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An allocation rule for dynamic random network formation processes*

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Abstract. Most allocation rules for network games presented in the literature assume that the network structure is fixed. We put explicit emphasis on the construction of networks and examine the dynamic formation of networks whose evolution across time periods is stochastic. Time-series of networks are studied that describe processes of network formation where links may appear or disappear at any period. Moreover, convergence to an efficient network is not necessarily prescribed. Transitions from one network to another are random and yield a Markov chain. We propose the link-based allocation rule for such dynamic random network formation processes and provide its axiomatic characterization. By considering a monotone game and a particular (natural) network formation process we recover the link-based flexible network allocation rule of Jackson (2005a).

JEL Classification: C71, D85

Keywords: dynamic networks, network game, link-based allocation rule, Markov chain, characterization

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1 Introduction

1.1 Aim of the paper

Interactions can be naturally modeled by networks, and consequently, successfully studied with the support of network theory. Interacting individuals can be viewed as players that are linked in a network and contribute to a total productive value or utility of the network. One of the key questions in a cooperative setting is how to divide the value generated by the network between the players. It means defining an allocation rule that will specify for each member his share of the value of the network. Different proposals concerning this topic, and both cooperative and non-cooperative foundations of network allocation rules, have been presented in the literature. For surveys of the vast literature on the issues we refer, e.g., to Slikker and van den Nouweland (2001), Dutta and Jackson (2003), van den Nouweland (2005), Jackson (2005b, 2008). Also some of the more recent works that will be mentioned in this section provide (short) surveys of different allocation rules.

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One of the crucial features of real-life social and economic interactions is the fact that they are usually not static. Although dynamic networks, i.e., networks that evolve over time, can be particularly useful for modeling such interactions, insufficient attention is still paid to network dynamics. In particular, studying allocation rules usually avoids dynamic aspects of network formation. In the present work, we aim at studying how to distribute between players the value generated by dynamic networks. The main contribution of the paper is to introduce and characterize axiomatically an allocation rule for dynamic random network formation processes.

Before describing more in detail our dynamic framework and the allocation rule, first we present a very brief overview of the related literature.

1.2 Related literature

In the seminal work by Myerson (1977) a cooperative game with transferable utilities has been supplemented by a network structure which can be seen as the communication lines between players. In the literature that followed his work the terms communication structures and communication games are usually used. Myerson (1977) introduced and characterized an allocation rule, an extension of the Shapley value (Shapley (1953)) to communication games, which is now called the Myerson value (see also, e.g., Myerson (1980), Aumann and Myerson (1988)). Meessen (1988) proposed an alternative rule for communication situations, called the position value, which has been then characterized by Borm et al. (1992) on the class of communication situations in which the graph is cycle-free. Slikker (2005a,b) provided two characterizations of this value without such restrictions on the graph. Hamiache (1999) presented another rule for communication situations; see also Bilbao et al. (2006). Allocation rules for hypergraph communication situations were studied by van den Nouweland et al. (1992). While Dutta et al. (1998) considered a class of external allocation rules that contains the Myerson value where forming communication links is costless, Slikker and van den Nouweland (2000b) used a natural extension of the Myerson value to determine the payoffs to the players in communication situations with costs for establishing links.

Jackson and Wolinsky (1996) introduced a class of games (called network games) where the value generated depends directly on the network structure, and showed that the Myerson value has a direct extension from communication games to network games. They defined the egalitarian allocation rule and the component-wise egalitarian allocation rule. In this framework, benefits and costs of forming links are assumed and the focus is on pairwise stable and efficient networks. For the potential conflict between stability and efficiency of networks, see, e.g., Dutta and Mutuswami (1997) and Dutta and Jackson (2000). Jackson and van den Nouweland (2005) considered the strongly stable networks, i.e., networks that are stable against changes in links by any coalition of players. Caulier et al. (2013) showed how efficiency and stability can be reconciled in a setting where players are organized in a network and a coalitional structure. They developed the notion of contractually stable networks where the consent of coalitional partners is needed in order to modify the structure. Slikker (2007) characterized axiomatically the Myerson value, the position value and the component-wise egalitarian solution, and also proposed three non-cooperative bargaining procedures that result in the same payoffs as the three rules. For other works concerning non-cooperative foundations of allocation rules, we should mention, e.g., Pérez-Castrillo and Wettstein (2001) who presented an implementation of the Shapley value, and Pérez-Castrillo and Wettstein (2005) who described a mechanism that ends in the Myerson value of the monotonic cover of the value function.

Caulier (2010) considered a rule that provides payoffs to links, where these payoffs are then divided equally over players. He followed Shapley's characterization for this allocation procedure to the links. The weighted Shapley values defined by Kalai and Samet (1987) was generalized to communication situations in Haeringer (1999) and to hierarchical structures in Slikker and van den Nouweland (2000a). van den Nouweland and Slikker (2012) characterized the position value for network situations where no condition on the underlying network is required. Ghintran (2013) generalized the position value by taking into account the negotiation powers of players on the allocation of the worth. She characterized this new weighted position value for communication situations with cycle-free networks. For a similar approach, see also, e.g., Haeringer (2006) who considered allocation rules for cooperative games with transferable utilities. Ghintran et al. (2012) generalized the position value defined for the class of deterministic communication situations, to the class of generalized probabilistic communication situations, and provided two characterizations of this allocation rule.

Although there exists the vast literature on modeling interactions by social and economic networks, the issue of network dynamics has still not received enough attention. Some examples of the dynamic approach to interactions are presented, e.g., in some extensions of the Jackson-Wolinsky connections model to a dynamic framework; see e.g., Jackson and Watts (2002), Watts (2001, 2002). Konishi and Ray (2003) considered a dynamic model of coalition formation when players are farsighted. A model of dynamic network formation with farsighted players was studied by Dutta et al. (2005). They defined a concept of equilibrium which takes into account farsighted behavior of players and allows for limited cooperation amongst them. Page et al. (2005) introduced a dynamic framework of network formation and analyzed farsightedly consistent directed networks. They studied the notion of a supernetwork which is a collection of directed networks and represents coalitional preferences and rules governing network formation. Page and Wooders (2009) introduced a model of network formation with a set of feasible networks, player preferences, rules of network formation and a dominance relation on feasible networks. The authors characterized sets of network outcomes that are likely to emerge and persist. Also Herings et al. (2009) addressed the question which networks one might expect to emerge in the long run when players are farsighted. They provided a full characterization of unique pairwise farsightedly stable sets of networks. Page and Wooders (2007) modeled club structures as bipartite networks and formulated the problem of club formation as a game of network formation. They identified club networks that are stable if players are farsighted, and club networks that are stable if players are myopic. Page and Wooders (2010) formulated club formation with multiple memberships as a noncooperative game of network formation and identified conditions sufficient to guarantee that the game has a potential function.

Despite the growing literature on dynamic networks, most allocation rules for network games assume that the network structure is fixed. The key question is therefore how to distribute between players the value generated by a dynamic network. Jackson (2005a) introduced a new class of allocation rules that take into account the potential alternative constructions of the network, by assuming that the efficient network will eventually emerge. He considered the so called player-based flexible network allocation rule and the link-based flexible network allocation rule. However, in the approach used in Jackson (2005a), even if the allocation is being decided upon when the network is formed or can still be changed, the framework is still static, since the dynamics is just introduced in the fictive construction/decomposition of the structure.

1.3 Summary of the paper

In this paper, we put explicit emphasis on the construction of networks and examine the dynamic formation of networks whose evolution across time periods is stochastic. For a similar approach but used for coalition processes, see Faigle and Grabisch (2012, 2013). We study time-series of networks that describe processes of network formation where several players or links may appear or disappear at any period. Convergence to one of the efficient networks does not necessarily need to be prescribed. One of the basic notions in our framework is the notion of a scenario of network formation processes which is simply a sequence of networks that are observed at subsequent time periods. We restrict our analysis to finite scenarios. A two-network sequence in a scenario is called a transition and is elementary if the two networks differ from each other only by one link. Transitions from one network to another are random and yield a Markov chain. A scenario allocation rule assigns to every value function a vector of allocations for every player and every scenario. In order to specify how the value generated by a dynamic process is distributed among players, an allocation rule for dynamic network processes is defined as an expected value over all possible scenarios of the scenario allocation rules. In the paper, we are interested in the perspective of assigning values to links rather than players, and consequently, our allocation rule is related to the link-based flexible network allocation rule by Jackson (2005a). We introduce the *link-based allocation rule for dynamic random network formation processes*, denoted as the *LBD allocation rule*. We focus on transitions, i.e., the LBD scenario allocation rule is equal to the sum of the transition allocation rules over the transitions that form the given scenario. The symmetric difference of two networks is the set of links where these networks differ. A player is inactive in a transition if he is not adjacent to the symmetric difference of the networks forming the transition. Otherwise, the player is active. For any elementary transition, the difference between the values generated by its new and old networks is divided equally between the two active players. Every inactive player gets zero. If the transition is not elementary, then we take the average over all possible shortest paths formed by elementary transitions of the given transition. We show that for a monotone game, the link-based flexible network allocation rule of Jackson (2005a) coincides with our allocation rule associated to the so called natural network formation process. In such a process, all scenarios are equally probable, we start with the empty network and add one link at each step until we get the complete network.

We provide an axiomatic characterization of the LBD scenario allocation rule which is based on a set of eight very natural axioms: concatenation, efficiency, inactive player, linearity for transitions, null link axiom, equal division, symmetry and antisymmetry for entering/leaving links. The concatenation axiom says that a scenario allocation rule of the concatenation of two concatenable scenarios (i.e., two sequences such that the last network of the one scenario is equal to the first network of the other scenario) is equal to the sum of scenario allocation rules of the two concatenable scenarios. Efficiency means that for any finite scenario, the sum of the scenario allocations for all players is equal to the difference between the values generated by the last and the first network that form the given scenario. Our two next axioms say that an inactive player in a transition gets always zero, and that the transition allocation rule is a linear operator for any transition. A link is null for a value function if adding that link to any network not containing that link does not change the value. According to the null link axiom, if in a transition all links in the symmetric difference to which a player is adjacent are null, then this player gets zero. According to the equal division axiom, in a transition where only one link which appears (resp., disappears) is nonnull and is not adjacent to any nonnull disappearing (resp., appearing) link, the two players forming that link get the same allocation. The symmetry axiom concerns only players in the same situation: either they have some new links (i.e., they are adjacent to some entering links) or they lose some links (they are adjacent to some leaving links). Players having both entering and leaving links are not concerned. Also, null links count for nothing. Suppose then that all entering nonnull links are symmetric (i.e., they have the same marginal contribution). Then the symmetry axiom says that each player receives an amount proportional to the number of links he is adjacent. Two distinct links are antisymmetric for a value function if adding them both to any graph not containing these links does not change the value. The antisymmetry for entering/leaving links says that if all pairs of entering links are symmetric (similarly for all pairs of leaving links), and if there exists an entering link and a leaving link which are antisymmetric, then every player gets an amount proportional to the number of links he is adjacent.

The paper is organized as follows. In Section 2 we recapitulate basic concepts on networks that will be used in the paper. In Section 3 our framework and the link-based allocation rule for dynamic network processes are introduced. In 4 we establish the axiomatic characterization for this new allocation rule. Section 5 presents some concluding remarks and Section 6 contains proofs of the main results.

2 Preliminaries on networks and allocation rules

In this section we recall some preliminaries and standard notations concerning networks and allocation rules. Some notations related exclusively to our dynamic model will be introduced in the next section.

Consider a finite set of players $N = \{1, \dots, n\}$ connected in some network relationship. A *network* is a set of pairs ij of players[1] with $i, j \in N$, $i \neq j$,[2] where $ij \in N$ indicates the presence of a link between players i and j. Networks under consideration are undirected.

Two particular network relationships among players in N are easily identified: the *empty network* $g^{\emptyset}$ without any link between players, and the *complete network* g^N which is the set of all possible subsets of N of size 2. Let G be the set of all possible network relationships among players in N, i.e., $G = \{g | g \subseteq g^N\}$.[3] We use the following standard set operations

$$g \cup g' = \{ij \mid ij \in g \text{ or } ij \in g'\}$$

$$g \cap g' = \{ij \mid ij \in g \text{ and } ij \in g'\}$$

[1] For convenience we use the shorthand notation ij for the pair $\{i, j\}$.
[2] Loop ii is not a possibility in this setting.
[3] Since N is fixed, in order to simplify the notation, we will use G instead of $G(N)$.

$$g \setminus g' = \{ij \mid ij \in g \text{ and } ij \notin g'\}$$

By $g + ij$ we denote the network obtained by adding link ij to an existing network g. Similarly, $g - ij$ is the network obtained by deleting link ij from an existing network g. For two networks g and g', $g \Delta g'$ denotes the symmetric difference, which is the set of links where g and g' differ, i.e.,

$$g \Delta g' = (g \cup g') \setminus (g \cap g')$$

Let $L_i(g)$ denote the set of links that player i is involved in, and let $\ell(g)$ be the total number of links in g, i.e.,

$$L_i(g) = \{ij \mid j \in N \text{ and } ij \in g\}, \quad \ell_i(g) = |L_i(g)|, \quad \ell(g) = \frac{1}{2}\sum_i \ell_i(g)$$

A *value function* on networks is a mapping $v : G \to \mathbb{R}$, assigning a real number to any network. This could be for example the benefit or some worth generated by the network. For simplicity, we denote $v(\{ij\})$ and $v(\{ij, \dots, kl\})$ by $v(ij)$ and $v(ij, \dots, kl)$, respectively. Usually, a pair (N, v) consisting of a player set N and a value function $v \in V$ is called a *network game*. We denote by $V(N)$ the set of all possible value functions on N, or more simply V if N is understood.

A value function v is *monotonic* if $v(g') \leq v(g)$ if $g' \subseteq g$. Adding links to a network is not detrimental to the value.

Given a value function $v \in V$, the *monotonic cover* of v is the value function $\hat{v}$ such that $\hat{v}(g) = \max_{g' \subseteq g} v(g')$. The idea is that the players in a given network g may use the available links in any way they want in order to maximize the value generated. Note that v is monotonic if and only if $v = \hat{v}$.

An important example of (monotonic) value functions are unanimity games. For any network $g \in G$, its associated *unanimity game* u_g is defined by

$$u_g(g') = \begin{cases} 1, & \text{if } g' \supseteq g \\ 0, & \text{otherwise} \end{cases}$$

As for TU-games, unanimity games form a basis of the set of network games for a fixed N (simply because the set of links plays exactly the rôle of N).

A network g is *efficient* relative to v if it maximizes v, i.e., $v(g) \geq v(g')$ for all $g' \in G$.

An *allocation rule* for a network game (N, v) specifies how the value generated by any network g is allocated among players. Specifically, an allocation rule is a function $Y : G \times V \to \mathbb{R}^n$ such that $\sum_i Y_i(g, v) = v(g)$ for all v and g.

Jackson (2005a) proposes, in particular, the player-based flexible network allocation rule and the link-based flexible network allocation rule. Consider any $v \in V$ and a network $g \in G$. The *link-based flexible network allocation rule* is defined by

$$Y_i^{LBFN}(g, v) = \frac{v(g)}{\hat{v}(g^N)} \sum_{j \neq i} \left[\sum_{g' \subseteq g^N - ij} \frac{1}{2}(\hat{v}(g' + ij) - \hat{v}(g')) \left(\frac{\ell(g')!(\ell(g^N) - \ell(g') - 1)!}{\ell(g^N)!} \right) \right] \tag{1}$$

Note that if g is efficient, then $v(g) = \hat{v}(g) = \hat{v}(g^N)$. Hence the normalization factor $\frac{v(g)}{\hat{v}(g^N)}$ disappears.

Let us make some comments about this rule, which will motivate the construction of our new rule. As far as we know, the LBFN rule is one of the few examples of an allocation rule trying to take into account some dynamics of the network formation. It is called "flexible" because it is considered that the network g under consideration, which is not necessarily efficient, should evolve towards some efficient network. However, note that the way this evolution is realized is ignored by the rule, as well as the true final efficient network, which is not necessarily g^N. Also, remark that g itself appears in the formula only as a normalization factor, but not directly in the computation. Lastly, we observe that it is $\hat{v}$ instead of v which is used in the formula. However, infinitely many games have the same monotonic cover, hence the information conveyed by v is partially ignored in the computation.

Based on these observations, we would like to propose an allocation rule which assumes that the network is not fixed and does not suppose that eventually the complete network or an efficient network will form. Rather, we consider that the evolution is free. Second, this rule should take fully into account the information contained in v, as well as the evolution of the network from the beginning to the end. As a conclusion, instead of $Y(g, v)$ or $Y(g^N, \hat{v})$ we should have $Y(\mathcal{G}, v)$ where $\mathcal{G}$ is a "scenario" of network evolution, or better $Y(\mathbf{U}, v)$, where $\mathbf{U}$ is a Markov chain or some stochastic process ruling the evolution of the network. This motivates the construction we propose in the following sections.

3 Markovian network processes

3.1 Description of the dynamic network process

We model a stochastic dynamic process of network formation, where at each discrete period of time a network is identified with some probability. One may think of a random meeting process where players randomly bump into each other in a pairwise fashion at each discrete period of time. In this setting, we view the formation of networks as a process that evolves over discrete time. A *scenario* of network formation process is a sequence of networks

$$\mathcal{G} = g_0, g_1, g_2, g_3 \dots$$

with $g_t \in G$ for $t = 0, 1, 2, \dots$ with the meaning that g_t is the network observed at time period t. Note that a given network may appear several times in a sequence. A special case of scenario, called a *normal scenario*, is the one in which $g_0 = g^{\emptyset}$. A scenario is *finite* if the process eventually stops after a finite number of periods or converges to a given network, i.e., if there is $T \in \mathbb{N}$ such that $g_t = g_T$ for each $t \geq T$. In this paper, we focus on finite scenarios, and by $\mathfrak{S}$ we denote the set of all finite scenarios. For a discussion of a possibility to consider infinite scenarios in Markovian coalition processes, see Faigle and Grabisch (2012).

Any two-element subsequence g, g' in $\mathcal{G}$ is called a *transition* and will be denoted by $g \to g'$. A transition $g \to g'$ is *elementary* if g' differs from g only by one link.

Example 1 Consider a small group consisting of 3 researchers in a lab: Agnieszka (A), Jean-François (J) and Michel (M), i.e., we have $N = \{A, J, M\}$. The lab wants to promote collaboration among its members by announcing regular calls for 2-person projects. A link between any two researchers means submitting a project by these two researchers for getting a grant. An example of a normal finite scenario is

$$\mathcal{G} = g^{\emptyset}, \{AM\}, \{AJ, JM\}, g^N, \{AM\}, \{AJ, AM\}$$

where each network appearing in this scenario corresponds to the set of the projects submitted for a given call. Consequently, in the scenario $\mathcal{G}$, first Agnieszka and Michel decide to submit a project, but they do not resubmit anything for the next call, where two other projects are submitted: one by Agnieszka and Jean-François, and another one by Jean-François and Michel. Then, at the next call, every pair submits a project. Next, only Agnieszka and Michel submit a project, and finally for the last call, Agnieszka and Jean-François as well as Agnieszka and Michel submit two projects.

In the scenario $\mathcal{G}$, three transitions are elementary, i.e.,

$$g^{\emptyset} \to \{AM\}, \quad \{AJ, JM\} \to g^N, \quad \{AM\} \to \{AJ, AM\}$$

and the remaining two transitions are not elementary, i.e.,

$$\{AM\} \to \{AJ, JM\}, \quad g^N \to \{AM\}$$

A value function v can be defined, where $v(g)$ depends on the CV's of the researchers involved and the quality of their cooperation. An allocation rule considering v and $\mathcal{G}$ would define a reward to individual researchers, for their involvement in enhancing cooperation in the lab over a certain period. We can assume that $v(g^{\emptyset}) = 0$.

We assume that the probability of transition from one network to another depends only on the current period, and not on the whole history of transitions. Consequently, a scenario is ruled by a Markov chain with G as the set of states. If the scenario has attained a network g at period t, then the probability to observe a network g' in period $t+1$, i.e., the probability of transition $g \to g'$ is given by $u^t_{gg'}$. We consider stationary Markov processes whose transition probabilities $u^t_{gg'}$ are independent on t and denote them simply by $u_{gg'}$. The *transition matrix* is therefore given by

$$\mathbf{U} := [u_{gg'}]_{g,g' \in G}$$

The matrix $\mathbf{U}$ is row-stochastic, i.e., $u_{gg'} \geq 0$ and $\sum_{g' \in \mathcal{G}} u_{gg'} = 1$ for each $g \in G$.

If $\mathcal{G} = g_0, g_1, g_2 \dots g_T$ is a finite normal scenario with the transition matrix $\mathbf{U}$, then the probability $\Pr(\mathcal{G})$ of occurrence of scenario $\mathcal{G}$ is given by

$$\Pr(\mathcal{G}) = \Pr(g_0) \prod_{t=1}^{T} u_{g_{t-1}g_t} \tag{2}$$

where $\Pr(g_0)$ is the probability of occurrence of the initial state g_0.

With a given transition matrix $\mathbf{U}$ we associate its *directed transition graph* $\mathit{\Gamma}$ whose vertices are elements $g \in G$, and there is a directed edge from g to g' in $\mathit{\Gamma}$ if and only if $u_{gg'} > 0$ in $\mathbf{U}$.

3.2 The link-based allocation rule for dynamic network processes

Next, we will introduce our allocation rule for the dynamic network process described above. First, we start with general definitions.

Definition 1 Let V be the set of value functions and $\mathfrak{G}$ be the set of scenarios. A *scenario allocation rule* is a mapping $\psi : V \to \mathbb{R}^{n\times|\mathfrak{G}|}$, with components of $\psi(v)$ denoted by $\psi_i^{\mathcal{G}}$ for player i and scenario $\mathcal{G}$.

Definition 2 An *allocation rule* for dynamic network processes is a mapping $\Psi : V \to \mathbb{R}^n$ defined as the expected value over all possible scenarios of the scenario allocation rules, i.e.,

$$\Psi(v) := \sum_{\mathcal{G}\in\mathfrak{G}} \Pr(\mathcal{G})\psi^{\mathcal{G}}(v)$$

where $\Pr(\mathcal{G})$ is the probability of occurrence of scenario $\mathcal{G}$.

We introduce and characterize the so-called *link-based allocation rule for dynamic network processes*, denoted by Φ and abbreviated by the *LBD (allocation) rule* (LBD = link-based dynamic). Accordingly, $\phi^{\mathcal{G}}$ denotes the *LBD scenario allocation rule.*

We focus on transitions and study the properties of the following allocation rule:

- If the transition $g \to g'$ is elementary (i.e., g' differs from g only by one link, say ij), then
$$\phi_i^{g\to g'} = \phi_j^{g\to g'} = \frac{1}{2}(v(g') - v(g))$$
and for all other players $\phi_k^{g\to g'} = 0$.
- If the transition is not elementary, we take the average over all possible shortest paths formed of elementary transitions from g to g'.

We introduce some additional concepts and notations.

Definition 3 A player i is *adjacent to graph* g if he is adjacent to some link in g, i.e., there exists $j \in N$ such that $ij \in g$. We denote it by $i \rightharpoonup g$ or $g \leftharpoonup i$. Similarly, a link ij is *adjacent to a graph* g if either $i \rightharpoonup g$ or $j \rightharpoonup g$. We denote it by $ij \rightharpoonup g$ or $g \leftharpoonup ij$.

In order to obtain compact formulations, we introduce the signed version of links and set of links. Considering a transition $g \to g'$, entering links $\lambda \in g' \setminus g$ have a positive sign, while leaving links have a negative sign. The signed version of a link λ is denoted by $\varepsilon\lambda$, with $\varepsilon = -$ or $+$ depending whether λ is leaving or entering. Similarly, if h is a set of links (graph), εh is the set of signed links[4]. As a consequence, writing $g + \varepsilon ij$ means $g + ij$ if $\varepsilon = +$ (entering) and $g - ij$ if $\varepsilon = -$ (leaving). Similarly, $g + \varepsilon h$ stands for the less readable $(g \cup h^+) \setminus h^-$ with h^+ (resp. h^-) the set of entering (resp. leaving) links in h. Note that the "+" operation acts like the usual addition, e.g., $g + ij - ij = g$, except that it is idempotent: $g + ij + ij = g + ij$, $g - ij - ij = g - ij$. Also, note that $g - \varepsilon ij = g + ij$ if $\varepsilon = -$.

In summary, entering and leaving links form the set $g\Delta g'$, which could be called the set of *active links* in transition $g \to g'$. Accordingly, one can define active players.

[4] We are conscious that the notation is ambiguous, but it should cause no problem since sets of links are denoted by g, g', h, h', etc.

Definition 4 A player i is said to be *active in a transition* $g \to g'$ if he is adjacent to $g\varDelta g'$ (in symbols: $i \rightharpoonup g\varDelta g'$), otherwise i is said to be *inactive* in the transition $g \to g'$.

Under these conventions we can introduce the following definition:

Definition 5 The *link-based allocation rule for dynamic network processes (the LBD allocation rule)* $\varPhi$ is defined as

$$\varPhi(v) := \sum_{\mathcal{G} \in \mathfrak{G}} \Pr(\mathcal{G}) \phi^{\mathcal{G}}(v) \tag{3}$$

and $\phi^{\mathcal{G}}$ is the *LBD scenario allocation rule* given by

$$\phi^{\mathcal{G}}(v) = \sum_{k=0}^{T-1} \phi^{g_k \to g_{k+1}}(v) \tag{4}$$

with $\mathcal{G} = g_0, g_1, g_2 \dots g_T$ and

$$\phi_i^{g \to g'}(v) = \begin{cases} \frac{1}{2|g\varDelta g'|!} \sum_\sigma \sum_{ik \in g\varDelta g'} (v(g + \varepsilon h_\sigma^{ik}) - v(g + \varepsilon h_\sigma^{ik} - \varepsilon ik)), & \text{if } i \rightharpoonup g\varDelta g' \\ 0, & \text{otherwise} \end{cases} \tag{5}$$

where σ is any permutation on the set of links $g\varDelta g' = \{\lambda_1, \dots, \lambda_{|g\varDelta g'|}\}$, and h_σ^{ik} is the first set in the sequence $\{\lambda_{\sigma(1)}\}, \{\lambda_{\sigma(1)}, \lambda_{\sigma(2)}\}, \dots, g\varDelta g'$ containing ik.

Note that inactive players receive 0.

Remark 1 It is important to note that the two levels (the allocation rule level and the scenario allocation rule level) work independently. That is, on the allocation rule level, the transitions between networks obey a Markov chain, which determines the probability of transitions of a network g into another network g'. This stochastic process typically results of the noncooperative behavior of the players, trying to maximize their own utility function. In a sense, it comes from the players taken as *individuals*. By contrast, on the scenario rule level, for a given transition $g \to g'$, one has to define in a proper way how the benefit/loss of the transition (i.e., $v(g') - v(g)$) is shared among the players who are active in the transition. This could be seen as a basic rule imposed by the "network", i.e., by the players considered as a *society*, and which therefore operates on a different level. If one imposes symmetry in the sharing (no player has a special advantage, only v and the structure of the network matter), our axiomatization shows that we are led to compute, for a nonelementary transition, the (unweighted) average over all possibilities of starting from g and arriving at g' by elementary transitions, *regardless* of the probabilities of those elementary transitions stemming from the Markov chain.

Example 2 Consider the scenario $\mathcal{G}$ given in Example 1. By virtue of (4), we have

$$\phi^{\mathcal{G}}(v) = \phi^{g^\emptyset \to \{AM\}}(v) + \phi^{\{AM\} \to \{AJ,JM\}}(v) + \phi^{\{AJ,JM\} \to g^N}(v) + \phi^{g^N \to \{AM\}}(v) + \phi^{\{AM\} \to \{AJ,AM\}}(v)$$

The symmetric differences and players adjacent to them are the following:

$$g^\emptyset \varDelta \{AM\} = \{AM\}, \quad \{AM\}\varDelta\{AJ, JM\} = g^N, \quad \{AJ, JM\}\varDelta g^N = \{AM\}$$

$$g^N \Delta\{AM\} = \{AJ, JM\}, \quad \{AM\}\Delta\{AJ, AM\} = \{AJ\}$$
$$A, M \rightharpoonup \{AM\}, \quad A, J, M \rightharpoonup g^N$$
$$A, J, M \rightharpoonup \{AJ, JM\}, \quad A, J \rightharpoonup \{AJ\}$$

Let us apply (5) to the example. First, for the elementary transitions we have

$$\phi_A^{g^\emptyset \to \{AM\}}(v) = \phi_M^{g^\emptyset \to \{AM\}}(v) = \frac{v(AM)}{2}, \quad \phi_J^{g^\emptyset \to \{AM\}}(v) = 0$$

$$\phi_A^{\{AJ,JM\} \to g^N}(v) = \phi_M^{\{AJ,JM\} \to g^N}(v) = \frac{v(g^N) - v(AJ, JM)}{2}, \quad \phi_J^{\{AJ,JM\} \to g^N}(v) = 0$$

$$\phi_A^{\{AM\} \to \{AJ,AM\}}(v) = \phi_J^{\{AM\} \to \{AJ,AM\}}(v) = \frac{v(AJ, AM) - v(AM)}{2}, \quad \phi_M^{\{AM\} \to \{AJ,AM\}}(v) = 0$$

Consider now the (not elementary) transition $\{AM\} \to \{AJ, JM\}$. There are six shortest sequences of elementary transitions between $\{AM\}$ and $\{AJ, JM\}$, as presented in Figure 1. The red/green/blue arrows in this figure indicate the transitions where $A/J/M$ is active.

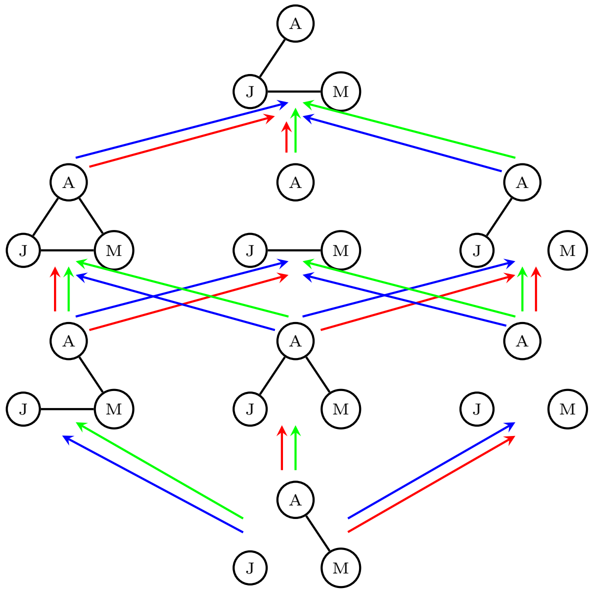

Fig. 1. The shortest sequences of elementary transitions between $\{AM\}$ and $\{AJ, JM\}$ and their active players (red/green/blue arrows indicate the transitions where $A/J/M$ is active)

We have then

$$\phi_A^{\{AM\} \to \{AJ,JM\}}(v) = \frac{1}{12}[v(AJ) - v(AM) - v(AM) + v(AJ, JM) - v(JM) + v(AJ) - v(AM) +$$
$$v(AJ, AM) - v(AM) + v(AJ, JM) - v(g^N) + v(AJ, JM) - v(AM, JM) + v(AJ, JM) - v(AM, JM)]$$
$$= \frac{1}{12}\left[v(AJ, AM) - v(JM) - v(g^N) + 2(v(AJ) - v(AM, JM)) + 4(v(AJ, JM) - v(AM))\right]$$

$$\phi_J^{\{AM\}\to\{AJ,JM\}}(v) = \frac{1}{12}[v(AJ,JM)+v(AJ,JM)+v(AJ,AM)-v(AM)+v(AJ,JM)-v(AJ)$$

$$+v(g^N)-v(AM)+v(AM,JM)-v(AM)+v(AJ,JM)-v(JM)+v(g^N)-v(AM)] =$$

$$= \frac{1}{12}\left[2v(g^N)+4v(AJ,JM)+v(AJ,AM)+v(AM,JM)-v(JM)-v(AJ)-4v(AM)\right]$$

$$\phi_M^{\{AM\}\to\{AJ,JM\}}(v) = \frac{1}{12}[-v(AM)+v(AJ,JM)-v(AJ)+2(v(JM)-v(AM))+$$

$$+2(v(AJ,JM)-v(AJ,AM))+v(AM,JM)-v(AM)+v(AJ,JM)-v(g^N)] =$$

$$= \frac{1}{12}\left[v(AM,JM)+2(v(JM)-v(AJ,AM))+4(v(AJ,JM)-v(AM))-v(g^N)-v(AJ)\right]$$

In a similar way we can calculate $\phi_i^{g^N\to\{AM\}}(v)$ for $i \in N$. We have then

$$\phi_A^{g^N\to\{AM\}}(v) = \frac{1}{4}\left[-v(g^N)+v(AM,JM)-v(AJ,AM)+v(AM)\right]$$

$$\phi_J^{g^N\to\{AM\}}(v) = \frac{1}{2}\left[v(AM)-v(g^N)\right]$$

$$\phi_M^{g^N\to\{AM\}}(v) = \frac{1}{4}\left[-v(AM,JM)+v(AM)-v(g^N))+v(AJ,AM)\right]$$

Note that for our scenario $\mathcal{G}$ we have $\phi_A^{\mathcal{G}}+\phi_J^{\mathcal{G}}+\phi_M^{\mathcal{G}} = v(AJ,AM)$. In the next section we will show that the LBD scenario allocation rule is indeed efficient.

A first remarkable result is that we can recover the link-based flexible network allocation rule of Jackson (2005a) by considering a special process, called the *natural process*.

Definition 6 The *natural process* is a process of network formation defined by:

- the process starts with the empty network, i.e., $g_0 = g^{\emptyset}$
- we add one link at each step until we obtain the complete graph g^N
- all scenarios are equally probable, i.e., $\Pr(\mathcal{G}) = \frac{1}{\eta!}$ for each $\mathcal{G} \in \mathfrak{S}$, where $\eta = \binom{n}{2}$.

Proposition 1 *Assume v is a monotone game. Then for $g = g^N$, the Y^{LBFN} rule coincides with Φ associated to the natural process.*

Proof: When v is monotone, the Y^{LBFN} rule reduces to

$$Y_i^{LBFN}(g^N,v) = \sum_{j\neq i}\Big[\sum_{g'\subseteq g^N-ij}\frac{1}{2}(v(g'+ij)-v(g'))\Big(\frac{\ell(g')!(\eta!-\ell(g')-1)!}{\eta!}\Big)\Big]$$

Take player i and count his contribution Δ_i in each scenario. Consider scenario $\mathcal{G}$. Since g^N is complete, player i has degree $n-1$, and therefore Δ_i is non null in exactly $n-1$

transitions of the scenario where links ij are added, $j = 1, \dots, i-1, i+1, \dots, n$. When the link ij is added, we find

$$\Delta_i = \frac{1}{2}\Big(v(g' + ij) - v(g')\Big)$$

where g' is the last graph without ij in $\mathcal{G}$. For a fixed g', there are $\ell(g')!$ paths from the empty graph to g', and $(\eta - \ell(g') - 1)!$ paths from $g' + ij$ to the complete graph g^N, hence the result. ■

4 Axiomatization of the LBD scenario allocation rule

Next, we provide an axiomatic characterization of the LBD scenario allocation rule. As introduced in Definition 1, $\psi^{\mathcal{G}}$ denotes a scenario allocation rule for scenario $\mathcal{G}$. In Section 1 we have already introduced the meaning of the axioms that characterize the LBD scenario allocation rule. In this section, we provide the formal descriptions of these axioms.

Definition 7 Two sequences $\mathcal{G} = g_0, \dots, g_q, \mathcal{G}' = g'_0, \dots, g'_r$ are said to be *concatenable* if $g_q = g'_0$, in which case their concatenation is the sequence

$$\mathcal{G} \oplus \mathcal{G}' := g_0, \dots, g_q, g'_1, \dots, g'_r.$$

Concatenation (C): Let $\mathcal{G}, \mathcal{G}'$ be two concatenable sequences. Then

$$\psi^{\mathcal{G} \oplus \mathcal{G}'} = \psi^{\mathcal{G}} + \psi^{\mathcal{G}'}.$$

Axiom (C) allows to restrict our attention to transitions. Indeed,

$$\psi^{\mathcal{G}} = \sum_{k=0}^{T-1} \psi^{g_k \to g_{k+1}}$$

holds for every sequence $\mathcal{G} = g_0, g_1, \dots, g_T$.

Efficiency (E): For any finite scenario $\mathcal{G} = g_0, g_1, \dots, g_T$ it holds $\sum_{i \in N} \psi_i^{\mathcal{G}} = v(g_T) - v(g_0)$.

Inactive player (IP): If i is inactive in $g \to g'$, then $\psi_i^{g \to g'}(v) = 0$ for any v.

Linearity for transitions (L): $v \mapsto \psi^{g \to g'}(v)$ is a linear operator for any transition $g \to g'$.

Definition 8 A link ij is *null* for v if $v(g + ij) = v(g)$ for every $g \not\ni ij$.

Null link axiom (NL): If in a transition $g \to g'$ all links in $g \Delta g'$ to which player i is adjacent are null for v, then $\psi_i^{g \to g'}(v) = 0$.

For any graph g and game v, the graph g^* is the graph where all links which are null for v have been deleted. This notation is used in the following axioms.

Equal division (ED): In a transition $g \to g'$ where only one link $ij \in g \setminus g'$ (resp., $ij \in g' \setminus g$) is nonnull and $ij \not\rightharpoonup (g' \setminus g)^*$ (resp., $ij \not\rightharpoonup (g \setminus g')^*$), there is equal division of the contribution of the link to i and j: $\psi_i^{g \to g'} = \psi_j^{g \to g'}$.

Definition 9 Two distinct links ij, kl are *symmetric* for v if for every graph g not containing them, it holds $v(g + ij) = v(g + kl)$.

Symmetry (S): Consider the transition $g \to g'$ and a game v. If all links in $(g \setminus g')^*$ (resp., $(g' \setminus g)^*$) are symmetric for v (assuming there are at least two symmetric links), then for all $i \rightharpoonup (g \setminus g')^*$, $i \not\rightharpoonup (g' \setminus g)^*$ (resp., $i \rightharpoonup (g' \setminus g)^*$, $i \not\rightharpoonup (g \setminus g')^*$), we have

$$\psi_i^{g \to g'}(v) \sim \ell_i((g \Delta g')^*),$$

where $\sim$ means "proportional".

Remark 2 The (NL), (ED) and (S) axioms can be combined into a single one. Indeed, if all null links are discarded, a player satisfying the conditions of the (NL) axiom (i.e., all adjacent links in $g \Delta g'$ are null) has degree 0 in $(g \Delta g')^*$. Now, if only one link ij is nonnull in $g \setminus g'$ and not adjacent to nonnull links in $g' \setminus g$, the degrees of i and j are equal to 1, hence they receive the same. One could call it the degree axiom (D) and it would read:

Degree axiom (D): Consider the transition $g \to g'$, a game v and the graph $(g \Delta g')^*$. If for all distinct links $\lambda, \lambda' \in (g \setminus g')^*$ (resp., $(g' \setminus g)^*$), λ and λ' are symmetric, then for all $i \rightharpoonup (g \setminus g')^*$, $i \not\rightharpoonup (g' \setminus g)^*$ (resp., $i \rightharpoonup (g' \setminus g)^*$, $i \not\rightharpoonup (g \setminus g')^*$), we have

$$\psi_i^{g \to g'}(v) \sim \ell_i((g \Delta g')^*).$$

Definition 10 Two distinct links ij, kl are *antisymmetric* for v if $v(g + ij + kl) = v(g)$ for every graph g not containing them.

Antisymmetry for entering/leaving links (ASEL): Consider the transition $g \to g'$, a game v and the graph $(g \Delta g')^*$. If all pairs of distinct links in $(g \setminus g')^*$ and in $(g' \setminus g)^*$ are symmetric for v, and if there exists a pair of links $\lambda \in (g \setminus g')^*, \lambda' \in (g' \setminus g)^*$ which are antisymmetric for v, then for all $i \rightharpoonup (g \Delta g')^*$,

$$\psi_i^{g \to g'}(v) \sim \ell_i((g \Delta g')^*).$$

Antisymmetric links have a counterbalancing effect, in the sense that they annihilate each other when entering together a network, which can be interpreted by saying that they bring the same contribution but of opposite sign. Therefore, if one is leaving and the other entering, their contribution in the scenario becomes equal and of the same sign.

A characterization of asymmetric links can be obtained directly from the corresponding result for TU-games. We find:

Lemma 1 *Distinct links λ, λ' are antisymmetric for v if and only if*

$$m^v(g + \lambda + \lambda') = -m^v(g + \lambda) - m^v(g + \lambda'), \quad \forall g \not\ni \lambda, \lambda',$$

where m^v is the Möbius transform of v.

Similarly we can prove the following lemma:

Lemma 2 *Distinct links λ, λ' are symmetric for v if and only if they are symmetric for m^v:*

$$m^v(g+\lambda) = m^v(g+\lambda'), \quad \forall g \not\ni \lambda, \lambda'.$$

Proof: λ, λ' are symmetric for v if $v(g+\lambda) = v(g+\lambda')$ for all $g \not\ni \lambda, \lambda'$, which is equivalent to

$$0 = \sum_{h \subseteq g+\lambda} m^v(h) - \sum_{h \subseteq g+\lambda'} m^v(h) = \sum_{h \subseteq g} (m^v(h+\lambda) - m^v(h+\lambda')), \quad \forall g \not\ni \lambda, \lambda'.$$

For $g = g^\emptyset$ this gives $m^v(\lambda) = m^v(\lambda')$. For $g = \{\lambda''\}$ we obtain $m^v(\lambda''+\lambda) = m^v(\lambda''+\lambda')$. Continuing the process we find the desired result. ■

Corollary 1 *Suppose λ, λ' are antisymmetric for v, and that λ, λ'' are symmetric for v. Then λ', λ'' are antisymmetric for v.*

This shows that in the situation described by axiom (ASEL), any entering link is antisymmetric with any leaving link.

Proposition 2 *The LBD scenario allocation rule satisfies (C), (E), (IP), (L), (NL), (ED), (S) and (ASEL).*

For the proof, see the Appendix.

Theorem 1 *The LBD scenario allocation rule is the unique allocation rule satisfying (C), (L), (IP), (E), (S), (NL), (ASEL) and (ED).*

For the proof, we use the following lemma.

Lemma 3 *Consider the unanimity game u_h, $h \neq \emptyset$. Then the following holds:*

(i) A link ij is nonnull for u_h if and only if $ij \in h$;
(ii) A pair of links ij, kl is symmetric for u_h if and only if both $ij, kl \in h$ or both $ij, kl \notin h$.

Proof: (i) Take $ij \in h$. Then ij is nonnull since $u_h(h) \neq u_h(h-ij)$. Conversely, take $ij \notin h$. Then $u_h(g) = u_h(g-ij)$ for every g.

(ii) Assume $|h| \geq 2$ and consider $ij, kl \in h$ two distinct links. Then for any g not containing them, $u_h(g+ij) = u_h(g+kl) = 0$. Now assume $|g^N \setminus h| \geq 2$ and consider $ij, kl \notin h$ two distinct links. Then for any g not containing them, $u_h(g+ij) = u_h(g+kl) = u_h(g)$. Conversely, for $h \neq g^N$, take $ij \in h$ and $kl \notin h$. Then $1 = u_h(h-ij+ij) \neq u_h(h-ij+kl) = 0$, hence these links are not symmetric. ■

For the proof of Theorem 1, see the Appendix.

5 Concluding remarks

We have considered the dynamic random network formation processes, where links may appear and disappear at any time, and a scenario of the process, i.e., a sequence of networks, forms a Markov chain. We have established the characterization of the scenario allocation rule for dynamic network processes based on the set of natural axioms. We have shown that if a monotone game is considered, then the link-based flexible network allocation rule of Jackson (2005a) coincides with our LBD link-based allocation rule associated to the natural process.

Our framework of dynamic network formation can naturally model situations with the set of active players changing over time, where individuals appear or disappear during the process: some players may become involved in later periods of the network formation process, some others can be 'active' all the time, some individuals may appear only for a short period of time and disappear forever, or appear again after some time of "silence", etc. When allocating value generated by the dynamic networks we take into account all players that were ever involved in the dynamic network formation process. Hence, many real-life dynamic interactions with appearing/disappearing actors can naturally be modeled by our framework.

There are several directions for follow-up research on this subject. While we have presented the LBD allocation rule which is 'fair' in the sense that it is symmetric, we could consider a weighted version of the rule by introducing additionally weights to players when allocating the value among them. For instance, in our Example 1, a rule to reward researchers' involvement in enhancing cooperation could violate symmetry, by taking into account the individuals' frequency of professional missions or their needs for more sophisticated research equipments. Another extension of the present work could include a strategic version of the framework and an endogenous model of a stochastic network formation process.

6 Appendix - Proofs

Proof of Proposition 2

(C) is immediate from (4). (IP), (L) and (NL) are immediate from (5).

(E): By concatenation, it suffices to prove the result for transitions. Consider an elementary transition $g \to g'$, with link ij added or deleted. Then

$$\sum_{k \in N} \phi_k^{g \to g'}(v) = \phi_i^{g \to g'}(v) + \phi_j^{g \to g'}(v) = 2 \times \frac{1}{2}(v(g') - v(g)).$$

Now, suppose that the transition is not elementary, with η created/deleted links. Along each of the $\eta!$ paths from g to g', efficiency holds, therefore

$$\sum_{k \in N} \phi_k^{g \to g'} = \frac{1}{\eta!} \eta! \left(v(g') - v(g)\right) = v(g') - v(g).$$

(S): Let us consider that either $|g \setminus g'| \geq 2$ or $|g' \setminus g| \geq 2$.

1. We begin by showing that all null links can be discarded without change: $\phi^{g\to g'}(v) = \phi^{g\to g'\setminus ij}(v)$ if $ij \in g' \setminus g$ is null for v (a similar reasoning holds for $ij \in g \setminus g'$). Take such a link ij. We have by (5)

$$\begin{aligned}\phi_i^{g\to g'}(v) = & \frac{1}{2|g\Delta g'|!}\sum_{\sigma}\big(v(g+\varepsilon h_\sigma^{ij}) - v(g+h_\sigma^{ij}-\varepsilon ij)\big)\\ & + \sum_{ik\in g\Delta g', k\neq j}\frac{1}{2|g\Delta g'|!}\sum_{\sigma}\big(v(g+\varepsilon h_\sigma^{ik}) - v(g+\varepsilon h_\sigma^{ik}-\varepsilon ik)\big)\end{aligned}$$

with notation of (5).

Since ij is null, the first term in the above equation vanishes. For the second term, we have

$$\frac{1}{2|g\Delta g'|!}\sum_{\sigma \text{ on } g\Delta g'}\big(v(g+\varepsilon h_\sigma^{ik}) - v(g+\varepsilon h_\sigma^{ik}-\varepsilon ik)\big) = \\ \frac{|g\Delta g'|}{2|g\Delta g'|!}\sum_{\sigma' \text{ on } g\Delta(g'-ij)}\big(v(g+\varepsilon h_{\sigma'}^{ik}) - v(g+\varepsilon h_{\sigma'}^{ik}-\varepsilon ik)\big)$$

(note that $h_{\sigma'}^{ik}$ is a set in the sequence $\{\lambda_{\sigma'(1)}\}, \{\lambda_{\sigma'(1)}, \lambda_{\sigma'(2)}\}, \ldots, g'\Delta(g-ij)$). In summary we get

$$\phi_i^{g\to g'}(v) = \frac{1}{2|g\Delta(g'-ij)|!}\sum_{\sigma \text{ on } g\Delta(g'-ij)}\sum_{ik\in g\Delta(g'-ij)}\big(v(g+\varepsilon h_\sigma^{ik}) - v(g+\varepsilon h_\sigma^{ik}-\varepsilon ik)\big) = \phi_i^{g\to g'-ij}.$$

2. We suppose now that no null link exists in $g\Delta g'$ and that $|g'\setminus g| \geq 2$, with all links in $g'\setminus g$ being symmetric. By symmetry we can set $\nu_0(h) = v(g-h)$, $\nu_1(h) = v(g-h+ij)$, $\nu_2(h) = v(g-h+ij+kl)$, ..., $\nu_{|g'\setminus g|}(h) = v(g'-h)$, with $ij, kl, \ldots \in g'\setminus g$, and h a set of links in $g\setminus g'$. Then for any i such that $i \rightharpoonup g'\setminus g$ and $i \not\rightharpoonup g\setminus g'$, we get

$$\begin{aligned}\phi_i^{g\to g'}(v) &= \frac{1}{2|g\Delta g'|!}\sum_{\sigma}\sum_{ik\in g'\setminus g}\big(v(g+\varepsilon h_\sigma^{ik}) - v(g+\varepsilon h_\sigma^{ik}-ik)\big) \qquad (6)\\ &= \frac{1}{2|g\Delta g'|!}\sum_{ik\in g'\setminus g}\Big(\sum_{h^-\subseteq g\setminus g'}\sum_{h^+\subseteq g'\setminus g-ik}\sum_{\sigma_{h^-,h^+}}\big(v(g-h^-+h^++ik) - v(g-h^-+h^+)\big)\Big)\\ &= \frac{1}{2|g\Delta g'|!}\sum_{ik\in g'\setminus g}\Big(\sum_{h^-\subseteq g\setminus g'}\sum_{h^+\subseteq g'\setminus g-ik}\sum_{\sigma_{h^-,h^+}}\big(\nu_{|h^+|+1}(h^-) - \nu_{|h^+|}(h^-)\big)\Big),\end{aligned}$$

where σ_{h^-,h^+} is any permutation putting first links in $h^-\cup h^+$ in any order. Observe that

$$K = \sum_{h^-\subseteq g\setminus g'}\sum_{h^+\subseteq g'\setminus g-ik}\sum_{\sigma_{h^-,h^+}}\big(\nu_{|h^+|+1}(h^-) - \nu_{|h^+|}(h^-)\big)$$

is constant for every $ik \in g'\setminus g$. Therefore

$$\phi_i^{g\to g'}(v) = \frac{d_{g'\setminus g}(i)}{2|g\Delta g'|!}K,$$

the desired result.

(ED): similar to (S). In the case of (ED), we have $h^+ = \emptyset$ in the above equations, and letting $(g' \setminus g)^* = \{ij\}$, the degrees of i and j are equal to 1.

(ASEL): Consider that links in $(g \setminus g')^*$ and in $(g' \setminus g)^*$ are symmetric for v, and that any two links $\lambda \in (g \setminus g')^*, \lambda' \in (g' \setminus g)^*$ are antisymmetric for v. Take $i, j \in N$ such that $(g \setminus g')^* \not\leftharpoondown i \rightharpoonup (g' \setminus g)^*$, and $(g' \setminus g)^* \not\leftharpoondown j \rightharpoonup (g \setminus g')^*$. We have by (5):

$$\begin{aligned}
\phi_i^{g\to g'}(v) =& \frac{1}{2|(g\varDelta g')^*|} \sum_{ik\in(g'\setminus g)^*} \sum_{m=0}^{|(g\varDelta g')^*|} \sum_{\substack{h^-\subseteq(g\setminus g')^* \\ h^+\subseteq(g'\setminus g)^*-ik \\ |h^+\cup h^-|=m}} m!(|(g\varDelta g')^*|-m-1)!\Big(v(g+h^+-h^-+ik) \\
&- v(g+h^+-h^-)\Big) \\
\phi_j^{g\to g'}(v) =& \frac{1}{2|(g\varDelta g')^*|} \sum_{j\ell\in(g\setminus g')^*} \sum_{m=0}^{|(g\varDelta g')^*|} \sum_{\substack{h^-\subseteq(g\setminus g')^*-j\ell \\ h^+\subseteq(g'\setminus g)^* \\ |h^+\cup h^-|=m}} m!(|(g\varDelta g')^*|-m-1)!\Big(v(g+h^+-h^--j\ell) \\
&- v(g+h^+-h^-)\Big),
\end{aligned}$$

which we rewrite simply as

$$\begin{aligned}
\phi_i^{g\to g'}(v) &= \frac{1}{2|(g\varDelta g')^*|} \sum_{ik\in(g'\setminus g)^*} \phi_{i,ik}^{g\to g'}(v) \\
\phi_j^{g\to g'}(v) &= \frac{1}{2|(g\varDelta g')^*|} \sum_{j\ell\in(g\setminus g')^*} \phi_{j,j\ell}^{g\to g'}(v).
\end{aligned}$$

Let us prove that $\phi^{g\to g'}_{i,ik}(v) = \phi^{g\to g'}_{j,j\ell}(v)$ using the fact that ik and $j\ell$ are antisymmetric for v, that is, $v(g''+ik) = v(g''-j\ell)$ for every graph $j\ell \in g'' \not\ni ik$:

$$\begin{aligned}
\phi^{g\to g'}_{j,j\ell}(v) = \sum_{m=0}^{|(g\Delta g')^*|} m!(|(g\Delta g')^*|-m-1)!\Bigg(& \sum_{\substack{h^-\subseteq(g\backslash g')^*-j\ell\\ h^+\subseteq(g'\backslash g)^*\\ h^+\not\ni ik\\ |h^+\cup h^-|=m}} \Big(v(g+h^+-h^--j\ell)-v(g+h^+-h^-)\Big)\\
& + \sum_{\substack{h^-\subseteq(g\backslash g')^*-j\ell\\ h^+\subseteq(g'\backslash g)^*\\ h^+\ni ik\\ |h^+\cup h^-|=m}} \Big(v(g+h^+-h^--j\ell)-v(g+h^+-h^-)\Big)\Bigg)\\
= \sum_{m=0}^{|(g\Delta g')^*|} m!(|(g\Delta g')^*|-m-1)!\Bigg(& \sum_{\substack{h^-\subseteq(g\backslash g')^*-j\ell\\ h^+\subseteq(g'\backslash g)^*\\ h^+\not\ni ik\\ |h^+\cup h^-|=m}} \Big(v(g+h^++ik-h^-)-v(g+h^+-h^-)\Big)\\
& + \sum_{\substack{h^-\subseteq(g\backslash g')^*-j\ell\\ h^+\subseteq(g'\backslash g)^*\\ h^+\ni ik\\ |h^+\cup h^-|=m}} \Big(v(g+h^+-h^--j\ell)-v(g+h^+-ik-h^--j\ell)\Big)\Bigg).
\end{aligned}$$

After a slight rewriting of $\phi^{g\to g'}_{i,ik}(v)$:

$$\begin{aligned}
\phi^{g\to g'}_{i,ik}(v) = \sum_{m=0}^{|(g\Delta g')^*|} m!(|(g\Delta g')^*|-m-1)!\Bigg(& \sum_{\substack{h^-\subseteq(g\backslash g')^*\\ h^+\subseteq(g'\backslash g)^*-ik\\ h^-\not\ni j\ell\\ |h^+\cup h^-|=m}} \Big(v(g+h^+-h^-+ik)-v(g+h^+-h^-)\Big)\\
& + \sum_{\substack{h^-\subseteq(g\backslash g')^*\\ h^+\subseteq(g'\backslash g)^*-ik\\ h^-\ni j\ell\\ |h^+\cup h^-|=m}} \Big(v(g+h^+-h^-+ik)-v(g+h^+-h^-)\Big)\Bigg)\\
= \sum_{m=0}^{|(g\Delta g')^*|} m!(|(g\Delta g')^*|-m-1)!\Bigg(& \sum_{\substack{h^-\subseteq(g\backslash g')^*\\ h^+\subseteq(g'\backslash g)^*-ik\\ h^-\not\ni j\ell\\ |h^+\cup h^-|=m}} \Big(v(g+h^++ik-h^-)-v(g+h^+-h^-)\Big)\\
& + \sum_{\substack{h^-\subseteq(g\backslash g')^*-j\ell\\ h^+\subseteq(g'\backslash g)^*\\ h^+\ni ik\\ |h^+\cup h^-|=m}} \Big(v(g+h^+-h^--j\ell)-v(g+h^+-ik-h^--j\ell)\Big)\Bigg)
\end{aligned}$$

we see that $\phi^{g\to g'}_{i,ik}(v) = \phi^{g\to g'}_{j,j\ell}(v)$, as desired. Now, by symmetry, $\phi^{g\to g'}_{i,ik}(v) = \phi^{g\to g'}_{i,ik'}(v)$ for every two links $ik, ik' \in (g' \setminus g)^*$, and similarly for $\phi^{g\to g'}_{j,j\ell}(v)$, which proves the result.

We address now the case where a node i is adjacent to links both in $(g' \setminus g)^*$ and $(g \setminus g')^*$. In this case, we have:

$$\begin{aligned}\phi^{g\to g'}_i(v) =& \frac{1}{2|(g\Delta g')^*|}\Bigg(\sum_{ik\in(g'\setminus g)^*}\sum_{m=0}^{|(g\Delta g')^*|}\sum_{\substack{h^-\subseteq(g\setminus g')^*\\ h^+\subseteq(g'\setminus g)^*-ik\\ |h^+\cup h^-|=m}} m!(|(g\Delta g')^*|-m-1)!\Big(v(g+h^+-h^-+ik)\\ &-v(g+h^+-h^-)\Big)\\ &+\sum_{i\ell\in(g\setminus g')^*}\sum_{m=0}^{|(g\Delta g')^*|}\sum_{\substack{h^-\subseteq(g\setminus g')^*-i\ell\\ h^+\subseteq(g'\setminus g)^*\\ |h^+\cup h^-|=m}} m!(|(g\Delta g')^*|-m-1)!\Big(v(g+h^+-h^- -i\ell)\\ &-v(g+h^+-h^-)\Big)\Bigg) = \frac{1}{2|(g\Delta g')^*|}\Big(\sum_{ik\in(g'\setminus g)^*}\phi^{g\to g'}_{i,ik}(v) + \sum_{i\ell\in(g\setminus g')^*}\phi^{g\to g'}_{i,i\ell}(v)\Big).\end{aligned}$$

Proceeding exactly as above shows that $\phi^{g\to g'}_{i,ik}(v) = \phi^{g\to g'}_{i,i\ell}(v)$. Now, by symmetry of all links in $(g' \setminus g)^*$, $\phi^{g\to g'}_{i,ik}(v) = \phi^{g\to g'}_{i,ik'}(v)$ for any $ik, ik' \in (g' \setminus g)^*$, and similarly for links in $(g \setminus g')^*$, which proves the result. ■

Proof of Theorem 1

We know by Proposition 2 that the LBD rule satisfies all these axioms. It remains to show uniqueness. By (L) and (C), it suffices to prove that for any unanimity game u_h, any transition $g \to g'$, $\psi^{g\to g'}(u_h)$ is uniquely determined.

We consider the unanimity game u_h, $h \subseteq g^N$. Inactive players are those not adjacent to $g\Delta g'$ and receive 0 by (IP). Now, by Lemma 3, links not in h are null. Therefore, if an active player is not adjacent to a link both in h and in $g\Delta g'$, by (NL) he receives 0. Then by (E), (NL) and (IP) we have in any situation:

$$u_h(g') - u_h(g) = \sum_{i \rightharpoonup h\cap(g\Delta g')} \psi^{g\to g'}_i(u_h), \tag{7}$$

and $\psi^{g\to g'}_i(u_h) = 0$ if $i \not\rightharpoonup h \cap (g\Delta g')$.

1. We suppose $g \subseteq g'$. From (7), we have:

(i) If $h \cap (g' \setminus g) = \emptyset$, $\psi^{g\to g'}_i(u_h) = 0$ for all i;
(ii) If $h \cap (g' \setminus g) = \{ij\}$ (only one nonnull link), by (ED) it follows that

$$\psi^{g\to g'}_i(u_h) = \psi^{g\to g'}_j(u_h) = \frac{u_h(g') - u_h(g)}{2},$$

and the other players adjacent to $g' \setminus g$ receive 0;

(iii) If $|h \cap (g' \setminus g)| > 1$, by Lemma 3, all links in $h \cap (g' \setminus g)$ are symmetric. Hence by (S), we find for any player i adjacent to $(g' \setminus g)^* = h \cap (g' \setminus g)$:

$$\psi_i^{g \to g'}(u_h) = \frac{d_{h\cap(g'\setminus g)}(i)(u_h(g') - u_h(g))}{\sum_{j \rightharpoonup h\cap(g'\setminus g)} d_{h\cap(g'\setminus g)}(j)}, \tag{8}$$

and the others receive 0.

Finally,

$$u_h(g') - u_h(g) = \begin{cases} 1, & \text{if } h \subseteq g' \text{ and } h \nsubseteq g \\ 0, & \text{otherwise.} \end{cases} \tag{9}$$

Combining the above results and (9), $\psi^{g \to g'}(u_h)$ is uniquely determined.

2. The case $g' \subseteq g$ proceeds similarly.

3. We suppose $g \setminus g' \neq \emptyset$ and $g' \setminus g \neq \emptyset$. Observe that

$$u_h(g') - u_h(g) = \begin{cases} 1, & \text{if } h \subseteq g' \text{ and } h \nsubseteq g \cap g' \\ -1, & \text{if } h \subseteq g \text{ and } h \nsubseteq g \cap g' \\ 0, & \text{otherwise.} \end{cases} \tag{10}$$

3.1. Suppose $h \subseteq g'$ and $h \nsubseteq g \cap g'$, hence $h \cap (g \Delta g') \neq \emptyset$ and $h \cap (g \Delta g') = h \cap (g' \setminus g)$. Using (7), we can proceed as in case 1:

(i) If $h \cap (g' \setminus g) = \{ij\}$, by (ED) it follows that

$$\psi_i^{g \to g'}(u_h) = \psi_j^{g \to g'}(u_h) = \frac{1}{2}.$$

(ii) If $|h \cap (g' \setminus g)| > 1$, we find for any player i adjacent to $(g' \setminus g)^* = h \cap (g \Delta g')$:

$$\psi_i^{g \to g'}(u_h) = \frac{d_{h\cap(g\Delta g')}(i)}{\sum_{j \rightharpoonup h\cap(g\Delta g')} d_{h\cap(g\Delta g')}(j)}, \tag{11}$$

and the others receive 0.

3.2. The case $h \subseteq g$ and $h \nsubseteq g \cap g'$ proceeds similarly and yields

$$\psi_i^{g \to g'}(u_h) = -\frac{d_{h\cap(g\Delta g')}(i)}{\sum_{j \rightharpoonup h\cap(g\Delta g')} d_{h\cap(g\Delta g')}(j)}. \tag{12}$$

3.3 Suppose $h \subseteq g \cap g'$. Then all links in $g \Delta g'$ are null, hence

$$\psi_i^{g \to g'}(u_h) = 0, \quad \forall i \in N. \tag{13}$$

3.4 Suppose $h \nsubseteq g$ and $h \nsubseteq g'$. Equation (7) becomes:

$$\sum_{i \rightharpoonup h\cap(g\Delta g')} \psi_i^{g \to g'}(u_h) = \sum_{\substack{i \rightharpoonup h\cap(g'\setminus g) \\ \& \ i \not\rightharpoonup h\cap(g\setminus g')}} \psi_i^{g \to g'}(u_h) + \sum_{\substack{i \rightharpoonup h\cap(g\setminus g') \\ \& \ i \not\rightharpoonup h\cap(g'\setminus g)}} \psi_i^{g \to g'}(u_h) + \sum_{\substack{i \rightharpoonup h\cap(g'\setminus g) \\ \& \ i \rightharpoonup h\cap(g\setminus g')}} \psi_i^{g \to g'}(u_h) = 0. \tag{14}$$

3.4.1. Suppose $h \cap (g' \setminus g) = \emptyset$. Then (14) reduces to

$$\sum_{i \rightharpoonup h\cap(g\setminus g')} \psi_i^{g\to g'}(u_h) = 0.$$

By Lemma 3(ii), all links in $h \cap (g \setminus g')$ are symmetric if there are at least two. Then (ED) or (S) imply $\psi_i^{g\to g'}(u_h) = 0$ for all $i \rightharpoonup h \cap (g\Delta g')$, and by (IP)

$$\psi_i^{g\to g'}(u_h) = 0, \quad \forall i \in N. \tag{15}$$

3.4.2. The case $h \cap (g \setminus g') = \emptyset$ proceeds similarly, and we also find (15).

3.4.3 Suppose $h \cap (g' \setminus g) \neq \emptyset$ and $h \cap (g \setminus g') \neq \emptyset$. We proceed by induction on $|h \cap (g' \setminus g)|$.

We consider first that $h \cap (g' \setminus g) = \{ij\}$ and the game $v = u_h - u_{h-ij}$. Applying Lemma 1, it is easy to check that ij and $k\ell$ are antisymmetric for v, for any $k\ell \in h\cap(g\setminus g')$. Also, all links in $h \cap (g \setminus g')$ are symmetric for v when there are at least two links. Then it follows from (ASEL) that

$$\frac{\psi_i^{g\to g'}(v)}{d_{h\cap(g\Delta g')}(i)} = \frac{\psi_j^{g\to g'}(v)}{d_{h\cap(g\Delta g')}(j)} = \frac{\psi_k^{g\to g'}(v)}{d_{h\cap(g\Delta g')}(k)} = \frac{\psi_\ell^{g\to g'}(v)}{d_{h\cap(g\Delta g')}(\ell)}$$

which by linearity (L) turns into

$$\frac{\psi_i^{g\to g'}(u_h) - \psi_i^{g\to g'}(u_{h-ij})}{d_{h\cap(g\Delta g')}(i)} = \frac{\psi_j^{g\to g'}(u_h) - \psi_j^{g\to g'}(u_{h-ij})}{d_{h\cap(g\Delta g')}(j)} = \frac{\psi_k^{g\to g'}(u_h) - \psi_k^{g\to g'}(u_{h-ij})}{d_{h\cap(g\Delta g')}(k)}$$
$$= \frac{\psi_\ell^{g\to g'}(u_h) - \psi_\ell^{g\to g'}(u_{h-ij})}{d_{h\cap(g\Delta g')}(\ell)}.$$

Since $(h - ij) \cap (g' \setminus g) = \emptyset$, we are back to case 3.4.1., hence $\psi^{g\to g'}(u_{h-ij}) = 0$ and we obtain

$$\frac{\psi_i^{g\to g'}(u_h)}{d_{h\cap(g\Delta g')}(i)} = \frac{\psi_j^{g\to g'}(u_h)}{d_{h\cap(g\Delta g')}(j)} = \frac{\psi_k^{g\to g'}(u_h)}{d_{h\cap(g\Delta g')}(k)} = \frac{\psi_\ell^{g\to g'}(u_h)}{d_{h\cap(g\Delta g')}(\ell)}. \tag{16}$$

Now, consider again equation (14). Observe that $\psi_i^{g\to g'}(u_h), \psi_j^{g\to g'}(u_h), \psi_k^{g\to g'}(u_h)$ and $\psi_\ell^{g\to g'}(u_h)$ are variables in this equation. Moreover, we can apply (S) or (ED) to the nodes i' in the second term (those adjacent to $h \cap (g \setminus g')$ but not to $h \cap (g' \setminus g)$), while the first and third term can only concern i or j or both. It follows that (14) contains only the variables $\psi_i^{g\to g'}(u_h), \psi_j^{g\to g'}(u_h), \psi_k^{g\to g'}(u_h)$ and $\psi_\ell^{g\to g'}(u_h)$. Substituting into it the different equalities in (16) determine these variables uniquely.

Suppose now that $\psi_i^{g\to g'}(u_h)$ is known till $|h \cap (g' \setminus g)| = m < |g' \setminus g|$ and let us determine $\psi^{g\to g'}(u_h)$ when $|h \cap (g' \setminus g)| = m + 1$. Consider the game

$$v = \sum_{h' \subseteq h\cap(g'\setminus g)} (-1)^{|h'|} u_{h-h'}.$$

We claim that any $ij \in h \cap (g' \setminus g)$ and any $k\ell \in h \cap (g \setminus g')$ are antisymmetric for v, and that moreover all $ij \in h \cap (g' \setminus g)$ are symmetric for v, and so are all links in $h \cap (g \setminus g')$.

Proof of the Claim: The values of the Möbius transform of v are 1 for h, -1 for $h-ij$, 1 for $h-ij-i'j'$, -1 for $h-ij-i'j'-i''j''$, etc., with $ij, i'j', i''j'' \in h\cap(g'\setminus g)$, and 0 otherwise. Let us check antisymmetry for $ij \in h\cap(g'\setminus g)$ and $k\ell \in h\cap(g\setminus g')$ by Lemma 1. We must check that

$$m^v(g+ij+k\ell) = -m^v(g+ij) - m^v(g+k\ell), \quad \forall g \not\ni ij, k\ell.$$

Observe that $m^v(g+ij) = 0$ for every $g \not\ni ij, k\ell$. By construction either $m^v(g+ij+kl)$ and $m^v(g+k\ell)$ are both 0 (if $g \neq h-h'-k\ell$ for some $h' \subseteq h\cap(g'\setminus g)$ containing ij), or $m^v(g+ij+k\ell) = -m^v(g+k\ell)$ holds. In both cases, the equality holds. Now, by Lemma 2, symmetry holds for any two links in $h\cap(g'\setminus g)$ if m^v is symmetric for these links, which is the case by construction. Finally, any two links in $h\cap(g\setminus g')$ are symmetric for v because they are symmetric for each $u_{h-h'}$, $h' \subseteq h\cap(g\setminus g')$.

Hence (ASEL) can be applied to any pair of links $ij \in h\cap(g'\setminus g)$ and $k\ell \in h\cap(g\setminus g')$, and since by induction hypothesis, all $\psi^{g\to g'}(u_{h-h'})$, $h' \neq \emptyset$ are determined, it follows that (16) holds, for any $ij \in h\cap(g'\setminus g)$ and any $k\ell \in h\cap(g\setminus g')$. Finally, the successive substitutions into (14) determine $\psi^{g\to g'}(u_h)$ uniquely. ■

Bibliography

R. Aumann and R. Myerson. Endogenous formation of links between players and coalitions: an application of the Shapley value. In A. Roth, editor, *The Shapley Value*, pages 175–191. Cambridge University Press, 1988.

J. Bilbao, N. Jiménez, and J. López. A note on a value with incomplete communication. *Games and Economic Behavior*, 54(2):419–429, 2006.

P. Borm, G. Owen, and S. Tijs. On the position value for communication situations. *SIAM Journal on Discrete Mathematics*, 5:305–320, 1992.

J.-F. Caulier. *Contribution to Collective Choice Theory.* Académie Universitaire Louvain, Belgium, 2010. Ph.D. Thesis, Facultés Universitaires Saint Louis.

J.-F. Caulier, A. Mauleon, and V. Vannetelbosch. Contractually stable networks. *International Journal of Game Theory*, 42:483–499, 2013.

B. Dutta and M. O. Jackson. The stability and efficiency of directed communication networks. *Review of Economic Design*, 5:251–272, 2000.

B. Dutta and M. O. Jackson. *Networks and Groups: Models of Strategic Formation.* Springer-Verlag, Heidelberg, 2003.

B. Dutta and S. Mutuswami. Stable networks. *Journal of Economic Theory*, 76:322–344, 1997.

B. Dutta, A. van den Nouweland, and S. Tijs. Link formation in cooperative situations. *International Journal of Game Theory*, 27:245–256, 1998.

B. Dutta, S. Ghosal, and D. Ray. Farsighted network formation. *Journal of Economic Theory*, 122(2):143–164, 2005.

U. Faigle and M. Grabisch. Values for Markovian coalition processes. *Economic Theory*, 51:505–538, 2012.

U. Faigle and M. Grabisch. A note on values for Markovian coalition processes. *Economic Theory Bulletin*, 2013. Forthcoming.

A. Ghintran. Weighted position values. *Mathematical Social Sciences*, 65:157–163, 2013.

A. Ghintran, E. González-Aranguena, and C. Manuel. A probabilistic position value. *Annals of Operations Research*, 201:183–196, 2012.

G. Haeringer. Weighted Myerson value. *International Game Theory Review*, 1(2):187–192, 1999.

G. Haeringer. A new weight scheme for the Shapley value. *Mathematical Social Sciences*, 52(1):88–98, 2006.

G. Hamiache. A value with incomplete communication. *Games and Economic Behavior*, 26(1):59–78, 1999.

P. J.-J. Herings, A. Mauleon, and V. Vannetelbosch. Farsightedly stable networks. *Games and Economic Behavior*, 67:526–541, 2009.

M. O. Jackson. Allocation rules for network games. *Games and Economic Behavior*, 51: 128–154, 2005a.

M. O. Jackson. A survey of network formation models: stability and efficiency. In G. Demange and M. Wooders, editors, *Group Formation in Economics: Networks, Clubs, and Coalitions*, pages 11–49. Cambridge University Press, Cambridge, UK, 2005b.

M. O. Jackson. *Social and Economic Networks.* Princeton University Press, 2008.

M. O. Jackson and A. van den Nouweland. Strongly stable networks. *Games and Economic Behavior*, 51:420–444, 2005.

M. O. Jackson and A. Watts. The evolution of social and economic networks. *Journal of Economic Theory*, 106(2):265–295, 2002.

M. O. Jackson and A. Wolinsky. A strategic model of social and economic networks. *Journal of Economic Theory*, 71:44–74, 1996.

E. Kalai and D. Samet. On weighted Shapley values. *International Journal of Game Theory*, 16(3):205–222, 1987.

H. Konishi and D. Ray. Coalition formation as a dynamic process. *Journal of Economic Theory*, 110:1–41, 2003.

R. Meessen. *Communication Games*. Master Thesis, University of Nijmegen, The Netherlands, 1988.

R. B. Myerson. Graphs and cooperation in games. *Mathematics of Operations Research*, 2:225–229, 1977.

R. B. Myerson. Conference structures and fair allocation rules. *International Journal of Game Theory*, 9(3):169–182, 1980.

F. H. Page and M. Wooders. Networks and clubs. *Journal of Economic Behavior and Organization*, 64:406–425, 2007.

F. H. Page and M. Wooders. Strategic basins of attraction, the path dominance core, and network formation games. *Games and Economic Behavior*, 66:462–487, 2009.

F. H. Page and M. Wooders. Club networks with multiple memberships and noncooperative stability. *Games and Economic Behavior*, 70:12–20, 2010.

F. H. Page, M. Wooders, and S. Kamat. Networks and farsighted stability. *Journal of Economic Theory*, 120:257–269, 2005.

D. Pérez-Castrillo and D. Wettstein. Bidding for the surplus: a non-cooperative approach to the Shapley value. *Journal of Economic Theory*, 100(2):274–294, 2001.

D. Pérez-Castrillo and D. Wettstein. Forming efficient networks. *Economic Letters*, 87(1): 83–87, 2005.

L. S. Shapley. A value for n-person games. In H. W. Kuhn and A. W. Tucker, editors, *Annals of Mathematics Studies*, 28, pages 307–317. Princeton University Press, 1953.

M. Slikker. A characterization of the position value. *International Journal of Game Theory*, 33(4):505–514, 2005a.

M. Slikker. Link monotonic allocation schemes. *International Game Theory Review*, 7(4): 473–489, 2005b.

M. Slikker. Bidding for surplus in network allocation problems. *Journal of Economic Theory*, 137:493–511, 2007.

M. Slikker and A. van den Nouweland. Communication situations with asymmetric players. *Mathematical Methods of Operations Research*, 52:39–56, 2000a.

M. Slikker and A. van den Nouweland. Network formation with costs for establishing links. *Review of Economic Design*, 5(3):333–362, 2000b.

M. Slikker and A. van den Nouweland. *Social and Economic Networks in Cooperative Game Theory*. Kluwer Academic Publishers, Boston, 2001.

A. van den Nouweland. Models of network formation in cooperative games. In G. Demange and M. Wooders, editors, *Group Formation in Economics: Networks, Clubs, and Coalitions*. Cambridge University Press, Cambridge, UK, 2005.

A. van den Nouweland and M. Slikker. An axiomatic characterization of the position value for network situations. *Mathematical Social Sciences*, 64:266–271, 2012.

A. van den Nouweland, P. Borm, and S. Tijs. Allocation rules for hypergraph communication situations. *International Journal of Game Theory*, 20:255–268, 1992.

A. Watts. A dynamic model of network formation. *Games and Economic Behavior*, 34: 331–341, 2001.

A. Watts. Non-myopic formation of circle networks. *Economic Letters*, 74:277–282, 2002.