ and the relation H_c as its transitive closure, we get that observations $(p_t, x_t)_{t=1, \dots, n}$ satisfy GARP iff for any $i, j = 1, \dots, n$, $x_i H_c x_j \Rightarrow p_j \cdot x_i \geq p_j \cdot x_j$, namely, $(\{x_i\} + \mathbb{R}_{++}^K) \cap C_j = \emptyset$. While the latter formulation of GARP is only relevant in the case of perfectly divisible goods, it shows that DARP and GARP express exactly the same consistency property, the only difference being discrete space versus continuous space.⁴ It is thus tempting to say that DARP appears as the right version of GARP when goods are indivisible.

⁴This also implies that DARP is computationally equivalent to GARP. It amounts to computing the transitive closure H , which can be done efficiently by using Warshall's algorithm (Varian, 1982).

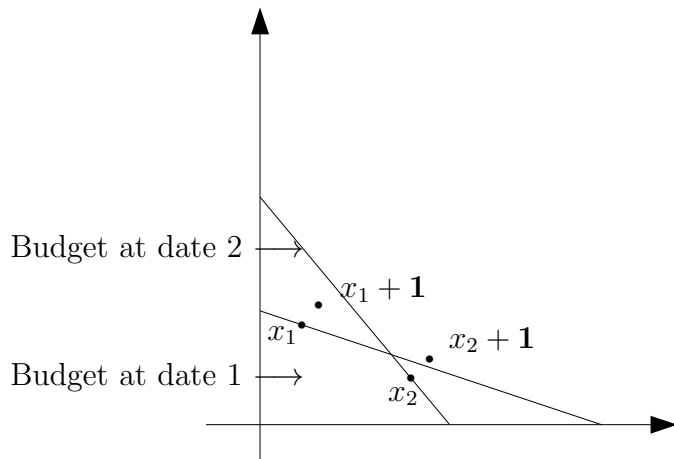


Figure 2: Violation of DARP (and GARP)

As already mentioned in section 1.1, Cosaert and Demuynck (2014)’s data consist, at every date $t = 1, \dots, n$, of a choice among finitely many consumption bundles $\{b_t^1, \dots, b_t^{N_t}\}$ with $b_t^k \in \mathbb{R}_+^K$, $k = 1, \dots, N_t$. Their framework is thus more general than ours since B_t is clearly finite. Their “weakly monotone axiom of revealed preference” (WMARP) can be formulated by first considering auxiliary budget sets $B'_t = \{x \in \mathbb{R}_+^K : \exists b_t^k \text{ such that } x \leq b_t^k\}$. These budget sets are of course not finite nor linear, but one can still define a direct preference relation R' by $x_i R' x_j$ if $x_j \in B'_j$ and deduce its transitive closure H' . WMARP then requires that $x_i R' x_j \Rightarrow x_i \notin \text{int} B'_j$. It is readily checked that WMARP is just GARP for the general budgets B'_t (see Forges and Minelli, 2009) and that if, at every date t , the finite budget sets are generated by discrete linear budget sets B_t , as in the current paper, WMARP and DARP are equivalent.

When WMARP holds, Cosaert and Demuynck (2014) construct a utility function over the whole \mathbb{R}_+^K , by adapting Afriat’s methodology to nonlinear budget sets, exactly as in Forges and Minelli (2009). This utility function has no particular property beyond monotonicity. As it will become clear in section 4, our application of Afriat’s methodology

keeps track of the linear structure behind the discrete budget sets, which allows us to derive a quasi-concave utility function.

The previous discussion suggests that the reason which makes GARP, in its standard form, inappropriate to deal with indivisible goods, does not come from the indivisibilities themselves but rather from the nonlinearities that they generate. By making further assumptions on the model, Polisson and Quah (2013) nevertheless reconcile the standard GARP with indivisible goods.

In the situation depicted by figure 1, suppose that the consumer behaves according to Eq.(1), then he is necessarily indifferent between x_1 and x_2 . It follows that his choices do not meet cost efficiency since $p_1 \cdot x_2 < p_1 \cdot x_1$. Such a situation is excluded by allowing either for an implicit continuous good (Polisson and Quah, 2013) or for money (Brown and Calsamiglia, 2007; Sákovics, 2013), or by imposing cost efficiency (Fujishige and Yang, 2012). But, nothing prevents rationality as given by Eq.(1) only to coexist with cost inefficiencies if goods are available only in discrete quantities. In figure 1, the consumer has no way to take advantage from the monetary gain that results from choosing x_2 instead of x_1 , at date 1.

In what follows we will show that DARP is the appropriate condition for consumption data to be consistent with utility maximization, in the sense of Eq.(1), as long as all goods are traded in discrete quantities.

3. RESULTS

The next proposition is our first main result.

Proposition 1 *The observations $(x_t, p_t, r_t)_{t=1, \dots, n}$ satisfy DARP if, and only if, there exists a discrete quasi-concave and monotonic rationalization of the observations.*

One shortcoming of the result is that we do not obtain a strictly monotonic rationalization. The following example shows that, under DARP, there is no hope in general to get such a property.

Example Let $[x_1 = (5, 4); p_1 = (1, 3, 2)]$ and $[x_2 = (7, 3); p_2 = (3, 2)]$ be the set of observed bundles and prices, and assume that $r_1 = p_1 \cdot x_1$ and $r_2 = p_2 \cdot x_2$. The observations satisfy DARP and, by proposition 1, there exists a monotonic rationalization u which necessarily satisfies $u(x_1) = u(x_2)$ since x_1 and x_2 are both affordable at both dates. Note that $y := x_2 + (1, 0) = (8, 3)$ is also affordable at date 1. Therefore, the utility function u cannot be strictly monotonic, otherwise $u(y) > u(x_2) = u(x_1)$ would contradict that x_1 has been purchased at date 1. Figure 1 illustrates this example.

Let us now define a stronger axiom than DARP to obtain strict monotonicity. Given the set of observations, let $c(t)$ be one of the cheapest goods at date t , that is, $c(t) \in \operatorname{argmin}\{p_t^g : g = 1, \dots, K\}$.

Definition 3 *The observations $(x_t, p_t, r_t)_{t=1, \dots, n}$ satisfy DARP* if for any $i, j = 1, \dots, n$*

$$x_i H x_j \Rightarrow x_i + \mathbf{e}^{c(j)} \notin B_j.$$

In figure 1 the bundle y is affordable at date 1, hence the set of observations does not pass DARP*. Using DARP* we obtain our second main result, which is the analog to proposition 1 with the additional property of strict monotonicity.

Proposition 2 *The observations $(x_t, p_t, r_t)_{t=1, \dots, n}$ satisfy DARP* if, and only if, there exists a discrete quasi-concave and strictly monotonic rationalization of the observations.*

To see why DARP and DARP* refer to one or the other version of monotonicity observe simply that DARP* amounts to: for any $i, j = 1, \dots, n$, $x_i H x_j$ implies $(\{x_i\} + \mathbb{N}^K \setminus \{0\}) \cap B_j = \emptyset$, while DARP requires $(\{x_i\} + \mathbb{N}_{++}^K) \cap B_j = \emptyset$.⁵

Finally, it is worth pointing out that DARP and DARP* can be restated in terms of an operational property of cyclical consistency, as it is also the case for GARP.

⁵Such a distinction has no bite if one considers GARP, as long as prices are positive, since $(\{x_i\} + \mathbb{R}_{++}^K) \cap B_j = \emptyset$ is equivalent to $(\{x_i\} + \mathbb{R}_+^K \setminus \{0\}) \cap B_j = \emptyset$. In other words, one gets strict monotonicity for free in the continuous case.

Definition 4 An $n \times n$ real matrix $A = (a_{jk})_{j,k=1,\dots,n}$ is cyclically consistent if, for any ordered subset $\{j, k, \ell, \dots, r\} \subset \{1, \dots, n\}$, $a_{jk} \leq 0, a_{k\ell} \leq 0, \dots, a_{rj} \leq 0$ implies all terms are 0.

Given the observations $(x_t, p_t, r_t)_{t=1,\dots,n}$, let $\beta = (\beta_{jk})_{j,k=1,\dots,n}$ be the $n \times n$ matrix where: $\beta_{jk} \in \{-1, 0, +1\}$; $\beta_{jk} \leq 0$ if $x_k \in B_j$, with strict inequality iff $x_k + \mathbf{1} \in B_j$; $\beta_{jk} = +1$ if $x_k \notin B_j$. Let $\tilde{\beta}$ be the matrix where $\tilde{\beta}_{jk} \in \{-1, 0, +1\}$; $\tilde{\beta}_{jk} \leq 0$ if $x_k \in B_j$, with strict inequality iff $x_k + \mathbf{e}^{c(j)} \in B_j$; $\tilde{\beta}_{jk} = +1$ if $x_k \notin B_j$.

The next proposition will be useful in the proofs of propositions 1 and 2 and can be established by standard arguments (see, e.g., Forges and Minelli, 2009, p.138).⁶

Proposition 3 The observations $(x_t, p_t, r_t)_{t=1,\dots,n}$ satisfy DARP (resp. DARP*) if, and only if, the matrix β (resp. $\tilde{\beta}$) is cyclically consistent.⁷

4. PROOFS OF PROPOSITIONS 1 AND 2

An interesting (and perhaps unexpected) feature of the proof of our main results (propositions 1 and 2) is that the construction of an explicit utility function from Afriat's inequalities goes through in our discrete framework. To make explicit the comparison with the competitive case we can sketch the proof of proposition 1 in the case where $p_t \cdot x_t = r_t$, $t = 1, \dots, n$.

By proposition 3, we know that DARP is equivalent to cyclical consistency of a matrix β with elements in $\{-1, 0, 1\}$ only. In addition, cyclical consistency is preserved by considering any matrix having entries with identical signs and zeros. We make use of this degree of freedom to deduce the existence of a solution for adequately chosen Afriat's

⁶Note that, under DARP (resp. DARP*), $\beta_{tt} = 0$ (resp. $\tilde{\beta}_{tt} = 0$).

⁷The formulation in terms of matrix α representing the data is not typical of the discrete consumption space. In the continuous case, Forges and Minelli (2009), Ekeland and Galichon (2013) and Forges and Iehlé (2013) already provide insights on the formulation of GARP in terms of a data matrix. In that case the entries of the matrix α are $\alpha_{jk} \in \{-1, 0, +1\}$; $\alpha_{jk} \leq 0$ if $x_k \in B_j$, with strict inequality iff $p_j \cdot x_k < p_j \cdot x_j$; and $\alpha_{jk} = +1$ if $x_k \notin B_j$.

inequalities. We obtain then ψ_1, \dots, ψ_n and $\delta_1, \dots, \delta_n > 0$ such that $u : \mathbb{N}^K \rightarrow \mathbb{R}$ defined as follows is a well-behaved rationalization of the observations:

$$u(x) = \min \left\{ \psi_1 + \delta_1 p_1 \cdot (x - x_1) \mathbb{1}_{A_1^c}(x), \dots, \psi_n + \delta_n p_n \cdot (x - x_n) \mathbb{1}_{A_n^c}(x) \right\}$$

where $A_t = \{x \in \mathbb{N}^K : p_t \cdot x_t - p_t \cdot \mathbf{1} < p_t \cdot x \leq p_t \cdot x_t\}$.

A potential difficulty with our construction is that the desirable properties of a utility function are not *a priori* granted here, contrary to the standard competitive case where the utility function can be taken as the infimum of linear and increasing functions.⁸

To establish proposition 2, we can proceed identically by using the entries of the data matrix $\tilde{\beta}$. In that case the resulting utility function is strictly monotonic.

4.1. Proof of proposition 1

We first show that DARP is necessary for utility maximization. Let u be a monotonic rationalization of the observations $(x_t, p_t, r_t)_{t=1, \dots, n}$ and $\{j, k, \ell, \dots, r\}$ be an ordered subset of $\{1, \dots, n\}$ such that $\beta_{jk} \leq 0, \beta_{k\ell} \leq 0, \dots, \beta_{rj} \leq 0$. For any two consecutive elements $a, b \in \{j, k, \ell, \dots, r\}$, $u(x_a) \geq u(x_b)$. The entire sequence of inequalities implies that $u(x_a) = u(x_b)$ for every $a, b \in \{j, k, \ell, \dots, r\}$. By construction, if $\beta_{ab} < 0$ for two consecutive elements $a, b \in \{j, k, \ell, \dots, r\}$ then $x_b + \mathbf{1} \in B_a$ and $u(x_b + \mathbf{1}) > u(x_b) = u(x_a)$ since u is monotonic. It contradicts that x_a has been purchased at date a . Hence DARP is satisfied from Proposition 3.

We show now the converse implication. Suppose that $(x_t, p_t, r_t)_{t=1, \dots, n}$ satisfies DARP and define a new data matrix γ , with entries $\gamma_{jk} = |\beta_{jk}| (p_j \cdot x_k - r_j)$. From Proposition 3, β satisfies cyclical consistency. It is an easy matter to check that it is also the case for

⁸In the competitive case the rationalization $\bar{u} : \mathbb{R}_+^K \rightarrow \mathbb{R}$ can be constructed as follows: $\bar{u}(x) = \min \{ \bar{\psi}_1 + \bar{\delta}_1 p_1 \cdot (x - x_1), \dots, \bar{\psi}_n + \bar{\delta}_n p_n \cdot (x - x_n) \}$ for some $\bar{\psi}_1, \dots, \bar{\psi}_n$ and $\bar{\delta}_1, \dots, \bar{\delta}_n > 0$. In the case of finite choice sets, Cosaert and Demuyne (2014) use a different strategy to construct the rationalization. In our setting it comes down to the following function $\underline{u} : \mathbb{N}^K \rightarrow \mathbb{R}$ (not necessarily discrete quasi-concave): $\underline{u}(x) = \min \{ \underline{\psi}_1 + \underline{\delta}_1 \min_{y_1 \in B_1} \max_{i=1, \dots, K} e^i \cdot (x - y_1), \dots, \underline{\psi}_n + \underline{\delta}_n \min_{y_n \in B_n} \max_{i=1, \dots, K} e^i \cdot (x - y_n) \}$ for some $\underline{\psi}_1, \dots, \underline{\psi}_n$ and $\underline{\delta}_1, \dots, \underline{\delta}_n > 0$.

the matrix γ since, in particular, $\gamma_{jk} \leq 0$ iff $p_j \cdot x_k - r_j \leq 0$ iff $\beta_{jk} \leq 0$. We can obtain therefore the following Afriat's inequalities (see, e.g., Fostel *et al.*, 2004).⁹ There exist ψ_1, \dots, ψ_n and $\delta_1, \dots, \delta_n > 0$ such that

$$\psi_k \leq \psi_j + \delta_j \gamma_{jk} \quad \forall j, k = 1, \dots, n. \quad (*)$$

For every $t = 1, \dots, n$, define

$$A_t = \{x \in \mathbb{N}^K : r_t - p_t \cdot \mathbf{1} < p_t \cdot x \leq r_t\}.$$

Let $u : \mathbb{N}^K \rightarrow \mathbb{R}$ be defined as follows:

$$u(x) = \min\{\psi_1 + \delta_1(p_1 \cdot x - r_1)\mathbb{1}_{A_1^c}(x), \dots, \psi_n + \delta_n(p_n \cdot x - r_n)\mathbb{1}_{A_n^c}(x)\}.$$

Let us show that the function u rationalizes the experiment.

From the definition of β , it holds that $u(x_t) = \min\{\psi_1 + \delta_1(p_1 \cdot x_t - r_1)\mathbb{1}_{A_1^c}(x_t), \dots, \psi_n + \delta_n(p_n \cdot x_t - r_n)\mathbb{1}_{A_n^c}(x_t)\} = \min\{\psi_1 + \delta_1(p_1 \cdot x_t - r_1)|\beta_{1t}|, \dots, \psi_n + \delta_n(p_n \cdot x_t - r_n)|\beta_{nt}|\} = \min\{\psi_1 + \delta_1\gamma_{1t}, \dots, \psi_n + \delta_n\gamma_{nt}\}$ for any $t = 1, \dots, n$.

Since $\gamma_{tt} = 0$, from the Afriat's inequalities, Eq.(*), we get that $u(x_t) = \psi_t$.

Next, we consider $x \in B_t$. It holds that

$$\begin{aligned} u(x) &= \min\{\psi_1 + \delta_1(p_1 \cdot x - r_1)\mathbb{1}_{A_1^c}(x), \dots, \psi_n + \delta_n(p_n \cdot x - r_n)\mathbb{1}_{A_n^c}(x)\} \\ &\leq \psi_t + \delta_t(p_t \cdot x - r_t)\mathbb{1}_{A_t^c}(x) \\ &\leq \psi_t = u(x_t) \end{aligned}$$

by using $\delta_t, \mathbb{1}_{A_t}(x) \geq 0$ and $p_t \cdot x - r_t \leq 0$.

Hence u rationalizes the data. Clearly the function is monotonic (but not strictly monotonic). The next lemma concludes the proof of proposition 1.

Lemma 1 *The function u is discrete quasi-concave.*

⁹The existence of a solution to the Afriat's inequalities relies only on the cyclical consistency of the matrix.

Proof To make use of customary arguments, let us assume first that the mapping u , constructed before, is defined on $X = \mathbb{R}_+^K$ instead of \mathbb{N}^K (idem for the sets $A_t, t = 1, \dots, n$). We prove that u is quasi-concave. Note that quasi-concavity is invariant to pointwise infimum of quasi-concave functions. Hence it suffices to prove quasi-concavity for the functions

$$f_t(x) := (p_t \cdot x - r_t) \mathbb{1}_{A_t}(x), \quad t = 1, \dots, n.$$

It amounts to showing that for each $x, x' \in X$ and $\lambda \geq 0$, $f_t(\lambda x + (1 - \lambda)x') \geq \min\{f_t(x), f_t(x')\}$. Let $X_1, X_2, X_3 \in X$ be three disjoint sets defined as follows: $X_1 = \{x \in X : p_t \cdot x - (r_t - p_t \cdot \mathbf{1}) \leq 0\}$, $X_2 = A_t(x)$, $X_3 = \{x \in X : p_t \cdot x - r_t > 0\}$. It is an easy matter to check that $X_1, X_2, X_3, X_1 \cup X_2, X_2 \cup X_3$ are all convex sets (note also that $X = \cup_{i=1, \dots, 3} X_i$). We need to consider several cases.

If $x, x' \in X_i$, for some $i = 1, \dots, 3$, then $\lambda x + (1 - \lambda)x' \in X_i$ and the required inequality is verified since either $f_t = 0$ (case X_2) or f_t is affine (cases X_1 and X_3). If $x \in X_2$ and $x' \in X_3$, then $\lambda x + (1 - \lambda)x' \in X_2 \cup X_3$; it follows that $f_t(x') \geq f_t(\lambda x + (1 - \lambda)x') \geq f_t(x) = 0$ (either $f_t(\lambda x + (1 - \lambda)x') = 0$ if $\lambda x + (1 - \lambda)x' \in X_2$ or $f_t(\lambda x + (1 - \lambda)x') = (1 - \lambda)(p_t \cdot x' - r_t) > 0$ if $\lambda x + (1 - \lambda)x' \in X_3$). The same reasoning applies to the case $x \in X_2$ and $x' \in X_1$. Finally, it remains to consider the case $x \in X_1$ and $x' \in X_3$. Either $\lambda x + (1 - \lambda)x' \in X_1 \cup X_3$ and then $f_t(\lambda x + (1 - \lambda)x') = \lambda(p_t \cdot x - r_t) + (1 - \lambda)(p_t \cdot x' - r_t) \geq f_t(x)$ since $f_t(x) < p_t \cdot x' - r_t$; or $\lambda x + (1 - \lambda)x' \in X_2$, and then $f_t(\lambda x + (1 - \lambda)x') = 0 \geq f_t(x)$ since $x \in X_1$ implies $p_t \cdot (x + \mathbf{1}) - r_t \leq 0$ which implies in turn $f_t(x) \leq 0$.

Hence u is quasi-concave on \mathbb{R}_+^K . Clearly, the restriction of u to \mathbb{N}^K is a fortiori discrete quasi-concave. □

4.2. Proof of proposition 2

The proof is readily analogous to the one of proposition 1 and amounts to replacing the vector $\mathbf{1}$ by $\mathbf{e}^{c(t)}$ at each date t . We omit that tedious repetition. However let us show explicitly how we obtain a rationalization that is strictly monotonic, in the *only if part* of the proof.

Following the arguments of the proof of proposition 1, one is led to the following construction of the rationalization :

$$\tilde{u}(x) = \min\{\tilde{\psi}_1 + \tilde{\delta}_1 \tilde{f}_1(x) \dots, \tilde{\psi}_n + \tilde{\delta}_n \tilde{f}_n(x)\}$$

where, for every $t = 1, \dots, n$,

$$\tilde{\psi}_t \geq 0, \tilde{\delta}_t > 0,$$

$$\tilde{f}_t(x) := [p_t \cdot x - r_t] \mathbb{1}_{\tilde{A}_t^c}(x),$$

and

$$\tilde{A}_t := \{x \in \mathbb{N}^K : r_t - p_t^{c(t)} < p_t \cdot x \leq r_t\}.$$

Lemma 2 *The function \tilde{u} is strictly monotonic.*

Proof To check strict monotonicity it suffices to prove it for the functions $\tilde{f}_t(x)$, $t = 1, \dots, n$. Suppose that $x \in \tilde{A}_t^c$ and $x' \succeq x$. If $x' \in \tilde{A}_t^c$ then clearly $\tilde{f}_t(x') = p_t \cdot x' - r_t > p_t \cdot x - r_t = \tilde{f}_t(x)$ since prices are positive. If $x' \in \tilde{A}_t$ it must be the case that x belongs to the lower part of \tilde{A}_t^c , i.e. $p_t \cdot x \leq r_t - p_t^{c(t)}$, thus $\tilde{f}_t(x') = 0 > p_t \cdot x - r_t = \tilde{f}_t(x)$. Suppose that $x \in \tilde{A}_t$ and $x' \succeq x$, then it follows that $p_t \cdot x' \geq p_t \cdot x + p_t^{c(t)}$. Since $x \in \tilde{A}_t$, we get that $p_t \cdot x' > r_t - p_t^{c(t)} + p_t^{c(t)}$, that is, $p_t \cdot x' > r_t$. Hence we have just shown that $\tilde{f}_t(x') > 0$. The result is proved since $\tilde{f}_t(x) = 0$. \square

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