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Crisp Fair Gambles

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Abstract

Axiomatic models of decision under ambiguity with a non-unique prior allow for the existence of Crisp Fair Gambles: acts whose expected utility is nul whichever of the priors is used. But, in these models, the DM has to be indifferent to the addition of such acts. Their existence is then at odds with a preference taking into account the variance of the prospects. In this paper we study some geometrical and topological properties of the set of priors that would rule out the existence of Crisp Fair Gambles, properties which have consequences on what can be an unambiguous financial asset.

JEL classification: D81, G11.

Keywords: Monotone Mean–Variance Preferences, Ambiguity, Set of Priors, Crisp acts, Unambiguous asset.

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1 Introduction

This work is motivated by recent papers ([Maccheroni, Marinacci, Rustichini, and Taboga, 2009](#) and [Černý, Maccheroni, Marinacci, and Rustichini, 2012](#)) which have shown some unsuspected links between Mean–Variance (MV) preferences and axiomatic models of decision under ambiguity.

[Markowitz \(1952\)](#)' work has brought a new perspective on the choice of the optimal portfolio and this point of view, the MV preference, remain the cornerstone of the theory. But the criterion itself is as disputed as it is ubiquitous for it has some shortcomings. Especially, while MV preferences are used to model rational behaviour, they fail to respect monotonicity, a widely admitted tenet of rationality which imposes that, if a prospect dominates state–by–state another one, it should be preferred by the decision maker (DM). This is why monotone criteria compatible with MV preferences attract such interest.

[Maccheroni, Marinacci, and Rustichini \(2006, henceforth MMR\)](#) and [Maccheroni et al. \(2009\)](#) prove that a specification of their Variational Preferences, named Monotone Mean Variance (MMV) Preferences, is the minimal monotone functional that extends the MV preferences and agrees with them on their domain of monotonicity. Formally let $(\Omega, \Sigma, \mathbb{P})$ be a probability space and consider $L^2(\mathbb{P})$, the set of all uncertain prospects with bounded variance. For any $f \in L^2(\mathbb{P})$, the MV preference relation is represented by the function

$$V_{\text{MV}}^\theta(f) = \mathbf{E}_{\mathbb{P}}[f] - \frac{\theta}{2} \mathbf{var}_{\mathbb{P}}[f]$$

with $\theta > 0$, which is monotonic on the set

$$\mathcal{G}^\theta = \left\{ f \in L^2(\mathbb{P}) : f - \mathbf{E}_{\mathbb{P}}[f] \leq \frac{1}{\theta} \right\}.$$

The MMV preference relation is represented by the function

$$V_{\text{MMV}}^\theta(f) = \min_{\mathbf{Q} \in \Delta^2(\mathbb{P})} \left\{ \mathbf{E}_{\mathbf{Q}}[f] - \frac{1}{2\theta} C(\mathbf{Q} \parallel \mathbb{P}) \right\}$$

where $\Delta^2(\mathbb{P})$ is the set of all probability measures with square integrable densities with respect to \mathbb{P} and

$$C(\mathbf{Q}||\mathbb{P}) = \begin{cases} \mathbf{E}_{\mathbb{P}} \left[\left(\frac{d\mathbf{Q}}{d\mathbb{P}} \right)^2 \right] - 1 & \text{if } \mathbf{Q} \in \Delta^2(\mathbb{P}) \\ +\infty & \text{otherwise} \end{cases}$$

is the relative Gini concentration index or χ^2 distance. The property is then that $V_{\text{MMV}}^\theta(f) = V_{\text{MV}}^\theta(f)$ for any $f \in \mathcal{G}^\theta$ and V_{MMV}^θ is the function which gives the lowest possible evaluation (the most cautious) outside of \mathcal{G}^θ while being monotonous over $L^2(\mathbb{P})$.

This means that a DM ranking prospects under ambiguity using the MMV criterion is in fact using a MV preference, at least in \mathcal{G}^θ . Hence building her evaluations taking into account a set of probabilities $\Delta^2(\mathbb{P})$, she ends up ranking prospects based on the mean and variance computed under a unique \mathbb{P} . This link is also confirmed by [Černý et al. \(2012\)](#) who have shown that optimal portfolios for the MMV criterion are closely related to optimal portfolios for the Expected Utility criterion with a truncated quadratic utility function. Our aim is then to investigate one aspect of these links between the modelling of ambiguity and aversion to variance.

While [Strzalecki \(2011\)](#) has succeeded in constructing the axiomatic foundations of the entropic or multiplier preferences, the MMV preferences have not yet been fully axiomatised. Nonetheless they are a Variational Preference hence satisfy the axioms of the latter whose behavioural signification should not be ignored. Especially, we are interested in the existence of a set of priors C underlying the decision process. It first emerged solely as a mathematical artefact in the seminal paper by [Gilboa and Schmeidler \(1989\)](#) on Maxmin Expected Utility (MEU) although a cognitive interpretation has been appealing from the beginning¹. The behavioural interpretation of C as the set of scenarios that the DM deems possible has since then been reinforced and justified in several directions. [Ghirardato, Maccheroni, and Marinacci \(2004\)](#), henceforth [GMM](#)) introduce the *unambiguous preference* to formalise the idea that the ranking between two acts can be unaffected by hedging considerations. It is then the maximal restriction of the initial preference which

¹See Chapter 17 in [Gilboa \(2009\)](#).

satisfies the Independence axiom. It is incomplete hence has a [Bewley \(1986\)](#) representation by an unanimity criterion over a set of probabilities which is unique and independent of the choice of normalisation for the utility function. Therefore “it is natural to refer to each prior $\mathbf{P} \in C$ as a “possible scenario” that the DM envisions”². Also, C is shown to be the Clarke differential at the origin of the decision criterion, a result which can somewhat leads to the interpretation of the set of priors as the collection of “all the probabilistic scenarios that could rationalise the DM’s evaluation of acts”³. This result is generalised in [Ghirardato and Siniscalchi \(2012\)](#) to non homogeneous functions where the subdifferential at an arbitrary point is not necessarily the same as the subdifferential at the origin, hence giving a more “local” interpretation of the result. Finally, [Cerrei-Vioglio, Maccheroni, Marinacci, and Montrucchio \(2011b\)](#) introduce the Uncertainty Averse preference as a generalisation of the Variational Preferences and prove that, with these representations, the DM does not take into account the probabilities which are not in the set of priors given by the unambiguous preference⁴. This important result implies that the MMV preference seen as a specification of the Variational Preferences should be written as

$$V_{\text{MMV}}^\theta(f) = \min_{\mathbf{Q} \in C \cap \Delta^2(\mathbb{P})} \left\{ \mathbf{E}_{\mathbf{Q}}[f] - \frac{1}{2\theta} C(\mathbf{Q} \parallel \mathbf{P}) \right\}$$

From a different, “normative viewpoint”, [Gilboa, Maccheroni, Marinacci, and Schmeidler \(2010\)](#) characterise a DM by two preference relations: one leading to *objectively rational* choices that “she can convince others she is right in making them” and a second leading to *subjectively rational* choices that “other cannot convince the DM that she is wrong in making them”. The first one is incomplete and admit a [Bewley](#) representation, the second one is supposed to be a MEU. Both of these representations involve a set of priors which are proven to be the same when two axioms of “Consistency” and “Caution” are imposed between the two preference relations. [Cerrei-Vioglio \(2011\)](#) extends the scope

²[Ghirardato et al. \(2004, p. 145\)](#).

³[Ghirardato et al. \(2004, p. 136\)](#).

⁴To be precise, they prove that the closure of the domain of the conjugate, or t -conjugate in the quasi concave case, is the set of priors. In [Appendix A.4](#) we propose a simple proof of the inclusion of these domains in the set of priors

of this result, replacing the MEU preference by an Uncertainty Averse preference. Then the priors taken into account by the subjective rationality preference are those given by the incomplete objective rationality preference.

All the preceding results support the case for the cognitive interpretation of the set of priors. To get to the subject of this paper, it must be added that all the above models have been set up in a generalised Anscombe-Aumann framework which has proven to be particularly well suited for the development of non Expected Utility models. The combination of this framework and of a set of priors C imply that acts can be classified in two types: *crisp* acts, which behave like constant acts and cannot provide any hedging opportunity, are characterised by an Expected Utility which remain constant whichever of the scenario in C is used, and acts which have variable utility profiles. Note that, as developed in [Cerreia-Vioglio, Ghirardato, Maccheroni, Marinacci, and Siniscalchi \(2011a\)](#), henceforth [CGMMS](#)), the set of crisp acts can be strictly larger than the set of *unambiguous* acts if the latter are understood to have the stronger property of being robust to permutations of the payoffs on the partition of events they define. Nonetheless crisp acts are linked to the geometry of the set of priors as will be shown in this paper.

We introduce a subset of the crisp acts, that we name the *crisp fair gambles*, which are defined by an expected utility which is 0 under all the probabilities in the set of priors C , that is, if k is such a crisp fair gamble, $\mathbf{E}_{\mathbf{Q}}[u \circ k] = \mathbf{E}_{\mathbf{P}}[u \circ k]$ for all \mathbf{P} and $\mathbf{Q} \in C$. It can be seen that the existence of such acts is a problem for the definition of the MMV as a specification of the Variational Preferences. Indeed, suppose that f , k and $f + k$ are in \mathcal{G}^θ , then

$$V_{\text{MMV}}^\theta(f + k) = \min_{\mathbf{Q} \in C \cap \Delta^2(\mathbb{P})} \left\{ \mathbf{E}_{\mathbf{Q}}[f] + \mathbf{E}_{\mathbf{Q}}[k] - \frac{1}{2\theta} C(\mathbf{Q} \parallel \mathbf{P}) \right\} = V_{\text{MMV}}^\theta(f)$$

Therefore the DM would be indifferent to the addition of a crisp fair gamble while the standard MV Preference wouldn't as $\mathbf{var}[f + k] \neq \mathbf{var}[f]$.

Therefore, in this paper we study the conditions for the existence or non existence of these crisp fair gamble and what they mean in a financial setting. For this we establish

some links between the geometry of the set of priors and the subspace of crisp fair gambles in finite and infinite dimension. Section 2 recalls some results from CGMMS, the setup in which our results are established. Section 3 poses the definition of the crisp fair gamble and gives their first properties. The main results are in Section 4 where geometrical and topological properties of the set of priors and the subspace of crisp fair gambles are proved. Finally Section 5 gives a link with unambiguous acts.

2 The Setup: Rational Preferences under Ambiguity

In this section we recall results from CGMMS where the authors generalise earlier results on the identification of a set of priors and on unambiguous acts and events. These results are extended to a preference relation satisfying minimal assumptions, the MBA (Monotonic, Bernoullian, Archimedean) preference, which encompasses most of the previously axiomatised preferences, such as the Uncertainty Averse preferences, the Variational preferences or the Vector Expected Utility (Siniscalchi, 2009).

The MBA preferences are set in an Anscombe-Aumann environment: given a state space Ω endowed with an algebra Σ , and X a convex subset of a vector space, simple acts are Σ -measurable functions $f: \Omega \rightarrow X$ such that $f(\Omega)$ is finite. The set of all acts is denoted by \mathcal{F} . Given an $x \in X$, define $x \in \mathcal{F}$ to be the constant act such that $x(\omega) = x$ for all $\omega \in \Omega$. With the usual slight abuse of notation, we can then identify X with the subset of constant acts in \mathcal{F} .

$B_0(\Sigma, I)$ is the space of simple Σ -measurable function on Ω with values in $I \subset \mathbb{R}$. We write $B_0(\Sigma)$ instead of $B_0(\Sigma, \mathbb{R})$ for the space of finite linear combinations of characteristic functions of sets in Σ . $ba(\Sigma)$, $ba_1(\Sigma)$ and $ca_1(\Sigma)$ denote respectively the spaces of finitely additive measures, finitely additive probabilities and countably additive probabilities on Σ . These spaces are endowed with the total variation norm. Denote by $B(\Sigma)$ the space of all uniform limits of functions in $B_0(\Sigma)$. Endowed with the supnorm it is a Banach space whose topological dual is isometrically isomorphic to $ba(\Sigma)$. We will write the duality pairing as $\langle a, \mu \rangle = \int a d\mu$, the hyperplane $H_{\mu, \alpha} = \{a \mid \langle a, \mu \rangle = \alpha\}$ and the closed half

space $H_{\mu, \alpha}^+ = \{a \mid \langle a, \mu \rangle \geq \alpha\}$. We denote by $\mathbf{1}_E$ the characteristic function of the event $E \in \Sigma$.

The DM preference is modelled by a binary relation \succsim over \mathcal{F} . This binary relation is called an *MBA preference* when it satisfies the following set of axioms:

AXIOM 1 (MBA PREFERENCE). *The preference relation \succsim*

(i) *is a Weak Order: \succsim is non-trivial, complete and transitive on \mathcal{F} ,*

(ii) *is Monotonic: if $f, g \in \mathcal{F}$ and $f(\omega) \succsim g(\omega)$ for all $\omega \in \Omega$, then $f \succsim g$,*

(iii) *satisfies Risk Independence: if $x, y, z \in X$ and $\lambda \in (0, 1]$, then $x \succ y$ implies $\lambda x + (1 - \lambda)z \succ \lambda y + (1 - \lambda)z$,*

(iv) *is Archimedean: If $f, g, h \in \mathcal{F}$ and $f \succ g \succ h$, then there are $\alpha, \beta \in (0, 1)$ such that $\alpha f + (1 - \alpha)h \succ g \succ \beta f + (1 - \beta)h$.*

and it can be represented as follows:

Proposition 2.1 (CGMMS, Proposition 1). *A binary relation \succsim is an MBA preference if and only if there exist a non-constant, affine function $u: X \rightarrow \mathbb{R}$ and a normalized, monotonic, continuous functional $I_u: B_0(\Sigma, u(X)) \rightarrow \mathbb{R}$ such that for each $f, g \in \mathcal{F}$*

$$f \succsim g \iff I_u(u \circ f) \geq I_u(u \circ g). \quad (1)$$

The unambiguous preference relation was initially defined by GMM as the largest subset of \succsim , seen as a subset of $\mathcal{F} \times \mathcal{F}$, which satisfies the von Neumann–Morgenstern Independence axiom. This relation has a representation in the form of a unanimity criterion à la Bewley (1986). These definition and property are extended in CGMMS to the MBA preferences:

Definition 2.2 (Unambiguous preference). *Let $f, g \in \mathcal{F}$. Say that f is unambiguously preferred to g , denoted $f \succ^* g$, if and only if, for all $h \in \mathcal{F}$ and all $\lambda \in (0, 1]$, $\lambda f + (1 - \lambda)h \succ \lambda g + (1 - \lambda)h$.*

Proposition 2.3 (CGMMS, Proposition 2). *Let \succsim be an MBA preference. Then there exists a non-empty, unique and closed set $C \subset ba_1(\Sigma)$ such that for each $f, g \in \mathcal{F}$*

$$f \succsim^* g \iff \int u \circ f \, d\mathbf{P} \geq \int u \circ g \, d\mathbf{P} \text{ for all } \mathbf{P} \in C. \quad (2)$$

where u is the function obtained in [Proposition 2.1](#). Moreover C is independent of the choice of normalisation of u .

For a function $a \in B_0(\Sigma, I)$, we will denote by $\underline{C}(a) \stackrel{\text{def}}{=} \min_{\mathbf{P} \in C} \int a \, d\mathbf{P}$ and $\overline{C}(a) \stackrel{\text{def}}{=} \max_{\mathbf{P} \in C} \int a \, d\mathbf{P}$. It holds that ([CGMMS](#), Corollary 3):

$$\underline{C}(a) \leq I_u(a) \leq \overline{C}(a) \quad (3)$$

Now that we have the unambiguous preference relation and the set of priors we can introduce the *crisp* acts:

Definition 2.4 (Crisp act). *An act is crisp if and only if it is unambiguously indifferent to a constant act.*

It is an immediate consequence of the previous propositions that, for an MBA preference, an act $k \in C$ is crisp if and only if $\underline{C}(u \circ k) = \overline{C}(u \circ k)$.

We close this section with [CGMMS](#) definitions of unambiguous acts and events and with some properties that we will need for our discussion in [Section 5](#). Their [example 3](#) illustrates that a crisp act may not be unambiguous if it is expected that two acts which induce the same partition of the state space Ω have to be either both ambiguous or both unambiguous.

Definition 2.5 (\succsim -permutation). *An act $g \in \mathcal{F}$ is a \succsim -permutation of another act $f \in \mathcal{F}$ if:*

(i) *for each $\omega \in \Omega$ there exists $\omega' \in \Omega$ such that $f(\omega) \sim g(\omega')$;*

(ii) *for each $\omega \in \Omega$ there exists $\omega' \in \Omega$ such that $g(\omega) \sim f(\omega')$;*

(iii) for each $\omega, \omega' \in \Omega$, $f(\omega) \sim f(\omega')$ if and only if $g(\omega) \sim g(\omega')$.

Definition 2.6 (\succsim -reduction). An act $g \in \mathcal{F}$ is reduced if $g(\omega) \sim g(\omega')$ implies $g(\omega) = g(\omega')$. A \succsim -reduction g of f is a reduced act $g = \{x_1, A_1; \dots; x_n, A_n\}$, with $x_1 \succ x_2 \succ \dots \succ x_n$ and $\{A_1, \dots, A_n\}$ a partition of Ω in Σ such that $g(\omega) \sim f(\omega)$ for all $\omega \in \Omega$.

Definition 2.7 (Unambiguous act). An act $f \in \mathcal{F}$ is unambiguous if every \succsim -permutation of f is crisp.

Definition 2.8 (Unambiguous event). An event $E \in \Sigma$ is unambiguous if the unambiguous acts are measurable with respect to E .

The class of all unambiguous events is denoted by Λ . It is a finite λ -system (CGMMS, Corollary 14) that is: (i) $\Omega \in \Lambda$; (ii) if $A \in \Lambda$ then $A^c \in \Lambda$; (iii) if $A, B \in \Lambda$ and $A \cap B = \emptyset$ then $A \cup B \in \Lambda$. We retain the following intuitive characterisations for an unambiguous event and an unambiguous act:

Proposition 2.9 (CGMMS, Proposition 14). An event $A \in \Sigma$ is unambiguous if and only if $\mathbf{P}(A) = \mathbf{Q}(A)$ for all $\mathbf{P}, \mathbf{Q} \in \mathcal{C}$.

Proposition 2.10 (CGMMS, Appendix D.3 (vii)). $f \in \mathcal{F}$ is unambiguous if and only if there exists a \succsim -reduction $\{x_i, A_i\}_{i=1}^n$ of f with $\{A_1, \dots, A_n\}$ a partition of Ω in Λ .

3 Crisp fair gambles

From now on, we suppose that the set of priors \mathcal{C} and the class of unambiguous events Λ have been obtained from an MBA preference, and that a vNM utility function has been chosen so that $0 \in u(X)$.

The unambiguous preference relation is not necessarily antisymmetric, that is $f \succsim^* g$ and $g \succsim^* f$ does not imply that $f = g$ but only implies that $\int u \circ f \, d\mathbf{P} = \int u \circ g \, d\mathbf{P}$ for all $\mathbf{P} \in \mathcal{C}$. If f and g are not equal and if the difference $f - g$ or $g - f$ is an act k in \mathcal{F} ,

this act is such that $\int u \circ k \, d\mathbf{P} = 0$ for all $\mathbf{P} \in \mathbf{C}$. This justifies the following definition:

Definition 3.1. *A crisp fair gamble is a crisp act whose expected utility is 0 under all probabilities in the set of priors.*

The first properties of crisp fair gambles are summarised in the following proposition.

Proposition 3.2. *For any $f, k \in \mathcal{F}$ such that k is a crisp fair gamble:*

(i) *for any $\lambda \in (0, 1]$, $\lambda f + (1 - \lambda)k \sim \lambda f$;*

(ii) *for any $\lambda \in \mathbb{R}$ such that $f + \lambda k \in \mathcal{F}$, $f + \lambda k \sim f$;*

(iii) *especially, for any $\lambda \in (0, 1]$ and for any $\lambda \in \mathbb{R}$ such that $\lambda k \in \mathcal{F}$, $k \sim \lambda k$.*

Proof: (i) Denote by x_0 the constant act in \mathcal{F} such that $u(x_0) = 0$. By definition, k is a crisp fair gamble if and only if $k \sim^* x_0$. That is for all $f \in \mathcal{F}$ and $\lambda \in (0, 1]$, $\lambda f + (1 - \lambda)k \sim \lambda f + (1 - \lambda)x_0$. Using representation (2), this is equivalent to $I_u(\lambda u \circ f + (1 - \lambda)u \circ k) = I_u(\lambda u \circ f + (1 - \lambda)u(x_0)) = I_u(\lambda u \circ f)$, that is $\lambda f + (1 - \lambda)k \sim \lambda f$.

(ii) For all $\mathbf{P} \in \mathbf{C}$, $\int u \circ (f + \lambda k) \, d\mathbf{P} = \int u \circ f \, d\mathbf{P}$, that is $f + \lambda k \sim^* f$, which implies $f + \lambda k \sim f$ (taking $\lambda = 1$ in Definition 2.2).

(iii) Take $f = k$ in the two previous propositions. ■

The DM is therefore indifferent to the scaling of a crisp fair gamble and to the addition of any crisp fair gamble. But the variance of λk is different from the variance of k and the variance of $f + k$ is different from the variance of f . Therefore, as discussed in the introduction, this behaviour is at odds with Mean-Variance preferences and more generally with what can be assumed from the rational behaviour of an investor in risky assets. It then seems desirable to impose that :

AXIOM 2 (NO CRISP FAIR GAMBLE). *The only crisp fair gamble is the constant act x_0 whose utility is 0.*

We want to stress that this does not seem a very demanding axiom in a financial context where the concern is for monetary outcomes only. These, in an Anscombe-Aumann type framework, have to be modelled as degenerate horse lotteries or *purely subjective* acts. We use here [MMR](#), Section 3.6's definition of *monetary acts*: X is the set of all finitely supported probabilities on \mathbb{R} and a monetary act f represents a random variable S by associating to any state $\omega \in \Omega$ a degenerate lottery $\delta_{S(\omega)}$. The vNM utility function u is the expected utility according to this probability, that is: $u \circ f(\omega) = \int_{\mathbb{R}} \tilde{u}(x) \delta_{S(\omega)} = \tilde{u}(S(\omega))$ where \tilde{u} is the utility function for monetary prizes. If, for example, \tilde{u} is quadratic, a crisp fair gamble is an act such that $\mathbf{E}_{\mathbf{P}}[S] = \alpha \mathbf{E}_{\mathbf{P}}[S^2]$ for all $\mathbf{P} \in \mathbf{C}$, with α a parameter independent of the choice of the prior. It looks very unlikely that an asset can have this property if the set of priors is not reduced to a singleton or if the random variable is not constant.

In the remaining of this paper, we explore some implications of the No crisp fair gamble Axiom by proving some relations between crisp acts and the geometry of the set of priors.

4 Crisp fair gambles in the space of utility profiles

We will assume from now on that the space of utility profiles is $B_0(\Sigma)$. This allows to avoid some technicalities and is motivated by two considerations.

First, the function I_u of [Proposition 2.1](#) is defined over the set of utility profiles $B_0(\Sigma, u(X))$ which is a subset of $B_0(\Sigma)$. When I_u is homogeneous, as is the case with the Maxmin Expected Utility of [Gilboa and Schmeidler \(1989\)](#), this function extends uniquely to $B_0(\Sigma)$ and to $B(\Sigma)$ by continuity. When I_u is not homogeneous but is constant additive, as is the case with for Variational Preferences, it has a unique minimal extension to $B_0(\Sigma, u(X)) + \mathbb{R}$ ([Dolecki and Greco, 1995](#)) but this is not enough to prove the unicity of the representation. To prove this unicity with the Variational Preferences, and the more general Uncertainty Averse Preferences, where I_u is not even a niveloid, the following axiom, which implies that $u(X) = \mathbb{R}$, need to be satisfied :

AXIOM 3 (UNBOUNDEDNESS). For every $x \succ y$ in X there are $z, z' \in X$ such that

$$\frac{1}{2}z + \frac{1}{2}y \succ x \succ y \succ \frac{1}{2}x + \frac{1}{2}z'.$$

Second, our final interest is in monetary acts, as they have been defined in the previous section, for which this axiom vacuously holds and for which the set of acts \mathcal{F} itself can be identified with $B_0(\Sigma)$.

4.1 Decomposition of crisp acts

Define the sets:

$$\begin{aligned} \mathbf{K} &\stackrel{\text{def}}{=} \{a \in B_0(\Sigma) \mid \text{there exists } \alpha \in \mathbb{R} \text{ such that } \langle a, \mathbf{P} \rangle = \alpha \text{ for all } \mathbf{P} \in \mathbf{C}\} \\ \mathbf{P} &\stackrel{\text{def}}{=} \{a \in B_0(\Sigma) \mid \langle a, \mathbf{P} \rangle \geq 0 \text{ for all } \mathbf{P} \in \mathbf{C}\} \end{aligned}$$

It is immediate that \mathbf{K} is a subspace of $B_0(\Sigma)$ and that $f \in \mathcal{F}$ is crisp if and only if $u \circ f \in \mathbf{K}$.

\mathbf{P} can be written as the intersection of closed half spaces: $\mathbf{P} = \bigcap_{\mathbf{P} \in \mathbf{C}} \mathbf{H}_{\mathbf{P},0}^+$, therefore it is a closed and convex set. It is also a cone which contains the origin, then, when considered as the cone of positive elements, it defines a partial order on $B_0(\Sigma)$ (Ekeland and Témam, 1999, Chapter III.5):

$$a \geq^* b \iff a - b \in \mathbf{P}$$

It holds that $f \succ^* g$ if and only if $u \circ f \geq^* u \circ g$.

Define the set:

$$\mathbf{L} \stackrel{\text{def}}{=} \mathbf{P} \cap -\mathbf{P}.$$

It is the lineality space of \mathbf{P} , that is the largest subspace contained in \mathbf{P} (Rockafellar, 1970, p. 65). It holds that k is a crisp fair gamble if and only if $u \circ k \in \mathbf{L}$. The assumption that the only crisp fair gamble is the constant act x_0 is then equivalent to the assumption that the cone \mathbf{P} is pointed ($l \in \mathbf{P} \cap -\mathbf{P}$ implies that $l = 0 \cdot \mathbf{1}_\Omega$) and that the partial order \geq^*

defined by this pointed cone \mathbf{P} is antisymmetric.

As an easy consequence of [Proposition 3.2](#) we obtain the following property:

Corollary 4.1. *The function $I_u: B_0(\Sigma) \rightarrow \mathbb{R}$ of [Proposition 2.1](#) is constant in any directions in \mathbf{L} and especially is null over \mathbf{L} .*

Proof: Let $a \in B_0(\Sigma)$ and $l \in \mathbf{L}$. There are $f, k \in \mathcal{F}$ such that $a = u \circ f$ and $k = u \circ l$ is a crisp fair gamble. Therefore by [Proposition 3.2](#), $f + k \sim f$ that is $I_u(a + l) = I_u(a)$. Moreover, as $k \sim x_0$ and I_u is normalized, $I_u(l) = u(x_0) = 0$. ■

With another slight abuse of notation, denote by $X \stackrel{\text{def}}{=} \mathbb{R} \cdot \mathbf{1}_\Omega$ the subspace of constant functions in $B_0(\Sigma)$.

Proposition 4.2. *The subspace of crisp utility profiles is the (internal) direct sum of the subspace of crisp fair gambles utility profiles and of the subspace of constant utility profiles:*

$$\mathbf{K} = \mathbf{L} \oplus X.$$

Proof: First $\mathbf{L} \cap X = \{0 \cdot \mathbf{1}_\Omega\}$. Second, for any $a \in \mathbf{K}$, there exists by definition $\alpha \in \mathbb{R}$ such that for all $\mathbf{P} \in \mathbf{C}$, $\langle a, \mathbf{P} \rangle = \alpha$. Define $l = a - \alpha \mathbf{1}_\Omega$ which is such that $\langle l, \mathbf{P} \rangle = 0$ for all $\mathbf{P} \in \mathbf{C}$ hence $l \in \mathbf{L}$. Now if there exist $l, l' \in \mathbf{L}$ and $\alpha, \alpha' \in \mathbb{R}$ such that $a = l + \alpha \mathbf{1}_\Omega = l' + \alpha' \mathbf{1}_\Omega$ we would immediately have for any $\mathbf{P} \in \mathbf{C}$, $\langle a, \mathbf{P} \rangle = \alpha = \alpha'$ hence $a - a = l - l' = 0 \cdot \mathbf{1}_\Omega$. Therefore the decomposition is unique and \mathbf{K} is the algebraic internal direct sum of \mathbf{L} and X . But \mathbf{L} is the intersection of closed sets and X is a finite dimensional subspace of a normed space hence both are closed, which concludes ([Megginson, 1998](#), Proposition 1.8.7 and Definition 1.8.8). ■

This implies that any crisp act can be uniquely decomposed as the sum of a crisp fair gamble and a constant act. Therefore, imposing the No crisp fair gamble Axiom, that is restricting the subspace of crisp utility profiles to the utility profile of the constant act x_0 , which is the null function in $B_0(\Sigma)$, is equivalent to imposing that the only crisp acts are the constant acts.

4.2 Crisp acts and the set of priors in finite dimension

In this section, we assume that the state space is finite dimensional with $|\Omega| = n$. Monetary acts and probabilities are then represented by vectors in \mathbb{R}^n . We can state our main result:

Theorem 4.3. *Let V be the subspace parallel to the affine hull of the set of priors.*

$$(i) \ L = (\text{span } C)^\perp,$$

$$(ii) \ K = V^\perp,$$

Proof: See appendix [A.1](#). ■

The first point of this theorem states that the subspace of crisp fair gambles utility profiles and the linear span of the set of priors are orthogonal complements in \mathbb{R}^n , therefore non constant crisp fair gambles exist if and only if the set of priors is contained in a subspace of a strictly lower dimension than n . The second, and equivalent, point states that the subspace of crisp utility profiles and the subspace parallel to the affine hull of the set of priors are orthogonal complements, therefore, non constant crisp fair gambles exist if and only if the affine hull of the set of priors is strictly included in the affine hull of the $n - 1$ simplex. To conclude, the No crisp fair gamble Axiom imposes that there exist no directions along which the set of priors, seen as a subset of the affine hull of the $n - 1$ simplex, is “flat”.

4.3 Examples in finite dimension

Ellsberg’s urn. The three colors Ellsberg’s urn contains 30 red balls and 60 blue and green balls. Utility profiles of acts are vectors $x = (x_R, x_B, x_G)^\top$ in \mathbb{R}^3 . The set of priors is described by $C = \text{co}\left\{\left(\frac{1}{3}, \underline{\alpha}, \frac{2}{3} - \underline{\alpha}\right)^\top, \left(\frac{1}{3}, \bar{\alpha}, \frac{2}{3} - \bar{\alpha}\right)^\top\right\}$ for any $\underline{\alpha} < \bar{\alpha}$ with $\underline{\alpha}, \bar{\alpha} \in [0, 2/3]$.

To verify point (i) of [Theorem 4.3](#), we calculate on the one hand the linear span of C

which is the set of points such that there exist $a, b \in \mathbb{R}$ with:

$$\begin{cases} x_R = \frac{1}{3}(a + b) \\ x_B = a\underline{\alpha} + b\bar{\alpha} \\ x_G = \frac{2}{3}(a + b) - a\underline{\alpha} - b\bar{\alpha} \end{cases} \iff x_R - \frac{1}{2}x_B - \frac{1}{2}x_G = 0$$

that is span C is in the plane orthogonal to the vector $(1, -\frac{1}{2}, -\frac{1}{2})^\top$. On the other hand, the cone of positive elements is

$$\mathbf{P} = \{x \mid \frac{1}{3}x_R + \frac{2}{3}x_G + \alpha(x_B - x_G) \geq 0, \forall \alpha \in [\underline{\alpha}, \bar{\alpha}]\}$$

and its lineality space is

$$\begin{aligned} \mathbf{L} &= \{x \mid \frac{1}{3}x_R + \frac{2}{3}x_G + \alpha(x_B - x_G) = 0, \forall \alpha \in [\underline{\alpha}, \bar{\alpha}]\} \\ &= \{x \mid x_R = -2x_G \text{ and } x_B = x_G\} \\ &= \text{span} \left\{ (1, -\frac{1}{2}, -\frac{1}{2})^\top \right\}. \end{aligned}$$

hence it holds that $\mathbf{L} = (\text{span C})^\perp$.

To verify point (ii) of [Theorem 4.3](#), we calculate on the one hand the affine hull of \mathbf{C} which is obtained by the equations used for the linear span with the added condition that $a + b = 1$. The affine hull is then the set of points such that $x_R = \frac{1}{3}$ and $x_B + x_G = \frac{2}{3}$ which is directed by the subspace (the line) given by $x_R = 0$ and $x_B = -x_G$. On the other hand, the subspace of crisp utility profiles is obtained by the equation $\mathbf{L} \oplus X$ that is

$$\mathbf{K} = \text{span} \left\{ (1, -\frac{1}{2}, -\frac{1}{2})^\top, (1, 1, 1)^\top \right\}$$

which is the plane orthogonal to $(0, 1, -1)^\top$ hence it holds that $\mathbf{K} = V^\perp$.

CGMMS's example 3 Also set in a three states space where utility profiles of acts are vectors $x = (x_1, x_2, x_3)^\top$ in \mathbb{R}^3 , the set of priors is given by $\mathbf{C} = \text{co}\left\{ (\frac{1}{3}, \frac{1}{4}, \frac{5}{12})^\top, (\frac{1}{4}, \frac{5}{12}, \frac{1}{3})^\top \right\}$

from which we obtain that

$$\begin{aligned}\text{span } C &= \{x \mid 13x_1 + x_2 - 11x_3 = 0\} \\ V &= \text{span} \{(1, -2, 1)^\top\} \\ L &= \text{span} \{(13, 1, -11)^\top\} \\ K &= \text{span} \{(13, 1, -11)^\top, (1, 1, 1)^\top\}\end{aligned}$$

which agree with point (i) and (ii) of [Theorem 4.3](#).

4.4 Crisp acts and the set of priors, the infinite dimensional case

We now extend this theorem to the infinite dimensional case. This extension is motivated by applications to finance of the axiomatic models of decision under ambiguity, applications for which the functional spaces used cannot be restricted to the finite dimensional ones.

We recall that the space $ba(\Sigma)$ endowed with the total variation norm is a Banach space and that the norm $\|\mu\|_{ba} = \sup_{E \in \Sigma} |\mu(E)|$ is equivalent to this total variation norm ([Dunford and Schwartz, 1988](#), p. 161). The weak* topology on $ba(\Sigma)$ is the topology $\sigma(ba(\Sigma), B_0(\Sigma))$, that is the smallest topology for which the linear functionals $\mu \mapsto \langle a, \mu \rangle$ are continuous for all $a \in B_0(\Sigma)$. Next are some definitions which extend the idea of orthogonality to dually paired spaces.

Definition 4.4 (Annihilators). *The annihilator of C in $B_0(\Sigma)$ is the set:*

$${}^\perp C = \{a \in B_0(\Sigma) \mid \langle a, \mathbf{P} \rangle = 0 \text{ for each } \mathbf{P} \in C\}.$$

The annihilator of L in $ba(\Sigma)$ is the set:

$$L^\perp = \{\mu \in ba(\Sigma) \mid \langle l, \mu \rangle = 0 \text{ for each } l \in L\}.$$

From the definition it is immediate that ${}^\perp C = L$. [Theorem 4.6](#) gives an expression of L^\perp first in the case where the priors in C are finitely additive, and secondly in the case

where they are countably additive probabilities. We recall that countable additivity of the priors can be obtained by the standard monotone continuity axiom (Chateauneuf et al., 2005) in all the decision models that have been cited in this paper. This second expression is easier to interpret and rely on the following theorem which proves that $\text{span } C$ is weakly* closed when the priors are σ -additive.

Theorem 4.5. *If C is a weak* closed convex set of elements of $ca_1(\Sigma)$, then:*

- (i) $\text{aff } C$, the affine hull of the set C , is closed in the weak* topology.
- (ii) V , the subspace parallel to the affine hull of C , is closed in the weak* topology.
- (iii) $\text{span } C$, the linear space spanned by the set C , is closed in the weak* topology.

Proof: See appendix A.2. ■

Theorem 4.6. *Let C be a weak* closed convex subset of $ba_1(\Sigma)$ and L be the annihilator of C in $B_0(\Sigma)$, then*

- (i) $L = {}^\perp(\text{span } C)$,
- (ii) $L^\perp = \overline{\text{span } C}^{w*}$.

Let C be a weak closed convex subset of $ca_1(\Sigma)$. Denote by V the subspace which directs the affine hull of C .*

- (iii) $L^\perp = \text{span } C$,
- (iv) $K = {}^\perp V$ and $K^\perp = V$.

Proof: See appendix A.3. ■

This theorem parallels Theorem 4.3 replacing orthogonal complements by annihilators. The idea that the No crisp fair gamble axiom imposes to the set of priors to extend in all directions can best be illustrated by the following corollary.

Corollary 4.7. *The No crisp fair gamble axiom holds if and only if $\overline{\text{span } C}^{w*}$ and $ba(\Sigma)$ are isometrically isomorphic.*

Proof: [Axiom 2](#) is equivalent to $L = \{0\}$, then the corollary is a consequence of point (iii) of [Theorem 4.6](#) and of the existence of an isometric isomorphism between $(B(\Sigma)/L)^* = (B(\Sigma))^* \simeq ba(\Sigma)$ and L^\perp ([Megginson, 1998](#), Theorem 1.10.17). ■

5 The link with unambiguous acts

Define the set:

$$\mathbf{U} \stackrel{\text{def}}{=} \left\{ a \in B_0(\Sigma) \left| a = \sum_{i \in I} a_i \mathbf{1}_{A_i}, \text{ with } \begin{array}{l} \{a_i\}_{i \in I} \text{ a finite family of reals} \\ \{A_i\}_{i \in I} \text{ a finite partition of } \Omega \text{ in } \Lambda \end{array} \right. \right\}$$

f is unambiguous if and only if $u \circ f \in \mathbf{U}$.

Because for all $A_i \in \Lambda$, $\mathbf{1}_{A_i} \in \mathbf{K}$, \mathbf{U} is a subset of \mathbf{K} but the inclusion can be strict as shown by [CGMMS's](#) example 3 (reproduced in section [4.3](#) below). It is also a subset of the linear span $\text{span}\{\mathbf{1}_{A_i}, A_i \in \Lambda\}$. In the general case, \mathbf{U} is a symmetric cone, not convex, nonetheless, we have the following property:

Proposition 5.1. *\mathbf{U} is a subspace of $B_0(\Sigma)$ if and only if Λ is an algebra. Then $\mathbf{U} = \text{span}\{\mathbf{1}_{A_i}, A_i \in \Lambda\}$.*

Proof: Let $a = \sum_{i \in I} a_i \mathbf{1}_{A_i}$ and $b = \sum_{j \in J} b_j \mathbf{1}_{B_j}$ be two elements of \mathbf{U} with $\{A_i\}_{i \in I}$ and $\{B_j\}_{j \in J}$ two finite partitions of Ω in Λ . Then $a + b = \sum_{i \in I} \sum_{j \in J} (a_i + b_j) \mathbf{1}_{A_i \cap B_j}$ where $\{A_i \cap B_j\}_{i \in I, j \in J}$ is also a finite partition of Ω . Therefore $a + b$ is in \mathbf{U} if and only if $A_i \cap B_j$ is in Λ for all i and j which makes it a π -system hence an algebra ([Aliprantis and Border, 2006](#), Lemma 4.10). Now let $a \in \text{span}\{\mathbf{1}_{A_i}, A_i \in \Lambda\}$, that is $a = \sum_{i \in I} a_i \mathbf{1}_{A_i}$ and let $\{B_j\}_{j \in J}$ be the partition of Ω generated by a . Each B_j writes as some finite unions and intersections of A_i hence it is in the algebra Λ . ■

As an example, it is known that Λ is an algebra if \succsim is a CEU preference ([Nehring, 1999](#), Theorem 2). To illustrate the link between the crisp and unambiguous acts, we can complete the examples of section [4.3](#). For the Ellsberg's urn, the class of unambiguous events is $\Lambda = \{\Omega, \emptyset, \{x_R\}, \{x_B, x_G\}\}$ which is an algebra. The set, which is then a subspace,

of unambiguous acts is:

$$U = \{a\mathbf{1}_{\{x_R\}} + b\mathbf{1}_{\{x_B, x_G\}}, (a, b) \in \mathbb{R}^2\} = \text{vect} \{(1, 0, 0)^\top, (0, 1, 1)^\top\}.$$

This is the plane orthogonal to $(0, 1, -1)^\top$ and K and U are the same subspace. In the case of CGMMS's example 3, there are no non trivial unambiguous events and $\Lambda = \{\Omega, \emptyset\}$ so that U is reduced to the constant acts while there are crisp fair gambles.

Therefore the link with the crisp fair gambles is not an equivalence but we can state the following necessary condition:

Proposition 5.2. *In order to have no crisp fair gambles, it is necessary that all the non trivial events in Σ are ambiguous.*

Proof: As $X \subset U \subset K$, if the subspace K is reduced to the constant line, U is also reduced to the same subspace of dimension 1 and Λ can only be the algebra $\{\Omega, \emptyset\}$. ■

In a financial setting, if there exist an event $E \in \Sigma$ such that $\mathbf{P}(E) = \pi$ for all $\mathbf{P} \in C$ with $\pi \neq 0$ and $\pi \neq 1$, then the contingent claim with payoff $\pi^{-1}\mathbf{1}_E - (1 - \pi)^{-1}\mathbf{1}_{E^c}$ in the utility space is a crisp fair gamble. A DM with an MBA preference is indifferent to holding this act and holding the constant act with nul utility, while the variance of the former is $(\pi(1 - \pi))^{-1}$ while the variance of the latter is 0. If we then impose that $\Lambda = \{\Omega, \emptyset\}$, the only unambiguous asset is the constant riskless asset, that is all risky assets are ambiguous or equivalently there are no risky but unambiguous asset.

6 Conclusion

Axiomatic models of decision under ambiguity with a non-unique prior allow for the existence of crisp fair gambles: acts whose expected utility is nul whichever of the priors is used. But, in these models, the DM has to be indifferent to the addition of such acts. Their existence is then at odds with a preference taking into account the variance of the prospects.

As a consequence, we would like to impose that these crisp fair gambles do not exist, at least when considering financial applications such as the Monotone Mean–Variance preferences. In this paper, we have shown that it is then necessary that there are no unambiguous events and that the set of priors has no direction of flatness. The knowledge of the environment and the attitude of the DM must not allow the use of Expected Utility for any other prospects than the constant ones. In the theory of finance, this has a direct consequence for the introduction of ambiguity in the models as only the riskless asset can be unambiguous, or can be perceived by the DM as being unambiguous.

A Proofs

Remark A.1 (Description of affine sets). Any affine set can be obtained from a subspace and a translation : let V be the subspace parallel the affine hull of the set C : $V = \text{aff } C - \text{aff } C = \{x^* - y^* \mid x^* \in \text{aff } C, y^* \in \text{aff } C\}$. Let x_0^* be any element of C . Any $v^* \in V$ is of the form $v^* = \sum_{i \in I} \alpha_i x_i^* - \sum_{j \in J} \beta_j y_j^*$ where I and J are finite subset of \mathbb{N} and $\sum_{i \in I} \alpha_i = \sum_{j \in J} \beta_j = 1$. But v^* can also be written as $w^* - x_0^*$ with $w^* = \sum_{i \in I} \alpha_i x_i^* - \sum_{j \in J} \beta_j y_j^* + x_0^*$ an element of $\text{aff } C$ as the sum of the coefficients is equal to one. Therefore $V = \text{aff } C - x_0^*$ for any $x_0^* \in C$.

A.1 Proof of Theorem 4.3

(i) Let $l \in L$ and $x^* \in \text{span } C$, that is there exist $\{\alpha_i\}_{i \in I}$ a finite family of reals and $\{p_i\}_{i \in I}$ a finite family of elements of C such that $x^* = \sum_{i \in I} \alpha_i p_i$. Then $l \cdot x^* = \sum_{i \in I} \alpha_i (p_i \cdot l) = 0$ and $L \subset (\text{span } C)^\perp$. Now take $x \in (\text{span } C)^\perp$, then $x \cdot x^* = 0$ for any $x^* \in \text{span } C$ especially $x \cdot p = 0$ for all $p \in C$ hence $(\text{span } C)^\perp \subset L$.

(ii) Let $x \in K$, there exists $\gamma \in \mathbb{R}$ such that $x \cdot p = \gamma$ for any $p \in C$. Any x^* in V can be written $x_1^* - x_2^*$ with x_1^* and x_2^* in $\text{aff } C$. Therefore there exist $\{\alpha_i\}_{i \in I}$ and $\{\beta_j\}_{j \in J}$ two finite families of reals such that $\sum_{i \in I} \alpha_i = 1$, and $\sum_{j \in J} \beta_j = 1$ and $\{p_i\}_{i \in I}$ and $\{q_j\}_{j \in J}$ two finite families of elements of C such that $x_1^* = \sum_{i \in I} \alpha_i p_i$ and $x_2^* = \sum_{j \in J} \beta_j q_j$. Then $x \cdot x^* = \sum_{i \in I} \alpha_i (x \cdot p_i) - \sum_{j \in J} \beta_j (x \cdot q_j) = \gamma \sum_{i \in I} \alpha_i - \gamma \sum_{j \in J} \beta_j = 0$ and $x \in V^\perp$. Now

suppose $x \in V^\perp$, then $x \cdot x^* = 0$ for any x^* in V . Let p_0 be any prior in C and set $\gamma = x \cdot p_0$. For any $p \in C$, $p - p_0$ is in V , hence $x \cdot (p - p_0) = 0$, that is $x \cdot p = \gamma$ and $x \in K$. ■

A.2 Proof of Theorem 4.5

The proof of this theorem needs the following lemmata.

Lemma A.2. *Let C be a non-singleton weak* closed convex set of elements of $ba_1(\Sigma)$. Denote by $\text{ri } C$ the relative weak* interior of the set C , i.e. its interior in the relative topology of $\text{aff } C$ generated by the weak* topology of $ba(\Sigma)$. Denote by $\text{rbd } C \stackrel{\text{def}}{=} C \setminus \text{ri } C$, its relative boundary. Let \mathbf{P}_0 be any probability in $\text{ri } C$.*

(i) *For any $\mathbf{P} \neq \mathbf{P}_0$ in C and $0 \leq \lambda < 1$, the probability $\mathbf{Q} = \lambda \mathbf{P} + (1 - \lambda) \mathbf{P}_0$ is in $\text{ri } C$.*

(ii) *Let $\mathbf{P} \neq \mathbf{P}_0$ be a probability in C . The half-ray emanating from \mathbf{P}_0 : $\{\lambda \mathbf{P} + (1 - \lambda) \mathbf{P}_0$ with $\lambda \geq 0\}$ has a unique intersection with the relative boundary $\text{rbd } C$.*

(iii) *For any measure μ in $\text{aff } C$, there exist a probability $\mathbf{P} \in \text{rbd } C$ and an $\alpha \in \mathbb{R}^+$, such that $\mu = \alpha \mathbf{P} + (1 - \alpha) \mathbf{P}_0$ or $\mu - \mathbf{P}_0 = \alpha(\mathbf{P} - \mathbf{P}_0)$.*

Proof: A basis for the weak* topology is given by the sets of the form $B(\mu, A) = \{\nu \mid \nu \in ba(\Sigma), |\langle \nu - \mu, a \rangle| < 1 \text{ for each } a \in A\}$ with $\mu \in ba(\Sigma)$ and A a finite subset of $B(\Sigma)$ (Megginson, 1998, Proposition 2.4.12). Therefore if $\mathbf{P}_0 \in \text{ri } C$, there exists A_0 , a finite subset of $B(\Sigma)$, such that $(B(\mathbf{P}_0, A_0) \cap \text{aff } C) \subset C$.

(i) For $\lambda \in [0, 1)$, define the set $U \stackrel{\text{def}}{=} (1 - \lambda)B(\mathbf{P}_0, A_0) + \lambda \mathbf{P}$, which is a subset of the convex C . Any $\mathbf{Q}' \in U$ is such that $\mathbf{Q}' = (1 - \lambda)\mathbf{P}' + \lambda \mathbf{P}$ with $\mathbf{P}' \in B(\mathbf{P}_0, A_0)$, that is $|\langle \mathbf{P}' - \mathbf{P}_0, a \rangle| < 1$ for each $a \in A_0$. This implies that $|\langle (1 - \lambda)\mathbf{P}' - (1 - \lambda)\mathbf{P}_0, a \rangle| < (1 - \lambda)$, hence $|\langle (1 - \lambda)\mathbf{P}' + \lambda \mathbf{P} - \lambda \mathbf{P} - (1 - \lambda)\mathbf{P}_0, a \rangle| < (1 - \lambda)$ or $\left| \left\langle \mathbf{Q}' - \mathbf{Q}, \frac{1}{1 - \lambda} a \right\rangle \right| < 1$, therefore $U = B\left(\mathbf{Q}, \frac{1}{1 - \lambda} A_0\right) \subset C$ and $\mathbf{Q} \in \text{ri } C$.

(ii) First note that the half-line is in the affine hull of C . Then, the previous point proves that this half-line has at most one intersection with the relative boundary of C . Finally, let $E \in \Sigma$ be such that $\mathbf{P}(E) \neq \mathbf{P}_0(E)$, and suppose $\mathbf{P}(E) - \mathbf{P}_0(E) < 0$ (otherwise

consider E^c) to show that, for a sufficiently large λ , $\mathbf{Q}(E) = \lambda(\mathbf{P}(E) - \mathbf{P}_0(E)) + \mathbf{P}_0(E) < 0$, hence $\mathbf{Q} \notin ba_1(\Sigma) \supset C$ which concludes.

(iii) Let $\mu \in \text{aff } C$. If $\mu = \mathbf{P}_0$, the result holds with $\alpha = 0$ and any \mathbf{P} in the relative boundary of C . Now suppose $\mu \neq \mathbf{P}_0$. We want to find a $\lambda > 0$ such that the set function $\mathbf{Q} = \lambda\mu + (1 - \lambda)\mathbf{P}_0$, which is in $\text{aff } C$, is in $B(\mathbf{P}_0, A_0)$, that is for all $a \in A_0$, $|\langle \lambda\mu + (1 - \lambda)\mathbf{P}_0 - \mathbf{P}_0, a \rangle| = |\lambda| \cdot |\langle \mu - \mathbf{P}_0, a \rangle| < 1$. This is obtained for any $0 < \lambda < (\sup_{a \in A_0} |\langle \mu - \mathbf{P}_0, a \rangle|)^{-1}$. Then $\mathbf{Q} \in (B(\mathbf{P}_0, A_0) \cap \text{aff } C) \subset C$. From the previous point, there exists a unique $\lambda' > 0$ such that $\mathbf{P} = \lambda'\mathbf{Q} + (1 - \lambda')\mathbf{P}_0 \in \text{rbd } C$. We finally have $\mu = \frac{1}{\lambda}\mathbf{Q} + (1 - \frac{1}{\lambda})\mathbf{P}_0$ and $\mathbf{Q} = \frac{1}{\lambda'}\mathbf{P} + (1 - \frac{1}{\lambda'})\mathbf{P}_0$. Setting $\alpha = \frac{1}{\lambda\lambda'}$ gives the result. ■

Lemma A.3. *Let C be a weak* closed convex set of elements of $ca_1(\Sigma)$:*

(i) *C and $\text{rbd } C$ are weak* sequentially compact.*

(ii) *Any weak* convergent sequence in $\text{aff } C$ or $\text{span } C$ has its limit in $ca(\Sigma)$.*

(iii) *Weak* convergence in C , $\text{rbd } C$, $\text{aff } C$ or $\text{span } C$ is equivalent to set-wise convergence.*

Proof: (i) As $\|\mathbf{P}\|_{ba} = 1$ for all $\mathbf{P} \in C$, C and $\text{rbd } C$ are norm bounded. By definition C is weak* closed and $\text{rbd } C = C \setminus \text{ri } C = C \cap \overline{\text{aff } C \setminus C}^{w*}$ being the intersection of weak* closed set, is also weak* closed, therefore both are weakly* compact (Dunford and Schwartz, 1988, Corollary V.4.3). As they are subsets of $ca(\Sigma)$ this is equivalent to their weak* sequential compactness (Gänssler, 1971, Corollary 2.17).

(ii) Let $\{\mu_n\}$ be a sequence in $\text{aff } C$ which converges weakly* to $\mu \in ba(\Sigma)$. It is clear that any $\mu_n \in \text{aff } C$ being the finite sum of σ -additive measures is itself σ -additive hence $\{\mu_n\} \subset \text{aff } C \subset ca(\Sigma)$. The weak* convergence of $\{\mu_n\}$ means that, for all $a \in B(\Sigma)$, $\{\langle \mu_n, a \rangle\}$ converges to $\langle \mu, a \rangle$. Take the functions a to be the characteristic functions of sets in Σ to obtain that, for each $E \in \Sigma$, $\mu(E) = \lim_n \mu_n(E)$ exists. A corollary of the Vitali-Hahn-Saks Theorem (Dunford and Schwartz, 1988, Corollary III.7.4) then concludes that μ is countably additive and that the countable additivity of μ_n is uniform in $n = 1, 2, \dots$.

(iii) Proposition 2.15 in Gänssler (1971) states that in $ca(\Sigma)$, weak* convergence is equivalent to set-wise convergence, which concludes with the previous point. ■

Proof (of Theorem 4.5): First, note that if the set of priors is a singleton: $C = \{\mathbf{P}_0\}$ then $\text{aff} C = \text{span} C = \{\mathbf{P}_0\}$ and the results hold trivially. We now suppose that there is more than one prior in C .

(i) Let $\{\mu_n\}$ be a sequence in $\text{aff} C$ which converges weakly* to $\mu \in \text{ca}(\Sigma)$. Let $\mathbf{P}_0 \in \text{ri} C$ that we can choose to be different from μ if needed. By Lemma A.2, there exist a sequence $\{\alpha_n\}$ in \mathbb{R} and a sequence $\{\mathbf{P}_n\}$ in $\text{rbd} C$ such that, for all n , $\mu_n = \mathbf{P}_0 + \alpha_n(\mathbf{P}_n - \mathbf{P}_0)$. By the weak* sequential compactness of $\text{rbd} C$, there exists a subsequence $\{\mathbf{P}_k\}$ of $\{\mathbf{P}_n\}$ weakly* convergent to $\mathbf{P} \in \text{rbd} C$. For any set function ν , denote by $\hat{\nu} \stackrel{\text{def}}{=} \nu - \mathbf{P}_0$, so that $\hat{\mu}_k = \alpha_k \hat{\mathbf{P}}_k \xrightarrow{w^*} \hat{\mu}$ and $\hat{\mathbf{P}}_k \xrightarrow{w^*} \hat{\mathbf{P}}$. Considering the characteristic functions of sets in Σ , this implies that for all $E \in \Sigma$, $\hat{\mu}_k(E) \rightarrow \hat{\mu}(E)$ and $\hat{\mathbf{P}}_k(E) \rightarrow \hat{\mathbf{P}}(E)$.

Let $\Sigma' \stackrel{\text{def}}{=} \{E \in \Sigma \mid \hat{\mathbf{P}}(E) \neq 0\}$. As $\mathbf{P}_0 \in \text{ri} C$ and $\mathbf{P} \in \text{rbd} C$, Σ' is not empty.

Let $E \in \Sigma'$, choose $m > 0$ such that $|\hat{\mathbf{P}}(E)| - m > 0$ and define $M = \frac{|\hat{\mu}(E)| + m}{|\hat{\mathbf{P}}(E)| - m}$. For any $\varepsilon > 0$ such that $\varepsilon \leq m$, the convergences of $\{\hat{\mu}_k(E)\}$ and of $\{\hat{\mathbf{P}}_k(E)\}$ imply that there exist k_1 and k_2 such that $|\hat{\mathbf{P}}_k(E) - \hat{\mathbf{P}}(E)| < \varepsilon$ for all $k \geq k_1$ and $|\hat{\mu}_k(E) - \hat{\mu}(E)| < \varepsilon$ for all $k \geq k_2$. Hence, for all $k \geq \max(k_1, k_2)$, $||\hat{\mathbf{P}}_k(E)| - |\hat{\mathbf{P}}(E)|| < m$ and $||\hat{\mu}_k(E)| - |\hat{\mu}(E)|| < m$, that is $|\hat{\mathbf{P}}_k(E)| > |\hat{\mathbf{P}}(E)| - m$ and $|\alpha_k \hat{\mathbf{P}}_k(E)| < |\hat{\mu}(E)| + m$ or $|\alpha_k| (|\hat{\mathbf{P}}(E)| - m) < |\alpha_k \hat{\mathbf{P}}_k(E)| < |\hat{\mu}(E)| + m$ that is $|\alpha_k| < M$. We also have

$$|\alpha_k \hat{\mathbf{P}}(E) - \hat{\mu}(E)| \leq |\alpha_k| \cdot |\hat{\mathbf{P}}(E) - \hat{\mathbf{P}}_k(E)| + |\alpha_k \hat{\mathbf{P}}_k(E) - \hat{\mu}(E)| < (M + 1)\varepsilon$$

hence α_k converges to $\alpha = \hat{\mu}(E)/\hat{\mathbf{P}}(E)$. This is true for all $E \in \Sigma'$, therefore α has to be independent of E . Finally $\alpha_k \hat{\mathbf{P}}_k(E)$ converges to $\alpha \hat{\mathbf{P}}(E)$ for all $E \in \Sigma'$.

Now for all $E \in \Sigma \setminus \Sigma'$, $\{\hat{\mathbf{P}}_k(E)\}$ converges to 0. With the convergence of $\{\alpha_k\}$, this implies that $\{\hat{\mu}_k(E)\}$ converges also to $0 = \alpha \hat{\mathbf{P}}(E)$. Finally, we have shown that for all $E \in \Sigma$, $\{\hat{\mu}_k(E)\}$ converges to $\alpha \hat{\mathbf{P}}(E)$, that is $\{\mu_k\}$ set-wise converges to $\mathbf{P}_0 + \alpha(\mathbf{P} - \mathbf{P}_0)$, hence, by Lemma A.3, weak* converges to $\mathbf{P}_0 + \alpha(\mathbf{P} - \mathbf{P}_0)$. Therefore $\mu = \alpha \mathbf{P} + (1 - \alpha)\mathbf{P}_0$ which is in $\text{aff} C$ and $\text{aff} C$ is weakly* closed.

(ii) From remark A.1, V can be written as $V = \text{aff} C - \mathbf{P}_0 = \{\mu \in \text{ba}(\Sigma) \mid \exists \nu \in \text{aff} C, \mu = \nu - \mathbf{P}_0\}$ with \mathbf{P}_0 any element of C . But then, from the previous point, V is

the sum of a weak* closed set and a point therefore it is weakly* closed (Megginson, 1998, Proposition 2.2.9(c)).

(iii) It is a consequence of the Krein-Šmulian Theorem that the linear space spanned by a weak* closed convex subset of a Banach space is closed in the weak* topology if and only if it is closed in the norm topology (Dunford and Schwartz, 1988, Corollary V.5.9). We therefore need to prove that $\text{span } C$ is closed in the norm topology.

Let $\{\mu_n\}$ be a sequence in $\text{span } C$ which converges (strongly) to $\mu \in ba(\Sigma)$. For each n , there exist $\{\alpha_n^i\}_{i \in I_n}$, a finite family of reals and $\{\mathbf{P}_n^i\}_{i \in I_n}$, a finite family of probabilities measures in C , such that $\mu_n = \sum_{i \in I_n} \alpha_n^i \mathbf{P}_n^i$. Denote by $\alpha_n = \sum_{i \in I_n} \alpha_n^i$. The convergence of $\{\mu_n\}$ imply that $\|\sum_{i \in I_n} \alpha_n^i \mathbf{P}_n^i - \mu\|_{ba} = \sup_{E \in \Sigma} |\sum_{i \in I_n} \alpha_n^i \mathbf{P}_n^i(E) - \mu(E)|$ converges to 0. This has to be true for the event Ω hence $\{\alpha_n\}$ converges to $\alpha = \mu(\Omega)$.

If $\alpha \neq 0$, there exists a sufficiently large $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$, $\alpha_n \neq 0$. Write $\mu_n = \alpha_n \sum_{i \in I_n} \frac{\alpha_n^i}{\alpha_n} \mathbf{P}_n^i = \alpha_n \mathbf{P}_n$, with $\mathbf{P}_n \in \text{aff } C$. It holds that

$$\|\alpha \mathbf{P}_n - \mu\|_{ba} = \|\alpha \mathbf{P}_n - \alpha_n \mathbf{P}_n + \alpha_n \mathbf{P}_n - \mu\|_{ba} \leq |\alpha - \alpha_n| \|\mathbf{P}_n\|_{ba} + \|\mu_n - \mu\|_{ba}$$

But $\|\mathbf{P}_n\|_{ba} = 1$ and for all $\varepsilon > 0$, there exist $n_1 \in \mathbb{N}$ and $n_2 \in \mathbb{N}$ such that for all $n \geq n_1$, $|\alpha - \alpha_n| < \varepsilon/2$ and for all $n \geq n_2$, $\|\mu_n - \mu\|_{ba} < \varepsilon/2$. Therefore, for all $n \geq \max(n_0, n_1, n_2)$, $\|\alpha \mathbf{P}_n - \mu\|_{ba} < \varepsilon$ and $\{\mathbf{P}_n\}$ converges strongly to $\frac{1}{\alpha} \mu$ which is then in the closure of $\text{aff } C$. By the previous point, $\text{aff } C$ is weak* closed hence closed for the norm topology (which includes all the weak* open and closed sets) hence $\mu \in \alpha \text{aff } C \subset \text{span } C$.

If $\alpha = 0$, take a $\mathbf{P}_0 \in C$ and define $\bar{\mu}_n \stackrel{\text{def}}{=} \mu_n + \mathbf{P}_0$. The sequence $\{\bar{\mu}_n\}$ converges to $\bar{\mu} \stackrel{\text{def}}{=} \mu + \mathbf{P}_0$. Denote by $\bar{\alpha}_n = \sum_{i \in I_n} \alpha_n^i + 1$ which converges to $\bar{\alpha} = 1$ and writes $\bar{\mu}_n = \bar{\alpha}_n \left(\sum_{i \in I_n} \frac{\alpha_n^i}{\bar{\alpha}_n} \mathbf{P}_n^i + \frac{1}{\bar{\alpha}_n} \mathbf{P}_0 \right) = \bar{\alpha}_n \bar{\mathbf{P}}_n$ with $\bar{\mathbf{P}}_n \in \text{aff } C$. The same reasoning as above shows that $\bar{\mathbf{P}}_n$ converges to a $\bar{\mathbf{P}} \in \text{aff } C$ hence $\mu \in \text{aff } C - \mathbf{P}_0 \subset \text{span } C$ (μ is indeed in the subspace parallel to $\text{aff } C$).

Therefore $\text{span } C$ is closed in the norm topology which concludes. ■

A.3 Proof of Theorem 4.6

Proof: (i) Take $l \in L$ and $\mu = \sum_{i=1}^n \alpha_i \mathbf{P}_i \in \text{span } C$. First $\langle l, \sum_{i=1}^n \alpha_i \mathbf{P}_i \rangle = \sum_{i=1}^n \alpha_i \langle l, \mathbf{P}_i \rangle = 0$, hence $L \subset {}^\perp(\text{span } C)$. Secondly, if $a \in {}^\perp(\text{span } C)$ then $\langle l, \sum_{i=1}^n \alpha_i \mathbf{P}_i \rangle = 0$ for all finite sequences $\{\alpha_i\}$ and $\mathbf{P}_i \in C$, especially $\langle l, \mathbf{P}_i \rangle = 0$ for all $\mathbf{P}_i \in C$, hence ${}^\perp(\text{span } C) \subset L$.

(ii) $L^\perp = ({}^\perp(\text{span } C))^\perp = \overline{\text{span}\{C\}}^{w^*}$, the weak* closure of the linear hull of C (Megginson, 1998, Proposition 2.6.6).

(iii) This is an application of Theorem 4.5 to the previous point.

(iv) From Proposition 4.2 and point (iii), $K = {}^\perp(\text{span } C) \oplus X$ so we want to prove that ${}^\perp(\text{span } C) \oplus X = {}^\perp V$.

Let $a \in {}^\perp(\text{span } C) \oplus X$. There exist $\delta \in \mathbb{R}$ and $b \in B_0(\Sigma)$ such that $a = b + \delta \mathbf{1}_\Omega$ with $\langle b, \mu \rangle = 0$ for all $\mu \in \text{span } C$, that is, for all μ of the form $\sum_{i \in I} \alpha_i \mathbf{P}_i$, with I a finite subset of \mathbb{N} and for all $i \in I$, $\alpha_i \in \mathbb{R}$ and $\mathbf{P}_i \in C$. Take a $\nu \in V$: ν writes $\sum_{j \in J} \beta_j \mathbf{P}_j - \sum_{k \in K} \gamma_k \mathbf{P}_k$, with J and K two finite subset of \mathbb{N} and, for all $j \in J$ and $k \in K$, $\beta_j, \gamma_k \in \mathbb{R}$, with $\sum_{j \in J} \beta_j = 1$, $\sum_{k \in K} \gamma_k = 1$, and $\mathbf{P}_j, \mathbf{P}_k \in C$. We have

$$\begin{aligned} \langle a, \nu \rangle &= \left\langle b + \delta \mathbf{1}_\Omega, \sum_{j \in J} \beta_j \mathbf{P}_j - \sum_{k \in K} \gamma_k \mathbf{P}_k \right\rangle \\ &= \left\langle b, \sum_{j \in J} \beta_j \mathbf{P}_j - \sum_{k \in K} \gamma_k \mathbf{P}_k \right\rangle + \delta \sum_{j \in J} \beta_j \langle \mathbf{1}_\Omega, \mathbf{P}_j \rangle - \delta \sum_{k \in K} \gamma_k \langle \mathbf{1}_\Omega, \mathbf{P}_k \rangle \\ &= 0 + \delta - \delta = 0 \end{aligned}$$

hence $a \in {}^\perp V$ and ${}^\perp(\text{span } C) \oplus X \subset {}^\perp V$.

Now let $a \in {}^\perp V$ and let \mathbf{P}_0 be a prior in C . For any $\mathbf{P} \in C$, $\mathbf{P} - \mathbf{P}_0 \in V$ hence $\langle a, \mathbf{P} - \mathbf{P}_0 \rangle = 0$. Let $\delta = \langle a, \mathbf{P}_0 \rangle$ and set $b = a - \delta \mathbf{1}_\Omega$ so that $\langle b, \mathbf{P} \rangle = 0$ for any $\mathbf{P} \in C$. Take $\mu \in \text{span } C$:

$$\langle b, \mu \rangle = \left\langle b, \sum_{i \in I} \alpha_i \mathbf{P}_i \right\rangle = \sum_{i \in I} \alpha_i \langle b, \mathbf{P}_i \rangle = 0$$

hence $b \in {}^\perp(\text{span } C)$, $a \in {}^\perp(\text{span } C) \oplus X$ and ${}^\perp V \subset {}^\perp(\text{span } C) \oplus X$ which gives the first equality.

The second one comes from the fact that V is weak* closed by Theorem 4.5 hence $(^\perp V)^\perp = V$. ■

A.4 A proof that the domain of the conjugate is included in the set of priors

C3M, Theorem 10 prove that, for Uncertainty Averse Preferences, the closure of the domain of the quasi-concave conjugate function $\text{dom}_\Delta G$ is the set of priors C . The Variational Preferences being the special case of additively separable Uncertainty Averse Preferences, this result implies that the domain of the concave Fenchel-Moreau conjugate $\text{dom } c$ is this same set of priors. We propose here a simple direct proof that $\text{dom } c \subset C$ and a slightly different proof from C3M that $\text{dom}_\Delta G \subset C$.

Proposition A.4. (i) $\text{dom } c \subset C$; (ii) $\text{dom}_\Delta G \subset C$.

Proof: In equation (3), the functions $\underline{C}(a)$ and $\overline{C}(a)$ are respectively the lower and the upper support functions of the set C .

(i) In standard convex analysis, it is known that $\underline{C}(a)$ is the concave conjugate of the (concave) indicator function of C defined from $ba(\Sigma)$ to $[-\infty, 0]$ by $\psi_C(\mu) = 0$ if $\mu \in C$, $\psi_C(\mu) = -\infty$ otherwise. C being (weak*-)closed and convex, it holds that the conjugate of $\underline{C}(a)$ is ψ_C (Aubin, 2007, § 2.4, Proposition 1). It is a straightforward consequence of the definition of the conjugate that $\underline{C} \leq I_u$ implies $(\underline{C})^* \geq (I_u)^*$ that is $\psi_C \geq -c$. Then $c(\mu) = +\infty$ for any μ not in C .

(ii) The t -quasi-conjugate of \underline{C} is defined for any $\mu \in ba(\Sigma)$ and $t \in \mathbb{R}$ by $G_\mu(t) = \sup_{a \in B_0(\Sigma)} \{\underline{C}(a) \mid \langle a, \mu \rangle \leq t\}$. If $t < 0$, as $G_\mu(t) = G_{-\mu}(-t)$, the following argument holds considering $\mu' = -\mu$, therefore we suppose that $t \geq 0$. If μ is not in C which is a weak*-closed and convex set, we can strictly separate the two by a linear functional: there exists $b \in B_0(\Sigma)$ and $\alpha \in \mathbb{R}$ such that $\langle b, \mu \rangle < \alpha < \langle b, \mathbf{P} \rangle$ for any $\mathbf{P} \in C$. Considering $b - \alpha \mathbf{1}_\Omega$, we can assume $\alpha = 0$ hence $\langle b, \mu \rangle < 0 < \underline{C}(b)$. Now for any $\lambda \in \mathbb{R}_*^+$, $\langle \lambda b, \mu \rangle < 0 \leq t$ while $\underline{C}(\lambda b) = \lambda \underline{C}(b)$ goes to infinity with λ which proves that $G_\mu(t) = +\infty$ for all $t \in \mathbb{R}$ and any μ not in C . ■

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