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A Model of Influence Based on Aggregation Functions*

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Abstract

The paper concerns a dynamic model of influence in which agents make a yes-no decision. Each agent has an initial opinion which he may change during different phases of interaction, due to mutual influence among agents. We investigate a model of influence based on aggregation functions. Each agent modifies his opinion independently of the others, by aggregating the current opinion of all agents. Our framework covers numerous existing models of opinion formation, since we allow for arbitrary aggregation functions. We provide a general analysis of convergence in the aggregation model and find all terminal classes and states. We show that possible terminal classes to which the process of influence may converge are terminal states (the consensus states and non trivial states), cyclic terminal classes, and unions of Boolean lattices (called regular terminal classes). An agent is influential for another agent if the opinion of the first one matters for the latter. A generalization of influential agent to an irreducible coalition whose opinion matters for an agent is called influential coalition. The graph (hypergraph) of influence is a graphical representation of influential agents (coalitions). Based on properties of the hypergraphs of influence we obtain conditions for the existence of the different kinds of terminal classes. An important family of aggregation functions – the family of symmetric decomposable models – is discussed. Finally, based on the results of the paper, we analyze the manager network of Krackhardt.

JEL Classification: C7, D7, D85

Keywords: influence, aggregation function, convergence, terminal class, influential coalition, hypergraph, social network

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1 Introduction

Influence and opinion formation are broadly studied in several scientific fields, e.g., in psychology, sociology, economics, marketing, mathematics, physics. Important contributions to the study of these issues can be found in the literature on dynamic aspects of influence; see, e.g., Jackson (2008) for an overview of dynamic models of imitation and social influence. One of the leading models of opinion and consensus formation is due to DeGroot (1974). In his model, every individual in a society has an initial opinion on a subject, represented by a number in $[0, 1]$, and he aggregates the opinions of other individuals through a weighted arithmetic mean. The interaction patterns are described by a stochastic matrix whose entries represent weights that an agent places on the current opinions of other agents in forming his own opinion for the next period. The opinions are updated over time. Results in Markov chain theory are easily adapted to the model. Several works have been devoted to the DeGroot framework and its different variations have been proposed. However, although the literature on influence and opinion formation is quite vast, most of the related works assume a convex combination as the way of aggregating opinions. In Section 7 we present a brief description of the DeGroot model as well as a survey of the related literature.

In this paper we investigate a new approach to influence based on *aggregation functions*. The point of departure is a one-step model (Grabisch and Rusinowska (2010, 2011)) in which agents make a yes-no decision on a certain issue. While each agent has his preliminary opinion (inclination), he may decide differently from that inclination, due to influence between agents. A transformation from the agents inclinations to their decisions is represented by an influence function. In Grabisch and Rusinowska (2010) we investigate some tools to analyze influence in this one-step framework. In particular, we define the influence indices to measure the influence of a coalition on an agent, introduce several influence functions and investigate the concept of a follower of a coalition. Also in Grabisch and Rusinowska (2011) the one-step model of influence is examined, but we establish there the exact relations between the key concepts of the influence model and the framework of command games by Hu and Shapley (2003a,b). In particular, the concept of an influence function is extensively studied. The present paper extends our previous research on influence in several aspects. While influence functions considered so far were any kind of deterministic function, with a decision process after a single step of influence, we consider now a dynamic influence mechanism which is assumed to be stochastic and to follow a stationary Markov chain. Moreover, the formation of opinions is based on aggregation functions.

Restricting the analysis to stationary Markov processes allows us to model the influence phenomenon that differs from learning process. Our framework can be seen as a reasonable approximation of reality when a committee of individuals is already established for a while. Its members know each other well, and while they can be sensitive to arguments of some members of the committee, they do not really follow influence attempts of other members. Then, most probably the process should be stationary, as learning is known to be more active at the early stage. Concerning the Markovian assumption, it is likely that for some reasons agents cannot or do not want to keep a long memory of all previous opinions. Discussions might become too long and tiring, many people express their opinions, so it becomes too complicated to recall all of them, or

discussed issues are not that important for an agent to memorize all previous meetings. Hence, we believe that in many decision making environments, it is essentially the last opinion which matters for influence.

There are three main contributions of the present paper to the study of influence, and basically three advantages of our framework over other existing models. First of all, we introduce and analyze a new framework of influence based on arbitrary aggregation functions, which to the best of our knowledge has not been proposed before. Each agent modifies his opinion by aggregating the current opinion of all agents (possibly including himself) according to his aggregation function. The framework covers numerous existing models of opinion formation, since we allow for arbitrary aggregation functions. We provide a general analysis of convergence in the aggregation model. First, we show that possible terminal classes to which the process of influence may converge are terminal states (the consensus states and non trivial states), cyclic terminal classes, and unions of Boolean lattices (called regular terminal classes). Next, we use the concepts of *influential agent* and *graph of influence*. Roughly speaking, an agent j is yes- (or no-) influential for agent i if the opinion of j matters for i . The graph of yes-influence (no-influence) is a directed graph whose nodes are the agents and there is an arc from j to i if j is yes-influential (no-influential) for i . A direct generalization of these notions leads to *influential coalitions* and *hypergraphs of yes- (or no-) influence*. It appears that the qualitative description of the convergence is entirely described by the hypergraphs of influence. Based on properties of the hypergraphs and influential coalitions we determine conditions for the existence of the different types of terminal classes. Furthermore, we study a specific family of aggregation models – the family of symmetric decomposable models, in which all influential coalitions are singletons and the graphs of yes- and no-influence coincide. Terminal classes in such models are analyzed.

The second advantage of the present model concerns the reduction of complexity. We assume that the influence mechanism is a Markovian process. Consequently, for the analysis of the qualitative convergence in a model with n agents we need the information on all entries of the $2^n \times 2^n$ (reduced) transition matrix. While the Markovian model of influence is exponentially complex, the subfamily based on aggregation functions is of polynomial complexity. Indeed, in order to determine all terminal classes in the aggregation model we only need to know the hypergraphs of yes- and no-influence, whose maximal size is $2n \binom{n}{\lfloor n/2 \rfloor}$. Note that the size difference between the Markovian model of influence and the aggregation model drastically increases with n .

The third advantage of the present model is related to practical considerations and applicability of the model. When we know exactly how each agent aggregates the current opinions of others when modifying his own opinion for the next step and how they are correlated, we can provide the full analysis of convergence. However, in practice, we frequently do not know how the aggregation is done by the agents. In our model, for the analysis of the qualitative convergence we do not need the full information on the agents' aggregation functions. What we only need to know are all influential coalitions, but this information can usually be obtained by observing the influence process.

In order to show the advantages of the aggregation framework over other existing models we study an empirical example based on the advice network of Krackhardt (1987). He collected data from managers of a small manufacturing firm in the US about who sought advice from whom. Based on these data, Jackson (2008) developed a social influence

matrix as defined in the context of the DeGroot model. We apply our approach to the same advice network of Krackhardt (1987) and provide the convergence analysis of the example. Moreover, for simplicity of the illustration and discussion of our results, we additionally analyze a smaller example of the advice network with 3 managers who have to decide whether to introduce a new technology in the company. The discussion on that issue may take many rounds and every manager may seek advice from the others before each round. Apart from the classical approach of weighted averaging aggregation, one can easily imagine different ways of aggregating the opinions by the managers.

The remainder of the paper is the following. In Section 2 we introduce the model. The concepts of influential agents (influential coalitions) and (hyper)graphs of influence are defined in Section 3. In Section 4 we provide a general analysis of convergence in the aggregation model. In Section 5 a specific family of aggregation functions – symmetric decomposable model – is discussed. In Section 6 we analyze the empirical example of Krackhardt (1987). Section 7 contains a brief survey of the related literature. In Section 8 we conclude with some practical considerations. The proofs of our results are placed in the Appendix.

2 Modelling influence by aggregation functions

2.1 Description of the model

We consider a set $N := \{1, \dots, n\}$ of agents having to make a yes-no decision for approving a bill, a project, a candidate, etc. (typically these agents form a committee). Each agent is supposed to have an initial opinion (called *inclination*), but during the different phases of the discussion, agents may change their initial opinion due to mutual influence among agents. (Grabisch and Rusinowska, 2010).

Given a situation S , there is a certain probability $b_{S,T}$ that the set of ‘yes’-voters after one step of influence is T . If we assume that the process of influence may iterate (several rounds in the discussion), we obtain a stochastic process, called the *influence process*¹, depicting the evolution of the coalition of ‘yes’-agents along time. We make here the following simplifying assumptions, which seem reasonable in our context of influence:

- (i) The process is Markovian, i.e., the probability $b_{S,T}$ depends on S (the present situation) and T (the future situation), and not on the whole history.
- (ii) The process is stationary, i.e., $b_{S,T}$ is constant over time.

States of this finite Markovian process are therefore all subsets $S \subseteq N$, representing the set of ‘yes’-agents; its *transition matrix* $\mathbf{B} := [b_{S,T}]_{S,T \subseteq N}$ is a $2^n \times 2^n$ row-stochastic matrix.

Markov theory provides a complete description of the convergence of the process through the matrix \mathbf{B} . This description can be *qualitative*, that is, it gives the terminal states and classes together with their type (periodic or not) in which the process will eventually converge (see definitions below), or *quantitative*, i.e., it gives in addition the

¹If the process is deterministic, we call it simply an influence function $B : 2^N \rightarrow 2^N$ (see (Grabisch and Rusinowska, 2010)).

probability that the process converges in a given terminal class, as well as the probability of each state in a terminal class. While the full quantitative description requires the knowledge of \mathbf{B} , it is known from Markov chain theory that the qualitative description needs only the knowledge of the *reduced matrix* $\tilde{\mathbf{B}}$, where

$$\tilde{b}_{S,T} = \begin{cases} 1, & \text{if } b_{S,T} > 0 \\ 0, & \text{otherwise.} \end{cases}$$

The reduced matrix can be equivalently represented by a graph $\Gamma = (2^N, E)$, called the *transition graph*, where E is the set of arcs. Γ is a directed graph (digraph), whose vertices are all possible coalitions, and an arc (S, T) from state S to state T exists if and only if $\tilde{b}_{S,T} = 1$.

A *path* in Γ from S to T is a sequence of states $S = S_0, S_1, S_2, \dots, S_{k-1}, S_k = T$ such that $(S_i, S_{i+1}) \in E$ for $i = 0, \dots, k-1$.

Definition 1. Let \mathcal{C} be a nonempty collection of states. \mathcal{C} is *strongly connected* if either $\mathcal{C} = \{S\}$, or for every distinct $S, T \in \mathcal{C}$, there is a path in \mathcal{C} from S to T and from T to S . Moreover, \mathcal{C} is a *class*² if it is strongly connected and maximal for this property (i.e., any addition of a new state in \mathcal{C} makes this property false).

A class is either *transient* or *terminal*. It is transient if there are states $S \in \mathcal{C}$ and $T \notin \mathcal{C}$ such that $(S, T) \in E$ (we call it an *outgoing arc*). Therefore, a terminal class has no outgoing arc, and therefore the process always converges to one of the terminal classes. If a terminal class is reduced to a single state, we call it a *terminal state*. A terminal class \mathcal{C} is *periodic* if there exists a partition $\{P_1, \dots, P_q\}$ of \mathcal{C} such that for $r = 1, \dots, q$, if the process is in a state belonging to P_r at time t , it will be in a state of P_{r+1} (letting $P_{q+1} = P_1$) at time $t + 1$. Otherwise, \mathcal{C} is said to be *aperiodic*.

The qualitative analysis of convergence therefore amounts to finding all terminal classes and terminal states, as well as their type (periodic or aperiodic).

Definition 2. We say that two influence processes are *qualitatively equivalent* if they have the same transition graphs (equivalently, the same reduced matrices), i.e., if they converge to the same terminal classes.

The influence process is a very general model, whose only restrictions are the Markovian and stationarity assumptions. The price to pay for this generality is the exponential complexity of the model: the transition matrix has size $2^n \times 2^n$, which makes it usable only for a small number of agents. This motivates the search for subfamilies of polynomial complexity, yet enough general to cover most of real situations. We propose here such a family, whose basic idea is very simple. It assumes that each agent modifies his opinion by aggregating the current opinion of all agents, possibly including himself. The precise way of aggregating opinions is characteristic to each agent, so that agents may have all different procedures for aggregating. The aggregation procedure is numerical, coding ‘yes’ by 1 and ‘no’ by 0. The result of aggregation is a number between 0 and 1, representing the probability that the considered agent says ‘yes’. For example, the simplest procedure of aggregation is to count the number of ‘yes’-agents and to divide by n : the more agents

²also called a *strongly connected component*.

say ‘yes’, the more you are inclined to say ‘yes’. One can also make a weighted count, putting weights on agents³, or imagine any kind of procedure, provided it is rational in the following sense: take S, S' two sets of ‘yes’-agents, and suppose that $S \subseteq S'$. Then the probability to say ‘yes’ for the S' situation should be at least equal to the probability for the S situation. This assumption supposes that influence is “positive”, that is, agents tend to follow the trend. One can consider as well “negative” influence, where agents modify their opinion in reaction to the opinion of the others: the more agents say ‘yes’, the more they are inclined to say ‘no’. In this case, just the opposite assumption on the aggregation procedure must be taken. In the rest of the paper, we deal only with positive influence.

Definition 3. An n -place aggregation function is any mapping $A : [0, 1]^n \rightarrow [0, 1]$ satisfying

- (i) $A(0, \dots, 0) = 0, A(1, \dots, 1) = 1$ (boundary conditions)
- (ii) If $\mathbf{x} \leq \mathbf{x}'$ then $A(\mathbf{x}) \leq A(\mathbf{x}')$ (nondecreasingness).

Aggregation functions are well-studied and there exist many families of them: all kinds of means (geometric, harmonic, quasi-arithmetic) and their weighted version, weighted ordered averages, any combination of minimum and maximum (lattice polynomials or Sugeno integrals), Choquet integrals, triangular norms, copulas, etc. (see Grabisch et al. (2009) and Section 5).

To each agent $i \in N$ we associate an aggregation function A_i , specifying the way agent i modifies his opinion from the opinion of the other agents and himself. We denote by $\mathbf{A} := (A_1, \dots, A_n)$ the vector of aggregation functions. Specifically, supposing that S is the set of agents saying ‘yes’, we compute $\mathbf{A}(1_S) = (A_1(1_S), \dots, A_n(1_S))$, where 1_S is the characteristic vector of S , i.e., $(1_S)_j = 1$ if $j \in S$ and $(1_S)_j = 0$ otherwise. Vector $\mathbf{A}(1_S)$ indicates the probability of each agent to say ‘yes’ after influence. In particular, if we assume that these probabilities are independent among agents, then the probability of transition from the yes-coalition S to the yes-coalition T is

$$b_{S,T} = \prod_{i \in T} A_i(1_S) \prod_{i \notin T} (1 - A_i(1_S)), \quad (1)$$

which determines \mathbf{B} . It follows that deterministic models correspond to aggregation functions satisfying $A_i(1_S) \in \{0, 1\}$ for all $i \in N$ (let us call them *Boolean aggregation functions*). An important remark is that the reduced matrix $\tilde{\mathbf{B}}$ is insensitive to possible correlation among agents. Indeed, $\tilde{b}_{S,T} = 1$ if and only if $A_i(1_S) > 0$ for all $i \in T$ and < 1 for all $i \notin T$, regardless of the correlation among agents.

2.2 Examples of aggregating opinions in the advice network

In order to illustrate the above notions, we consider an example of an advice network inspired by the example of Krackhardt (1987). There are n managers in a company

³These weights reflect to which extent an agent takes into account the opinion of the others. For example, in models of contagion (Morris, 2000), or diffusion (López-Pintado, 2010), each agent is only influenced by his neighbors. Note also that if two agents cannot communicate, this can be represented by zero weights.

who discuss whether to introduce a new technology in the company. The discussion takes many rounds. The initial opinions of all managers are common knowledge. How do managers form their opinion during the discussion? Do they change their opinion in different rounds? Are they influenced by other managers? Do they reach a consensus? To which terminal class does the influence process converge? Let us start with considering three very natural ways of forming an opinion. The full example of Krackhardt will be studied in Section 6.

Example 1 (The guru influence function). Let $\tilde{k} \in N$ be a particular manager called the guru, who has the property that every manager always follows the opinion of the guru. Therefore, the aggregation functions of all managers are identical and given by $A_i(1_S) = 1$ if $S \ni \tilde{k}$, and 0 otherwise.

The convergence of the process is extremely simple: it converges in one step to either N or \emptyset , depending whether the opinion of the guru is ‘yes’ or ‘no’.

Example 2 (The majority influence function). One of the natural ways of making a decision in an influence environment is to decide according to the inclination of the majority. In other words, if the majority of managers has a ‘yes’ inclination, then all managers decide ‘yes’, and if not, then all managers decide ‘no’. Let $n \geq q > \lfloor \frac{n}{2} \rfloor$ represent the majority threshold. Then, all managers have the same aggregation function which is Boolean: $A_i(1_S) = 1$ if $|S| \geq q$, and 0 otherwise.

Obviously, \emptyset and N are terminal states, and the convergence is also reached in one step.

Example 3 (The mass psychology influence process). Let ε denote either ‘yes’ or ‘no’, and $\bar{\varepsilon}$ denote the opposite of ε . Mass psychology influence (also called herd behavior, see Banerjee (1992)) means that if there is a sufficient number of managers with opinion ε , they will possibly attract some managers with inclination $\bar{\varepsilon}$ and make them change their opinion. Let $n \geq q > \lfloor \frac{n}{2} \rfloor$. The *mass psychology function* $\text{Mass}^{[q]}$ (Grabisch and Rusinowska, 2010) satisfies

$$\text{if } |S| \geq q, \text{ then } \text{Mass}^{[q]}(S) \supseteq S, \text{ and if } |N \setminus S| \geq q, \text{ then } \text{Mass}^{[q]}(S) \subseteq S.$$

An example of the stochastic version of this mass psychology function (with uniform distribution) ($n = 3, q = 2$) is given by the following transition matrix:

$$\text{Mass}^{[2]} = \begin{matrix} & \emptyset & 1 & 2 & 12 & 3 & 13 & 23 & 123 \\ \emptyset & \left[\begin{array}{ccccccccc} 1 & & & & & & & & \\ 0.5 & 0.5 & & & & & & & \\ 0.5 & & 0.5 & & & & & & \\ 12 & & & 0.5 & & & & & 0.5 \\ 3 & 0.5 & & & 0.5 & & & & \\ 13 & & & & & 0.5 & & & 0.5 \\ 23 & & & & & & 0.5 & 0.5 & \\ 123 & & & & & & & & 1 \end{array} \right] & \end{matrix} \quad (2)$$

where a “blank” entry means zero. The associated transition graph is given in Figure 1. Again, \emptyset and 123 are terminal states. For the mass psychology influence process in

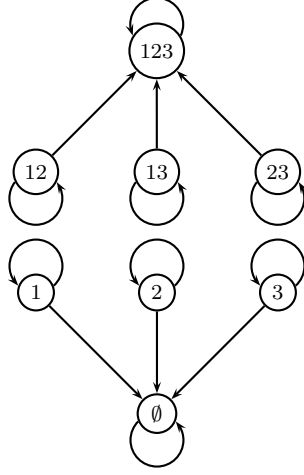


Figure 1: The transition graph of the mass psychology influence process $\text{Mass}^{[2]}$ for $n = 3$

the general case, with $n > 3$ and $q > \lfloor \frac{n}{2} \rfloor$, the following is easy to establish: If the initial state S_0 satisfies $|S_0| \geq q$, then it converges to N with probability 1 (under some mild conditions on the transition matrix). If $|S_0| \leq n - q$, then it converges to \emptyset with probability 1.

Suppose that for each situation ($S \subseteq N$, $\varepsilon = \text{'yes'}$ or 'no'), the probability $p_i^{S,\varepsilon}$ that manager $i \in N \setminus S$ changes his opinion is specified, and that managers in $N \setminus S$ change independently their opinion. Then this is equivalent to an aggregation model defined as follows, for every $S \subseteq N$:

$$A_i(1_S) = \begin{cases} 1, & \text{if } i \in S \text{ and } |N \setminus S| < q \\ p_i^{S,\text{yes}}, & \text{if } i \in N \setminus S \text{ and } |S| \geq q \\ 0, & \text{if } i \in N \setminus S \text{ and } |S| < q \\ p_i^{N \setminus S,\text{no}}, & \text{if } i \in S \text{ and } |N \setminus S| \geq q. \end{cases}$$

The transition matrix (2) is then recovered as follows:

$$\begin{array}{lll} A_1(1\ 0\ 0) = 0.5, & A_2(1\ 0\ 0) = 0, & A_3(1\ 0\ 0) = 0 \\ A_1(0\ 1\ 0) = 0, & A_2(0\ 1\ 0) = 0.5, & A_3(0\ 1\ 0) = 0 \\ A_1(0\ 0\ 1) = 0, & A_2(0\ 0\ 1) = 0, & A_3(0\ 0\ 1) = 0.5 \\ A_1(1\ 1\ 0) = 1, & A_2(1\ 1\ 0) = 1, & A_3(1\ 1\ 0) = 0.5 \\ A_1(1\ 0\ 1) = 1, & A_2(1\ 0\ 1) = 0.5, & A_3(1\ 0\ 1) = 1 \\ A_1(0\ 1\ 1) = 0.5, & A_2(0\ 1\ 1) = 1, & A_3(0\ 1\ 1) = 1. \end{array}$$

3 Influential coalitions and hypergraphs of influence

3.1 Influential players and influential coalitions

The following notions are fundamental in our analysis of convergence.

Definition 4. Consider an influence model based on aggregation functions $\mathbf{A} = (A_1, \dots, A_n)$.

- (i) Agent $j \in N$ is *yes-influential* in A_i if $A_i(1_j) > 0$.
- (ii) Agent $j \in N$ is *no-influential* in A_i if $A_i(1_{N \setminus j}) < 1$.
- (iii) The *graph of yes-influence* is a directed graph $G_{\mathbf{A}}^{\text{yes}} = (N, E)$ whose set of nodes is N , and there is an arc (j, i) from j to i if j is yes-influential in A_i . The *graph of no-influence* $G_{\mathbf{A}}^{\text{no}}$ is defined similarly.

We comment on these definitions. Roughly speaking, an agent j is (yes- or no-) influential for agent i if the opinion of j matters for i . Indeed, supposing $i \neq j$, even if everybody but agent j says ‘no’, there is a positive probability that agent i changes his mind due to the influence of j (and similarly if ‘yes’ is replaced by ‘no’, since the vector $1_{N \setminus j}$ depicts a situation where every agent says ‘yes’, except agent j). Note that the monotonicity of aggregation functions entails that when j is yes-influential in A_i , $A_i(1_S) > 0$ whenever $S \ni j$, and $A_i(1_S) < 1$ if $S \not\ni j$ when j is no-influential. Also, if all A_i are increasing functions, then the graphs of yes- and no-influence coincide and are complete with a loop on each node.

The graphs of influence give then a clear view of who influences whom. They give a formal and simple definition of a notion often used in the literature⁴. Figure 2 gives the yes-influential graph of our three examples (the no-influential graph is the same).

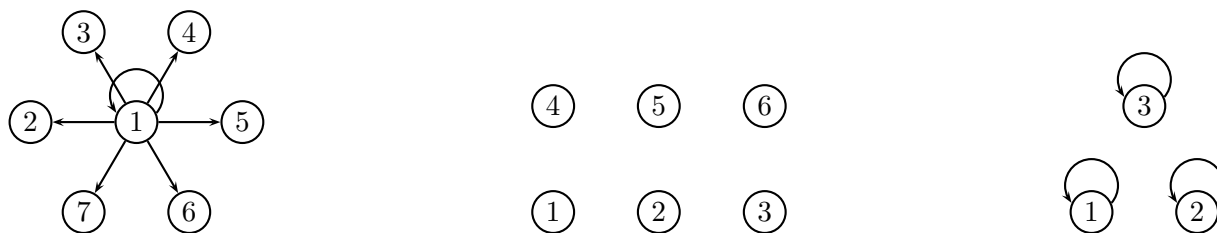


Figure 2: The graph of yes-influence of the guru influence function (left; $n = 7$, agent 1 is the guru), the majority influence function (middle, $n = 6$), and the mass psychology influence process (right, $n = 3$, corresponds to the transition matrix (2))

The graph of the guru influence function is a star, showing clearly the rôle of the guru. By contrast, the two other graphs do not reveal anything clear on the influence. This is because no particular agent is really influential in these models. Influence is done only by means of the number of people having the same opinion. This shows that graphs of influence, though convenient and intuitive, cannot explain all phenomena of influence, and that we need a more powerful concept.

Definition 5. Let A_i be the aggregation function of agent i . A nonempty coalition $S \subseteq N$ is *yes-influential* for i if

- (i) $A_i(1_S) > 0$
- (ii) For all $S' \subset S$, $A_i(1_{S'}) = 0$.

⁴Aracena et al. (2004) define the *connection graph*, which is very close to ours: there is an arc (i, j) if the (Boolean) aggregation function A_j depends on the input of i .

Similarly, a coalition S is *no-influential* for i if

- (i) $A_i(1_{N \setminus S}) < 1$
- (ii) For all $S' \subset S$, $A_i(1_{N \setminus S'}) = 1$.

Call $\mathcal{C}_i^{\text{yes}}$ and $\mathcal{C}_i^{\text{no}}$ the collections of yes- and no-influential coalitions for i . Note that these collections are never empty because if they do not contain any proper subset of N , they must contain N by virtue of the conditions $A_i(1_N) = 1$ and $A_i(1_\emptyset) = 0$. Also these collections are antichains, i.e., any two sets in $\mathcal{C}_i^{\text{yes}}$ or $\mathcal{C}_i^{\text{no}}$ are incomparable w.r.t. set inclusion.

Influential coalitions clearly generalize influential agents⁵. Coalition S is yes-influential for agent i if, when agents in S say ‘yes’ and every other agent says ‘no’, agent i has a positive probability to say ‘yes’ (and similarly for no-influential coalitions). Condition (ii) in each definition merely says that no subcoalition of S satisfies the first condition, which means that S has no superfluous agent.

It seems that there is no easy way to have a graphical representation of influential coalitions. Theoretical results on convergence in Section 4 will show that the notion of hypergraph is the most natural one, although admittedly difficult to visualize.

Definition 6. We define the following concepts:

- (i) A *hypergraph* (Berge, 1976) H is a pair (N, \mathcal{E}) where N is the set of nodes and \mathcal{E} the set of *hyperedges*, where an hyperedge $S \in \mathcal{E}$ is a nonempty subset of N . Note that if $|S| = 2$ for all $S \in \mathcal{E}$, then we have a classical graph.
- (ii) A *directed hypergraph* on N is a hypergraph on N where each hyperedge S is an ordered pair (S', S'') (which we call an *hyperarc* from S' to S''), with S', S'' being nonempty and $S' \cup S'' = S$. If in addition $S' \cap S'' = \emptyset$, the hyperarc is *normal*.
- (iii) A *directed hyperpath* from i to j is a sequence $i_0 S_1 i_1 S_2 i_2 \cdots i_{q-1} S_q i_q$, where $i_0 := i, i_1, \dots, i_{q-1}, j := i_q$ are nodes, $S_1 = (S'_1, S''_1), \dots, S_q = (S'_q, S''_q)$ are hyperarcs such that $S'_k \ni i_{k-1}$ and $S''_k \ni i_k$ for all $k = 1, \dots, q$. A *directed hypercycle* is a directed hyperpath with $i_0 = i_q$.
- (iv) The hypergraph is *strongly connected* if each pair of nodes i, j is connected through directed hyperpaths from i to j , and from j to i .
- (v) For any hypergraph H we define its restricted version \hat{H} by removing all nonnormal hyperarcs.
- (vi) Given an influence model based on aggregation functions \mathbf{A} , we define the *hypergraphs* $H_{\mathbf{A}}^{\text{yes}}, H_{\mathbf{A}}^{\text{no}}$ of *yes-influence* and *no-influence* as follows. For $H_{\mathbf{A}}^{\text{yes}}$, the set of nodes is N , and there is an hyperarc $(C, \{i\})$ for each $C \in \mathcal{C}_i^{\text{yes}}$ (similarly for $H_{\mathbf{A}}^{\text{no}}$).

⁵Influential coalitions give a qualitative description of influence. For having a quantitative point of view, we may consider the quantities $A_i(1_S)$ (probability that i says ‘yes’ if only S says ‘yes’) and $1 - A_i(1_{N \setminus S})$ (probability that i says ‘no’ if only S says ‘no’) as yes- and no-degree of influence of S over i , or generalize the notion of influence index proposed in Grabisch and Rusinowska (2010).

Since $\mathcal{C}_i^{\text{yes}}$ and $\mathcal{C}_i^{\text{no}}$ are antichains, there is no hyperarc included into another one, but the hyperarcs are not necessarily normal. Clearly, hypergraphs of influence generalize graphs of influence in the sense that if influential coalitions reduce to singletons (influential agents), then the hypergraphs reduce to graphs of influence. Note also that because of condition (ii) in Definition 5, in any case, the hypergraphs of influence *contain* their respective graphs of influence.

We illustrate these notions considering our previous examples. We find easily that:

- (i) For the guru function, only the guru is influential, and there is no other influential coalition. Therefore, the hypergraphs of influence reduce to the graphs of influence.
- (ii) For the majority function, each collection $\mathcal{C}_i^{\text{yes}}$ contains all coalitions of exactly q agents, while $\mathcal{C}_i^{\text{no}}$ contains all coalitions of exactly $n - q + 1$ agents.
- (iii) For the mass psychology influence process with $n = 3$ and transition matrix (2), we have $\mathcal{C}_1^{\text{yes}} = \mathcal{C}_1^{\text{no}} = \{1, 23\}$, $\mathcal{C}_2^{\text{yes}} = \mathcal{C}_2^{\text{no}} = \{2, 13\}$ and $\mathcal{C}_3^{\text{yes}} = \mathcal{C}_3^{\text{no}} = \{3, 12\}$. Hence, the yes- and no-influential hypergraphs are identical (see Figure 3).

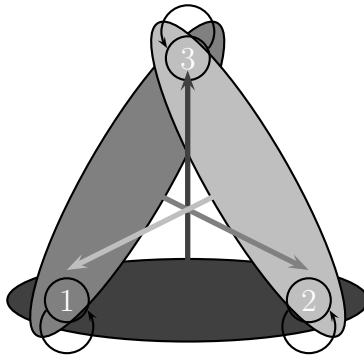


Figure 3: Hypergraph $H_{\mathbf{A}}^{\text{yes}}$ for the mass psychology influence process with transition matrix (2)

3.2 Equivalence between influence hypergraphs and the matrix $\tilde{\mathbf{B}}$

A fundamental question is: To what extent do the influence hypergraphs represent the Markov process of influence? The size of the transition matrix is 2^{2^n} , while the definition of the hypergraphs amounts to list $2n$ antichains, of size at most $\binom{n}{\lfloor \frac{n}{2} \rfloor}$. Table 1 shows

$n =$	2	3	4	5	6	7	8
size of the transition matrix	16	64	256	1024	4096	16384	65536
maximal size of the hypergraphs	8	18	48	100	240	490	1120

Table 1: Comparison of the sizes of the models

clearly that hypergraphs take much less space, so that we may be inclined to think that hypergraphs represent poorly the process of influence. Surprisingly, the next result shows that hypergraphs are equivalent to the reduced matrix $\tilde{\mathbf{B}}$, and therefore contain the entire qualitative description of the convergence.

Theorem 1. Consider an influence process \mathbf{B} based on aggregation functions \mathbf{A} . Then $\tilde{\mathbf{B}}$ can be reconstructed from the hypergraphs $H_{\mathbf{A}}^{\text{yes}}$ and $H_{\mathbf{A}}^{\text{no}}$ as follows: for any $S, T \in 2^N$, $\tilde{b}_{S,T} = 1$ if and only if

- (i) For each $i \in T$, there exists a nonempty $S'_i \subseteq S$ such that S'_i is yes-influential on i , i.e., $S'_i \in \mathcal{C}_i^{\text{yes}}$; and
- (ii) For each $i \notin T$, there exists a nonempty S''_i such that $S''_i \cap S = \emptyset$ and S''_i is no-influential on i , i.e., $S''_i \in \mathcal{C}_i^{\text{no}}$.

In particular, $\tilde{b}_{\emptyset,T} = 0$ for all $T \neq \emptyset$, $b_{\emptyset,\emptyset} = 1$, and $\tilde{b}_{N,T} = 0$ for all $T \neq N$, $b_{N,N} = 1$.

Proof. Observe from (1) that $b_{S,T} > 0$ if and only if $A_i(1_S) > 0$ for all $i \in T$ and $A_i(1_S) < 1$ for all $i \notin T$, and this remains true even if independence is not assumed. Now by monotonicity of the aggregation functions, $A_i(1_S) > 0$ is equivalent to say that there exists a yes-influential coalition S'_i for i , which is included in S . On the other hand, $A_i(1_S) < 1$ means that there exists some coalition $S'_i \neq N$ containing S such that $N \setminus S'_i$ is no-influential on i . Lastly, note that $(N \setminus S'_i) \cap S = \emptyset$, and letting $S''_i = N \setminus S'_i$ the proof is complete. \square

The main consequence of this result is that hypergraphs (or equivalently, the collections $\mathcal{C}_i^{\text{yes}}, \mathcal{C}_i^{\text{no}}$, $i \in N$) constitute a very efficient representation of the influence process, able to fully describe the qualitative convergence. Moreover, this description is valid even if statistical independence between the players does not hold.

We end this section by noting that if \mathbf{A} is Boolean (i.e., all aggregation functions are 0-1-valued), then the collections $\mathcal{C}_i^{\text{yes}}, \forall i \in N$, entirely determine \mathbf{A} (and therefore the transition matrix \mathbf{B} if the correlation among players is known). Indeed, $A_i(1_S) = 1$ if and only if $S \supseteq C$ for some $C \in \mathcal{C}_i^{\text{yes}}$, and is 0 otherwise. The same conclusion holds for the collections $\mathcal{C}_i^{\text{no}}, \forall i \in N$.

4 Convergence in the aggregation model

We provide here a general analysis of convergence of influence processes based on aggregation functions.

4.1 Types of terminal classes

In all examples given above, we can see that the process converges to the *consensus states* N or \emptyset , i.e., the states where all managers have the same opinion. It is easy to see that the two consensus states are *always* terminal states, even if other terminal classes may exist. Indeed, for every aggregation function $A_i(1_N) = 1$ and $A_i(0, \dots, 0) = 0$, hence $(1, \dots, 1)$ and $(0, \dots, 0)$ are fixed points of \mathbf{A} . Therefore, we call \emptyset and N *trivial terminal states*. Of course, other terminal classes are possible, as shown by the next theorem.

Theorem 2. Consider an influence process \mathbf{B} based on aggregation functions \mathbf{A} . Then terminal classes are:

- (i) either singletons $\{S\}$, $S \in 2^N$;
- (ii) or cycles of nonempty sets $\{S_1, \dots, S_k\}$ of any length $2 \leq k \leq \binom{n}{\lfloor n/2 \rfloor}$ (and therefore they are periodic of period k) with the condition that all sets are pairwise incomparable (by inclusion);
- (iii) or collections \mathcal{C} of nonempty sets with the property that $\mathcal{C} = \mathcal{C}_1 \cup \dots \cup \mathcal{C}_p$, where each subcollection \mathcal{C}_j is a Boolean lattice $[S_j, S_j \cup K_j]$, $S_j \neq \emptyset$, $S_j \cup K_j \neq N$, and at least one K_j is nonempty.

We call *cyclic terminal classes* those terminal classes of the second type and *regular terminal classes* those of the third type. Regular terminal classes can be periodic (see Example 5 below). Regular terminal classes formed by a single Boolean lattice $[S, S \cup K]$ are called *Boolean terminal classes*.

(see proof in the appendix)

We give some comments on this result, beginning with a simple corollary.

Corollary 1. If \mathbf{A} is Boolean (i.e., valued in $\{0, 1\}^n$), then terminal classes are either terminal states or cycles.

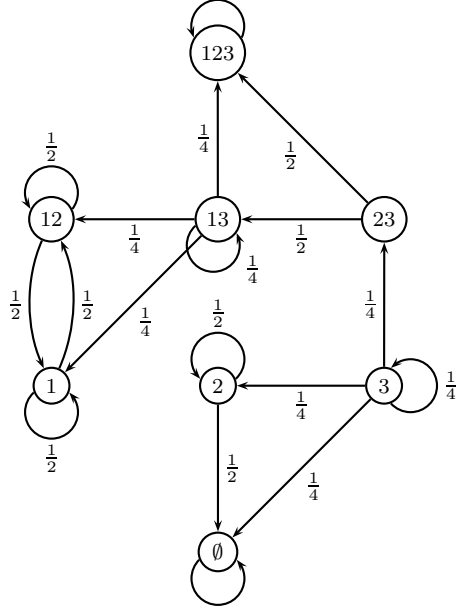
Proof. Let \mathbf{A} be Boolean and consider a terminal class $\{S_1, \dots, S_k\}$ with $k > 1$. Since every transition is deterministic, the only possibility is that there exists a cycle passing through each set of the class, hence the result. \square

Example 4 shows the existence of regular terminal classes.

Example 4. Consider $N = \{1, 2, 3\}$ and the following aggregation functions:

$$\begin{aligned}
 A_1(1\ 0\ 0) &= 1 & A_2(1\ 0\ 0) &= 0.5 & A_3(1\ 0\ 0) &= 0 \\
 A_1(0\ 1\ 0) &= 0 & A_2(0\ 1\ 0) &= 0.5 & A_3(0\ 1\ 0) &= 0 \\
 A_1(0\ 0\ 1) &= 0 & A_2(0\ 0\ 1) &= 0.5 & A_3(0\ 0\ 1) &= 0.5 \\
 A_1(1\ 1\ 0) &= 1 & A_2(1\ 1\ 0) &= 0.5 & A_3(1\ 1\ 0) &= 0 \\
 A_1(1\ 0\ 1) &= 1 & A_2(1\ 0\ 1) &= 0.5 & A_3(1\ 0\ 1) &= 0.5 \\
 A_1(0\ 1\ 1) &= 1 & A_2(0\ 1\ 1) &= 0.5 & A_3(0\ 1\ 1) &= 1.
 \end{aligned}$$

This gives the following digraph for the Markov chain:



Clearly, $\{1, 12\}$ is a regular terminal class.

The existence of cyclic terminal classes was already remarked by Aracena et al. (2004) for the case of regular aggregation functions, together with the condition of incomparability among sets in the cycle. We know by Sperner's lemma that the longest possible sequence of incomparable sets in 2^N has length $\binom{n}{\lfloor n/2 \rfloor}$, hence the upper limit of length.

Cyclic classes are not the only case of periodic terminal classes, as the following example shows.

Example 5. Consider $N = \{1, 2, 3\}$ and the following aggregation functions:

$$\mathbf{A}(100) = \mathbf{A}(110) = (0x1)$$

$$\mathbf{A}(001) = \mathbf{A}(011) = (1x0)$$

$$\mathbf{A}(010) = \mathbf{A}(000) = (000)$$

$$\mathbf{A}(101) = \mathbf{A}(111) = (111)$$

with arbitrary $0 < x < 1$. Then $\{1, 3, 12, 23\}$ forms a periodic terminal class of period 2 (see Figure 4, left). Now consider the following aggregation functions:

$$\mathbf{A}(100) = \mathbf{A}(110) = (001)$$

$$\mathbf{A}(001) = (1x0)$$

$$\mathbf{A}(010) = \mathbf{A}(000) = (000)$$

$$\mathbf{A}(101) = \mathbf{A}(011) = \mathbf{A}(111) = (111)$$

with arbitrary $0 < x < 1$. Then $\{1, 3, 12\}$ is a periodic class of period 2 with 3 sets (see Figure 4, right).

Generally speaking, to construct a periodic regular terminal class of period k , one needs k pairwise disjoint subcollections $[S_j, S_j \cup K_j]$, $j = 1, \dots, k$, with the usual restrictions on the S_j, K_j 's, and defines $A_i(1_S) = 1$ if $i \in S_{j+1}$, $A_i(1_S) = x \in]0, 1[$ if $i \in K_{j+1}$, and 0 otherwise, for all $S \in [S_j, S_j \cup K_j]$, identifying $k + 1$ with 1. Note that there is no need to have the K_j 's of equal size (see Figure 4, right).

Theorem 2 shows the different kinds of terminal classes. We study each type separately in the forthcoming sections.

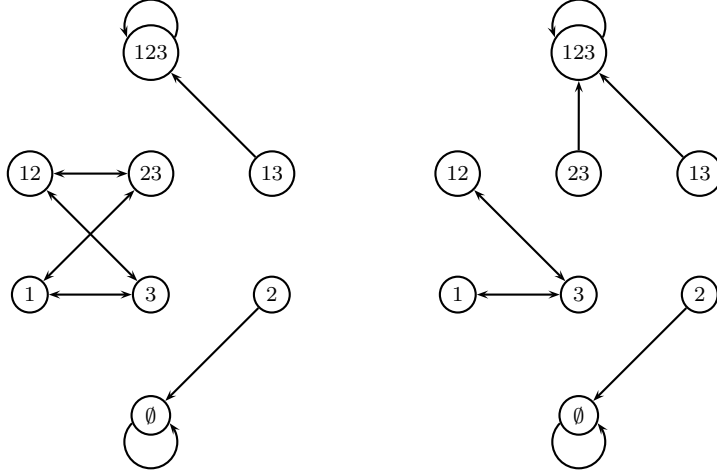


Figure 4: Examples of a periodic terminal class

4.2 Terminal states

Terminal states are relatively easy to study since these are fixed points of \mathbf{A} , i.e., S is a terminal state if and only if

$$A_i(1_S) = 1 \quad \forall i \in S, \quad A_i(1_S) = 0 \text{ otherwise.} \quad (3)$$

An interesting question is whether other terminal states than the trivial ones exist. Indeed, trivial terminal states are the only possible consensus stable situations. If another terminal state $S \neq \emptyset, N$ exists, it means that eventually the society of agents will be cut into two parts S and $N \setminus S$ of opposite opinions. It is therefore fundamental to know under which conditions this situation can arise.

We say that a coalition S has an ingoing hyperarc⁶ $T = (T', T'')$ in some hypergraph if $T' \subseteq N \setminus S$ and $T'' \subseteq S$ (and *vice versa* for outgoing).

Theorem 3. Consider an influence process \mathbf{B} based on aggregation functions \mathbf{A} . A nonempty subset $S \subset N$ is a (non trivial) terminal state if and only if it has no ingoing hyperarc in the hypergraph $(\hat{H}_{\mathbf{A}}^{\text{yes}})^* \cup \hat{H}_{\mathbf{A}}^{\text{no}}$, where $(\hat{H}_{\mathbf{A}}^{\text{yes}})^*$ is the hypergraph $\hat{H}_{\mathbf{A}}^{\text{yes}}$ with all hyperarcs inverted.

Corollary 2. Consider an influence process \mathbf{B} based on aggregation functions \mathbf{A} . The following holds.

- (i) If the hypergraph $(\hat{H}_{\mathbf{A}}^{\text{yes}})^* \cup \hat{H}_{\mathbf{A}}^{\text{no}}$ is not strongly connected, then there exists a nontrivial terminal state.
- (ii) If the graph $(G_{\mathbf{A}}^{\text{yes}})^* \cup G_{\mathbf{A}}^{\text{no}}$ is strongly connected, then there is no nontrivial terminal state.
- (iii) Suppose the hypergraphs $H_{\mathbf{A}}^{\text{yes}}, H_{\mathbf{A}}^{\text{no}}$ reduce to ordinary graphs. Then $(G_{\mathbf{A}}^{\text{yes}})^* \cup G_{\mathbf{A}}^{\text{no}}$ is strongly connected if and only if there is no nontrivial terminal state.

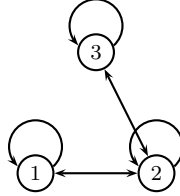
⁶Note that this definition makes sense only for normal hyperarcs.

(see proofs in the appendix)

Note that the converse of (i) is not true in general, but (ii) says that the converse is true for the “graph part” of $(\hat{H}_{\mathbf{A}}^{\text{yes}})^* \cup \hat{H}_{\mathbf{A}}^{\text{no}}$. The combination of these two results yields immediately (iii).

Let us apply these results to our previous examples.

- (i) In the guru example, $G_{\mathbf{A}}^{\text{yes}}$ and $G_{\mathbf{A}}^{\text{no}}$ are a star centered on \tilde{k} with arcs going from \tilde{k} into all agents. Hence, $(G_{\mathbf{A}}^{\text{yes}})^* \cup G_{\mathbf{A}}^{\text{no}}$ is strongly connected, and by Corollary 2 (ii), there is no nontrivial terminal state.
- (ii) Let us show by Theorem 3 that for the majority function there is no nontrivial terminal state. Due to symmetry, it suffices to show that any set S has some ingoing arc in $\hat{H}_{\mathbf{A}}^{\text{no}}$ or some outgoing arc in $\hat{H}_{\mathbf{A}}^{\text{yes}}$. The latter is true if $|S| \geq q$ since it contains an influential set of size q . If $|S| < q$, then $|N \setminus S| > n - q$, hence $N \setminus S$ contains a no-influential set, which influences any element of S in particular.
- (iii) Theorem 3 applied to the mass psychology example (see Figure 3) easily yields that no nontrivial terminal states can exist.
- (iv) In Example 4, we see that agent 1 is yes-influential in A_1 , all agents are yes- and no-influential in A_2 , and agent 3 is yes-influential in A_3 , while agents 2,3 are no-influential in A_3 . The graph $(G_{\mathbf{A}}^{\text{yes}})^* \cup G_{\mathbf{A}}^{\text{no}}$ is depicted below.



Clearly, the graph is strongly connected. Hence by Corollary 2 (ii), there is no nontrivial terminal state.

4.3 Regular terminal classes

We turn to the study of regular terminal classes. As for nontrivial terminal states, it is important to know when they exist. Indeed, if a Boolean terminal class $[S, S \cup K]$ exists, it means that no consensus is reached, but all agents in S agree to say ‘yes’, while all agents in $N \setminus (S \cup K)$ agree to say ‘no’. The agents in K oscillate between ‘yes’ and ‘no’ in all possible ways, without ending. If there are several Boolean lattices in the class, the interpretation is more complex but similar, and depends on how exactly are the transitions between the lattices.

The following result permits to check if a strongly connected set of states (see Definition 1) is a Boolean terminal class.

Theorem 4. Consider an influence process \mathbf{B} based on aggregation functions \mathbf{A} , and let $[S, S \cup K]$ be strongly connected in the transition graph Γ , where $S \neq \emptyset, K \neq \emptyset, S \cup K \neq N$. It is a Boolean terminal class if and only if the following two conditions are satisfied:

- (i) There is no ingoing hyperarc of $\hat{H}_{\mathbf{A}}^{\text{no}}$ into S

(ii) There is no outgoing hyperarc of $\hat{H}_{\mathbf{A}}^{\text{yes}}$ from $S \cup K$.

(see proof in the appendix)

Theorem 4 shows that S forms an isolated group, receiving no no-influence, and yes-influencing only the agents in K . Hence, in some sense $S \cup K$ forms a subsociety ruled by S .

For regular classes which are not Boolean, it is possible to get results if one considers the “graph part” of the hypergraphs (i.e., influential agents). First we set some notations. A regular terminal class has the form $\mathcal{C} = \bigcup_{k=1}^p [S_k, S_k \cup K_k]$, with the restrictions on S_k, K_k as in Theorem 2. Let us introduce the *lower and upper bounds* of \mathcal{C} :

$$S_* = \bigcap_{k=1}^p S_k, \quad S^* = \bigcup_{k=1}^p (S_k \cup K_k).$$

Clearly, $[S_*, S^*] \supseteq \mathcal{C}$. Note that $S^* \setminus S_* \neq \emptyset$ since \mathcal{C} is not a singleton, but $S_* = \emptyset, S^* = N$ are possible (see Figure 4, left). If $S_* \neq \emptyset$ and $S^* \neq N$, we say that the class is *normal*.

Lemma 1. If there is a normal regular terminal class with upper and lower bounds S^*, S_* , then subset S_* has no ingoing arc in $G_{\mathbf{A}}^{\text{no}}$, and S^* has no outgoing arc in $G_{\mathbf{A}}^{\text{yes}}$.

(see proof in the appendix) From this we get a simple condition to test the existence of normal regular terminal classes. To this end, we introduce for an agent $i \in N$ its *closure* in $G_{\mathbf{A}}^{\text{no}}$, denoted by $\text{cl}(i)$, which is the set of agents which can reach i by a path in $G_{\mathbf{A}}^{\text{no}}$. By convention, $i \in \text{cl}(i)$.

Theorem 5. Consider an influence process \mathbf{B} based on aggregation functions \mathbf{A} . Then there is no normal regular terminal class if for each $i \in N$, every agent outside $\text{cl}(i)$ can be reached by a path from $\text{cl}(i)$ in $G_{\mathbf{A}}^{\text{yes}}$.

(see proof in the appendix)

There are two simple particular cases with influential agents where regular terminal classes cannot occur: when either $G_{\mathbf{A}}^{\text{yes}}$ or $G_{\mathbf{A}}^{\text{no}}$ is strongly connected (by Theorem 5), or when there exists one agent who is no-influential for all agents, including himself (by Lemma 1). $G_{\mathbf{A}}^{\text{yes}}$ or $G_{\mathbf{A}}^{\text{no}}$ strongly connected means that more or less directly, all agents are yes- or no- influential for all agents, therefore a dichotomy among agents cannot occur. Also, if one agent is influential for all the others, a consensus will finally emerge around this agent.

4.4 Cyclic terminal classes

We turn to the study of cyclic classes. Since the presence of such classes is not a desirable feature (no convergence of opinion can occur), we want to find some sufficient condition to avoid cyclic classes. We begin by giving some sufficient conditions in the general case. Although these are rather strong conditions, they are very easy to check.

Proposition 1. Consider an influence process \mathbf{B} based on aggregation functions \mathbf{A} . Then there is no cyclic class, if one of the following conditions is satisfied:

(i) There exists $j \in N$ such that A_j takes values 0,1 only for \emptyset, N .

(ii) There exists $j \in N$ such that all agents are yes- and no-influential for j .

(iii) i is yes- and no-influential for i , for all $i \in N$.

(see proof in the appendix)

We suppose now that \mathbf{A} is Boolean. The next result shows that any cycle in the transition graph induces a cycle in the hypergraph of yes-influence⁷.

Theorem 6. Let \mathbf{A} be Boolean, and $\{S_1, S_2, \dots, S_k\}$ be a cyclic class. Then there exists a hypercycle of length rk for some integer r , among agents in $S_1 \cup \dots \cup S_k$ in the yes-influence hypergraph $H_{\mathbf{A}}^{\text{yes}}$.

(see proof in the appendix)

The condition is not necessary, as shown by the example of the majority function. Indeed there are many cycles in the hypergraph of yes-influence⁸, although no cycle in the transition graph.

Corollary 3. Suppose that \mathbf{A} is Boolean. Then, there is no cyclic class if there is no cycle in $H_{\mathbf{A}}^{\text{yes}}$.

5 Symmetric decomposable models

We exhibit now an important family of aggregation functions and examine its properties.

Definition 7. An aggregation function A_i is said to be *decomposable* if all its yes- and no-influential coalitions are singletons. An aggregation function is symmetric if it has the same yes- and no-influential coalitions.

Similarly, an influence model based on aggregation functions is decomposable (resp., symmetric) if all its aggregation functions are decomposable (resp., symmetric).

Proposition 2. The family of generalized weighted means, defined by

$$M_f(x_1, \dots, x_n) = f^{-1}\left(\sum_{i=1}^n w_i f(x_i)\right), \quad (x_1, \dots, x_n) \in [0, 1]^n,$$

with $w_i \geq 0$, $i = 1, \dots, n$, $\sum_{i=1}^n w_i = 1$, and f a continuous automorphism on $[0, 1]$, is a family of decomposable symmetric aggregation functions. Moreover, an agent i is yes-influential if and only if he is no-influential if and only if $w_i > 0$.

Proof. Suppose that $f(1) = 1$ (the case $f(1) = 0$ works the same), and that $S \subset N$, $|S| > 1$ is yes-influential. Then $M_f(1_S) = f^{-1}(\sum_{i \in S} w_i) > 0$, which is equivalent to $\sum_{i \in S} w_i > 0$. On the other hand, for any $i \in S$ we have $M_f(1_i) = f^{-1}(w_i) = 0$, equivalent to $w_i = 0$, a contradiction. Similarly, one can show that no coalition of size greater than 1 is no-influential.

Now, i yes-influential iff $f^{-1}(w_i) > 0$ iff $w_i > 0$ iff $\sum_{j \neq i} w_j < 1$ iff $f^{-1}(\sum_{j \neq i} w_j) < 1$ iff i is no-influential. \square

⁷Recall from Section 3 that in the case of a Boolean \mathbf{A} , the yes-influence hypergraph determines entirely the function.

⁸For example, with $n = 3$, 12 influences 3, 23 influences 1, 13 influences 2, 12 influences 3, etc.

When $f = Id$, we recover the weighted arithmetic mean (WAM) used in the DeGroot model. Observe that Proposition 2 implies that *any* directed graph on N can be the influence graph of some WAM-based model.

Let us consider the class of *symmetric decomposable* aggregation models. We will simply write G from now on for the influence graph, and the undirected version of G will be denoted by G^0 . An arc from j to i in G will be denoted by $j \rightarrow i$. Denote by $\mathcal{SDB}(N)$ the set of 0-1 matrices being the reduced matrix of some symmetric decomposable model on N . It is useful to rewrite for this class of models the conditions in Theorem 1 since they become much simpler:

$$\tilde{b}_{S,T} = 1 \text{ if and only if } \begin{cases} \forall i \in T, & \exists j \in S \text{ such that } j \rightarrow i \text{ in } G; \text{ and} \\ \forall i \notin T, & \exists j \notin S \text{ such that } j \rightarrow i \text{ in } G. \end{cases} \quad (4)$$

The symmetry of (4) immediately implies that for every $S, T \in 2^N$,

$$\tilde{b}_{S,T} = 1 \Leftrightarrow \tilde{b}_{N \setminus S, N \setminus T} = 1. \quad (5)$$

The following result is fundamental.

Theorem 7. Any symmetric decomposable model is qualitatively equivalent to a unique WAM model. Conversely, any WAM model is qualitatively equivalent to some symmetric decomposable model.

(see proof in the appendix)

It follows that the study of terminal classes for symmetric decomposable models amounts to the study of WAM models. The existence of terminal states is easy to check by using Theorem 3 and Corollary 2: $S \neq \emptyset, N$ is a nontrivial terminal state if and only if S is a connected component in G^0 , and no nontrivial terminal state exists if and only if G^0 is connected.

The existence of regular terminal classes is more tricky. The subsequent results give some conditions for the existence of Boolean terminal classes.

Theorem 8. Let us consider a symmetric decomposable model. Consider $S, K \neq \emptyset$, $S \cup K \neq N$. Then $[S, S \cup K]$ is a Boolean terminal class if G satisfies the following six conditions:

- (i) The subgraph S has no ingoing arc;
- (ii) Each $i \in K$ has an arc $j \rightarrow i$ for some $j \in S$;
- (iii) Each $i \in S$ has an arc $j \rightarrow i$ with $j \in S$;

and conditions (i)', (ii)' and (iii)', where S is replaced by $N \setminus (S \cup K)$. Moreover, the conditions (i), (i)', (iii) and (iii)' above are also necessary. The following condition is necessary:

- (iv) For each $i \in K$, there is a path from S to i and a path from $N \setminus (S \cup K)$ to i .

(see proof in the appendix)

A configuration of the influence graph satisfying the six conditions is illustrated on Figure 5.

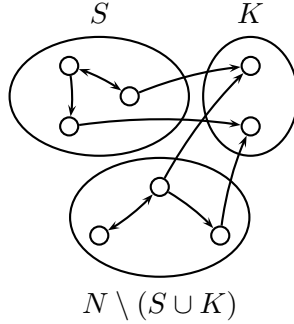


Figure 5: An example of structure of the influence graph having a Boolean terminal class

6 An empirical example

We study an empirical example based on the advice network of Krackhardt (1987) who collected data from managers of a small manufacturing firm in the US (100 employees and 21 managers) about who sought advice from whom. Based on these data, Jackson (2008) developed a social influence matrix as defined in the context of the DeGroot model (see Section 7 for a brief description). He proved that the managers converge to the same opinion in $[0, 1]$. We provide a different analysis, considering the advice network of Krackhardt as giving information on the influence between managers.

Among the different graphs given in Krackhardt (1987), we use the graph of the consensus structure, meaning that there is an arc from manager i to manager j if there is a majority of people who think that i is asking advice to j . We make the assumption that there are only influential agents and no influential coalition, that an agent i asking advice to agent j means that j influences i , and that there is no mean to distinguish from the data the yes- and no-influence. It follows that we are in a situation representable by a symmetric decomposable model, and that the influence graph is simply the graph of the consensus structure, with all arcs inverted. We may suppose that each agent takes also his own opinion into account, which means that each node of the graph has a loop (for clarity they are not figured on the graph) (see Fig. 6).

We analyze the convergence of the model using our results. First, there is no nontrivial terminal state because G^0 is connected (see Corollary 2 (iii)). Also, there is no cyclic class by the presence of loops on every node (see Proposition 1 (iii)).

Let us check if regular terminal classes may exist. We may for this use either Theorem 5 (sufficient condition for avoiding any normal regular class) or Theorem 8 to check if a given interval $[S, S \cup K]$ is a Boolean terminal class. In both cases, one has to find subgraphs in G with no ingoing arc, which amounts to construct the closure $\text{cl}(i)$ of each agent, and to find agents outside $\text{cl}(i)$ reachable from $\text{cl}(i)$ in G .

Starting with 2 (one of the nodes having the higher number of connections), 2 receives arcs from 4, 7, 14, 18 and 21. These nodes in turn receive arcs from 6, 8, 9, 10, 11, then we find 5, 12, 19, 20, and there is no new node any more. Hence $\text{cl}(2) = \{2, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 18, 19, 20, 21\}$. We find that all the remaining agents 1, 3, 13, 16, 17 receive an arc from $\text{cl}(2)$, hence the condition of Theorem 5 for $\text{cl}(2)$ is satisfied (see Figure 7). In terms of Theorem 8, setting $S = \text{cl}(2)$, we find

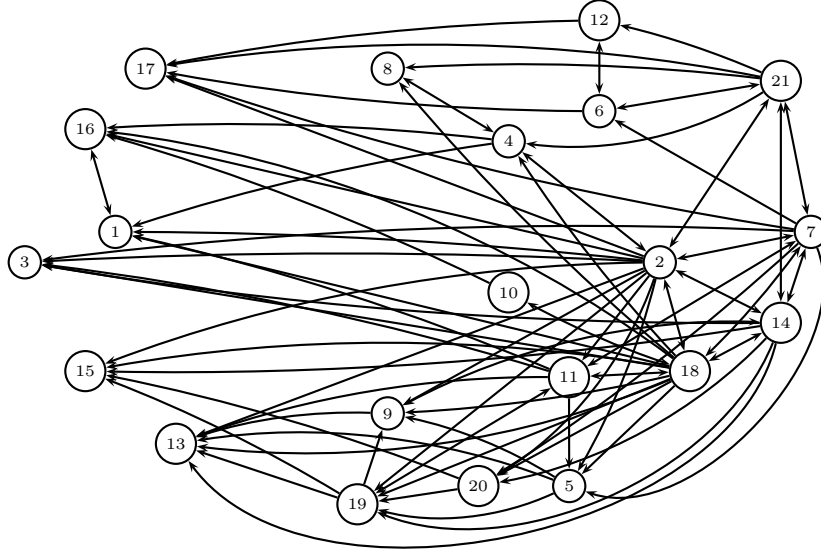


Figure 6: The graph of influence in the Krackhardt example

$K = \{1, 3, 13, 16, 17\} = N \setminus S$. It follows that $N \setminus (S \cup K) = \emptyset$, hence $[S, S \cup K]$ is not a terminal class.

Since agent 2 influences agents 1, 3, 4, 5, 7, 9, 11, 13, 14, 15, 16, 17, 18, 19, 20, 21, the closure of these agents contains $\text{cl}(2)$, hence the same conclusion is reached for these agents. For the remaining ones, we find that 6 is influenced by 7 who is influenced by 2, hence again $\text{cl}(6)$ contains $\text{cl}(2)$. Also, 8, 10 and 12 receive indirectly an influence from 2, hence their closure contain $\text{cl}(2)$. In summary, for each agent i , every agent outside $\text{cl}(i)$ receives an arc from $\text{cl}(i)$. By Theorem 5, we can conclude that no normal regular class exists.

7 Related literature

The idea of an aggregation model where each agent aggregates the opinions of the others is not that new, since it can be traced back at least to the model of DeGroot. We give a brief description of this model, pointing out the differences with our approach. In this model, the opinion of agent i at time t is a number $a_i(t) \in [0, 1]$. Each agent i aggregates the opinion of all agents by a weighted sum $\sum_j w_j^i a_j(t)$, hence the opinion vector $\mathbf{a}(t+1)$ at time $t+1$ is given by $\mathbf{a}(t+1) = \mathbf{W}\mathbf{a}(t)$, where $\mathbf{W} = [w_j^i]_{i,j \in N}$ is the matrix of weights. Since \mathbf{W} is a row-stochastic matrix, the study of convergence of the opinion relies on results on stochastic matrices, and it is proved that under mild assumptions, all agents converge to the same opinion $\alpha \in [0, 1]$. It should be noted that the DeGroot model is *not* a stochastic process since the state of opinion of each agent is purely deterministic. By contrast, our model is a stochastic process, where each agent has only two possible states of opinion (yes or no). The *probability* of saying ‘yes’ for a given agent at next step is obtained as an aggregation (in particular a weighted sum is possible) of the opinions of all agents (with ‘yes’ coded by 1, and ‘no’ by 0). The convergence of the opinions is

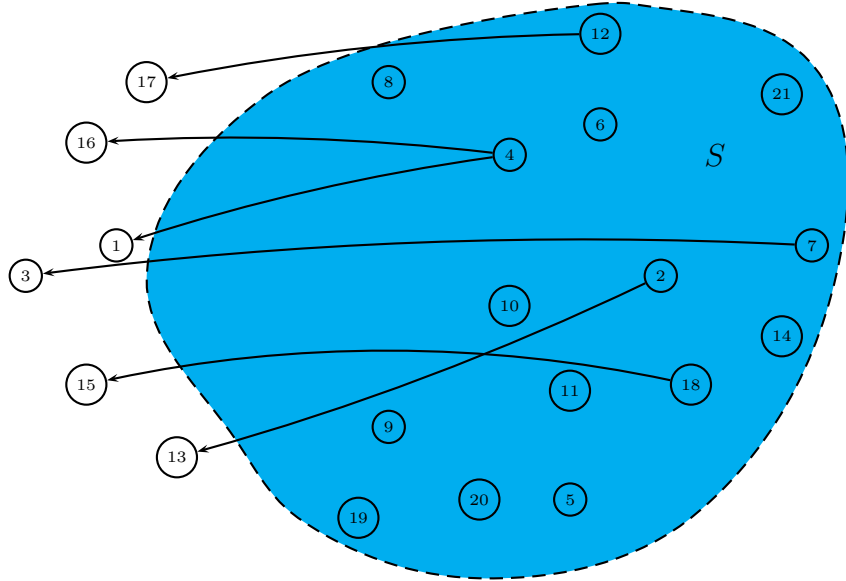


Figure 7: Closure of agent 2 (in grey). Each agent outside $cl(2)$ receives an arc from $cl(2)$.

ruled by the stochastic process, which is, in our study, a Markov chain⁹.

Although numerous works in the network literature deal with the DeGroot model and its variations, they usually assume that the aggregation of opinions is done through a convex combination; see e.g. Jackson (2008) for a survey. Our model covers many existing models of opinion formation, since we allow for arbitrary aggregation functions.

Jackson (2008) and Golub and Jackson (2010) examine a model, in which agents communicate in a social network and update their beliefs by repeatedly taking weighted averages of their neighbors' opinions. In the model of information transmission and opinion formation by DeMarzo et al. (2003) the agents in a network try to estimate some unknown parameter, which allows updating to vary over time, i.e., an agent may place more or less weight on his own belief over time. Pan (2010) proposes a variation of the DeGroot model in which the influence weight matrix is time-varying. Influence networks and the role of social influence in determining distinct collective outcomes is also examined in López-Pintado (2008, 2010); López-Pintado and Watts (2008). Calvó-Armengol and Jackson (2009) consider an overlapping-generations model which generates a Markov process. Agents representing some dynasties of a community take binary actions, and the state of the community contains the yes-no actions of all dynasties. Büchel et al. (2011) introduce a generalization of the DeGroot model and study the transmission of cultural traits from one generation to the next one. Büchel et al. (2012) analyze an influence model in which agents may misrepresent their opinion in a conforming or counter-conforming way. A model of strategic influence is studied in Galeotti and Goyal (2009).

Another stream of literature concerns diffusion through networks, like diffusion of innovation or spread of disease; see Jackson (2008) for an overview. Contagion – which

⁹Our proposal is closer to the one of Asavathiratham (2000), although there are important differences in the dynamic aspect.

occurs if an action can spread from a finite set of individuals to the whole population – is one of the phenomena related to the process of opinion formation. The model of influence based on aggregation function could be used to analyze social interactions with contagion features. However, since the present model of influence assumes that the set of agents is finite, to analyze frameworks of diffusion with infinitely many individuals, like e.g. the contagion model by Morris (2000), an analogous aggregation model of influence with an infinite set of agents should be used.

There is a numerous literature on social learning, in particular, in the context of social networks; see, e.g., Banerjee (1992); Ellison (1993); Bala and Goyal (1998, 2001); Gale and Kariv (2003); Banerjee and Fudenberg (2004). While in social learning models agents observe choices over time and update their beliefs accordingly, in the influence framework the choices depend on the influence of others.

Furthermore, we like to mention the percolation theory which describes the behavior of connected clusters in a random graph, and the Ising and Potts models in statistical mechanics. The Ising model consists of discrete variables called spins that can be in one of two states and interact at most with their nearest neighbors.

Although we are not aware of other studies in the field of influence and social networks that consider a general class of aggregation functions like in our model, it is possible to find related studies in discrete mathematics concerning mainly Boolean functions, often applied to bioinformatics. There is a remarkable paper by Aracena et al. (2004), that we have already cited above. Here, functions from $\{0, 1, \dots, m-1\}^n$ to $\{0, 1, \dots, m-1\}^n$ are considered, which corresponds in our framework to n agents having m possible answers, and the aggregation gives directly the opinions of the agents after influence (hence the case $m = 2$ is exactly our influence model with \mathbf{A} being Boolean). Aracena et al. (2004) study in particular the cyclic terminal classes, supposing some properties, like monotonicity of the aggregation functions and symmetry of the graph of influence. Rémy et al. (2008) provide also an interesting study of cyclic terminal classes when the aggregation functions are Boolean functions, with the restriction that transitions are of the type $S \rightarrow S \cup i$ or $S \rightarrow S \setminus i$, for $S \subseteq N$.

Lastly, we mention Mueller-Frank (2010) who provides a study of convergence applied to non-Bayesian learning in social networks. Here, aggregation functions which are continuous and have a special property called “constricting” are considered. Supposing a graph depicting neighboring relations among agents, A_i is constricting if $A_i(\mathbf{x})$ is comprised between the minimum and maximum of the x_j , $j \in N_i \cup i$, where N_i is the set of neighbors of i . It is shown that for such functions, the process converges to a fixed point.

All these studies consider particular cases of aggregation functions, and up to our knowledge there is no general study as the one we have undertaken in this paper.

8 Concluding remark

We like to elaborate a little bit more on the applicability of the aggregation model – the issue already mentioned in the introduction and shown in the empirical example of the advice network. Suppose one wants to model influence among a group of agents, just by observation. Our sole assumption is that each agent aggregates the opinion of the other agents plus himself to formulate his opinion, but we do not assume any knowledge about

the type of aggregation functions, nor about the kind of correlation existing among the agents. Theorem 1 tells us that the reduced matrix $\tilde{\mathbf{B}}$ (hence the qualitative analysis of convergence) can be recovered from the knowledge of the yes- and no-influential coalitions for each agent. Moreover, various theorems (3 to 6, 8) permit to check the existence of different types of terminal classes, only by the inspection of yes- and no-influential coalitions.

The aggregation model can therefore be quite useful in practical situations. Influence of agents on a given individual means nothing else than a particular way of aggregating the opinions of the agents. The only limitation of the aggregation model is that aggregation functions are monotonically increasing, therefore reactive behaviors (the more people say 'yes', the more I am inclined to say 'no') cannot be modeled in this framework. A much more challenging issue would be to consider that some agents are reactive, while the others are not. Now, it is unreasonable to assume (like it is done in the De Groot model) that the aggregation functions of the agents are known: this would mean that for each agent, the type of aggregation function (weighted average, ordered weighted average, etc.) is known, as well as the parameters of this function. Agents act unconsciously and do not have a precise idea – certainly not formalized to that level – how they are influenced. Similarly, it seems difficult to get some knowledge about the correlation between agents. On the other hand, it seems possible and much simpler to get some knowledge about who influences whom (like in the Krackhardt's example), and consequently, to determine influential coalitions.

References

- J. Aracena, J. Demongeot, and E. Goles. On limit cycles of monotone functions with symmetric connection graph. *Theoretical Computer Science*, 322:237–244, 2004.
- C. Asavathiratham. *Influence model: a tractable representation of networked Markov chains*. PhD thesis, Massachusetts Institute of Technology, Cambridge, MA, 2000.
- V. Bala and S. Goyal. Conformism and diversity under social learning. *Economic Theory*, 17:101–120, 2001.
- V. Bala and S. Goyal. Learning from neighbours. *The Review of Economic Studies*, 65(3): 595–621, 1998.
- A. V. Banerjee. A simple model of herd behavior. *Quarterly Journal of Economics*, 107(3):797–817, 1992.
- A. V. Banerjee and D. Fudenberg. Word-of-mouth learning. *Games and Economic Behavior*, 46:1–22, 2004.
- C. Berge. *Graphs and Hypergraphs*. North-Holland, Amsterdam, 2nd edition, 1976.
- B. Büchel, T. Hellmann, and M. Pichler. The dynamics of continuous cultural traits in social networks. Institute of Mathematical Economics Working Paper 457, 2011.
- B. Büchel, T. Hellmann, and S. Klößner. Opinion dynamics under conformity. Institute of Mathematical Economics Working Paper 469, 2012.

- A. Calvó-Armengol and M. O. Jackson. Like father, like son: social network externalities and parent-child correlation in behavior. *American Economic Journal: Microeconomics*, 1:124–150, 2009.
- M. H. DeGroot. Reaching a consensus. *Journal of the American Statistical Association*, 69:118–121, 1974.
- P. DeMarzo, D. Vayanos, and J. Zwiebel. Persuasion bias, social influence, and unidimensional opinions. *Quarterly Journal of Economics*, 118:909–968, 2003.
- G. Ellison. Learning, local interaction, and coordination. *Econometrica*, 61:1047–1072, 1993.
- D. Gale and S. Kariv. Bayesian learning in social networks. *Games and Economic Behavior*, 45(2):329–346, 2003.
- A. Galeotti and S. Goyal. Influencing the influencers: a theory of strategic diffusion. *The RAND Journal of Economics*, 40(3):509–532, 2009.
- B. Golub and M. O. Jackson. Naïve learning in social networks and the wisdom of crowds. *American Economic Journal: Microeconomics*, 2(1):112–149, 2010.
- M. Grabisch and A. Rusinowska. A model of influence in a social network. *Theory and Decision*, 69(1):69–96, 2010.
- M. Grabisch and A. Rusinowska. Influence functions, followers and command games. *Games and Economic Behavior*, 72(1):123–138, 2011.
- M. Grabisch, J.-L. Marichal, R. Mesiar, and E. Pap. *Aggregation Functions*. Number 127 in *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, 2009.
- X. Hu and L. S. Shapley. On authority distributions in organizations: equilibrium. *Games and Economic Behavior*, 45:132–152, 2003a.
- X. Hu and L. S. Shapley. On authority distributions in organizations: controls. *Games and Economic Behavior*, 45:153–170, 2003b.
- M. O. Jackson. *Social and Economic Networks*. Princeton University Press, 2008.
- D. Krackhardt. Cognitive social structures. *Social Networks*, 9:109–134, 1987.
- D. López-Pintado. Diffusion in complex social networks. *Games and Economic Behavior*, 62:573–590, 2008.
- D. López-Pintado. Influence networks. Working Paper Series, WP ECON 10.06, 2010.
- D. López-Pintado and D. J. Watts. Social influence, binary decisions and collective dynamics. *Rationality and Society*, 20(4):399–443, 2008.
- S. Morris. Contagion. *The Review of Economic Studies*, 67:57–78, 2000.

M. Mueller-Frank. A fixed point convergence theorem in Euclidean spaces and its application to non-Bayesian learning in social networks. <http://ssrn.com/abstract=1690925>, 2010.

Z. Pan. Trust, influence, and convergence of behavior in social networks. *Mathematical Social Sciences*, 60:69–78, 2010.

E. Rémy, P. Ruet, and D. Thieffry. Graphic requirements for multistability and attractive cycles in a Boolean dynamical framework. Working Paper, 2008.

Appendix - Proofs of theorems

Proof of Theorem 2

Throughout this section we use the convenient shorthand

$$(1_S, x_K) := (0 \cdots 0 \underbrace{1 \cdots 1}_S \underbrace{x_1 x_2 \cdots x_k}_K)$$

for vectors in $[0, 1]^n$. The following straightforward lemma is central for proving the result.

Lemma 2. From each set $S \in 2^N$, the number of possible transitions is of the form 2^k , for some $k \in \{0, 1, \dots, n\}$, where k is the number of components in $\mathbf{A}(1_S)$ different from 0 and 1. More precisely, if $\mathbf{A}(1_S) = (1_T, x_K)$ with $x_i \in]0, 1[$ for all $i \in K$, then S has a transition to any set in the Boolean lattice $[T, T \cup K] := \{S' \in 2^N \mid T \subseteq S' \subseteq T \cup K\}$.

Proof. (of Theorem 2) Consider a terminal class \mathcal{C} , and $S \in \mathcal{C}$. Unless \mathcal{C} is reduced to a single state, S cannot be empty because from the empty set there is no other transition than to itself, and so \mathcal{C} would not be a class. One and only one of the following cases can happen:

- (i) $\mathbf{A}(1_S) = 1_S$. Then S is a terminal state, i.e., $\mathcal{C} = \{S\}$.
- (ii) $\mathbf{A}(1_S) = 1_T$, $T \neq S$. There is transition from S to T with certainty. If for all sets in \mathcal{C} the transitions are certain, the only possibility is that \mathcal{C} is a cycle $S \rightarrow T \rightarrow \dots \rightarrow S$.
- (iii) $\mathbf{A}(1_S) = (1_T, x_K)$, with $x_i \in]0, 1[$ for any $i \in K$, $|K| = k$. From Lemma 2, there are 2^k transitions, which form the Boolean lattice $[T, T \cup K]$. Then necessarily, $[T, T \cup K]$ is included in \mathcal{C} , for if some set $L \in [T, T \cup K]$ does not belong to \mathcal{C} , there would be a transition from S to L , i.e., an arc outgoing from the class, contradicting that it is a terminal class.

1. The first case (terminal states) being clear, let us study the second case (cycles), and put $\mathcal{C} = \{S_1, \dots, S_k\}$. If the sequence S_1, \dots, S_k with $k \geq 2$ is a cycle, we must have:

$$\begin{aligned} \mathbf{A}(1_{S_1}) &= 1_{S_2} \\ \mathbf{A}(1_{S_2}) &= 1_{S_3} \\ &\vdots \\ \mathbf{A}(1_{S_k}) &= 1_{S_1}. \end{aligned}$$

Suppose that the vectors $1_{S_1}, \dots, 1_{S_k}$ are incomparable (i.e., no relation of inclusion occurs among the S_i 's). Then no condition due to the nondecreasingness of the A_i 's applies, and therefore there is no contradiction among the above equations. Conversely, suppose there exist S_i, S_j in the sequence such that $S_i \subseteq S_j$. By monotonicity of A this implies that $S_{i+1} = A(S_i) \subseteq A(S_j) = S_{j+1}$, etc. This causes a contradiction since we finally arrive at $S_i \subset S_j \subseteq \dots \subseteq S_i$ (letting $S_{k+1} := S_1$, etc.). By Sperner's lemma, we know that the longest possible sequence of incomparable sets has length $\binom{n}{\lfloor \frac{n}{2} \rfloor}$, hence the bound on k .

2. We suppose that \mathcal{C} is a terminal class, which is neither a cycle nor a singleton. Then there exists a set $S' \in \mathcal{C}$ with several possible transitions, i.e., $\mathbf{A}(1_{S'}) = (1_{S_1}, x_{K'_1})$, with $x_i \in]0, 1[$, for all $i \in K'_1$. We have $S_1 \neq \emptyset$, otherwise a transition to \emptyset would be possible, i.e., \mathcal{C} is not terminal. Similarly, $S_1 \cup K'_1 \neq N$, otherwise a transition to N would be possible. There may be several such S' with same S_1 but different K'_1 . Call S one of them with the largest K'_1 , and put $\mathbf{A}(1_S) = (1_{S_1}, x_{K_1})$, $|K_1| =: k_1$. Then the Boolean lattice $\mathcal{C}_1 := [S_1, S_1 \cup K_1]$ is the set of transitions from S and is included in \mathcal{C} .

2.1. Suppose that for any $T \in \mathcal{C}_1$, we have $\mathbf{A}(1_T) = (1_{S_1}, x_{K_1}^T)$, with $x_i^T \in [0, 1]$, $i \in K_1$. Then all transitions from a set of \mathcal{C}_1 remain in \mathcal{C}_1 . It follows from the assumption on \mathcal{C} that \mathcal{C}_1 is a terminal class, and therefore $\mathcal{C} = \mathcal{C}_1$. Note that monotonicity of the aggregation functions entails that for any $T, T' \in \mathcal{C}_1$,

$$T \subseteq T' \Leftrightarrow x_{K_1}^T \leq x_{K_1}^{T'},$$

i.e., the vectors $(x_{K_1}^T)_{T \in \mathcal{C}_1}$ form a Boolean lattice isomorphic to $[S_1, S_1 \cup K_1]$.

Note that the $x_{K_1}^T$ must be such that \mathcal{C}_1 is strongly connected. This is achieved, e.g., if $x_i^T \in]0, 1[$, for all $T \in \mathcal{C}_1$.

2.2 Suppose on the contrary that $\mathcal{C} \neq \mathcal{C}_1$, implying that there is some set $S' \in \mathcal{C}$ with a different transition, say $\mathbf{A}(1_{S'}) = (1_{S_2}, x_{K'_2})$, with $S_2 \not\supseteq S_1$ or $S_2 \cup K'_2 \not\subseteq S_1 \cup K_1$. Again, among all S' with same S_2 but different K'_2 , choose S with largest K'_2 and put $\mathbf{A}(1_S) = (1_{S_2}, x_{K_2})$, $|K_2| =: k_2$. Then the Boolean lattice $\mathcal{C}_2 := [S_2, S_2 \cup K_2]$ is the set of transitions from S and is included in \mathcal{C} .

Examining the transitions of all sets in \mathcal{C} we eventually conclude that $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \dots \cup \mathcal{C}_p$, where each \mathcal{C}_j is a Boolean lattice $[S_j, S_j \cup K_j]$, defined as above. Note that at least one K_j must be nonempty, otherwise each set has only one transition, and then the only way to have a class is to have a cycle, a case which is excluded here.

There are two cases concerning transitions among the subcollections \mathcal{C}_j . Let us take w.l.o.g. \mathcal{C}_1 as starting point.

2.2.1. Suppose that for all $T \in \mathcal{C}_1$ we have $\mathbf{A}(1_T) = (1_{S_2}, x_{K_2}^T)$, with $x_i^T \in [0, 1]$, $i \in K_2$, i.e., from any set of \mathcal{C}_1 there are only transitions to members of \mathcal{C}_2 . Note that in this case there is no special relation between S_1 and S_2 .

2.2.2. Suppose on the contrary that in \mathcal{C}_1 there is a set S with transition to, say, \mathcal{C}_2 , and another one T with transition to \mathcal{C}_3 . Then monotonicity relations among the sets in \mathcal{C}_1 induce monotonicity relations between S_2, S_3 and $S_2 \cup K_2, S_3 \cup K_3$.

If $S \subset T$, then monotonicity of the aggregation functions entails $\mathbf{A}(1_S) \leq \mathbf{A}(1_T)$. Supposing $\mathbf{A}(1_S) = (1_{S_2}, x_{K_2}^S)$, $x_i^S \in]0, 1[$ for all i in K_2 and similarly for T , this implies $S_2 \subseteq S_3$, $S_2 \cup K_2 \subseteq S_3 \cup K_3$ and $x_i^S \leq x_i^T$ for all i in $K_2 \cap K_3$.

If S and T are incomparable, then S_2 and S_3 can be also incomparable. In this case, for $S \cap T$ and $S \cup T$ the transitions must satisfy $\mathbf{A}(1_{S \cap T}) \leq (1_{S_2 \cap S_3}, x_{K_2 \cap K_3}^{S \cap T})$, and similarly for $\mathbf{A}(1_{S \cup T})$. Note that necessarily $S_2 \cap S_3 \neq \emptyset$ and $S_2 \cup K_2 \cup S_3 \cup K_3 \neq N$, otherwise the class would not be terminal. Also, from \mathcal{C}_1 there will be transitions to 4 different subcollections, with bottom elements S_2, S_3 , a subset of $S_2 \cap S_3$ (say, $S_{2 \wedge 3}$) and a superset of $S_2 \cup S_3$, say $S_{2 \vee 3}$, respectively. Observe that the union of these subcollections has a least and a greatest element, which are $S_{2 \wedge 3}$ and $S_{2 \vee 3}$, respectively. \square

Proof of Theorem 3 and Corollary 2

We need first a basic lemma.

Lemma 3. Take $S \subset N$, $S \neq \emptyset$. Then S cannot be a terminal state if and only if, either there exist $S' \subseteq N \setminus S$, $i \in S$ such that S' is no-influential for i , or there exist $S' \subseteq S$, $i \notin S$ such that S' is yes-influential for i .

Proof. (of the Lemma) Sufficiency: Take $S' \subseteq N \setminus S$, $i \in S$, and suppose that S' is no-influential in A_i . Then $A_i(1_S) \leq A_i(1_{N \setminus S'}) < 1$, hence S is not a terminal state by (3). Now, suppose that $S' \subseteq S$, $i \notin S$, S' yes-influential in A_i . Then $0 < A_i(1_{S'}) \leq A_i(1_S)$, which implies that S is not a terminal state.

Necessity: Suppose that S is not a terminal state. Then by (3), either there is some $i \in S$ such that $A_i(1_S) < 1$ or some $i \notin S$ such that $A_i(1_S) > 0$.

Suppose that for some $i \in S$, $A_i(1_S) < 1$. Take the largest $S' \supseteq S$ such that $A_i(1_{S'}) < 1$. Then $N \setminus S'$ is no-influential for i . Similarly, suppose now that for some $i \notin S$, $A_i(1_S) > 0$. Take the smallest $S' \subseteq S$ such that $A_i(1_{S'}) > 0$. Then S' is influential for i . \square

Proof. (of Theorem 3) By Lemma 3, we know that $S \neq \emptyset$, N is eliminated (from the set of terminal states) if either there exist $S' \subseteq N \setminus S$, $i \in S$ such that $S' \in \mathcal{C}_i^{\text{no}}$, or $S' \subseteq S$, $i \notin S$ such that $S' \in \mathcal{C}_i^{\text{yes}}$. Putting the other way round, we have established two rules $\mathcal{R}^{\text{yes}}, \mathcal{R}^{\text{no}}$:

$$\begin{aligned} \mathcal{R}^{\text{yes}} : & \quad C \in \mathcal{C}_i^{\text{yes}} \text{ rules out every coalition } S \in 2^N \text{ such that } S \supseteq C \text{ and } S \not\ni i \\ \mathcal{R}^{\text{no}} : & \quad C \in \mathcal{C}_i^{\text{no}} \text{ rules out every coalition } S \in 2^N \text{ such that } S \subseteq N \setminus C \text{ and } S \ni i. \end{aligned}$$

Note that if $C \ni i$ and $C \in \mathcal{C}_i^{\text{yes}}$, it does not rule out any coalition (same for $\mathcal{C}_i^{\text{no}}$). This shows that we can restrict to normal hyperarcs.

Take any $S \subseteq N, \emptyset$. If S is eliminated by \mathcal{R}^{no} , it means that there is a hyperarc of $\hat{H}_{\mathbf{A}}^{\text{no}}$ ingoing¹⁰ into S . Similarly, if S is eliminated by \mathcal{R}^{yes} , there is some hyperarc of $\hat{H}_{\mathbf{A}}^{\text{yes}}$ outgoing from S . Therefore, S is eliminated by \mathcal{R}^{yes} or \mathcal{R}^{no} if and only if S has

¹⁰See definition above Theorem 3.

an ingoing hyperarc in the hypergraph $H := (\hat{H}_{\mathbf{A}}^{\text{yes}})^* \cup \hat{H}_{\mathbf{A}}^{\text{no}}$. By Lemma 3, we conclude that S is not a terminal state if and only if S has an ingoing hyperarc in the hypergraph H . \square

Proof. (of Corollary 2)

- (i) For a given node $i \in N$, $R(i)$ denotes the set of nodes which can reach i by a directed hyperpath.

Suppose that H is not strongly connected. Then there exist $i, j \in N$ such that $R(i) \not\ni j$. If $R(i) = \emptyset$, it means that $\{i\}$ has no ingoing hyperarc. Otherwise we have $R(i) \neq \emptyset, N$, and by definition $R(i)$ has no ingoing hyperarc. In both cases there exists a set $S \neq \emptyset, N$ with no ingoing hyperarc.

The converse is false because the absence of ingoing hyperarcs does not mean the absence of a path.

- (ii) Suppose there exists a set $S \neq \emptyset, N$ with no ingoing hyperarc. In particular, it has no ingoing arc in the graph $G := (G_{\mathbf{A}}^{\text{yes}})^* \cup G_{\mathbf{A}}^{\text{no}}$. Clearly nodes in $N \setminus S \neq \emptyset$ are not connected to those of S in G , therefore G is not strongly connected.

- (iii) Clear from (i) and (ii).

\square

Proof of Theorems 4 and 5, and Lemma 1

Proof. (of Theorem 4) Suppose $[S, S \cup K]$ is strongly connected in Γ . By definition of an ingoing and an outgoing arc, we have to show that $[S, S \cup K]$ is a Boolean terminal class if and only if for all $S' \subseteq N \setminus S$ and $i \in S$, S' is not no-influential for i , and for all $S' \subseteq S \cup K$ and $i \notin S \cup K$, S' is not yes-influential for i (we call this condition $(*)$).

Necessity: Take $S' \subseteq N \setminus S$, $i \in S$, and suppose that S' is no-influential in A_i . Then $A_i(1_S) \leq A_i(1_{N \setminus S'}) < 1$, hence $[S, S \cup K]$ cannot be a Boolean terminal class, for any K . Now, suppose that $S' \subseteq S \cup K$, $i \notin S \cup K$, S' is yes-influential in A_i . Then $0 < A_i(1_{S'}) \leq A_i(1_{S \cup K})$, which implies that $[S, S \cup K]$ is not a terminal class.

Sufficiency: Suppose that $[S, S \cup K]$ is not a Boolean terminal class. Since $[S, S \cup K]$ is strongly connected, this means that either $[S, S \cup K]$ is not minimal for this property, or there is a possible transition outside the interval, which in fact amounts to the same. Then there exists $S' \in [S, S \cup K]$ such that either $A_i(1_{S'}) < 1$ for some $i \in S$, or $A_i(1_{S'}) > 0$ for some $i \notin S \cup K$. Suppose first that for some $i \in S$, $A_i(1_{S'}) < 1$. Take the largest $S'' \supseteq S'$ s.t. $A_i(1_{S''}) < 1$. Then $N \setminus S'' \subseteq N \setminus S$ and is no-influential for i . Now, suppose that for some $i \notin S \cup K$, $A_i(1_{S'}) > 0$. Take the smallest $S'' \subseteq S'$ such that $A_i(1_{S''}) > 0$. Then $S'' \subseteq S \cup K$ and is yes-influential for i . In both cases, $(*)$ is violated. \square

We provide a counterexample with $N = \{1, 2, 3, 4, 5\}$, showing that without supposing that $[S, S \cup K]$ is strongly connected, the theorem does not hold, i.e., conditions (i) and

(ii) alone cannot imply that the interval $[S, S \cup K]$ is strongly connected. Consider \mathbf{A} partially specified by:

$$\begin{aligned} \mathbf{A}(1_1) &= (1, x, x, 0, 0), & \mathbf{A}(1_{123}) &= (1, 1, x, x, 0) \\ \mathbf{A}(1_{12}) &= (1, 1, x, 0, 0), & \mathbf{A}(1_{124}) &= (1, 1, 1, 0, 0) \\ \mathbf{A}(1_{13}) &= (1, 1, x, 0, 0), & \mathbf{A}(1_{134}) &= (1, 1, 1, 0, 0) \\ \mathbf{A}(1_{14}) &= (1, 1, 1, 0, 0), & \mathbf{A}(1_{1234}) &= (1, 1, 1, x, 0) \end{aligned}$$

with $x > 0$. Then the interval $[1, 1234]$ satisfies conditions (i) and (ii), but it is not strongly connected in Γ (see the transition graph Γ on Figure 8).

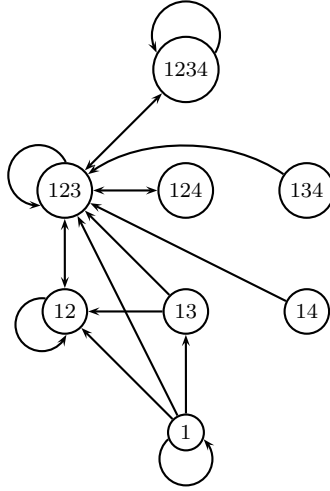


Figure 8: Transition graph of the counterexample

Proof. (of Lemma 1) Suppose that \mathcal{C} is a regular terminal class with bounds S_*, S^* . Then for all $S' \in \mathcal{C}$, we must have

$$A_i(1_{S'}) = \begin{cases} 1, & \text{if } i \in S_* \quad (*) \\ 0, & \text{if } i \notin S^*. \quad (**) \end{cases}$$

Suppose $i \in S_*$ and A_i has a no-influential player $j \in N \setminus S_*$. Take $S' \in \mathcal{C}$ such that $j \notin S'$. This is always possible since necessarily one of the S_k 's does not contain j (for if $j \in S_k$ for all $k = 1, \dots, p$, then $j \in S_*$). This implies $1 > A_i(1_{N \setminus j}) \geq A_i(1_{S'})$, which contradicts (*). Hence \mathcal{C} is not terminal.

Suppose now that $i \notin S^*$ and A_i has a yes-influential player $j \in S^*$. Take $S' \in \mathcal{C}$ such that $j \in S'$. Again this is always possible since $j \in S^*$ implies $j \in S_k \cup K_k$ for some k . Then $A_i(1_{S'}) \geq A_i(1_j) > 0$, which contradicts (**), proving that \mathcal{C} is not terminal. \square

Proof. (of Theorem 5) By applying the above lemma to every pair S_*, S^* with $S_* \neq \emptyset$, $S^* \neq N$, $S_* \subset S^*$, one can exclude any normal regular terminal class not reduced to the trivial terminal classes. Put differently, any such pair S_*, S^* must be excluded by the application of the following rules:

\mathcal{R}^{yes} : j yes-influential for i rules out every regular class with upper bound S^* , such that $i \notin S^*, j \in S^*$ (arc outgoing from S^* in the graph of yes-influence).

\mathcal{R}^{no} : j no-influential for i rules out every regular class with lower bound S_* , such that $i \in S_*, j \notin S_*$ (arc going into S_* in the graph of no-influence).

We prove that rules $\mathcal{R}^{\text{yes}}, \mathcal{R}^{\text{no}}$ rule out all normal regular classes if and only if for each agent i , every agent j outside $\text{cl}(i)$ can be reached by a yes-path.

Necessity: Suppose there exist $i, j \in N$ such that i cannot be reached by a yes-path from $\text{cl}(j)$. Consider $R(i)$, the set of nodes that can reach i by a yes-path. Note that $R(i) \cap \text{cl}(j) = \emptyset$. Then a regular terminal class with lower and upper bounds $\text{cl}(j), N \setminus R(i)$ is possible, because $\text{cl}(j)$ is not ruled out by \mathcal{R}^{no} (no ingoing no-arc), and $N \setminus R(i)$ can be ruled out only by a yes-arc going into $R(i)$, which does not exist by definition of $R(i)$.

Sufficiency: Take any normal regular class with bounds S_*, S^* , i.e., with $1 \leq |S_*| < n - 1$ and $2 \leq |S^*| < n$. If S_* has an ingoing no-arc, it is ruled out by \mathcal{R}^{no} . If it has no ingoing no-arc, then it is either a closure of some agent, or a union of closures (indeed, for each $i \in S_*$, S_* must contain $\text{cl}(i)$). By hypothesis every node outside S_* (and therefore outside S^*) is linked to S_* by a yes-path. Since $N \setminus S^*$ is nonempty by the normality assumption, taking any node i in $N \setminus S^*$, there is a yes-path from S_* to i , hence necessarily a yes-arc outgoing from S^* . Therefore, S^* is ruled out by \mathcal{R}^{yes} . \square

Proofs of Proposition 1 and Theorem 6

We need the following lemma for the proof of Proposition 1 (iii).

Lemma 4. Suppose there is a cyclic class with S_1, S_2 consecutive. Then A_i cannot have a no-influential player $j \in N \setminus S_1$ if $i \in S_2$, or a yes-influential player $j \in S_1$ if $i \notin S_2$.

Proof. (of the Lemma) Since S_2 is the successor of S_1 in the cycle, we have $A_i(1_{S_1}) = 1$ if $i \in S_2$ and 0 otherwise. Suppose that $j \in S_1$ is yes-influential in A_i with $i \notin S_2$. Then $0 < A_i(1_j) \leq A_i(1_{S_1}) = 0$, a contradiction. Similarly, if $j \notin S_1$ is no-influential in A_i with $i \in S_2$, we have $1 = A_i(1_{S_1}) \leq A_i(1_{N \setminus j}) < 1$, again a contradiction. \square

Proof. (of Proposition 1)

- (i) Because of A_j , for no set $S \neq \emptyset, N$, there can be a transition to a set T with probability 1, which forbids the existence of cycles.
- (ii) If all agents are yes- and no-influential in A_j , then $0 < A_j(1_i) < 1$ for all $i \in N$. Indeed, $1 = A_j(1_i) \leq A_j(1_{N \setminus k})$ for any $k \neq i$, contradicting the fact that k is no-influential in A_j . Similarly, $0 < A_j(1_{N \setminus i}) < 1$ holds for any $i \in N$. It follows by monotonicity that $A_j(1_S) \neq 0, 1$ for all $S \neq \emptyset, N$, which by (i) proves the result.
- (iii) From Lemma 4, i yes- and no-influential in A_i forbids any cycle with consecutive S_1, S_2 such that $S_1 \ni i$ and $S_2 \not\ni i$, or $S_1 \not\ni i$ and $S_2 \ni i$. Since S_1, S_2 must be incomparable, $S_1 = \{i\}$ is ruled out, as well as $S_1 = N \setminus i$. Take $S_1 \subset N, S_1 \neq \emptyset$. Then $S_2 = L \cup T$, with $L \subseteq N \setminus S_1, L \neq \emptyset$ and $T \subseteq S_1$. Take $i \in S_1$. The fact that i is influential in A_i forbids $T \not\ni i$, for all L . Since this holds for all $i \in S_1$, it follows that no T is possible, hence no S_2 .

□

Proof. (of Theorem 6) Let us consider a cycle $\mathcal{C} = \{S_1, \dots, S_k\}$, setting $S_{k+1} = S_1$, in the transition graph. Take $j_{k,1} \in S_k \setminus S_1$, which is always possible since sets are incomparable in a cycle. Since $A_{j_{k,1}}(1_{S_{k-1}}) = 1$ and aggregation functions are 0-1-valued, there exists $C_{k-1,1} \subseteq S_{k-1}$ which is influential for $j_{k,1}$. Note that $C_{k-1,1} \subseteq S_{k-1} \cap S_k$ is impossible, otherwise one would have $1 = A_{j_{k,1}}(1_{C_{k-1,1}}) \leq A_{j_{k,1}}(1_{S_k})$, contradicting the fact that the transition S_k to S_1 is certain. Now take some $j_{k-1,1} \in C_{k-1,1} \setminus S_k$. By the same reasoning, it exists $C_{k-2,1} \subseteq S_{k-2}$ which is influential for $j_{k-1,1}$. Let us denote this by $C_{k-2,1} \rightarrow j_{k-1,1} \in C_{k-1,1} \rightarrow j_{k,1}$. We can repeat this procedure till the set S_1 , which gives

$$C_{1,1} \rightarrow j_{2,1} \in C_{2,1} \rightarrow j_{3,1} \in C_{3,1} \rightarrow \dots \rightarrow C_{k-2,1} \rightarrow j_{k-1,1} \in C_{k-1,1} \rightarrow j_{k,1}.$$

The predecessor of S_1 in the cycle being S_k , there exists also a set $C_{k,2} \subseteq S_k$ which is influential for some $j_{1,1}$ chosen in $C_{1,1} \setminus S_2$. Suppose that $j_{k,1} \in C_{k,2}$. Then we obtain

$$j_{k,1} \in C_{k,2} \rightarrow j_{1,1} \in C_{1,1} \rightarrow j_{2,1} \in C_{2,1} \rightarrow j_{3,1} \in C_{3,1} \rightarrow \dots \rightarrow C_{k-2,1} \rightarrow j_{k-1,1} \in C_{k-1,1} \rightarrow j_{k,1},$$

which is a cycle in the hypergraph $H_{\mathbf{A}}^{\text{yes}}$ of length k . Suppose on the contrary that $j_{k,1} \notin C_{k,2}$. Then choose some $j_{k,2} \in C_{k,2} \setminus S_1$, and there will exist a set $C_{k-1,2} \subseteq S_{k-1}$ influencing $j_{k,2}$. If $j_{k-1,1} \in C_{k-1,2}$, we get also a cycle of length k . Suppose this is not the case, and continue the procedure, creating sets $C_{i,\ell} \subseteq S_i$ and choosing elements $j_{i,\ell}$ in $C_{i,\ell} \setminus S_i$, different from the previous ones $j_{i,\ell-1}, \dots, j_{i,1}$. By finiteness of the sets $S_i \setminus S_{i+1}$ for all $i = 1, \dots, k$, it must happen that at some stage $C_{i,\ell} \ni j_{i,\ell'}$ for $\ell' < \ell$, creating a cycle of length $(\ell - \ell')k$. □

Proof of Theorem 7

Let \mathbf{A} be a symmetric decomposable model, and let Γ be its transition graph (equivalently its reduced matrix $\tilde{\mathbf{B}}$).

Claim 1: Two distinct matrices $\tilde{\mathbf{B}}, \tilde{\mathbf{B}}' \in \mathcal{SDB}$ induce two distinct influence graphs G, G' . Indeed, if $\tilde{\mathbf{B}}, \tilde{\mathbf{B}}'$ are distinct, there exist $S, T \in 2^N$ such that $b_{S,T} = 1$ and $b'_{S,T} = 0$. By (4) this implies that for all $i \in T$ there is an arc $j \rightarrow i$ in G with $j \in S$, and for all $i \notin T$, an arc $j \rightarrow i$ with $j \notin S$, while in G' , either there exists some $i \in T$ with no arc $j \rightarrow i$ such that $j \in S$, or there is some $i \notin T$ with no arc from $N \setminus S$. Hence the two graphs differ.

Claim 2: Two distinct influence graphs G, G' correspond to two distinct reduced matrices $\tilde{\mathbf{B}}, \tilde{\mathbf{B}}' \in \mathcal{SDB}$.

Proof of Claim 2: Since $G \neq G'$, suppose w.l.o.g. that there is an arc $i \rightarrow j$ in G' which is not in G . It follows from the first condition of (4) that $b_{i,T} = 0$ for all $T \ni j$.

Choose $T \subseteq N \setminus j$ such that $b_{i,T} > 0$. Such a T necessarily exists otherwise there would be no transition from the state $\{i\}$. We claim that $b'_{i,T \cup j} > 0$, showing that $\tilde{\mathbf{B}}$ and $\tilde{\mathbf{B}}'$ differ, since we have $b_{i,T \cup j} = 0$.

To prove $b'_{i,T \cup j} > 0$ we first need to prove that for each $k \in T \cup j$, the arc $i \rightarrow k$ exists. We know that for $k = j$ it exists. If an arc exists for all other $k \in T$ we are done. Otherwise, by (4) again, $b'_{i,T} = 0$, so that $\tilde{\mathbf{B}} \neq \tilde{\mathbf{B}}'$. Second, we need to prove that for

each $k \notin T \cup j$, there is an arc $l \rightarrow k$ for some $l \neq i$. If this is true, we are done. If not, some $k \in N \setminus (T \cup j)$ has no arc from an element $l \neq i$. Again, this implies that $b'_{i,T} = 0$, hence the two matrices differ.

As a consequence of the two claims, there is a one-to-one correspondence between $SDB(N)$ and the set of directed graphs on N .

By the above remark, there is a one-to-one correspondence between the set of directed graphs and the set of WAM models. Since the correspondence between symmetric decomposable models and $SDB(N)$ is onto, the theorem is proved.

Proof of Theorem 8

We begin by proving a simple fact.

Lemma 5. Consider a symmetric decomposable aggregation model. Then for any $\emptyset \neq S \subset N$, any $\emptyset \neq K \subset N \setminus S$, $[S, S \cup K]$ is a Boolean terminal class if and only if $[N \setminus (S \cup K), N \setminus S]$ is a Boolean terminal class.

Proof. (of the Lemma) First we prove that $[S, S \cup K]$ is a class iff $[N \setminus (S \cup K), N \setminus S]$ is a class. $[S, S \cup K]$ is a class iff for all $T, T' \in [S, S \cup K]$, there is a path from T to T' in the transition graph, i.e., a sequence $T = T_1, T_2, \dots, T_k = T'$ with $b_{T_i, T_{i+1}} > 0$ for $i = 1, \dots, k-1$. Since by (5) $b_{T_i, T_{i+1}} > 0$ iff $b_{N \setminus T_i, N \setminus T_{i+1}} > 0$, this is equivalent to have a path from $N \setminus T$ to $N \setminus T'$ for every $T, T' \in [S, S \cup K]$, which is equivalent to say that $[N \setminus (S \cup K), N \setminus S]$ is a class.

Second, we prove that $[S, S \cup K]$ is terminal iff $[N \setminus (S \cup K), N \setminus S]$ is. We know by Th. 5 of the paper that $[S, S \cup K]$ is terminal iff there is no ingoing arc into S and no outgoing arc from $S \cup K$ in the influential graph. These conditions are equivalent to say that there is no outgoing arc from $N \setminus S$ and no ingoing arc into $N \setminus (S \cup K)$, which is equivalent to the desired result. \square

Proof. (of Theorem 8) 1. Assume that the six conditions hold. We prove first that $[S, S \cup K]$ is strongly connected in the transition graph Γ . It suffices to show that $b_{T, T'} > 0$ for any $T, T' \in [S, S \cup K]$. By (4), we have to prove that (a) any $i \in T'$ has an ingoing arc from T , and (b) any $i \in N \setminus T'$ has an ingoing arc from $N \setminus T$.

(a): T, T' contain S and possibly elements of K . For $i \in S$, by condition (iii), i has an ingoing arc from S , thus from T . For $i \in K$, use condition (ii) instead.

(b): $N \setminus T'$ and $N \setminus T$ contain $N \setminus (S \cup K)$ and possibly elements of K . For all $i \in N \setminus (S \cup K)$, there is an ingoing arc from $N \setminus (S \cup K)$ by (iii)'. Consider then $i \in K \cap (N \setminus T')$, it has also an arc from $N \setminus (S \cup K)$ by (ii)'.

Hence $[S, S \cup K]$ is strongly connected. By Theorem 4 it remains to prove that there is no ingoing arc into S , and no outgoing arc from $S \cup K$. These conditions are fulfilled by conditions (i) and (i)'.

2. We know by Theorem 4 and Lemma 5 that (i) and (i)' are necessary.

Suppose that some $i \in S$ has no ingoing arc from S . Then $b_{S, T} = 0$ for any $T \in [S, S \cup K]$ since $b_{S, T} > 0$ implies that each $i \in S$ has an ingoing arc from S by (4). Therefore, $[S, S \cup K]$ is not a class since it is not strongly connected. This shows that (iii) is necessary.

By symmetry (see Lemma 5), (iii') is also necessary.

3. We prove that (iv) is necessary for S . Define K_0 the (possibly empty) set of elements of K which are connected to S by some path. Suppose that $K \setminus K_0 \neq \emptyset$, i.e., there are elements in K not connected to S . We claim that there is no transition from some T in $[S, S \cup K_0]$ to some T' in $[S, S \cup K]$ containing an element of $K \setminus K_0$, which suffices to prove that $[S, S \cup K]$ is not a class. Indeed, if such a transition would exist, by (4) there would be an arc $j \rightarrow k$ with $j \in K_0$ and $k \in K \setminus K_0$. Such an arc cannot exist by definition of K_0 . Hence, we have necessarily $K_0 = K$.

Now, by Lemma 5, (iv) is also necessary for $N \setminus (S \cup K)$. □