Evolutionary beliefs and financial markets

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Evolutionary Beliefs and Financial Markets[*][†]

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Abstract

Why do investors keep different opinions even though they learn from their own failures and successes? Why do investors keep different opinions even though they observe each other and learn from their relative failures and successes? We analyze beliefs dynamics when beliefs result from a very general learning process that favors beliefs leading to higher absolute or relative utility levels. We show that such a process converges to the Nash equilibrium in a game of strategic belief choices. The asymptotic beliefs are subjective and heterogeneous across the agents. Optimism (resp. overconfidence) as well as pessimism (resp. doubt) both emerge from the learning process. Furthermore, we obtain a positive correlation between pessimism (resp. doubt) and risk-tolerance. Under reasonable assumptions, beliefs exhibit a pessimistic bias and, as a consequence, the risk premium is higher than in a standard setting.

Keywords: belief formation, heterogeneous beliefs, evolutionary game theory, pessimism, risk premium

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1 Introduction

In the classical financial economics theory, decision makers are assumed to have homogenous and rational expectations. This assumption has been the basis for many developments in finance like the portfolio selection model (Markowitz, 1952) and the CAPM (Sharpe 1964 and Lintner 1965). Homogeneous expectations might be justified through a learning argument: agents should see that other people are being more or less successful and they should adopt the same beliefs as the most successful ones. Rational expectations have been often justified through an evolutionary argument: agents who make inaccurate predictions are driven out of the market by those who are more accurate[1] (see e.g. Sandroni, 2000).

However, it suffices to observe the heterogeneity of analysts or professional forecasters forecasts or more generally of experts opinions to realize that the both assumptions of homogeneous beliefs and rational beliefs are not realistic.

A question that naturally arises is then the following: Why do investors keep different opinions even though they learn from their own failures and successes? Why do investors keep different opinions even though they observe each other and learn from their relative failures and successes? We analyze beliefs dynamics when beliefs result from a very general learning process that favors beliefs leading to higher absolute or relative utility levels. We consider both the situation where learning favors beliefs leading to higher utility levels than other beliefs and the situation where learning favors beliefs leading to higher utility levels that the other agents. Surprisingly, we show through arguments from evolutionary game theory[2] that interactions and learning do not lead to homogeneous beliefs among the agents. They rather lead to heterogeneous and subjective beliefs.

Our model is embedded in a simple, standard Walrasian equilibrium problem with risky assets and beliefs are about the risk distribution. Our learning process is not based on statistical inference but on the analysis of relative successes and failures: each

[1]This argument is in the same vein as the arguments of Alchian (1950) and Friedman (1953) applied to the profit maximization assumption in a competitive market.

[2]The evolutionary paradigm is applied to the population of possible beliefs and not to the initial population of agents.

agent has an initial distribution of possible beliefs and randomly tries different beliefs according to this distribution. She/he dynamically adjusts her/his beliefs distribution by increasing (decreasing) the frequency of the beliefs that lead to higher (lower) utility levels at the Walrasian equilibrium. Beliefs that lead to high utility levels spread within the population of beliefs. Our agents gradually learn that a certain way of forming beliefs is more rewarding than others. They select the beliefs that are more beneficial for them in the spirit of the pragmatic beliefs concept of Hvide[3] (2002). We characterize the types of beliefs that survive especially in terms of optimism/overconfidence and we analyze the implications in terms of equilibrium characteristics.

We show that the asymptotical behaviour of our agents corresponds to the behaviour that they would adopt in a static game that is naturally associated to our learning process. This fictitious game corresponds to a situation where each agent adopts a belief to maximize her/his utility from trade, taking into account the effect her/his choice has on price and taking as given the strategy of the other agents[4].

Our findings are the following. First, we show that a learning process, which favours beliefs leading to higher utility levels at the Walrasian equilibrium leads to belief subjectivity and heterogeneity. The objective belief is not optimal, and agents differ in their beliefs. Indeed, optimism (resp. overconfidence) as well as pessimism (resp. doubt) both survive in the long run. Furthermore, we find a positive correlation between pessimism (resp. doubt) and risk-tolerance. The intuition is as follows. For a very risk-tolerant agent, his demand in the risky asset is positive, so that his expected utility from trade is decreasing in the price of the risky asset. A pessimistic belief is associated to a lower demand, hence to a lower price, and balances this benefit of pessimism against the costs of worse decision making. We obtain exactly the same conclusions when agents do not favor the beliefs leading to higher utility levels in absolute terms but rather favor the beliefs leading to higher utility levels than the other agents.

[3]The framework of Hvide (2002) is very different from ours (principal-agent approach in a job market model).

[4]This asymptotic behaviour corresponds then to a Nash equilibrium in demands, as presented in Kyle (1989), "perhaps the most obvious modification of the conventional competitive rational expectations concept. It preserves market clearing through a Walrasian mechanism and keeps the Nash flavour of a competitive equilibrium."

Our results explain why agents differ in their beliefs and why this divergence persists.

In an exponential utility and normal distribution setting, our learning process leads to a pessimistic bias in individual beliefs. Such a pessimistic bias has been observed in empirical studies in a purely behavioural setting (Ben Mansour et al., 2006), in a decision theory framework (Wakker, 2001) or in a market framework (Giordani and Söderlind, 2006). In particular, as underlined by Shefrin (2005) based on Wall $treet Week data “between 1983 and 2002, professional investors were unduly pessimistic, underestimating market returns”. As a consequence of the pessimistic bias, our evolutionary approach leads to a risk premium that is greater than in the standard rational expectations equilibrium[5]. We provide numerical simulations that show that the risk premium can be multiplied by 3 along the learning process and that the asymptotic beliefs and risk premium can be reached in a few months.

After focusing on such an exponential utility and normal distribution setting, we show that the main properties (beliefs heterogeneity, positive correlation between pessimism and risk tolerance) carry over to more general specifications of preferences and uncertainty. The positive correlation between pessimism and risk tolerance leads to a pessimistic bias. We also show that some of the properties are retained when allowing agents to differ by their level of doubt/overconfidence instead of their level of pessimism/optimism.

Our evolutionary explanation of beliefs should be contrasted with approaches in which forward-looking agents optimally distort beliefs and in which beliefs are of intrinsic value to agents, as with wishful thinking, self-esteem or fear of disappointment (Akerlof and Dickens, 1982, Benabou and Tirole, 2002, Brunnermeier and Parker, 2005, Gollier and Muermann, 2010). In these models, beliefs result from an individual optimization problem while our beliefs result from interaction. Our results differ from those obtained in such an *optimal* beliefs/illusions setting, in which in most cases there is no belief heterogeneity

[5] Abel (1989), Detemple-Murthy (1994), Gollier (2007) and Jouini-Napp (2006) have already underlined that pessimism and a positive correlation between risk tolerance and pessimism lead to an increase of the risk premium; in this paper we construct a model in which pessimism and the positive correlation endogeneously emerge at the equilibrium.

and an optimistic bias[6]. This bias results from the specific mental process they consider: since agents have a higher current felicity if they are optimistic, the optimal beliefs balance this benefit of optimism against the costs of worse decision making.

The paper is organized as follows. Section 2 introduces the concept of evolutionary beliefs and details the dynamics of beliefs formation when agents favor the beliefs that are more beneficial for them in absolute terms. We also explain the link between the evolutionary framework and a static concept of Nash equilibrium in demand schedules. In Section 3, explicit computations are provided in a setting with exponential utility functions and normal distributions in which the evolutionary variable is the expected payoff of the risky asset. In Section 4, we compare our results with those obtained in an *optimal* beliefs setting. In Section 5, qualitative results are provided in a setting with more general utility functions and distributions. Section 6 considers extensions of the model of Section 3 in essentially three directions; a model in which the agents compare their respective utility levels and favor the beliefs that permits to outperform the other agents, a model in which the evolutionary variable is the variance of the payoff of the risky asset and a model with multiple sources of risk. Section 7 concludes.

2 A model of evolutionary beliefs

We consider a standard equilibrium model, except that we allow for subjective beliefs and a possible evolution of beliefs through market interaction and learning. We start by describing the Walrasian equilibrium. We then introduce the learning procedure as well as the underlying 2-player game.

2.1 The Walrasian equilibrium

The economy is composed of two agents called Ann and Bob with real valued, increasing and strictly concave utility functions u_A and u_B defined on $\mathbb{R}_+$. The uncertainty is

[6]Gollier-Muermann (2010) consider a model of optimal beliefs with ex-ante savoring and ex-post disappointment. Depending upon the intensity of anticipatory feelings and disappointment they might also obtain a systematic pessimistic bias.

described by a probability space (Ω, F, P) where Ω is the set of states of nature, F is the $\sigma-$field of observable events and P is a probability measure giving the likelihood of occurrence of the different events in F. There is a single consumption good as well as a single risky asset in the economy, whose payoff at the end of the period is described by a random variable $\widetilde{x}$. We let p denote the unit price of the risky asset in terms of consumption good, which means that both agents can sell their property rights on the risky asset against the delivery of the sure quantity p at the end of the period. We assume that the agents have the same endowment, which consists of a half unit of the risky asset. The difference with the standard model stems from the fact that agents have possibly incorrect beliefs. We assume that the set of possible beliefs for Ann (Bob) is parametrized by γ (by θ) in a given set $\mathcal{B}$. Our agents then have different probability measures over (Ω, F) respectively denoted by Q_γ and Q_θ. We denote by E^γ and E^θ the expectation operators respectively associated to Q_γ and Q_θ. For a given random consumption $\widetilde{c}$, the subjective expected utility of Ann (Bob) is then given by $E^\gamma\left[u_A(\widetilde{c})\right]$ (by $E^\theta\left[u_B(\widetilde{c})\right]$). As in the standard portfolio problem, agents determine the optimal composition of their portfolio, in other words their optimal exposure to the risk.

Definition 1 *A Walrasian equilibrium*[7] $(p; (\alpha_A^*, \alpha_B^*))$ *is defined by a price* p *and quantities* (α_A^*, α_B^*) *of risky asset for each agent such that the quantity* α_i^* *is optimal for agent* i, $i \in \{A, B\}$, *under her/his budget constraint, i.e.*

$$\begin{aligned}
\alpha_A^* &= \arg\max_\alpha E^\gamma\left[u_A(\frac{1}{2}p + \alpha\left(\widetilde{x} - p\right))\right] \\
\alpha_B^* &= \arg\max_\alpha E^\theta\left[u_B(\frac{1}{2}p + \alpha\left(\widetilde{x} - p\right))\right]
\end{aligned}$$

and such that markets clear, i.e. $\alpha_A^* + \alpha_B^* = 1$.

The structure of the underlying Walrasian equilibrium problem is quite general and may be applied to several equilibrium problems in which heterogeneous risk-averse agents have to choose their optimal exposure to a given risk. For instance, when an insurance

[7]Walrasian equilibrium models with heterogeneous beliefs have been studied, among others, by Williams (1977), Abel (1989), Detemple-Murthy (1994), Calvet et al. (2002) and Jouini-Napp (2006, 2007).

company and a reinsurance company have to determine an optimal retention rate or when entrepreneurs have to fix the optimal proportion of equity to retain for a given project.

In the next, we denote by $u_A(\gamma, \theta, x) = u_A(\frac{1}{2}p + \alpha_A^* (x - p))$ and $u_B(\gamma, \theta, x) = u_B(\frac{1}{2}p + \alpha_A^* (x - p))$ the equilibrium ex-post utility levels of Ann and Bob when their beliefs are given by γ and θ and for a given realization x of $\widetilde{x}$.

2.2 The evolutionary framework

The situation can be characterized in terms of game theory with 2 players: Ann and Bob playing against each other and playing against Nature. Ann chooses a belief γ in $\mathcal{B}$, Bob chooses a belief θ in $\mathcal{B}$ then Nature chooses a realization x in Ω. We denote by $u_A(\gamma, \theta, x)$ and $u_B(\gamma, \theta, x)$ the utility levels of Ann and Bob at the Walrasian equilibrium.

For a given pair of beliefs (γ, θ) for Ann and Bob, the average utility level of Ann over the different states of Nature is given by $U_A(\gamma, \theta) = \int u_A(\gamma, \theta, \widetilde{x})\, dP$ and the average utility level of Bob is given by $U_B(\gamma, \theta) = \int u_B(\gamma, \theta, \widetilde{x})\, dP$.

Let us first assume that Bob has a constant belief θ^* and that Ann randomly tries different beliefs γ, each of them with a given frequency. Let us denote by Υ the distribution over $\mathcal{B}$ associated to those frequencies. In classical game theory, it is assumed that each agent knows all the game parameters (in particular, the objective probability P, the payoff function of the other agents) and that she/he may observe the strategies of the other agents. At this stage, we do not need to assume that the agents have access to such information: they might observe the other agents actions or not[8]. We simply assume that they use the following rule of thumb: they increase the frequency of the more rewarding beliefs and they decrease the frequency of the less rewarding beliefs. If Bob maintains his belief θ^*, Ann will increase (decrease) the frequency of the γs that lead to higher (lower) values for $U_A(\gamma, \theta^*)$. It is easy to see that such a dynamics converges in the set of best replies to θ^*, $BR_A(\theta^*) = \arg\max U_A(\gamma, \theta^*)$.

However, there is no reason for Bob to maintain his belief fixed. If Bob also plays different strategies with different frequencies described by a distribution Θ over $\mathcal{B}$, the

[8] In Section 6.1, we will explicitly consider the case where the agents compare their utility level to the utility level of the other agents selecting the beliefs that permit to them to outperform the other agents.

average utility level of Ann (over the states of nature and over the different strategies played by Bob) when she plays γ is given by $\int u_A(\gamma, \theta, \widetilde{x})\, d\Theta \otimes dP$. Since Nature chooses the state of the world after the choices of Ann and Bob and independently of them, the average utility level of Ann (over the states of nature and over the different strategies played by Bob) when she plays γ is equal to $\int U_A(\gamma, \theta)\, d\Theta$. Symmetrically, the average utility level of Bob when he plays θ is given by $\int U_B(\gamma, \theta)\, d\Upsilon$. Each of them modifies her/his frequencies (i.e. modifies Υ and Θ) according to the rule of thumb described above leading to a dynamical system that we now describe rigorously.

Let us denote by Υ_0 the initial distribution (over $\mathcal{B}$) of Ann and by Θ_0 the initial distribution (over $\mathcal{B}$) of Bob. Both distributions are assumed to have full support over $\mathcal{B}$ and to belong to the set $\Delta(\mathcal{B})$ of Borel probability distributions over $\mathcal{B}$. Similarly, we denote by Υ_t and Θ_t their respective distributions at date $t \geq 0$. At date t and for a given belief γ, Ann's average utility level is given by $\int U_A(\gamma, \theta)\, d\Theta_t$. At date t and for a given belief θ, Bob's average utility level is given by $\int U_B(\gamma, \theta)\, d\Upsilon_t$.

The frequency at which a given belief is chosen by Ann is modified according to the average utility level provided by this belief: the frequency of high utility level beliefs is increased and the frequency of low utility level beliefs is decreased. The simplest way to do so is to assume that, for a given Borel measurable set $\mathcal{A} \subset \mathcal{B}$, the weight of $\mathcal{A}$ increases proportionally to the quantity $\int_{\mathcal{A}\times\mathcal{B}} U_A(\gamma, \theta)\, d\Upsilon_t \otimes d\Theta_t - \int_{\mathcal{B}\times\mathcal{B}} U_A(\gamma, \theta)\, d\Upsilon_t \otimes d\Theta_t$ that measures how much the beliefs in $\mathcal{A}$ outperform in average the average belief in $\mathcal{B}$. Since Ann and Bob select their beliefs independently of each other (even if their beliefs distribution depends upon past beliefs of both agents) this last quantity is equal to $\int_{\mathcal{A}} d\Upsilon_t \int U_A(\gamma, \theta)\, d\Theta_t - \int\int U_A(\gamma, \theta)\, d\Upsilon_t d\Theta_t$. A symmetric reasoning on Bob leads to the following global dynamics

$$\frac{d}{dt}\Upsilon_t(\mathcal{A}) = \lambda\left(\int_{\mathcal{A}} d\Upsilon_t \int U_A(\gamma, \theta)\, d\Theta_t - \int\int U_A(\gamma, \theta)\, d\Upsilon_t d\Theta_t\right), \tag{1}$$

$$\frac{d}{dt}\Theta_t(\mathcal{A}) = \lambda'\left(\int_{\mathcal{A}} d\Theta_t \int U_B(\gamma, \theta)\, d\Upsilon_t - \int\int U_B(\gamma, \theta)\, d\Upsilon_t d\Theta_t\right) \tag{2}$$

where λ and λ' are given positive real numbers. Note that this leads to $\frac{d}{dt}\Upsilon_t(\mathcal{B}) = \frac{d}{dt}\Theta_t(\mathcal{B}) = 0$ which is natural since $\Upsilon_t(\mathcal{B}) = \Theta_t(\mathcal{B}) = 1$, for all t.

This dynamics is well known in evolutionary game theory and called the replicator dynamics (see e.g. Cressman et al., 2006). As underlined above, it favors the most rewarding beliefs. The most general formulation for a dynamics satisfying such a property has been introduced in Heifetz et al. (2007a,b). In the next, we adopt this general formulation by assuming that the distributions evolve as follows:

$$\begin{aligned} \frac{d}{dt}\Upsilon_t(\mathcal{A}) &= \int_{\mathcal{A}} V_A(\gamma, \Theta_t) d\Upsilon_t, \qquad \mathcal{A} \subset \mathcal{B}, \text{ Borel measurable,} && (3) \\ \frac{d}{dt}\Theta_t(\mathcal{A}) &= \int_{\mathcal{A}} V_B(\Upsilon_t, \theta) d\Theta_t, \qquad \mathcal{A} \subset \mathcal{B}, \text{ Borel measurable,} && (4) \end{aligned}$$

where V_A and V_B are continuous[9] growth-rate functions that satisfy

$$\begin{aligned} V_A(\gamma, \Theta_t) &> V_A(\tilde{\gamma}, \Theta_t) \Longleftrightarrow \int U_A(\gamma, \theta)\, d\Theta_t > \int U_A(\tilde{\gamma}, \theta)\, d\Theta_t, && (5) \\ V_B(\Upsilon_t, \theta) &> V_B(\Upsilon_t, \tilde{\theta}) \Longleftrightarrow \int U_B(\gamma, \theta)\, d\Upsilon_t > \int U_B(\gamma, \theta)\, d\Upsilon_t, && (6) \end{aligned}$$

as well as

$$\int V_A(\gamma, \Theta_t) d\Upsilon_t = 0 \qquad \text{and} \qquad \int V_B(\Upsilon_t, \theta) d\Theta_t = 0 \qquad \text{for each } t, \qquad (7)$$

in order to ensure that Υ_t and Θ_t remain probability measures for each t. This dynamics is similar to the dynamics considered in Heifetz et al. (2007a,b) and corresponds to the most general way to represent the fact that agents increase the frequency of the most rewarding beliefs at the expense of the beliefs that lead to lower average utility levels. In this sense, each agent constructs her/his beliefs pragmatically. We refer to Heifetz et al. (2007a,b) and Oechssler and Riedel (2001) for the technical conditions that guarantee that the system of Equations (3)-(4) has a well defined solution.

The function V plays the same role as the adjustment function used in the Walras

[9] V_A and V_B are respectively defined over $B \times \Delta(B)$ and $\Delta(B) \times B$. The continuity property is meant here in the sense of the product of the topology of B and of the weak-topology of $\Delta(B)$.

tâtonnement mechanism, reducing the price when the excess demand is positive and increasing it when the excess demand is negative. In our framework, an agent increases the frequency of a given belief when the utility provided by this belief exceeds the average utility over all possible beliefs. The replicator dynamics defined by Equations (1) and (2) is the simplest form for such a behavior and corresponds to a linearization of the function V.

We are interested in the asymptotic characteristics of the beliefs in this dynamic problem. More precisely, we are interested in the following questions. Does this model of evolutionary beliefs lead to subjectivity in beliefs? Does it generate heterogeneous beliefs? Is there a link between risk-tolerance and beliefs and what is the nature of this link? Is there a pessimistic/optimistic bias at the equilibrium at the individual as well as at the collective level? What are the consequences on the equilibrium characteristics and in particular on the risk premium?

2.3 The underlying 2-player game

When Bob keeps his belief fixed and when Ann follows a dynamics that favors the beliefs leading to high utility levels like the dynamics governed by Equation (1) or Equation (3) then the beliefs of Ann will converge in the set of best replies to Bob's strategy. When both agents adjust their beliefs, the replicator dynamics governed by Equations (1) and (2) is very close to the best reply dynamics. As underlined by Hofbauer et al. (2009), both dynamics have the same asymptotic behavior but contrarily to the best reply dynamics, the replicator dynamics does not need to assume that all agents are highly rational. Our dynamics is then expected to converge towards the Nash equilibrium of the game where each agent chooses her/his belief strategically in order to maximize her/his payoff function. This close connection between the asymptotic behavior of evolutionary games and Nash equilibria is well established in the case of the replicator dynamics and in the framework of symmetric games with finite strategic sets. This is a classical result in evolutionary game theory, often called "the Folk Theorem of evolutionary games" and we refer to Hofbauer and Sigmund (2003) for a survey.

In this section we will show that this connection holds in our non symmetric and non finite framework and when beliefs are governed by the more general dynamics described by Equations (3) and (4). This means that whatever the precise rule of beliefs revision is, beliefs converge to the Nash equilibrium beliefs as long as the revision process increases the frequency of the beliefs that lead to high utility levels at the expense of the beliefs that lead to low utility levels.

This result is very useful since it characterizes the asymptotic behavior of our evolutionary beliefs in terms of Nash equilibria of simple games. This permits to explicitly compute the asymptotic beliefs for some specifications of the agents utility functions and of the risky asset payoff distribution. This also permits to derive qualitative properties of the solution in the case of general utility functions and general payoff distributions.

Let us first rigorously describe the 2-player (fictitious) game under consideration. It is denoted by $\Gamma = (\mathcal{B}, \mathcal{B}, U_A, U_B)$ where $\mathcal{B}$ is the common strategic set of the agents and where U_A and U_B are their payoff functions. The game Γ corresponds to a game in which each agent chooses her/his belief in order to maximize her/his utility taking into account her/his own impact on equilibrium prices (and therefore on equilibrium utility levels).

Even though it could seem puzzling to maximize utility functions the agents do not really know (the expectations are taken under the objective probability) this maximization is only done in the fictitious game and is not assumed at Ann and Bob's level.

Definition 2 *A Nash equilibrium of the game Γ is defined as a pair of beliefs $(\gamma^*, \theta^*) \in \mathcal{B} \times \mathcal{B}$ such that*

$$\begin{aligned} U_A(\gamma^*, \theta^*) &\geq U_A(\gamma, \theta^*) \text{ for all } \gamma \in \mathcal{B}, \\ U_B(\gamma^*, \theta^*) &\geq U_B(\gamma^*, \theta) \text{ for all } \theta \in \mathcal{B}. \end{aligned}$$

The Nash equilibrium in the game Γ is the analogue of the notion of Nash equilibrium in demand schedules[10] (introduced by Kyle, 1989) restricted to demand schedules of a

[10] As underlined by Kyle (1989) this is "perhaps the most obvious modification of the conventional competitive rational expectations concept. It preserves market clearing through a Walrasian mechanism and keeps the Nash flavour of a competitive equilibrium."

specific form (namely, demand schedules that are parametrized by underlying beliefs and that correspond to the optimal demands of agents endowed with those beliefs).

Let us assume that the game Γ admits a Nash equilibrium (γ^*, θ^*); by definition we have $U_A(\gamma^*, \theta^*) \geq U_A(\gamma, \theta^*)$ for all $\gamma \in \mathcal{B}$ and then $V_A(\gamma^*, \delta_{\theta^*}) \geq V_A(\gamma, \delta_{\theta^*})$ where δ_{θ^*} is the Dirac measure at θ^*. The average value of $V_A(\gamma, \delta_{\theta^*})$ over $\{\gamma^*\}$ is then higher than the average value of $V_A(\gamma, \delta_{\theta^*})$ over any set $\mathcal{A} \subset \mathcal{B}$. This means that when Θ_t is kept constant equal to δ_{θ^*} then the selection dynamics described by (3) will favor the strategy γ^*. Similarly, when Υ_t is kept constant equal to δ_{γ^*} then the selection dynamics described by (4) will favor the strategy θ^*. The Nash equilibria of the game Γ are then good candidates for the asymptotic behaviour of the selection dynamics described by (3) and (4).

3 Exponential utility and normal distributions setting

For analytical tractability, we first consider exponential utility functions with normal distributions and provide explicit results.

3.1 The results

Agents have utility functions for consumption of the form $u_A(c) = -\exp\left(-\frac{c}{\theta_A}\right)$ and $u_B(c) = -\exp\left(-\frac{c}{\theta_B}\right)$, where $\theta_i > 0$ denotes the degree of (absolute) risk-tolerance of agent i. Moreover, we assume that $\widetilde{x}$ is normally distributed, with mean μ and variance σ^2.

Let us first determine the Walrasian equilibrium characteristics when Ann believes that $\widetilde{x}$ is normal with mean μ_A and variance σ^2 and Bob believes that $\widetilde{x}$ is normal with mean μ_B and variance σ^2. The parameters μ_A and μ_B play then the role of the parameters γ and θ of the previous section and are assumed to be drawn from a set $\mathcal{B} = \left[\underline{\mu}, \overline{\mu}\right]$ of possible beliefs that contains the objective value μ.

In such a setting and for a given p, the optimal demand $\alpha_i(p)$ of agent i is given by

$\alpha_i(p) = \theta_i \frac{\mu_i - p}{\sigma^2}$. The market clearing condition $\alpha_A(p) + \alpha_B(p) = 1$ then imposes that the equilibrium price $p(\mu_A, \mu_B)$ of the risky asset is given by

$$p(\mu_A, \mu_B) = \frac{\theta_A}{\theta_A + \theta_B}\mu_A + \frac{\theta_B}{\theta_A + \theta_B}\mu_B - \frac{\sigma^2}{\theta_A + \theta_B}, \tag{8}$$

which is the equilibrium price in an economy in which agents share the same expectations given by $\frac{\theta_A \mu_A + \theta_B \mu_B}{\theta_A + \theta_B}$. In the next we will call this average belief, the consensus belief. It corresponds to the belief that if held by both agents would lead to the same equilibrium prices. Replacing p by the expression of the equilibrium price, we obtain that the optimal demand α_i^* of agent i at the equilibrium is given by

$$\alpha_i^*(\mu_A, \mu_B) = \alpha_i(p(\mu_A, \mu_B), \mu_i) = \frac{\theta_i}{\overline{\theta}}\left[1 + \theta_j \frac{\mu_i - \mu_j}{\sigma^2}\right] \tag{9}$$

and the part of the risk borne by agent i depends upon both her/his level of risk-tolerance and her/his belief.

The resulting average ex-post utility levels are then given by

$$U_i(\mu_A, \mu_B) = E\left[u_i\left\{\frac{1}{2}p(\mu_i, \mu_j) + \{\widetilde{x} - p(\mu_i, \mu_j)\}\,\alpha_i^*(\mu_i, \mu_j)\right\}\right], \qquad i \in \{A, B\}.$$

Letting RP (resp. RP^{stdd}) denote the risk premium $\mu - p$ in this setting (resp. in the standard setting) we obtain

$$RP = \frac{\sigma^2}{\overline{\theta}} + \left(\mu - \frac{\theta_A \mu_A + \theta_B \mu_B}{\theta_A + \theta_B}\right) = RP^{stdd} + \left(\mu - \frac{\theta_A \mu_A + \theta_B \mu_B}{\theta_A + \theta_B}\right). \tag{10}$$

As underlined by e.g. Jouini and Napp (2007), this means that the risk premium in an economy with heterogeneous subjective beliefs is higher than in the standard rational expectations setting if and only if the consensus belief, which is the risk tolerance weighted average of the individual beliefs, is pessimistic, where pessimistic is meant in the sense that the mean of the risky asset's payoff is underestimated. It is therefore particularly interesting to explore when and why individuals are pessimistic, as well as the nature of the link between risk tolerance and pessimism. In the present paper, the individual beliefs are determined endogenously and we analyze their properties, especially in terms of pessimism, correlation between pessimism and risk tolerance and impact on the risk premium.

With these specifications and for given initial distributions Υ_0 and Θ_0 with full support over $\mathcal{B}$, the selection dynamics given by (3) and (4) is fully described. The game $\Gamma = (\mathcal{B}, \mathcal{B}, U_A, U_B)$ is also fully described.

Proposition 3 *Let us consider a model with two agents, exponential utility functions and a risky asset $\widetilde{x} \sim \mathcal{N}(\mu, \sigma^2)$ and let us assume that the possible beliefs are of the form $\widetilde{x} \sim \mathcal{N}(\mu', \sigma^2)$ with $\mu' \in \mathcal{B} = \left[\underline{\mu}, \overline{\mu}\right]$. We have*

1. *For given initial distributions Υ_0 and Θ_0 in $\Delta(\mathcal{B})$ with full support over $\mathcal{B}$, any dynamics that favors beliefs with high utility levels at the expense of beliefs of low utility levels (i.e. that satisfies Equations 3 to 6) converges to a pair of Dirac distributions $\left(\delta_{\hat{\mu}_A}, \delta_{\hat{\mu}_B}\right)$ where $(\widehat{\mu}_A, \widehat{\mu}_B)$ is the unique Nash equilibrium of the game $\Gamma = (\mathcal{B}, \mathcal{B}, U_A, U_B)$.*

2. *The surviving beliefs $(\widehat{\mu}_A, \widehat{\mu}_B)$ are given by*

$$\widehat{\mu}_A = \mu - \frac{\sigma^2}{4\theta_B(\theta_A + \theta_B)}(\theta_A - \theta_B), \qquad \widehat{\mu}_B = \mu - \frac{\sigma^2}{4\theta_A(\theta_A + \theta_B)}(\theta_B - \theta_A). \quad (11)$$

   Asymptotically, the more risk-tolerant agent is pessimistic, in the sense that she/he behaves as if the mean of $\widetilde{x}$ lied below its true value, and the less risk-tolerant agent is optimistic. Moreover, the more risk-tolerant agent is more pessimistic than the less risk-tolerant agent is optimistic and the unweighted average of the beliefs is pessimistic:

$$\frac{\widehat{\mu}_A + \widehat{\mu}_B}{2} = \mu - \frac{1}{8}\frac{(\theta_A - \theta_B)^2\sigma^2}{\theta_A\theta_B\overline{\theta}}. \quad (12)$$

3. *Asymptotically, the consensus belief is pessimistic, i.e. the average of the individual beliefs weighted by the risk-tolerance is pessimistic. More precisely,*

$$\frac{\theta_A\widehat{\mu}_A + \theta_B\widehat{\mu}_B}{\theta_A + \theta_B} = \mu - \frac{1}{4}\frac{(\theta_A - \theta_B)^2\sigma^2}{\theta_A\theta_B(\theta_A + \theta_B)}. \quad (13)$$

4. Asymptotically, the risk premium RP (resp. the price) is higher (resp. lower) than in the standard rational expectations equilibrium. More precisely

$$RP = RP^{stdd} + \left(\mu - \frac{\theta_A \widehat{\mu}_A + \theta_B \widehat{\mu}_B}{\theta_A + \theta_B} \right) = RP^{stdd} + \frac{1}{4} \frac{(\theta_A - \theta_B)^2 \sigma^2}{\theta_A \theta_B (\theta_A + \theta_B)}. \tag{14}$$

5. Asymptotically, the optimal demands are given by

$$\alpha_A^* = \frac{\theta_A}{\theta_A + \theta_B} + \frac{(\theta_B - \theta_A)}{4(\theta_A + \theta_B)}, \qquad \alpha_B^* = \frac{\theta_B}{\theta_A + \theta_B} + \frac{(\theta_A - \theta_B)}{4(\theta_A + \theta_B)}.$$

which means that the volumes of trade (and the risk sharing) are reduced compared to the standard setting. The more risk tolerant (resp. risk averse) agent selects a less (resp. more) risky portfolio.

Evolution selects only one belief for each agent and the pair of asymptotic beliefs corresponds to the unique Nash equilibrium of the game Γ. Therefore, the surviving belief for a given agent corresponds to the belief that would be chosen by the agent under consideration in a model where such an agent would take into account her/his impact on equilibrium prices (and therefore on equilibrium utility levels) and would choose her/his beliefs accordingly.

Note that our construction of evolutionary beliefs leads to subjective and heterogeneous beliefs. Indeed, evolutionary beliefs differ from the objective belief, Ann and Bob differ in their beliefs and, as expressed in Equations (11), belief heterogeneity takes its roots in the difference in risk-aversion levels. Besides, more than just being "heterogeneous", evolutionary beliefs are "antagonistic" in the sense that one of the agents is optimistic ($\widehat{\mu}_i > \mu$) and the other one is pessimistic ($\widehat{\mu}_i < \mu$).

With the 2-player game interpretation, the different qualitative results are easy to interpret.

The pessimism of the more risk-tolerant agent can be interpreted as follows. Suppose that Ann is more risk-tolerant. When agents only differ in their level of risk-aversion, the risky asset's demand for Ann is positive at the equilibrium. Her expected utility from trade is then decreasing in the price of the risky asset. The choice of a pessimistic belief is associated to a lower demand, hence to a lower price and a higher expected utility. The evolutionary belief balances this benefit of pessimism against the costs of worse decision making. The converse reasoning applies to Bob, who, at the equilibrium, has a negative demand in the risky asset and benefits from optimism. In this illustration, the more risk tolerant agent (Ann) has a positive demand while the more risk averse one (Bob) has a negative demand. This is due to the fact that both agents are assumed to have the same initial endowment. When this is not the case, it may occur that the more risk tolerant (more risk averse) agent has a negative (positive) demand. In fact, the optimal demands and beliefs appear as linear in the initial endowment and the case with equal endowment corresponds to the average situation. For instance, in a financial dynamic setting, the respective levels of wealth of the agents evolve through time and states of the world but the more risk tolerant agent will be, on average through time and states of the world, more pessimistic and the more risk averse agent will be, on average, more optimistic.

The positive correlation between pessimism and risk-tolerance might seem counterintuitive because the two concepts (pessimism and risk-aversion) appear at first sight as closely related. Investing a large amount in a risky asset may result indifferently from optimism or risk-tolerance. Similarly an entrepreneurial behavior may be explained by either concepts. However, this does not provide any hint about the complementarity or the substituability of these two concepts.

As a consequence of this positive correlation between pessimism (optimism) and risk-tolerance (risk-aversion), the more risk-tolerant will insure the less risk-tolerant less than in the standard setting, which induces less risk-sharing.

The consensus belief, which is given by the average of the individual beliefs weighted by the risk-tolerance, is pessimistic. Intuitively, the more risk-tolerant agents make the market, and the consensus belief reflects the characteristics of the more risk-tolerant. Since we have just seen that the more risk-tolerant is pessimistic, it is consistent to obtain a pessimistic consensus belief.

The risk premium is greater than in the standard rational expectations equilibrium, which is interesting in light of the risk premium puzzle. This is easily understandable, since, as we have seen, in equilibrium models with heterogeneous beliefs the risk premium is higher than in the standard setting if and only if the consensus belief is pessimistic. The reason why pessimism increases the risk premium is not that a pessimistic agent requires a higher risk premium. She/He requires the same risk premium but her/his pessimism leads her/him to underestimate the average rate of return of the risky asset. Thus the objective expectation of the equilibrium risk premium is greater than the agent's subjective expectation, hence is greater than the standard risk premium (see Abel, 2002, and Jouini and Napp, 2006).

To sum up, our construction of endogenous beliefs through an evolutionary approach leads to beliefs that are different from the objective belief, heterogeneous, and antagonistic (one is optimistic and the other is pessimistic). There is a positive correlation between risk-tolerance (resp. risk-aversion) and pessimism (resp. optimism), which leads to less risk-sharing and to a higher risk premium.

Our results are robust to variations in the total endowment. At first sight, a negative supply in the risky asset seems to lead to an optimistic bias. Indeed, in that case, our evolution process induces an upward bias on the mean of the risky asset distribution but this corresponds to a pessimistic bias on the total wealth of the economy. The unique situation where all the effects we exhibited disappear corresponds to the case where there is no aggregate risk (i.e. when the total supply in risky assets is equal to zero). Indeed, in such a framework, there is no trade at the Walrasian equilibrium and there is then no price effect and no utility gain associated to a deviation from the objective belief.

3.2 Numerical illustration

In this section, we provide a simple numerical illustration of how the individual beliefs converge towards the Nash equilibrium beliefs. Recall that the strategic set of each agent is the set of measures over $\mathcal{B}$. In order to make things tractable, we consider the case where $\mathcal{B}$ is reduced to 2 points. Since we want to characterize the convergence towards the Nash equilibrium beliefs, we take $\mathcal{B} = \{\widehat{\mu}_A, \widehat{\mu}_B\}$ with $\widehat{\mu}_A = \mu - \frac{\sigma^2}{4\theta_B(\theta_A+\theta_B)}(\theta_A - \theta_B)$ and $\widehat{\mu}_B = \mu - \frac{\sigma^2}{4\theta_A(\theta_A+\theta_B)}(\theta_B - \theta_A)$.

We assume that the initial strategy of Ann consists in a weight $\pi_A(0)$ on $\widehat{\mu}_A$ and $(1-\pi_A(0))$ on $\widehat{\mu}_B$ and that the initial strategy of Bob consists in a weight $\pi_B(0)$ on $\widehat{\mu}_A$ and $(1-\pi_B(0))$ on $\widehat{\mu}_B$. Our goal is to characterize the evolution of the pair $(\pi_A(t), \pi_B(t))$ and, in particular, to analyze the speed of convergence towards $(1,0)$ that corresponds to the Nash equilibrium, i.e. the situation where Ann only plays $\widehat{\mu}_A$ and Bob only plays $\widehat{\mu}_B$.

If we further assume that $\pi_A(0) = \pi_B(0) = \frac{\theta_B}{(\theta_A+\theta_B)}$ then the initial average belief of Ann (Bob) corresponds to the objectif belief μ and the evolutionary process will lead them to diverge from the objective belief to their respective Nash equilibrium beliefs.

When Ann choses $\mu_A \in \mathcal{B}$ and Bob choses $\mu_B \in \mathcal{B}$, the average ex-post utility levels are given by

$$
\begin{aligned}
U_A(\mu_A,\mu_B) &= -\exp\left(\frac{-\frac{1}{2}p(\mu_A,\mu_B) + \alpha_A(\mu_A,\mu_B)\left(\mu - p(\mu_A,\mu_B)\right) - \frac{1}{2\theta_A}\alpha_A^2(\mu_A,\mu_B)\sigma^2}{\theta_A}\right)\\
U_B(\mu_A,\mu_B) &= -\exp\left(\frac{-\frac{1}{2}p(\mu_A,\mu_B) + \alpha_B(\mu_A,\mu_B)\left(\mu - p(\mu_A,\mu_B)\right) - \frac{1}{2\theta_B}\alpha_B^2(\mu_A,\mu_B)\sigma^2}{\theta_B}\right)
\end{aligned}
$$

where, as previously, $p(\mu_A,\mu_B) = \frac{\theta_A}{\theta_A+\theta_B}\mu_A + \frac{\theta_B}{\theta_A+\theta_B}\mu_B - \frac{\sigma^2}{\theta_A+\theta_B}$, $\alpha_A(\mu_A,\mu_B) = \theta_A\frac{\mu_A - p}{\sigma^2}$ and $\alpha_B(\mu_A,\mu_B) = \theta_B\frac{\mu_B - p}{\sigma^2}$. We introduce the following log transformations of U_A and U_B respectively denoted by $U_A^{\log}$ and $U_B^{\log}$

$$
\begin{aligned}
U_A^{\log}(\mu_A,\mu_B) &= -\frac{1}{2}p(\mu_A,\mu_B) + \alpha_A(\mu_A,\mu_B)\left(\mu - p(\mu_A,\mu_B)\right) - \frac{1}{2\theta_A}\alpha_A^2(\mu_A,\mu_B)\sigma^2,\\
U_B^{\log}(\mu_A,\mu_B) &= -\frac{1}{2}p(\mu_A,\mu_B) + \alpha_B(\mu_A,\mu_B)\left(\mu - p(\mu_A,\mu_B)\right) - \frac{1}{2\theta_B}\alpha_B^2(\mu_A,\mu_B)\sigma^2.
\end{aligned}
$$

We now extend the domain of $U_A^{\log}$ (resp. $U_B^{\log}$) in order to take into account the fact that Bob (resp. Ann) may play a mixed strategy described by a distribution $(\pi_B, 1-\pi_B)$ (resp. $(\pi_A, 1-\pi_A)$) over $\mathcal{B}$. We then have the following natural extended definitions for $U_A^{\log}$ and $U_B^{\log}$

$$
\begin{aligned}
U_A^{\log}(\mu_A,(\pi_B,1-\pi_B)) &= \pi_B U_A^{\log}(\mu_A,\widehat{\mu}_A) + (1-\pi_B)U_A^{\log}(\mu_A,\widehat{\mu}_B),\\
U_B^{\log}(\left(\pi_A,1-\pi_A\right),\mu_B) &= \pi_A U_B^{\log}(\widehat{\mu}_A,\mu_B) + (1-\pi_A)U_B^{\log}(\widehat{\mu}_B,\mu_B).
\end{aligned}
$$

Let us now introduce the functions V_A and V_B that correspond to the replicator dynamics (Equations 1 and 2) or, in other words, to the linearization of Equations 3 and 4, for the utility functions $U_A^{\log}$ and $U_B^{\log}$. We have

$$\begin{aligned}
V_A(\mu_A, (\pi_B, 1-\pi_B)) &= U_A^{\log}(\mu_A, (\pi_B, 1-\pi_B)) \\
&\quad - \Big(\pi_A U_A^{\log}(\widehat{\mu}_A, (\pi_B, 1-\pi_B)) + (1-\pi_A)\, U_A^{\log}(\widehat{\mu}_B, (\pi_B, 1-\pi_B))\Big), \\
V_B((\pi_A, 1-\pi_A), \mu_B) &= U_B^{\log}((\pi_A, 1-\pi_A), \mu_B) \\
&\quad - \Big(\pi_B U_B^{\log}((\pi_A, 1-\pi_A), \widehat{\mu}_A) + (1-\pi_B)\, U_B^{\log}((\pi_A, 1-\pi_A), \widehat{\mu}_B)\Big).
\end{aligned}$$

Our evolution equations are then given by

$$\begin{aligned}
\frac{d\pi_A}{\pi_A} &= \lambda V_A(\widehat{\mu}_A, (\pi_B, 1-\pi_B))dt, \\
\frac{d\pi_B}{\pi_B} &= \lambda' V_B((\pi_A, 1-\pi_A), \widehat{\mu}_B)dt.
\end{aligned}$$

These equations reflect the fact that the weight on $\widehat{\mu}_A$ increases proportionally to the difference between the "utility" resulting from playing $\widehat{\mu}_A$ and the average "utility", where the utility is measured through a mean-variance criterium.

We show in the Appendix that (π_A, π_B) converges to $(1, 0)$ and that the convergence is exponential.

Let us now calibrate this example. The speed of convergence is directly governed by λ and λ'. Multiplying both parameters by 2 permits to converge twice as fast. At this stage we take $\lambda = \lambda' = 1$. We discuss at the end of this section how λ and λ' could be calibrated through experimental procedures. Let us assume that the initial wealth of each agent is \$10,000 and that the standard deviation of their gains or losses after 1 year is equal to \$200 (2% of their wealth). We have then $\mu = 0$ and $\sigma = 200$. If the relative risk aversion is, as usually assumed, between 1 and 10, we obtain that the levels of absolute risk aversion should be between 1,000 and 10,000. Let us take $\theta_A = 1,000$ and $\theta_B = 10,000$. The standard risk premium as expressed above is then given by $\frac{\sigma^2}{\theta_A+\theta_B} = 3.6$ which means

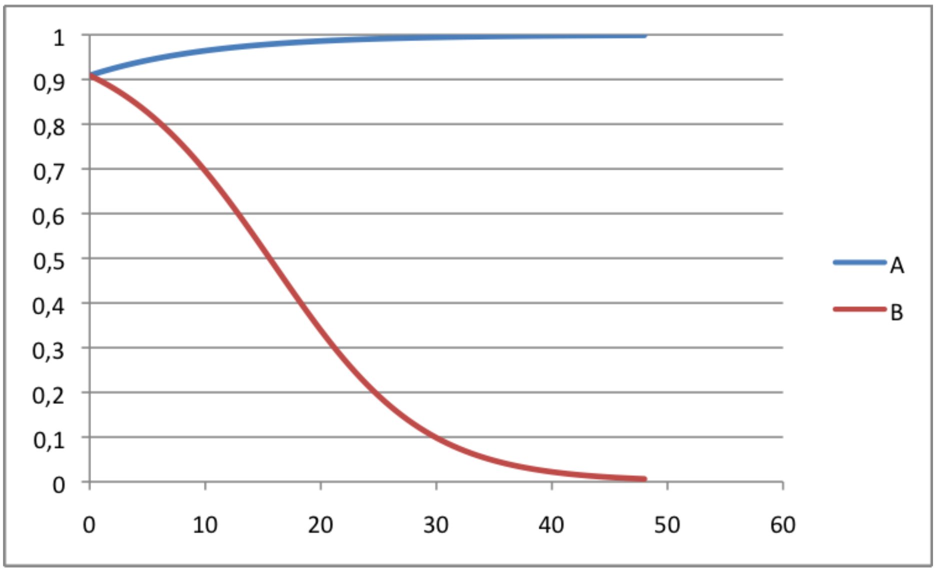

Figure 1: This Figure represents the evolution of π_A and π_B when $\theta_A = 1,000$, $\theta_B = 10,000$ and $\sigma = 200$. The starting point (t=0) corresponds to an average belief for each agent that coincides with the objective belief. The x-axis scale is in months.

that the agents are willing to pay \$3.6, at the equilibrium and in the standard setting, in order to insure themselves against a gain or a loss whose standard deviation is \$200. Numerical simulations give that after 46 months, we have $\pi_A(t) > 0.99$ and $\pi_B(t) < 0.01$. The risk premium is at $t = 46$ increased by $\frac{1}{4}\frac{(\theta_A-\theta_B)^2\sigma^2}{\theta_A\theta_B(\theta_A+\theta_B)} = 7.34$ which means that the risk premium is multiplied by 3. We can check that the risk premium is multiplied by 2 after only 17 months. Figures 1 to 3 respectively represent the evolution of π_A and π_B, of Ann's and Bob's beliefs and of the risk premium.

As underlined above, the speed of convergence is proportional to λ and λ'. Calibrating these parameters can be done through the following simple experiment. Two agents (a real one, referred to as the agent, and a computer simulated one) face a market with one asset whose total supply is 1 and whose final payoff is governed by a normal distribution $\mathcal{N}(\mu^*, 1)$. We assume that the risk tolerance level θ of the agent has already been estimated and we know that his/her demand function is of the form $\alpha(p) = \theta(\mu - p)$ where μ corresponds to his/her subjective belief. The agent is asked for his/her demand if the asset price is equal to p_1 (let us say $p_1 = 0.3$) and for his/her demand if the asset price is equal to p_2 (let us say $p_2 = 0.7$). His/her demand function is then linearly interpolated from these answers. The intercept of this function with the y-axis reveals the

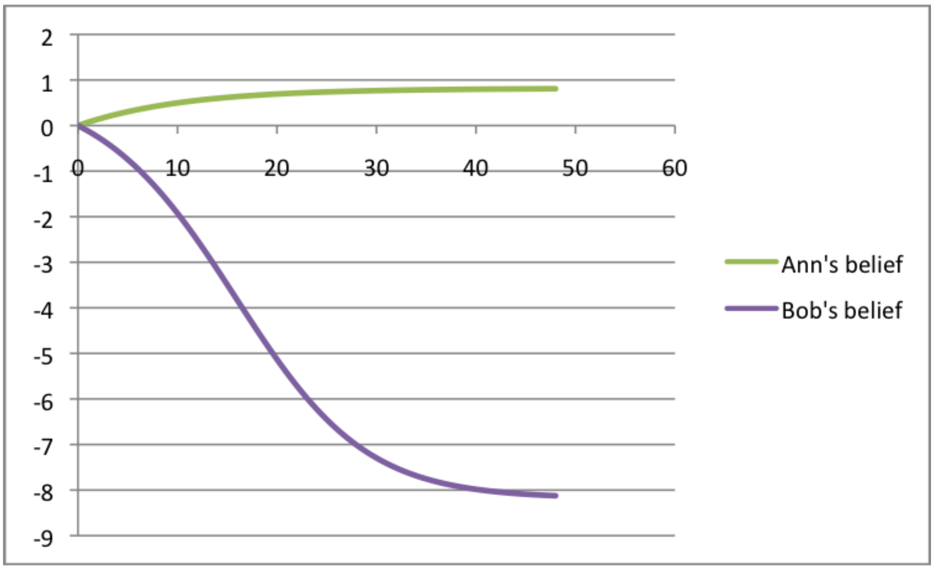

Figure 2: This Figure represents the evolution of Ann's and Bob's beliefs when $\theta_A = 1,000$, $\theta_B = 10,000$ and $\sigma = 200$. The starting point (t=0) corresponds to an average belief for each agent that coincides with the objective belief (0 average gain or loss). The x-axis scale is in months. Bob is more risk tolerant and becomes pessimistic while Ann becomes optimistic. It is easy to see that the average belief is pessimistic.

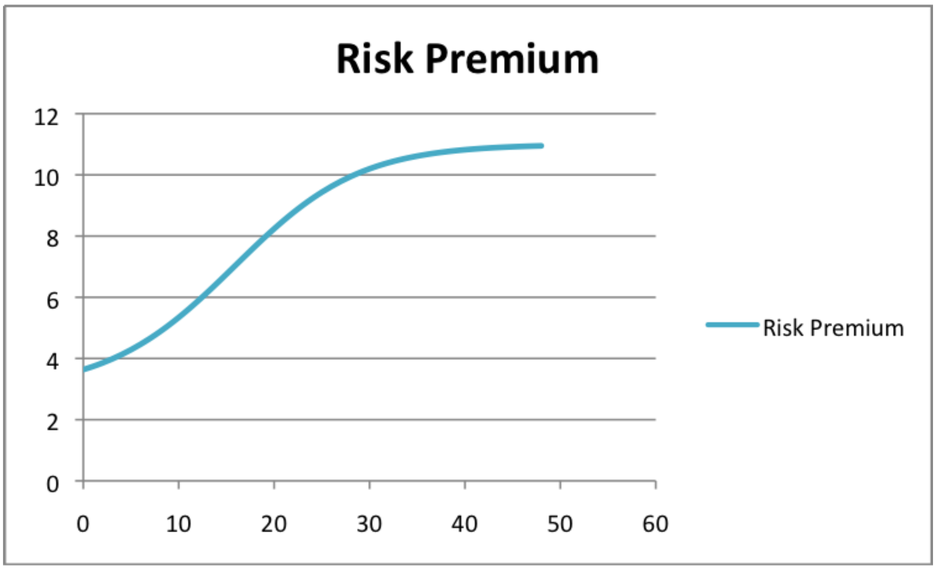

Figure 3: This Figure represents the risk premium evolution when $\theta_A = 1,000$, $\theta_B = 10,000$ and $\sigma = 200$. The starting point (t=0) corresponds to an average belief for each agent that coincides with the objective belief. The x-axis scale is in months. The risk premium is multiplied by 2 after 17 months and multiplied by 3 after 46 months.

agent's subjective belief μ. The equilibrium price is computed assuming that the demand function for the simulated agent is $\bar{d}(p) = \bar{\theta}\,(\bar{\mu} - p)$ which corresponds to a subjective belief $\bar{\mu}$ and to a risk tolerance level $\bar{\theta}$. The equilibrium price and the asset's payoff are then revealed and payments take place. If this sequence is played repeatedly, it is then possible to observe the evolution of agent's beliefs and to analyze its speed of convergence.

4 Evolutionary vs "optimal" beliefs

The construction of endogenous subjective beliefs that are solutions of a given utility maximization problem has been considered in recent literature by Brunnermeier and Parker (2005), Gollier (2005), Gollier and Muermann (2010), Brunnermeier et al. (2007). In our framework, the subjective beliefs are not optimal but *evolutionary* and asymptotically strategic. Indeed, they do not result from an individual utility maximization problem but from an evolutionary process and asymptotically from a Nash equilibrium in which each agent takes into account the impact of her/his choices on the equilibrium price and allocations. In a non-strategic setting where agents choose their belief in order to maximize a criterion related to their well-being, we show that the *optimal* belief must be optimistic for all agents and that all agents select a riskier portfolio. This result is intuitive whereas there is no immediate intuition for a given systematic bias in our evolutionary setting.

Let us compare our results with those that are obtained in an *optimal* framework. More precisely, adopting the same framework and notations as above, we consider the following concept of optimal beliefs, which corresponds to a simplified version of Brunnermeier and Parker (2005), Gollier (2005) and Brunnermeier et al. (2007).

Definition 4 *For a given price p, an optimal belief $\breve{\mu}_i(p)$ for agent i is defined as the solution of*

$$\arg\max_{\mu_i \in [\underline{\mu}, \bar{\mu}]} E_i \left[u_i \left(\frac{1}{2} p + \alpha_i(p, \mu_i)\,(\widetilde{x} - p) \right) \right]$$

where E_i is the expectation operator associated[11] *to the belief μ_i.*

[11] More precisely, E_i is the expectation operator associated to a probability P_i that represents agent i's belief and under which $\widetilde{x} \sim \mathcal{N}(\mu_i, \sigma^2)$.

The belief $\check{\mu}_i(p)$ is optimal in the sense that it maximizes over the set $\left[\underline{\mu}, \overline{\mu}\right]$ the well-being of agent i.

The original definition of optimal beliefs introduced by Brunnermeier and Parker (2005) and further studied by Brunnermeier et al. (2007) and Gollier (2005) considers a weighted average of the objectively expected utility and of the subjectively expected utility. Our definition is more simple but it is easy to check that the results obtained below, under Definition 4, remain valid under the original definition. This will be further discussed at the end of this section.

We can now define an associated equilibrium concept as follows.

Definition 5 *An equilibrium price with optimal beliefs is defined as a price $\check{p}$ such that agents have optimal demands and optimal beliefs and such that markets clear, i.e.*

$$\alpha_A(\check{p}, \check{\mu}_A(\check{p})) + \alpha_B(\check{p}, \check{\mu}_B(\check{p})) = 1.$$

Proposition 6 *In the setting of the previous section (exponential utility and normal distributions), we have*

1. *For a given price p, the optimal belief $\check{\mu}_i(p)$ for agent i solves*

$$\max_{\mu \in \left\{\underline{\mu}, \overline{\mu}\right\}} (\mu - p)^2.$$

2. *If $\frac{\sigma^2}{\theta_A + \theta_B} \geq \frac{\overline{\mu} - \underline{\mu}}{2}$, then the equilibrium is characterized by $\check{p} = \overline{\mu} - \frac{\sigma^2}{\theta_A + \theta_B}$ and $\check{\mu}_A(\check{p}) = \check{\mu}_B(\check{p}) = \overline{\mu}$. The agents share the same optimistic belief and the risk premium is lower than in the standard setting.*

3. *If $\frac{\theta_A \underline{\mu} + \theta_B \overline{\mu}}{\theta_A + \theta_B} - \frac{\sigma^2}{\theta_A + \theta_B} = \frac{\underline{\mu} + \overline{\mu}}{2}$, where $\theta_A < \theta_B$, then the equilibrium is characterized by $\check{p} = \frac{\underline{\mu} + \overline{\mu}}{2}$, $\check{\mu}_A(\check{p}) = \underline{\mu}$ and $\check{\mu}_B(\check{p}) = \overline{\mu}$. The more risk-tolerant agent is the more optimistic and the consensus belief $\frac{\theta_A \underline{\mu} + \theta_B \overline{\mu}}{\theta_A + \theta_B}$ is more optimistic than the equally weighted belief $\frac{\underline{\mu} + \overline{\mu}}{2}$.*

We have then two possible situations but both of them induce an optimistic bias at the aggregate level. Furthermore, unless $\frac{\theta_A \underline{\mu} + \theta_B \overline{\mu}}{\theta_A + \theta_B} - \frac{\sigma^2}{\theta_A + \theta_B} = \frac{\underline{\mu} + \overline{\mu}}{2}$, there is no belief heterogeneity and both agents are optimistic. In fact, even when $\frac{\theta_A\underline{\mu}+\theta_B\overline{\mu}}{\theta_A+\theta_B} - \frac{\sigma^2}{\theta_A+\theta_B} = \frac{\underline{\mu}+\overline{\mu}}{2}$, the more risk averse agent is not truly pessimistic. Indeed, it is easy to check that this agent is short in the risky asset and is then optimistic with respect to his own allocation, i.e. overestimates the return of his own portfolio.

Notice that for $\frac{\sigma^2}{\theta_A+\theta_B} < \frac{\overline{\mu}-\underline{\mu}}{2}$ and $\frac{\sigma^2}{\theta_A+\theta_B} \neq \frac{\underline{\mu}+\overline{\mu}}{2} - \frac{\theta_A\underline{\mu}+\theta_B\overline{\mu}}{\theta_A+\theta_B}$ where Ann is less risk-tolerant than Bob, there is no equilibrium. A natural extension of the model would consist in allowing for mixed strategies or for a continuum of agents. We retrieve then an equilibrium in which a proportion η of the agents choose the belief $\underline{\mu}$ and a proportion $(1-\eta)$ choose the belief $\overline{\mu}$. For instance, if we assume that the distribution of beliefs is independent of the distribution of risk-tolerances, the market clearing condition leads to

$$\eta\underline{\mu} + (1-\eta)\,\overline{\mu} - \frac{\sigma^2}{\int \theta_i di} = \frac{\underline{\mu}+\overline{\mu}}{2}. \tag{15}$$

The proportion η is then perfectly determined if $\frac{\sigma^2}{\int \theta_i di} \leq \frac{\overline{\mu}-\underline{\mu}}{2}$. The solution η is always lower than $\frac{1}{2}$ which means that the consensus belief is always optimistic. This equilibrium in which each agent is indifferent between two possible beliefs and in which the market clearing condition imposes the proportions of agents choosing each belief resembles the equilibrium obtained in Brunnermeier and Parker (2005).

These results are analogous to those of Brunnermeier and Parker (2005) even if in their case there is no aggregate risk[12].

We would obtain the same kind of results if we considered a weighted average of the objectively expected utility and the subjectively expected utility as in the original model of Brunnermeier and Parker (2005)

$$\max_{\mu_i \in K} \left\{ \beta E\left[u_i \left(\frac{1}{2}p + \alpha_i(p,\mu_i)\,(\widetilde{x}-p) \right) \right] + (1-\beta) E_i \left[u_i \left(\frac{1}{2}p + \alpha_i(p,\mu_i)\,(\widetilde{x}-p) \right) \right] \right\}.$$

For β large enough, in other words when the weight on the objective expectation

[12] In this case, there is no absolute concept of optimism or pessimism and both agents are optimistic with respect to their own equilibrium allocation.

is beyond a given threshold, then the agents share the same belief and this belief is optimistic. Otherwise, there is not a unique optimal belief, agents have extreme beliefs (i.e. $\underline{\mu}$ or $\overline{\mu}$), but the possible equilibria still lead to an optimistic average belief[13]. In all cases, the average optimal belief is optimistic leading to a lower risk premium. These results are similar to those obtained by Gollier (2005) in a general discrete distributions setting.

To conclude, in the *optimal* setting, agents' beliefs are always optimistic (with respect to their own allocations) and the risk premium is always lower than in the rational expectations setting. Furthermore, except for specific degenerate situations (see Equation 15), the agents share the same belief. The difference between *optimal* and *evolutionary* beliefs is now clear, since in the latter setting, there is belief heterogeneity, one agent is optimistic while the other is pessimistic and the risk premium is higher.

5 The general case

The purpose of this section is to analyze the robustness of the results of Section 3 to more general utility functions and distributions. In particular, we show that even with more general assumptions, the game Γ associated to our evolutionary process admits Nash equilibria that have the same qualitative properties as those exhibited in the exponential/normal framework.

For this purpose, we consider a family of beliefs $(P_{\tilde{x}}^{\mu})_{\mu \in [\underline{\mu},\overline{\mu}]}$, corresponding to the possible subjective distributions for $\tilde{x}$. We may assume without any loss of generality that the objective distribution corresponds to $\mu = 0$. We further assume that $0 \in [\underline{\mu},\overline{\mu}]$, i.e. the objective distribution lies in the set of possible/plausible beliefs for the agents. In the next, we also assume that all the considered expectations exist and are finite. For $\mu \in [\underline{\mu},\overline{\mu}]$, we let $f(.,\mu)$ denote the density function of $P_{\tilde{x}}^{\mu}$ with respect to the Lebesgue measure on $\mathbb{R}_+$ and we let E^{μ} denote the expectation operator under the density

[13] More precisely, for $\beta < \frac{1}{2}$, the agents have extreme beliefs, $\underline{\mu}$ or $\overline{\mu}$, as above and there might exist equilibria with heterogeneous optimal beliefs if the model parameters satisfy a condition like Equation (15). For $\beta > \frac{1}{2}$, i.e. when there is more weight on the objective expectation, and if $\overline{\mu}$ is sufficiently large ($\overline{\mu} > \overline{\mu}^*$ for some $\overline{\mu}^*$) the agents share the same belief and this belief is an interior point of $[\underline{\mu},\overline{\mu}]$. For $\beta = \frac{1}{2}$ or for $\overline{\mu} \leq \overline{\mu}^*$ the agents share the same belief $\overline{\mu}$.

$f(.,\mu)$, i.e. $E^{\mu}\left[g(\widetilde{x})\right] = \int g(s)f(s,\mu)dx$. We will simply denote by E (instead of E^0) the expectation operator associated to $P^0_{\widetilde{x}}$.

As in Section 3, our economy is composed of two agents, initially endowed with a half unit of the risky asset $\tilde{x}$ and we consider the game in which these agents can manipulate their beliefs and choose an optimal composition of their portfolio, taking into account the effect their trading has on price.

We make the following assumption.

Assumption (A)

- The utility functions u_A and u_B are increasing, strictly concave and twice continuously differentiable on $\mathbb{R}_+$,
- Inada conditions: $u_i'(0) = +\infty$ and $\lim_{x\to\infty} u_i'(x) = 0$,
- The family $(P^{\mu}_{\widetilde{x}})_{\mu\in\left[\underline{\mu},\overline{\mu}\right]}$ is increasing in the sense of the first-order stochastic dominance, i.e. for all $x > 0$ we have $\int_0^x f(s,\mu)ds \geq \int_0^x f(s,\mu')ds$ for $\mu' \geq \mu$ in $\left[\underline{\mu},\overline{\mu}\right]$.
- The functions $s \mapsto su_i'(s)$, $i \in \{A, B\}$, are increasing.

The first condition is standard. The second one guarantees interior solutions to the individual portfolio choice problem. The third condition ensures an order on the set of admissible beliefs. The setting of Section 3 satisfies this condition. More generally, any family of beliefs $(P^{\mu}_{\widetilde{x}})_{\mu\in\mathbb{R}_+}$ such that $f(s,\mu) = g(s-\mu)$ for a given distribution function g on $\mathbb{R}_+$ satisfies this monotonicity condition. Another example is provided by a family of log-normal distributions, $(\ln\mathcal{N}(\mu,\sigma^2))_{\mu\in\mathbb{R}}$. The fourth condition guarantees that a first-order stochastic dominance shift in the risky asset's payoffs increases the demand for the risky asset (see Gollier, 2001). The same portfolio property can be obtained without this condition if we replace the first-order stochastic dominance of the third condition by the monotone likelihood ratio order (Landsberger and Meilijson, 1990).

Let us first determine the Walrasian equilibrium characteristics. For a given belief $P^{\mu}_{\widetilde{x}}$ and for a given price p, the demand function of agent i for $i \in \{A, B\}$, is given by

$$\alpha_i(p,\mu) = \arg\max_{\alpha_i} E^{\mu}\left[u_i(\frac{1}{2}p + \alpha_i(\tilde{x} - p))\right].$$

For a pair of beliefs (μ_A, μ_B), the equilibrium price $p(\mu_A, \mu_B)$ is determined by the market-clearing condition $\alpha_A(p(\mu_A, \mu_B), \mu_A) + \alpha_B(p(\mu_A, \mu_B), \mu_B) = 1$ and the associated optimal demand for agent i is defined by $\alpha_i^*(\mu_A, \mu_B) = \alpha_i(p(\mu_A, \mu_B), \mu_i)$. The Walrasian equilibrium utility level for agent i is then given by

$$U_i(\mu_A, \mu_B) = E\left[u_i(\frac{1}{2}p(\mu_A, \mu_B) + \alpha_i^*(\mu_A, \mu_B)(\tilde{x} - p(\mu_A, \mu_B)))\right]. \tag{16}$$

Proposition 7 *Under Assumption (A),*

1. *The functions $\alpha_i(p, \mu)$, $p(\mu_A, \mu_B)$ and $\alpha_i^*(\mu_A, \mu_B)$ are well defined and satisfy $\frac{\partial \alpha_i}{\partial p}(p, \mu) \leq 0$, $\frac{\partial \alpha_i}{\partial \mu}(p, \mu) \geq 0$, $\frac{\partial p}{\partial \mu_i} \geq 0$, $\frac{\partial \alpha_i^*}{\partial \mu_i} \geq 0$, $\frac{\partial \alpha_i^*}{\partial \mu_j} \leq 0$, $i \in \{A, B\}$, $j \neq i$.*

2. *If the game $\Gamma = (\mathcal{B}, \mathcal{B}, U_A, U_B)$, with $\mathcal{B} = \left[\underline{\mu}, \overline{\mu}\right]$, admits an interior Nash equilibrium $(\hat{\mu}_A, \hat{\mu}_B)$, i.e. a Nash equilibrium such that $\hat{\mu}_i \in \left(\underline{\mu}, \overline{\mu}\right)$, then one of the agents (agent i) is pessimistic and the other one (agent j) is optimistic and we have $\alpha_j^*(\hat{\mu}_A, \hat{\mu}_B) \leq \frac{1}{2} \leq \alpha_i^*(\hat{\mu}_A, \hat{\mu}_B)$.*

3. *If one of the agents (say Ann) is more risk-averse than the other one in the sense of Arrow-Pratt, then $\alpha_A^*(\hat{\mu}_A, \hat{\mu}_B) \leq \frac{1}{2}$, and there is a positive correlation between pessimism and risk-tolerance.*

4. *If the game $\Gamma = (\mathcal{B}, \mathcal{B}, U_A, U_B)$ admits a unique interior Nash equilibrium $(\hat{\mu}_A, \hat{\mu}_B)$ and if the functions U_A and U_B have increasing differences, i.e. $U_i(\mu'_A, \mu'_B) - U_i(\mu_A, \mu'_B) \geq U_i(\mu'_A, \mu_B) - U_i(\mu_A, \mu_B)$ for all $\mu'_A \geq \mu_A$ and $\mu'_B \geq \mu_B$, then the evolutionary process defined by (3) and (4) converges to the pair of Dirac distributions $\left(\delta_{\hat{\mu}_A}, \delta_{\hat{\mu}_B}\right)$.*

As far as the Walrasian equilibrium is concerned, we obtain first that the optimal demand of the agents (as a function of the price and the belief) increases with the belief and decreases with the price, which are natural properties. As a consequence, the equilibrium price increases with the beliefs, which is also natural; if the asset is more "desirable", its equilibrium price increases. An increase in the belief of agent i has then two effects on her/his demand α_i^* at the equilibrium, a direct positive effect and an indirect negative effect due to the price increase. The global effect is positive. The effect of an increase of the belief of agent i on the equilibrium demand α_j^* of the other agent is negative because there is only one effect, namely the price effect.

We obtain that the heterogeneity of is robust to the choice of more general utility functions and distributions. Moreover, as in Section 3, one agent is optimistic and the other agent is pessimistic. The pessimistic agent is the one for which the net demand is positive. This result can be explained as before. For the agent who expresses a positive net demand for the risky asset, a pessimistic belief is associated to a lower price and a higher expected utility; the evolutionary belief balances this benefit of pessimism against the costs of worse decision making. The converse reasoning applies to the other agent, who, at the equilibrium, has a negative net demand in the risky asset and benefits from optimism.

The positive correlation between pessimism and risk-tolerance is also robust to this more general setting. When one of the agents is more risk-tolerant, her/his net demand is necessarily positive. Otherwise, she/he would have a negative demand which would lead to an optimistic belief while the other agent would be pessimistic, more risk-averse with a positive net demand. This is obviously impossible. The positive correlation follows.

These properties are then satisfied at any interior Nash equilibrium of the game $\Gamma = (\mathcal{B}, \mathcal{B}, U_A, U_B)$ and, in particular, at any Nash equilibrium of the game $\Gamma^\infty = (\mathbb{R}, \mathbb{R}, U_A, U_B)$.

Additional assumptions about the structure of the game are needed in order to characterize the asymptotic behaviour of the evolutionary process. Exploring the conditions under which the game Γ admits a unique Nash equilibrium or the conditions under which U_i has increasing differences goes beyond the scope of this paper. However, if such conditions are satisfied then the evolutionary process converges to a pair of heterogeneous and antagonistic beliefs that are positively correlated with the level of risk aversion when the utility functions are well ordered with respect to this criterion.

6 Extensions of the model

In this section, we analyze the robustness of our results (heterogeneity of beliefs, positive correlation between pessimism and risk-tolerance in the game Γ, convergence of the evolutionary process to the Nash equilibrium of the game Γ and, in the exponential utility/normal distributions setting, aggregate pessimism and higher risk premia) to other specifications of the model. In previous sections, agents modified their beliefs in order to favor the most rewarding beliefs in absolute terms. We may also consider the situation where the agents favor the more rewarding beliefs in relative terms, that is to say those beliefs that permit to perform better than the other agents. Until now, we have analyzed beliefs evolution when the set of possible beliefs is ordered for first-order stochastic dominance shifts. Such shifts correspond to changes on the mean for normal distributions and more generally can be interpreted in terms of optimism and pessimism. We can also be interested in beliefs evolution when the set of possible beliefs is ordered for mean preserving spreads. Such spreads correspond to changes on the variance for normal distributions and more generally can be interpreted in terms of overconfidence and doubt/underconfidence. For tractability reasons, we will analyze this impact in an exponential utility and normal distributions framework (as in Section 3). We will also analyze, in an analogous framework, if our results are robust to the introduction of multiple sources of risk.

6.1 Comparing with the other agents

Let us consider the situation where the agents watch each other and compare their utility level with the utility level of the other agent. We analyze the asymptotic behavior of a beliefs dynamics based on the rule of thumb that consists in favoring those beliefs that permit to reach a utility level that is higher than the utility level of the other agent at the expense of those beliefs that lead to a utility level that is lower than the utility level of the other agent.

In fact, it suffices to consider, as previously, dynamics that favor high utility level beliefs at the expense of low utility level beliefs when the utility functions $U_A(\gamma, \theta) = \int u_A(\gamma, \theta, \widetilde{x})\, dP$ and $U_B(\gamma, \theta) = \int u_B(\gamma, \theta, \widetilde{x})\, dP$ are replaced by the comparative utility functions $\overline{U}_A(\gamma, \theta, x) = \frac{U_A(\gamma,\theta)}{U_B(\gamma,\theta)}$ and $\overline{U}_B(\gamma, \theta, x) = \frac{U_B(\gamma,\theta)}{U_A(\gamma,\theta)}$.

The same reasoning as above permits to show that such a dynamics converges to the Nash equilibrium of the game where Ann and Bob are endowed with the functions $\overline{U}_A(\gamma, \theta)$ and $\overline{U}_B(\gamma, \theta)$. Under the specifications of Section 3 about the distribution of $\widetilde{x}$ and about u_A and u_B, we obtain

Proposition 8 *Let us consider a model with two agents, exponential utility functions and a risky asset $\widetilde{x} \sim \mathcal{N}(\mu, \sigma^2)$ and let us assume that the possible beliefs are of the form $\widetilde{x} \sim \mathcal{N}(\mu', \sigma^2)$ with $\mu' \in \mathcal{B} = \left[\underline{\mu}, \overline{\mu}\right]$. We have*

1. *For given initial distributions Υ_0 and Θ_0 in $\Delta(\mathcal{B})$ with full support over $\mathcal{B}$, any dynamics that favors, for each agent, beliefs that lead to higher utility levels with respect to the other agent converges to a pair of Dirac distributions $\left(\delta_{\widetilde{\mu}_A}, \delta_{\widetilde{\mu}_B}\right)$ where $(\widetilde{\mu}_A, \widetilde{\mu}_B)$ is the unique Nash equilibrium of the game $\Gamma = (\mathcal{B}, \mathcal{B}, \overline{U}_A, \overline{U}_B)$.*

2. *The surviving beliefs $(\widetilde{\mu}_A, \widetilde{\mu}_B)$ are given by*

$$\widetilde{\mu}_A = \mu + \frac{1}{4}\theta_B^{-1}\theta_A^{-1}(\theta_B - \theta_A)\sigma^2, \qquad \widetilde{\mu}_B = \mu - \frac{1}{4}\theta_B^{-1}\theta_A^{-1}(\theta_B - \theta_A)\sigma^2. \tag{17}$$

   Asymptotically, the more risk-tolerant agent is pessimistic, in the sense that she/he behaves as if the mean of $\widetilde{x}$ lied below its true value, and the less risk-tolerant agent is optimistic.

3. *Asymptotically, the consensus belief is pessimistic, i.e. the average of the individual beliefs weighted by the risk-tolerance is pessimistic. More precisely,*

$$\frac{\theta_A\widetilde{\mu}_A + \theta_B\widetilde{\mu}_B}{\theta_A + \theta_B} = \mu - \frac{1}{4}\frac{(\theta_A - \theta_B)^2}{\theta_A\theta_B(\theta_A + \theta_B)}\sigma^2. \tag{18}$$

4. *Asymptotically, the risk premium RP (resp. the price) is higher (resp. lower) than*

in the standard rational expectations equilibrium. More precisely

$$RP = RP^{stdd} + \left(\mu - \frac{\theta_A \widetilde{\mu}_A + \theta_B \widetilde{\mu}_B}{\theta_A + \theta_B}\right) = RP^{stdd} + \frac{1}{4} \frac{(\theta_A - \theta_B)^2 \sigma^2}{\theta_A \theta_B (\theta_A + \theta_B)}. \tag{19}$$

Remark that the average belief $\frac{\widetilde{\mu}_A + \widetilde{\mu}_B}{2}$ is now equal to μ; however the consensus belief remains unchanged with respect to the situation where each agent favors those beliefs that lead to higher utility levels in absolute terms (Equations 13 and 18). For the same reasons, the impact on the risk premium is also unchanged (Equations 14 and 19).

6.2 Disagreement on the variance

The model is the same as in Section 3 except that the beliefs are now about the variance of $\widetilde{x}$. Our aim is to analyze how these distributions evolve through interactions between the 2 groups and through the evolutionary Equations (3) and (4) introduced above. The payoff of the risky asset $\widetilde{x}$ is still normally distributed with mean μ and variance σ^2 and the set of beliefs is parametrized by $\sigma \in \mathcal{B} \overset{def}{=} [\underline{\sigma}, \overline{\sigma}]$. With this parametrization, an agent who has a belief $\sigma_i \in \mathcal{B}$ believes that $\widetilde{x}$ is normally distributed with mean μ and variance σ_i^2.

The Walrasian equilibrium characteristics are determined as follows. If Ann has a belief σ_A and Bob has a belief σ_B, then the optimal demand of agent i is given by $\alpha_i(p, \sigma_i) = \theta_i \frac{\mu - p}{\sigma_i^2}$ and the market clearing price is given by $p(\sigma_A, \sigma_B) = \mu - \left(\frac{\theta_A}{\sigma_A^2} + \frac{\theta_B}{\sigma_B^2}\right)^{-1}$. The bias with respect to the objective belief can here be interpreted as a form of doubt/underconfidence ($\sigma_i > \sigma$) or overconfidence ($\sigma_i < \sigma$) instead of the pessimism/optimism biases[14] of Section 3. Note that the obtained equilibrium price corresponds to the equilibrium price in an economy in which agents share the same belief, namely an harmonic average of the initial beliefs, weighted by the risk-tolerance[15]. In this setting, the equilibrium risk premium is given by $\mu - p(\sigma_A, \sigma_B) = \left(\frac{\theta_A}{\sigma_A^2} + \frac{\theta_B}{\sigma_B^2}\right)^{-1}$, which means that the

[14] See Abel (2002) for concepts of pessimism and doubt related to first and second order stochastic dominance.

[15] Walrasian equilibrium models in which agents have heterogeneous beliefs on the variance of the asset under consideration have been studied by, among others, Abel (1989, 2002), Duchin-Levy (2010), Fama-French (2007), Jouini-Napp (2006) and Yan (2010).

risk premium in an economy with heterogeneous subjective beliefs is higher than in the standard rational expectations setting if and only if the consensus belief exhibits doubt (i.e., $(\frac{\theta_A}{\sigma_A^2}+\frac{\theta_B}{\sigma_B^2})^{-1} > \frac{\sigma^2}{\theta_A+\theta_B}$). The Walrasian utility levels are then given by

$$U_i(\sigma_A,\sigma_B) = E\left[u_i\left\{\frac{1}{2}p\left(\sigma_A,\sigma_B\right)+\left\{\widetilde{x}-p\left(\sigma_A,\sigma_B\right)\right\}\alpha_i^*\left(\sigma_A,\sigma_B\right)\right\}\right],$$

where $\alpha_i^*\left(\sigma_A,\sigma_B\right)=\alpha_i\left(p\left(\sigma_A,\sigma_B\right),\sigma_i\right)$.

With these specifications and for given initial distributions Υ_0 and Θ_0 with full support over $\mathcal{B}$, the selection dynamics given by (3) and (4) is fully described. The game $\Gamma = (\mathcal{B},\mathcal{B},U_A,U_B)$ is also fully described.

Proposition 9 *1. For any initial distributions Υ_0 and Θ_0 in $\Delta(\mathcal{B})$ with full support over $\mathcal{B}$, the evolutionary process converges to a pair of Dirac distributions $(\delta_{\widehat{\sigma}_A},\delta_{\widehat{\sigma}_B})$ where $(\widehat{\sigma}_A,\widehat{\sigma}_B)$ is the unique Nash equilibrium of the game $\Gamma=(\mathcal{B},\mathcal{B},U_A,U_B)$.*

2. The asymptotic beliefs $(\widehat{\sigma}_A,\widehat{\sigma}_B)$ are given by

$$\widehat{\sigma}_A^2=\sigma^2\left(1+\frac{\theta_A-\theta_B}{4\theta_B}\right), \qquad \widehat{\sigma}_B^2=\sigma^2\left(1+\frac{\theta_B-\theta_A}{4\theta_A}\right). \tag{20}$$

The more risk-tolerant agent exhibits doubt, in the sense that she/he overestimates the variance of $\widetilde{x}$, and the less risk-tolerant agent is overconfident. Moreover, the more risk-tolerant agent exhibits more doubt than the less risk-tolerant agent exhibits overconfidence, and the unweighted (harmonic) average of the beliefs exhibits doubt:

$$2\left(\frac{1}{\widehat{\sigma}_A^2}+\frac{1}{\widehat{\sigma}_B^2}\right)^{-1}=\sigma^2\left(1+\frac{\left(\theta_A-\theta_B\right)^2}{2\left(\theta_A^2+\theta_B^2+6\theta_A\theta_B\right)}\right). \tag{21}$$

3. The consensus belief exhibits doubt, i.e. the harmonic average of the individual beliefs weighted by the risk-tolerance exhibits doubt. More precisely,

$$\left(\theta_A+\theta_B\right)\left(\frac{\theta_A}{\widehat{\sigma}_A^2}+\frac{\theta_B}{\widehat{\sigma}_B^2}\right)^{-1}=\sigma^2\left(1+\frac{3\left(\theta_A-\theta_B\right)^2}{16\theta_A\theta_B}\right).$$

4. *The risk premium (resp. the price) is higher (resp. lower) than in the standard rational expectations equilibrium. More precisely*

$$RP = RP^{stdd} + \left(\left(\frac{\theta_A}{\sigma_A^2} + \frac{\theta_B}{\sigma_B^2} \right)^{-1} - \frac{\sigma^2}{\theta_A + \theta_B} \right) = RP^{stdd} + \frac{3\left(\theta_A - \theta_B\right)^2}{16\theta_A \theta_B \left(\theta_A + \theta_B\right)} \sigma^2.$$

We then retrieve the same properties as in Section 3 except that pessimism is replaced by doubt. Note that behavioural studies in a non strategic and non market context generally exhibit overconfidence instead of doubt (Shiller, 2000, p.142). The evolutionary framework induces then an effect in the opposite direction and this might explain that Giordani and Söderlind (2006) "find little evidence of either overconfidence or doubt" in the survey of professional forecasters.

6.3 The model with two sources of risk

The model is essentially the same as in Section 3 except that we now suppose that there are two sources of risk in the economy, whose associated payoffs at the end of the period are respectively denoted by $\widetilde{x}$ and $\widetilde{y}$. We let p (resp. q) denote the price of $\widetilde{x}$ (resp. $\widetilde{y}$) and we assume that $\widetilde{x}$ and $\widetilde{y}$ are normally distributed, more precisely $\widetilde{x} \sim \mathcal{N}(\mu, \sigma^2)$ and $\widetilde{y} \sim \mathcal{N}(\nu, \varpi^2)$. We let ρ denote the correlation between $\widetilde{x}$ and $\widetilde{y}$, i.e., $\rho \equiv \frac{cov(\widetilde{x},\widetilde{y})}{\sigma\varpi}$. Each agent is initially endowed with one half unit of each risky asset.

We assume that the set of possible beliefs is given by $\mathcal{B} = \left[\underline{\mu}, \overline{\mu}\right] \times \left[\underline{\nu}, \overline{\nu}\right]$. For a given belief (μ_i, ν_i) for agent i and for a given price system (p, q), it is immediate to show that the demand functions α_i and β_i respectively in the first and the second asset are given by

$$\alpha_i(p, (\mu_i, \nu_i)) = \theta_i \frac{\mu_i - p}{\sigma^2 \left(1 - \rho^2\right)} - \theta_i \frac{\left(\nu_i - q\right)\rho}{\sigma\varpi\left(1 - \rho^2\right)}, \tag{22}$$

$$\beta_i(p, (\mu_i, \nu_i)) = \theta_i \frac{\nu_i - q}{\varpi^2 \left(1 - \rho^2\right)} - \theta_i \frac{\left(\mu_i - p\right)\rho}{\sigma\varpi\left(1 - \rho^2\right)}. \tag{23}$$

The Walrasian equilibrium price system $(p((\mu_A, \nu_A), (\mu_B, \nu_B)), q((\mu_A, \nu_A), (\mu_B, \nu_B)))$ is determined by the market-clearing conditions $\alpha_A(p, (\mu_A, \nu_A)) + \alpha_B(p, (\mu_B, \nu_B)) = 1$ and

$\beta_A(p,(\mu_A,\nu_A)) + \beta_B(p,(\mu_B,\nu_B)) = 1$, which leads to an equilibrium price for the first (second) asset that only depends on agents' beliefs about that asset

$$p(\mu_A,\mu_B) = \frac{\theta_A\mu_A + \theta_B\mu_B}{\overline{\theta}} - \frac{\sigma^2 + \sigma\varpi\rho}{\overline{\theta}}, \tag{24}$$
$$q(\nu_A,\nu_B) = \frac{\theta_A\nu_A + \theta_B\nu_B}{\overline{\theta}} - \frac{\varpi^2 + \sigma\varpi\rho}{\overline{\theta}}. \tag{25}$$

From Equations (22) to (25), we derive easily α_A^*, α_B^*, β_A^* and β_B^* the optimal quantities of assets at the equilibrium.

The Walrasian equilibrium utility level for agent i is then given by

$$U_i((\mu_A,\nu_A),(\mu_B,\nu_B)) = E\left[u_i(\frac{1}{2}p(\mu_A,\mu_B) + \frac{1}{2}q(\nu_A,\nu_B) + \alpha_i^*(\tilde{x} - p(\mu_A,\mu_B)) + \beta_i^*(\widetilde{y} - q(\nu_A,\nu_B)))\right]. \tag{26}$$

Proposition 10 *1. For any initial distributions Υ_0 and Θ_0 in $\Delta(\mathcal{B})$ with full support over $\mathcal{B}$, the evolutionary process converges to a pair of Dirac distributions $\left(\delta_{(\widehat{\mu}_A,\widehat{\nu}_A)},\delta_{(\widehat{\mu}_B,\widehat{\nu}_B)}\right)$ where $((\widehat{\mu}_A,\widehat{\nu}_A),(\widehat{\mu}_B,\widehat{\nu}_B))$ is the unique Nash equilibrium of the game $\Gamma = (\mathcal{B},\mathcal{B},U_A,U_B)$.*

2. The beliefs $(\widehat{\mu}_i,\widehat{\nu}_i)_{i=A,B}$ are given by

$$\widehat{\mu}_i = \mu - \frac{(\theta_i - \theta_j)(\sigma^2 + \sigma\varpi\rho)}{4\theta_j\overline{\theta}}, \qquad \widehat{\nu}_i = \nu - \frac{(\theta_i - \theta_j)(\varpi^2 + \sigma\varpi\rho)}{4\theta_j\overline{\theta}}.$$

3. The risk premia (resp. the prices) are given by

$$\mu - p = RP^{stdd}(\widetilde{x}) + \frac{1}{4}\frac{(\theta_A - \theta_B)^2(\sigma^2 + \sigma\varpi\rho)}{\theta_A\theta_B\overline{\theta}}$$
$$\nu - q = RP^{stdd}(\widetilde{y}) + \frac{1}{4}\frac{(\theta_A - \theta_B)^2(\varpi^2 + \sigma\varpi\rho)}{\theta_A\theta_B\overline{\theta}}$$

where $RP^{stdd}(\widetilde{x})$ and $RP^{stdd}(\widetilde{y})$ denote the standard risk-premium for $\widetilde{x}$ and $\widetilde{y}$ in an homogenous beliefs setting. The risk premium on $\widetilde{x}$ (resp. $\widetilde{y}$) is higher than in the standard setting if and only if $cov(\widetilde{x},\widetilde{x}+\widetilde{y}) > 0$ (resp. $cov(\widetilde{y},\widetilde{x}+\widetilde{y}) > 0$).

As far as the market portfolio $(\widetilde{x}+\widetilde{y})$ is concerned, the market risk-premium RP^M and the beliefs ξ_i^M on the average market return are given by

$$\begin{aligned}
\xi_i^M &= \xi - \frac{(\theta_i - \theta_j)\,\sigma_M^2}{4\theta_j\overline{\theta}} \\
RP^M &= RP^{stdd}(\widetilde{x}+\widetilde{y}) + \frac{1}{4}\frac{(\theta_A - \theta_B)^2\,\sigma_M^2}{\theta_A\theta_B\overline{\theta}}
\end{aligned}$$

where $\xi = \mu + \nu$ and $\sigma_M^2 = \varpi^2 + 2\rho\sigma\varpi + \sigma^2$ correspond respectively to the objective market portfolio return and variance. These formulas are exactly the same as in the one asset framework which means that the more risk tolerant (risk averse) agent is pessimistic (optimistic) at the aggregate level and the consensus belief is pessimistic at the aggregate level. The formulas for individual assets that are provided in the proposition are similar to those obtained in the one asset framework. However, for each asset, the variance term in the one-asset formula is replaced by the covariance of the considered asset payoffs with the market portfolio payoffs. Recall that in the CAPM setting, the equilibrium price for a given asset depends on the covariance of the payoffs of this asset with the payoffs of the market portfolio and not on the total variance of the asset payoffs. Since beliefs evolution is directly related to price formation, it is natural to obtain optimal beliefs that depend on the covariance with the market portfolio and not on the total variance. The aggregate level properties (pessimism, correlation between pessimism and risk tolerance, higher risk premium) are then retrieved at the individual assets level as far as these assets are positively correlated with the market portfolio.

It is interesting to note that these effects are more pronounced for the riskier asset. Intuitively, the evolutionary behaviour leads to more beliefs dispersion for the riskier asset and hence to a more pronounced impact on the market for the riskier asset.

7 Conclusion

The introduction of an evolutionary framework for belief formation provides a rationale for belief heterogeneity; an evolution process that favors the most rewarding beliefs leads to beliefs that are subjective, heterogeneous and antagonistic. The selection of evolutionary beliefs is governed by very precise rules. These beliefs must be related to the individual level of risk-aversion: the beliefs of more risk-averse agents exhibit optimism and/or overconfidence and the beliefs of more risk-tolerant agents exhibit pessimism and/or doubt. As a consequence, there is a positive correlation between pessimism/doubt and risk-tolerance. In a setting with exponential utility and normal distributions, the average belief exhibits pessimism and/or doubt as well as the consensus belief. This is compatible with the observation that subjects in experimental and empirical studies exhibit a dose of pessimism (Wakker, 2001, Ben Mansour et al., 2006, Giordani and Söderlind, 2006). This induced pessimism/doubt of investors leads to higher risk premia, which is interesting in light of the equity premium puzzle (Mehra and Prescott, 1985). In the insurance industry, our results lead to a situation where the more risk-averse agent (the insured) is optimistic and the less risk-averse agent (the insurer) is pessimistic. The average belief is pessimistic leading to a higher insurance premium, which might help to explain the purchase of vastly overpriced insurance in a range of situations (Cutler and Zeckhauser, 2004). In corporate finance, IPOs can be modeled as a decision for a risk-averse entrepreneur to sell shares of her/his firm to more risk-tolerant investors. The application of our results to such a setting leads to a pessimistic consensus belief. As a result, the firm is underpriced and the short run return is large, which is consistent with the empirical literature on IPOs (Ibbotson and Ritter, 1995).

Obviously, we do not pretend that evolutionary features, such as in our simple model, are the unique explanation for these puzzles, however, it is interesting to remark that our approach helps to explain these puzzles as well as belief heterogeneity without introducing any information asymmetry nor principal-agent features.

This work suggests further investigation in two main directions. First, in this paper we have let aside information asymmetry and heterogeneity in order to focus on the impact of market interactions on individual beliefs and from there on equilibrium prices and allocations. It would be useful to consider a more general model including a strategic use of private information combined with an evolutionary framework for belief formation.

Second, we have only considered totally ordered families of possible subjective distributions for the risky asset payoffs. In particular, all beliefs deformations can be interpreted in terms of pessimism/optimism or in terms of doubt/overconfidence. It would be interesting to consider more general possible deformations of the objective distribution in particular in terms of higher order moments (as skewness and kurtosis).

Appendix

Proof of Proposition 3

2. Let us first analyze the game Γ. The expected utility of agent i at the Walrasian equilibrium, given the belief μ_j of agent j, $j \neq i$, can be written

$$\begin{aligned} U_i(\mu_A, \mu_B) &= E\left[-\exp\left(-\frac{\frac{1}{2}p(\mu_A,\mu_B) + \alpha_i(\mu_A,\mu_B)(\widetilde{x} - p(\mu_A,\mu_B))}{\theta_i}\right)\right] \\ &= -\exp\left[-\left(\frac{\frac{1}{2}p(\mu_A,\mu_B) + \alpha_i(\mu_A,\mu_B)(\mu - p(\mu_A,\mu_B))}{\theta_i} - \frac{1}{2}\left(\frac{\alpha_i(\mu_A,\mu_B)}{\theta_i}\right)^2 \sigma^2\right)\right] \end{aligned}$$

where $\alpha_i(\mu_A,\mu_B) = \theta_i \frac{\mu_i - p(\mu_A,\mu_B)}{\sigma^2}$. We now have an explicit expression for the utility level functions of both agents. The problem of agent i is then to maximize this utility level with respect to μ_i, the strategy μ_j of the other player being given.

Maximizing $U_i(\mu_A,\mu_B)$ with respect to μ_i amounts to maximizing

$$W_i(\mu_A,\mu_B) = \frac{1}{2}p(\mu_A,\mu_B) + \alpha_A(\mu_A,\mu_B)(\mu - p(\mu_A,\mu_B)) - \frac{1}{2}\frac{1}{\theta_A}(\alpha_A(\mu_A,\mu_B))^2\sigma^2.$$

This program is concave and the maximum is reached for μ_i such that $\frac{dW_i}{d\mu_i}(\mu_i) = 0$. This corresponds to the best response $BR_i(\mu_j)$ of agent i when agent j plays μ_j and we have

$$BR_i(\mu_j) = \frac{2\mu\theta_j(\theta_A + \theta_B) + 2\theta_i\theta_j\mu_j + \sigma^2(\theta_j - \theta_i)}{4\theta_j\theta_i + 2\theta_i^2}. \tag{27}$$

We then solve for $(\widehat{\mu}_A, \widehat{\mu}_B)$ such that $BR_A(\widehat{\mu}_B) = \widehat{\mu}_A$ and $BR_B(\widehat{\mu}_A) = \widehat{\mu}_B$. We obtain Equations (11).

1. Let us now prove that this unique Nash equilibrium characterizes the limit of our evolutionary process. First, let us show that the set of strategies surviving iterated strict

dominance in the game Γ is reduced to the unique Nash equilibrium exhibited above. It is equivalent to prove this result for the game Γ or for the game $\widetilde{\Gamma}$ defined by the same strategic sets as Γ but with functionals W_A and W_B instead of U_A and U_B. It suffices then to show that $\widetilde{\Gamma}$ is a supermodular game (see e.g. Milgrom and Roberts, 1990, or Fudenberg and Tirole, 1991). Since our strategic sets are compact and our functions are continuous, it suffices to check that $W_i(\mu_A, \mu_B)$ has increasing differences in (μ_A, μ_B), for $i \in \{A, B\}$ or equivalently that $\frac{\partial^2 W_i}{\partial \mu_A \partial \mu_B}(\mu_A, \mu_B) \geq 0$. This is immediate.

Let us show now that any strictly dominated strategy is eliminated by the evolution process.

Let us consider μ_A that is stricly dominated by μ'_A from Ann's point of view. This means that $U_A(\mu_A, \mu_B) < U_A(\mu'_A, \mu_B)$ for all μ_B. We have then for every distribution Θ over $\mathcal{B}$, $\int U_A(\mu_A, \mu_B) d\Theta < \int U_A(\mu'_A, \mu_B) d\Theta$ and then $V_A(\mu_A, \Theta) < V_A(\mu'_A, \Theta)$. Since $\mathcal{B}$ is compact, $\Delta(\mathcal{B})$ is compact with respect to the weak-topology. We have then $V_A(\mu_A, \Theta) + k < V_A(\mu'_A, \Theta)$ for some $k > 0$. Let us now consider some open subset $\mathcal{A}$ (resp. $\mathcal{A}'$) of $\mathcal{B}$ containing μ_A (resp. μ'_A) such that $V_A(\mu, \Theta) + \frac{k}{2} < V_A(\mu', \Theta)$ for all $(\mu, \mu') \in \mathcal{B} \times \mathcal{B}$ and all $\Theta \in \Delta(\mathcal{B})$. We have

$$\frac{d}{dt} \ln \left(\frac{\Upsilon_t(\mathcal{A})}{\Upsilon_t(\mathcal{A}')} \right) = \frac{\int_{\mathcal{A}} V_A(\gamma, \Theta_t) d\Upsilon_t}{\Upsilon_t(\mathcal{A})} - \frac{\int_{\mathcal{A}'} V_A(\gamma, \Theta_t) d\Upsilon_t}{\Upsilon_t(\mathcal{A}')} < -\frac{k}{2}$$

which gives that $\frac{\Upsilon_t(\mathcal{A})}{\Upsilon_t(\mathcal{A}')} \to 0$ and then $\Upsilon_t(\mathcal{A}) \to 0$. For each strictly dominated strategy μ_A, there exists then an open set $\mathcal{A}$ such that $\mu_A \in \mathcal{A} \subset \mathcal{B}$ and such that $\Upsilon_t(\mathcal{A}) \to 0$. Adapting the proof of Theorem 1 in Heifetz et al. (2007b) to non-symmetric functions, we obtain that the asymptotic distribution concentrates all the mass on the unique strategy that is not eliminated by iterated strict dominance, that is to say $\widehat{\mu}_A$.

3. and 4. Straightforward using Equations (11). ■

Proof of the results of Section 3.2

The evolution can be rewritten as follows

$$\begin{aligned}
\frac{d\pi_A}{\pi_A(1-\pi_A)} &= \left(W_A(\widehat{\mu}_A, (\pi_B, 1-\pi_B)) - W_A(\widehat{\mu}_B, (\pi_B, 1-\pi_B))\right) dt \\
\frac{d\pi_B}{\pi_B(1-\pi_B)} &= \left(W_B((\pi_A, 1-\pi_A), \widehat{\mu}_A) - W_B((\pi_A, 1-\pi_A), \widehat{\mu}_B)\right) dt
\end{aligned}$$

and after computations

$$\begin{aligned}
\frac{d\pi_A}{\pi_A(1-\pi_A)} &= \frac{1}{32}\frac{(2\theta_A+\theta_B+2\pi_B\theta_A)}{(\theta_A+\theta_B)^2\theta_A\theta_B}(\theta_B-\theta_A)^2\sigma^2 dt, \\
\frac{d\pi_B}{\pi_B(1-\pi_B)} &= \frac{1}{32}\frac{(2\pi_A\theta_B-4\theta_B-\theta_A)}{(\theta_A+\theta_B)^2\theta_A\theta_B}(\theta_B-\theta_A)^2\sigma^2 dt.
\end{aligned}$$

Remark that if we introduce $\pi_B^* = 1-\pi_B$, the system (π_A, π_B^*) satisfies

$$\begin{aligned}
\frac{d\pi_A}{(1-\pi_A)\pi_A} &= \frac{1}{32}\frac{(4\theta_A+\theta_B-2\pi_B^*\theta_A)}{(\theta_A+\theta_B)^2\theta_A\theta_B}(\theta_B-\theta_A)^2\sigma^2 dt, \\
\frac{d\pi_B^*}{(1-\pi_B^*)\pi_B^*} &= \frac{1}{32}\frac{(4\theta_B+\theta_A-2\pi_A\theta_B)}{(\theta_A+\theta_B)^2\theta_A\theta_B}(\theta_B-\theta_A)^2\sigma^2 dt.
\end{aligned}$$

Since π_A, π_B and π_B^* are in $[0,1]$, the growth rate of $\frac{\pi_A}{1-\pi_A}$ that is given by $\frac{d}{dt}\ln\frac{\pi_A}{1-\pi_A} = \frac{d\pi_A}{(1-\pi_A)\pi_A dt}$, is bounded below by $K = \frac{1}{32}\frac{(2\theta_A+\theta_B)}{(\theta_A+\theta_B)^2\theta_A^2\theta_B}(\theta_B-\theta_A)^2\sigma^2 > 0$. A similar bound is obtained for π_B^*. Both $\frac{\pi_A(t)}{1-\pi_A(t)}$ and $\frac{\pi_B^*(t)}{1-\pi_B^*(t)}$ are then increasing and converge to ∞ which gives that (π_A, π_B) converges to $(1,0)$.

Furthermore, we have

$$\ln\frac{\pi_A(t)}{1-\pi_A(t)} \geq \ln\frac{\pi_A(0)}{1-\pi_A(0)} + Kt$$

which gives

$$1-\pi_A(t) \leq \frac{\pi_A(0)}{1-\pi_A(0)}\exp(-Kt).$$

The convergence of $\pi_A(t)$ towards 1 is then exponential and so is the convergence of $\pi_B^*(t)$ towards 1 or equivalently the convergence of $\pi_B(t)$ towards 0. ■

Proof of Proposition 6

The utility level of agent i is given by $E_i\left[u_i\left(\frac{1}{2}p+\alpha_i(p,\mu_i)(\widetilde{x}-p)\right)\right]$ with $\alpha_i(p,\mu_i) =$

$\theta_i \frac{\mu_i - p}{\sigma^2}$. Then, for a given p, the agent maximizes $\theta_i \frac{(\mu_i - p)^2}{\sigma^2}$.

When $p > \frac{\underline{\mu}+\overline{\mu}}{2}$, all the agents have the same belief $\underline{\mu}$ and the equilibrium price, if it exists, must satisfy $\overline{p} = \underline{\mu} - \frac{\sigma^2}{\theta_A+\theta_B}$ which is not compatible with the condition $p > \frac{\underline{\mu}+\overline{\mu}}{2}$. When $p < \frac{\underline{\mu}+\overline{\mu}}{2}$, all the agents have the same belief $\overline{\mu}$ and the equilibrium price, if it exists, must satisfy $\overline{p} = \overline{\mu} - \frac{\sigma^2}{\theta_A+\theta_B}$ which is compatible with the condition $p < \frac{\underline{\mu}+\overline{\mu}}{2}$ only if $\frac{\sigma^2}{\theta_A+\theta_B} > \frac{\overline{\mu}-\underline{\mu}}{2}$. When $p = \frac{\underline{\mu}+\overline{\mu}}{2}$, both agents may choose the same belief $\overline{\mu}$ leading to an equilibrium only if $\frac{\sigma^2}{\theta_A+\theta_B} = \frac{\overline{\mu}-\underline{\mu}}{2}$. They may also choose different beliefs. If Ann (Bob) chooses $\underline{\mu}$ (resp. $\overline{\mu}$), the market clearing condition leads to

$$\frac{\theta_A \underline{\mu} + \theta_B \overline{\mu}}{\theta_A + \theta_B} - \frac{\sigma^2}{\theta_A + \theta_2} = \frac{\underline{\mu} + \overline{\mu}}{2}. \tag{28}$$

If $\mu = \frac{\underline{\mu}+\overline{\mu}}{2}$ and $\theta_A < \theta_B$, then $\frac{\theta_A \underline{\mu}+\theta_B \overline{\mu}}{\theta_A+\theta_B} > \mu$. ■

Proof of Proposition 7

1. It is well known that due to Inada conditions, the demand function is characterized by the following first order condition

$$E^{\mu}\left[(\tilde{x} - p)u_i'(\alpha_i(p,\mu)(\tilde{x} - p) + \frac{1}{2}p)\right] = 0$$

and the partial derivatives of $\alpha_i(p,\mu)$ with respect to p and μ are given by

$$\begin{aligned}
\frac{\partial \alpha_i}{\partial p}(p,\mu) &= -\frac{E^{\mu}\left[\left(\frac{1}{2} - \alpha_i(p,\mu)\right)(\tilde{x} - p)u_i''(c(p,\mu)) - u_i'(c(p,\mu))\right]}{E^{\mu}\left[(\tilde{x} - p)^2 u_i'(c(p,\mu))\right]}, \\
\frac{\partial \alpha_i}{\partial \mu}(p,\mu) &= -\frac{\frac{\partial}{\partial \mu} E^{\mu}\left[(\tilde{x} - p)u_i'(c(p,\mu))\right]\Big|_{(\mu,p,\alpha_i(p,\mu))}}{E^{\mu}\left[(\tilde{x} - p)^2 u_i'(c(p,\mu))\right]},
\end{aligned}$$

with $c(p,\mu) = \alpha_i(p,\mu)(\tilde{x} - p) + \frac{1}{2}p$. Letting $\widetilde{c}$ denote $c(p,\mu)$, remark that

$$E^{\mu}\left[\left(\frac{1}{2} - \alpha_i(p,\mu)\right)(\tilde{x} - p)u_i''(\widetilde{c}) - u_i'(\widetilde{c})\right] = E^{\mu}\left[-u_i'(\widetilde{c}) - \widetilde{c}u_i''(\widetilde{c}) + \frac{1}{2}\widetilde{x}u_i''(\widetilde{c})\right].$$

Hence, $\frac{\partial \alpha_i}{\partial p}(p,\mu)$ is negative. Furthermore, $(\tilde{x} - p)u_i'(\widetilde{c}) = \frac{1}{\alpha_i(p,\mu)}\widetilde{c}u'(\widetilde{c}) - \frac{1}{2}\frac{p}{\alpha_i(p,\mu)}u'(\widetilde{c})$ and is then increasing. By the first-stochastic dominance property, we have $\frac{\partial \alpha_i}{\partial \mu}(p,\mu) \geq 0$.

We have then

$$\frac{\partial p}{\partial \mu_i} = -\frac{\frac{\partial \alpha_i}{\partial \mu_i}(p, \mu_i)}{\frac{\partial \alpha_A}{\partial p}(p, \mu_A) + \frac{\partial \alpha_B}{\partial p}(p, \mu_B)}$$

hence $\frac{\partial p}{\partial \mu_i} \geq 0$, $i = A, B$.

For $i \neq j$, we have $\frac{\partial \alpha_i^*(\mu_A,\mu_B)}{\partial \mu_i} = \frac{\partial \alpha_i(p,\mu_i)}{\partial \mu_i} + \frac{\partial \alpha_i(p,\mu_i)}{\partial p}\frac{\partial p}{\partial \mu_i} = \frac{\frac{\partial \alpha_i}{\partial \mu_i}(p,\mu_i)\frac{\partial \alpha_j}{\partial p}(p,\mu_j)}{\frac{\partial \alpha_A}{\partial p}(p,\mu_A)+\frac{\partial \alpha_B}{\partial p}(p,\mu_B)} \geq 0.$

2. If the game Γ has an interior Nash equilibrium $(\hat{\mu}_A, \hat{\mu}_B)$, the first-order condition for agent i at $(\hat{\mu}_A, \hat{\mu}_B)$ gives

$$E\left[(\frac{\partial \alpha_i^*}{\partial \mu_i}(\tilde{x}-p) + \left(\frac{1}{2} - \alpha_i^*\right)\frac{\partial p}{\partial \mu_i})u_i'(\alpha_i^*(\tilde{x}-p) + \frac{1}{2}p)\right] = 0.$$

If $\frac{1}{2} - \alpha_i^*(\hat{\mu}_A, \hat{\mu}_B) \leq 0$ then $\left(\frac{1}{2} - \alpha_i^*(\hat{\mu}_A, \hat{\mu}_B)\right)\frac{\partial p}{\partial \mu_i} \leq 0$, hence $E\left[(\tilde{x} - p\left(\hat{\mu}_A, \hat{\mu}_B\right))u_i'\right] \geq 0$. As previously, by the first-order stochastic dominance property we obtain $\hat{\mu}_i \leq 0$. Analogously $\frac{1}{2} - \alpha_i^*(\hat{\mu}_A, \hat{\mu}_B) \geq 0$ leads to $\hat{\mu}_i \geq 0$. We have then proved that the agent for which $\alpha_i^*(\hat{\mu}_A, \hat{\mu}_B) \geq \frac{1}{2}$ (resp. $\alpha_i^*(\hat{\mu}_A, \hat{\mu}_B) \leq \frac{1}{2}$) is pessimistic (resp. optimistic).

3. If one of the utility functions (let us say u_A) is more risk-averse than the other one in the sense of Arrow-Pratt, let us prove that $\alpha_A^*(\hat{\mu}_A, \hat{\mu}_B) \leq \frac{1}{2}$. If this is not the case, we have $\alpha_B^*(\hat{\mu}_A, \hat{\mu}_B) \leq \frac{1}{2}$ and Bob is optimistic while Ann is pessimistic. We have then

$$\frac{1}{2} < \alpha_A(p\left(\hat{\mu}_A, \hat{\mu}_B\right), \hat{\mu}_A) \leq \alpha_B(p\left(\hat{\mu}_A, \hat{\mu}_B\right), \hat{\mu}_A)$$

because Ann is more risk-averse. Furthermore we have $\alpha_B(p\left(\hat{\mu}_A, \hat{\mu}_B\right), \hat{\mu}_A) \leq \alpha_B(p\left(\hat{\mu}_A, \hat{\mu}_B\right), \hat{\mu}_B)$ because $\hat{\mu}_B$ is larger than $\hat{\mu}_A$. We would have then $\alpha_B^*(\hat{\mu}_A, \hat{\mu}_B) > \frac{1}{2}$ which contradicts our assumption.

4. See Proof of Proposition 3 (point 1). ■

Proof of Proposition 8

Let us consider the game where Ann and Bob are endowed with $\overline{U}_A$ and $\overline{U}_B$. We have

$$\begin{aligned}
\overline{U}_A(\mu_A, \mu_B) &= \exp(W_B(\mu_A, \mu_B) - W_A(\mu_A, \mu_B)) \\
\overline{U}_B(\mu_A, \mu_B) &= \exp(W_A(\mu_A, \mu_B) - W_B(\mu_A, \mu_B))
\end{aligned}$$

where W_A and W_B have been introduced in the proof of Proposition 3.

If we solve for $\frac{d\overline{U}_A}{d\mu_A} = \frac{d\overline{U}_B}{d\mu_B} = 0$, we obtain $\widetilde{\mu}_A = \mu + \frac{1}{4}\theta_B^{-1}\theta_A^{-1}(\theta_B - \theta_A)\sigma^2$ and $\widetilde{\mu}_B = \mu - \frac{1}{4}\theta_B^{-1}\theta_A^{-1}(\theta_B - \theta_A)\sigma^2$. ■

Proof of Proposition 9

2. Let us first characterize the Nash equilibrium in the game Γ. Agent i maximizes

$$W_i(\sigma_A, \sigma_B) = \frac{1}{2}p + \alpha_i(\mu - p) - \frac{1}{2}\frac{1}{\theta_i}(\alpha_i)^2\sigma^2$$

with respect to σ_i where p and α_i both depend on σ_i and are given by $p = \mu - \left(\frac{\theta_A}{\sigma_A^2} + \frac{\theta_B}{\sigma_B^2}\right)^{-1}$ and $\alpha_i = \theta_i\frac{\mu - p}{\sigma_i^2}$.

This problem is concave in $\left(\frac{\theta_A}{\sigma_A^2} + \frac{\theta_B}{\sigma_B^2}\right)^{-1}$. Setting $\frac{dW_i}{d\sigma_i}(\widehat{\sigma}_A, \widehat{\sigma}_B) = 0$ leads to

$$\widehat{\sigma}_i^2 = \frac{2}{3}\sigma^2 + \frac{1}{3}\left(\frac{\theta_i}{\theta_j}\widehat{\sigma}_j^2\right). \tag{29}$$

The resulting beliefs are then given by Equations (20). Equation (21) follows.

Since

$$\left|\left(\frac{\theta_A - \theta_B}{4\theta_B}\right)\left(\frac{4\theta_A}{\theta_B - \theta_A}\right)\right| = \frac{\theta_A}{\theta_B}$$

the more risk-tolerant agent exhibits more doubt than the less risk-tolerant agent exhibits overfidence.

1. In the "disagreement on the variance" case, the game is not supermodular. However, as in Section 3, the set of strategies surviving iterated strict dominance is reduced to the unique Nash equilibrium exhibited above. Indeed, let us denote by $[\underline{\sigma}_A^{2n}, \overline{\sigma}_A^{2n}]$ the set of strategies surviving after n rounds of strict dominance (these sets are constructed by induction and the construction will be clearer at the end of the argument) and let us denote by

$$\begin{aligned}
\underline{\sigma}_B^{2n+1} &= \arg\max A_2\left(\underline{\sigma}_A^{2n}, \sigma_B\right) = \sqrt{\frac{2}{3}\sigma^2 + \frac{1}{3}\left(\frac{\theta_B}{\theta_A}\left(\underline{\sigma}_A^{2n}\right)^2\right)} \\
\overline{\sigma}_B^{2n+1} &= \arg\max A_2\left(\overline{\sigma}_A^{2n}, \sigma_B\right) = \sqrt{\frac{2}{3}\sigma^2 + \frac{1}{3}\left(\frac{\theta_B}{\theta_A}\left(\overline{\sigma}_A^{2n}\right)^2\right)}.
\end{aligned}$$

It is immediate that $\underline{\sigma}_B^{2n+1} \leq \overline{\sigma}_B^{2n+1}$. Let us show that when $\sigma_A \in [\underline{\sigma}_A^{2n}, \overline{\sigma}_A^{2n}]$ any strategy for Bob that lies outside $\left[\underline{\sigma}_B^{2n+1}, \overline{\sigma}_B^{2n+1}\right]$ is stricly dominated by a strategy inside this interval. For a given $\sigma_A \in [\underline{\sigma}_A^{2n}, \overline{\sigma}_A^{2n}]$, the function $W_B(\sigma_A, \sigma_B)$ is increasing until $\sqrt{\frac{2}{3}\sigma^2 + \frac{1}{3}\left(\frac{\theta_B}{\theta_A}(\sigma_A)^2\right)}$ and then decreasing. Since $\sqrt{\frac{2}{3}\sigma^2 + \frac{1}{3}\left(\frac{\theta_B}{\theta_A}(\sigma_A)^2\right)} \in \left[\underline{\sigma}_B^{2n+1}, \overline{\sigma}_B^{2n+1}\right]$ this means that, for any $\sigma_B < \underline{\sigma}_B^{2n+1}$, we have $W_B(\sigma_A, \sigma_B) < W_B(\sigma_A, \underline{\sigma}_B^{2n+1})$ and σ_B is then strictly dominated by $\underline{\sigma}_B^{2n+1}$. Similarly, any $\sigma_B > \overline{\sigma}_B^{2n+1}$ is dominated by $\overline{\sigma}_B^{2n+1}$. The set $\left[\underline{\sigma}_A^{2n+2}, \overline{\sigma}_A^{2n+2}\right]$ is constructed similarly switching the roles between Ann and Bob. We have then decreasing sets of surviving strategies and it is easy to show that, at the limit, we must have $\underline{\sigma}_B^{\infty} = \sqrt{\frac{2}{3}\sigma^2 + \frac{1}{3}\left(\frac{\theta_B}{\theta_A}(\underline{\sigma}_A^{\infty})^2\right)}$ and $\underline{\sigma}_A^{\infty} = \sqrt{\frac{2}{3}\sigma^2 + \frac{1}{3}\left(\frac{\theta_A}{\theta_B}(\underline{\sigma}_B^{\infty})^2\right)}$. We know from 2. that these equations in $(\underline{\sigma}_A^{\infty}, \underline{\sigma}_B^{\infty})$ admit a unique solution $(\widehat{\sigma}_A, \widehat{\sigma}_B)$ and we have then $\underline{\sigma}_A^{\infty} = \widehat{\sigma}_A$ and $\underline{\sigma}_B^{\infty} = \widehat{\sigma}_B$. The same argument gives us $\overline{\sigma}_A^{\infty} = \widehat{\sigma}_A$ and $\overline{\sigma}_B^{\infty} = \widehat{\sigma}_B$ and the set of surviving strategies is reduced to $(\widehat{\sigma}_A, \widehat{\sigma}_B)$.

3. and 4. Straightforward using Equations (20). ■

Proof of Proposition 10

2. In the setting of the proposition, agent i maximizes

$$W_i(\mu_A, \nu_A, \mu_B, \nu_B) = \frac{1}{2}p + \alpha_i(\mu - p) + \frac{1}{2}q + \beta_i(\nu - q) - \frac{1}{2}\frac{1}{\theta_i}\left(\alpha_i^2\sigma^2 + \beta_i^2\varpi^2 + 2\alpha_i\beta_i\sigma\varpi\rho\right)$$

with respect to (μ_i, ν_i) taking $\left(\mu_j, \nu_j\right) = \left(\widehat{\mu}_j, \widehat{\nu}_j\right)$, $j \neq i$, as given. The maximization programs under consideration are concave. Setting $\frac{dW_i}{d\mu_i} = \frac{dW_i}{d\nu_i} = 0$, for $i = A, B$, leads to

$$\widehat{\mu}_i = \mu - \frac{(\theta_i - \theta_j)(\sigma^2 + \sigma\varpi\rho)}{4\theta_j\overline{\theta}}, \qquad \widehat{\nu}_i = \nu - \frac{(\theta_i - \theta_j)(\varpi^2 + \sigma\varpi\rho)}{4\theta_j\overline{\theta}}$$

which is the unique Nash equilibrium. We then conclude as in the proof of Proposition 3.

3. Straightforward using 2.

1. Let us show that the game Γ where Ann chooses her strategy $b^A = (\mu_A, \nu_A)$ in her strategic set $\mathcal{B} = \left[\underline{\mu}, \overline{\mu}\right] \times [\underline{\nu}, \overline{\nu}]$ and Bob chooses his strategy $b^B = (\mu_B, \nu_B)$ in the same strategic set, is a supermodulat game when $\mathcal{B}$ is endowed with the natural partial order $\preceq$ defined for $b = (\mu, \nu)$ and $b' = (\mu', \nu')$ by $b \preceq b'$ if $\mu \leq \mu'$ and $\nu \leq \nu'$. For this purpose,

it suffices to check that $W_i(b^A, b^B)$ has increasing differences, i.e. for all $b'_A \geq b_A$ and $b'_B \geq b_B$ we have $W_i(b'_A, b'_B) - W_i(b_A, b'_B) \geq W_i(b'_A, b_B) - W_i(b_A, b_B)$. Easy computations give us $\frac{\partial^2 W_i}{\partial \mu_i \partial \mu_j} \geq 0$, $\frac{\partial^2 W_i}{\partial \nu_i \partial \mu_j} \geq 0$, $\frac{\partial^2 W_i}{\partial \mu_i \partial \nu_j}$ and $\frac{\partial^2 W_i}{\partial \nu_i \partial \nu_j} \geq 0$ which is sufficient to prove that W_i has increasing differences. The rest of the proof is similar to the proof of Proposition 3.

■

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