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History of mathematical education in ancient, medieval and pre modern India (within the Chapter: Mathematics Education in Oriental Antiquity and Middle Ages)

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► To cite this version:

Agathe Keller. History of mathematical education in ancient, medieval and pre modern India (within the Chapter: Mathematics Education in Oriental Antiquity and Middle Ages). Karp and Schubring. Handbook on the History of Mathematics education, Springer, pp.70-83, 2014. halshs-01006132v1

HAL Id: halshs-01006132

<https://shs.hal.science/halshs-01006132v1>

Submitted on 10 Jun 2015 (v1), last revised 24 Jan 2017 (v3)

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Mathematics Education in Oriental Antiquity and Middle Ages

(Alexei Volkov has written the second part of this chapter on Mathematical Education in China)

Mathematical Education in Ancient, Medieval and Premodern India

pre print version

Agathe Keller

Introduction

Very little is known of the context in which much of ancient India's scholarly knowledge burgeoned. Part of this ignorance springs precisely from the fact that very little is known on elementary, higher or specialized education in ancient and medieval India. For ancient and medieval mathematics in the Indian sub-continent, most of the studied textual sources are in Sanskrit, a brahmanical language which came to be the scholarly language of an educated cosmopolitan elite. Sources in vernacular languages could provide information on the contents and means of mathematical knowledge transmission in wider circles, but concerning mathematics and astronomy little studied and quite late texts provide us with meager testimonies. Finally, archeology has until now given us little information on how mathematics was taught. In the following, to sketch an uncertain image of mathematical education in ancient, medieval and premodern India, we have tentatively structured this part along the three historical periods relevant for the history of mathematics in India: Vedic period (ca. 2500 BCE-500 BCE), Classical and Medieval period (-500 BCE-XIIth century), and Premodern period (XIIIth-XVIIIth century). However, truly, the continuous study of ancient texts throughout this period mingled with our ignorance tends to blend it all into one continuous mode of transmission.

Vedic India (ca. 2500 BCE -500 BCE.):

The oldest texts that have come down to us from the Indian subcontinent, the Vedas (ca. 2500-1700 BCE) gave rise to a set of scholarly commentaries that define the knowledge that should be imparted to a brahman, and how this knowledge should be transmitted. According to such texts, the high cast male (and in rare cases female) should live through four life stages, one being the state of *brahmacārin*, student. The pupil (*śiṣya*) and his teacher or master (*ācārya*, *guru*) practice restraint and yogic exercises to develop an inner energy (*tapas*) believed to be central to good learning. Important religious texts being oral texts, knowledge is seen as being acquired through heard (*śruti*)

and remembered (*smṛti*) texts. An important part of the education of a brahman consists in learning how to recite specific parts of the Vedas. It implies not only knowing these texts by heart but also knowing how to chant them according to strict metrical rules. In some cases, the recitation involves knowing how to chant Vedic verses following many systematic combinations of their syllables: first in order, then inverting one verse/syllable after another, then reciting it backwards etc. so that the recitation can in itself be seen as an application of a systematical «mathematical» combination.

To become a student, one had to find a teacher who would accept to perform an *upanayana* ceremony, by which the teacher becomes symbolically pregnant of his student, who is in between 8 and 12 years old. Women were not normally made to study, but there are known exceptions. Education then lasted at least twelve years, for the least. The teaching season opened by an *upakarman* ceremony on a full moon day in July or August and ended five to six months later. To end the period of apprenticeship a ceremony was held, where the student offered a present to his teacher. Teaching was probably not performed by an individual preceptor. Texts describe the benefit for a student to have several teachers. Further, small «assemblies» (*śakha*, *charaṇa*, *pariṣad*, etc.) were founded around the transmission of a certain number of texts, and specific interpretations, housing students and teachers together. Students had the duty to tend the houses, fires and cattle of the «assemblies» or of his teachers. Obviously such “assemblies” could gather several members of a same extended family, and, reciprocally, such education could be undertaken within a family unit. It was possible to stay a student during one’s whole life.

The verb *adhī-* usually used to indicate how a student should learn the Vedic text (itself referred sometimes as «one’s own lesson» *svādhyāya*) means both «learning by heart», but also «seeking». Vedic auxiliaries (*vedāṅga*) proclaim that they serve the purpose of seeking the meaning of the Vedic text. They consist of five topics: Phonetics, Metrology, Etymology, Ritual and Astral Science (*jyotiṣa*). Astral science as propounded by the *Jyotiṣavedāṅga* («Vedic auxiliary on Asterisms») (ca 1200 BCE) although not properly a mathematicised astronomy from the point of view of the movement of planets, still contains procedures involving elementary arithmetical operations, such as the Rule of Three. The *Śulbasūtras* («Rules of the Chord») (8th to 5th century B.C.E) are subparts of larger ritual texts, belonging each to different schools and ascribed to different authors (Baudhāyana, Apāstamba, Kātyāyana, Mānava). Secondary literature on the history of mathematics groups all *Śulbasūtras* together. They indeed have many rules and topics in common. They describe the construction of Vedic ritual altars and delimitate ritual grounds. They provide algorithms to construct with strings and poles, oriented geometrical figures (square, rectangles, right triangles, etc.) having a given size. Procedures describe how to transform a figure into another having the same (or

proportional) area (a rectangle into a square, an isosceles triangle into a square etc.). Rules to construct altars of given shapes (some quite complex as that of a hawk with open wings) with a fixed amount of bricks are given as well. Each *Śulbasūtra* often gives several separate procedures for a same aim. It also contains rules that have a general scope, such as a procedure for the Pythagorean Theorem. However, texts give little information on how they were to be transmitted. Each separate text itself discusses several ritual schools, thus exposing contradictory data on the length of this altar, or the size of that brick. The knowledge imparted by the *Śulbasūtras* was probably to be mastered by the *adhvaryu* priest, one of the well versed priests in charge of the Vedic sacrifice. According to a hypothesis of Chattopadhyaya 1986 the paving knowledge of the *Śulbasūtras* would have been inherited from the artisans who had constructed the remarkable prehistoric cities of the Indus civilizations. This supposes, quite provocatively, a transmission of knowledge from non-brahmins to brahmins. The knowledge imparted in the *śulbasūtras* is still brought to life today by a brahmin cast of Kerala, the Nambudiri's, who regularly perform vedic sacrifices. If priests know the theory of how altars and sacrificial areas should be delimited and constructed, the actual setting is constructed by specialized artisans of lower cast. Thus contemporary anthropological evidence would back up Chattopadhyaya's hypothesis of a collaboration of artisan's with priests, although that would involve the improbable supposition, that over thousands of years roles and channels of transmissions remained unchanged, or that, in this respect, randomly, the situation happens to be the same at the beginning of the XXIst century than it was three thousand years ago. Little then is known on the education of low cast Vedic education. Given the occupational definitions of casts, it is often imagined that the education was taken in charge either by guilds or within the family. Separate independent schools for astral science probably developed during the end of the Vedic period. Religious sects, contesting Vedic values proliferate. Among them appears Buddhism and Jainism. Each will develop a non-Sanskrit scholarly literature. No mathematical or astronomical buddhist text has been transmitted to us, although buddhist texts refer to astronomy, and even evoke counting machines. The four branches of Jaina canonical texts include principles of mathematics (*gaṇitānuयोग*), arithmetics (*saṃkhyāna*) and astral science (*jyotiṣa*). Such texts are known to us in later forms, as compilations made during the classical and middle ages. They form a separate part of the history of mathematics in India, although in constant dialog with Hindu lore. We are uncertain however how these texts were integrated in the curriculum of monks. They may have been at some specific moment in time part of the curricula. This does not mean that they were constantly studied. Thus from the Vedic period, is transmitted first a vision of how education is carried out: transmission is oral and seems to be- but in exceptional cases- an education transmitted within each one's

own cast, and profession. The function of scholarly mathematics- what has been transmitted to us that is- seems to have been used essentially for ritual, cosmological and religious purposes: to determine the moment of sacrifice, explain the universe, and to help build proper sacrificial altars and grounds.

Scholarly knowledge has also by now a standard given form: that of treatises described as being said orally whose aphorism (*sūtras*) should be as simple and concise as possible. They are sometimes studied with their written commentaries, often by another author. This form will have an everlasting imprint on the texts transmitted to us in the Classical and Medieval ages of India.

Classical and Medieval India (-500 BCE-XIIth century CE)

After the burgeoning of the late Vedic period a 1000 years silence follows in the textual transmission of Sanskrit texts in astral science and mathematics. Then, two kinds of Sanskrit mathematical texts come to light. The first, and largest in numbers are mathematical chapters of astronomical treatises. The second type are self proclaimed “mathematics for worldly affairs” (*loka vyavahāra*), and are often related to Jain lore.

Both kinds of texts are in the form of versified rules, more or less aphoristic, that transmit definitions and procedures. It is probable that these rules did not aim at describing precisely an algorithm to be carried out, or a defined object, but to coin the important and remarkable elements worth memorizing: the rest being completed either by wit or with the help of a commentary. Thus secondary literature has often read these treatises as student manuals. The commentary is consequently seen as something like «teacher’s notes». These commentaries in prose can stage dialogs, in which it is tempting to read a representation of an actual act of teaching. Indeed, the words to coin how these texts are transmitted and conveyed, allude to a broad educational context. They thus refer to themselves as showing (*pradrś-*), indicating (*upadiś-*), explaining (*pratipād-*), all these verbs can also mean «to teach».

In the Vth century C. E. a mathematical astronomy appears as an already completed body of knowledge, through two treatises that are self proclaimed compilations: the «Five astronomical treatises» (*Pañcasiddhānta*) of Varahamihira (476 CE) and the *Āryabhaṭīya* of Āryabhaṭa (499). The latter has a chapter devoted to mathematics in which a definition of the place value notation is given, derivations of sines are evoked, as well as different elements of arithmetics, algebra, and indeterminate analysis. These topics will be subsequently constantly re-explored. The *Āryabhaṭīya* will give rise to schools, commentaries and criticisms,-a steady tradition of Sanskrit scholarly astral science including mathematics.

Most of the texts devoted to “worldly mathematics” (also called «board mathematics» -*pāṭīgaṇita* probably a reference to the slab on which computations and drawings could be carried out) have been transmitted to us by chance. This is the case of the *Bakhshālī Manuscript* (bearing the name of the town where it was excavated), of uncertain date (8th to 12th century) found by a peasant digging a hole in his field near Peshwar (now in present day Pakistan) in 1881. Sometimes, we have but a unique manuscript belonging to a given library, such as Ṭhakkura Pheru’s *Gaṇitasārakaumudī*, «Moonlight of the essence of mathematics» (ca. 1310) or the Paṭan manuscript (later than the XIVth century), both Jain texts known in a unique recension. The most famous of all such texts being the texts compiled by Śrīdhāra (ca. 9th century) the *Trīśatika* («Three Hundred [Rules]») and the *Paṭīgaṇita* («Board Mathematics»). Much of these texts, often self proclaimed compilations, testify of vernacular lore of all kinds, and has strong ties with the Jain tradition, such as the very popular Sanskrit *Gaṇitasārasaṃgraha* («Collection of the essence of mathematics») by Mahāvīra (ca. fl. 850).

Both mathematical traditions refer to each other as separate but dependent forms. By the XIIth century a synthesis of both traditions seems to be attempted in certain circles. This is probably the case of Bhāskarācārya’s (b. 1114) mathematical texts the *Līlāvati* («With Fun» or maybe the name of the girl to whom the examples in this text are addressed)- devoted to arithmetics-, and the *Bījagaṇita* («Seed Mathematics»)- devoted to algebra- both integrated as chapters to his astronomical treatise, the *Siddhāntaśiromāṇi* («Crest-jewel (among) astronomical treatises»).

Teaching elementary mathematics

Very little testimony exists on what would have been the elements of mathematics taught to any child to whom an education was given.

Medieval non mathematical texts give us fleeting information on the mathematical education of non-brahmins. Thus Kauṭilya’s *Arthaśāstra* («Manual of Statecraft» -ca. 100 BCE-100 CE), a sort of Indian machiavelic law manual for Kings give information on regular education. We learn from this text that after the ceremony of tonsure, a child was taught writing and arithmetic (*saṃkhyāna*).

The future king was encouraged to learn accounting, so as not to be easily swindled. Further, the *Arthaśāstra* has detailed list of measuring units and their shifting values, evidently a King should master conversions.

The Pali Buddhist canon (through the *Gaṇakamoggallānasutta* of the *Majjhimanikāya*) compiled in the first centuries before the C. E. describes a brahmin calculator (*gaṇaka*) who took in live-in pupils (*antevāsin*). He started by teaching them how to count to a hundred.

As we will see, scholarly mathematical texts, whether devoted to «worldly mathematics» or to astral science, provide a classification of operations and topics. It is noteworthy that but for one exception, no text describes how one should carry out an addition or a subtraction with the decimal place value notation. More generally, the algorithms put forth in scholarly mathematical texts, seem to take for granted that additions and subtractions on higher numbers, multiplications, divisions, squares and cubes of digits were known. We do not know if children were made to learn tables of multiplication, but this could have been the case; as later (modern) manuscripts of vernacular tables of multiplication, squares and cubes are known.

According to Hayashi 2001, *saṃkhyāna* could also have included «a sort of statistical estimate of the quantity of nuts, crops, etc.»

Board mathematics: a prerequisite for mathematical astronomy?

Texts of “worldly mathematics” probably testify of what would have been a general mathematical culture. Not an elementary one, but not either the realm of high brow Sanskrit mathematical knowledge.

They have in common with the mathematical chapters of astronomical texts to present a structure of elementary and/or fundamental operations (*parikarman*) and of “practices” (*vyavahāra*). These express an implicit theory of how such mathematics is organized. They also provide a structured list in which knowledge can be coined and memorized.

Bhāskarācārya gives what has in the secondary literature become sort of a canonical subdivision of topics. Arithmetics (*rāśiganīta*) was subdivided by eight operations (*parikarman*), and eight practices (*vyavahāra*) who include topics that we would probably classify in geometry and trigonometry:

The operations were:	The practices were:
1. Addition (<i>saṅkalita</i>)	1. Mixtures (<i>miśraka</i>)
2. Subtraction (<i>vyavakalita</i>)	2. Series (<i>średhī</i>)
3. Multiplication (<i>pratyutpanna</i>)	3. Figures (<i>kṣetra</i>)
4. Division (<i>bhāgahāra</i>)	4. Excavations (<i>khata</i>)
5. Square (<i>varga</i>)	5. Stacks (<i>citi</i>)
6. Square-Root (<i>vargamūla</i>)	6. Sawings (<i>krakacika</i>)
7. Cube (<i>ghana</i>)	7. Grains (<i>rāśi</i>)

The operations were:	The practices were:
8. Cube-Root (<i>ghanamūla</i>)	8. Shadow (<i>chāyā</i>)

Each operation being defined for integers (actually referred to as regular numbers, *sañkhyā*), fractions (*bhinna*), and zero (*śunya*). Such an organisation was articulated with algebra (*bījagaṇita*). For Bhāskarācārya this topic was structured by six rules (*vidha*), which corresponds to the six first operations of arithmetics. These rules were applied to positives (*dhana*) and negatives (*rna*), zero, undetermined (e.g. «unknowns», *avyakta*), and surds (*karaṇī*). They were defined as providing sub-operations for rules concerning different topics such as indeterminate linear (*kuṭṭaka*) and quadratic (*varga prakṛti*) problems, and equations with one or more unknowns (*samakarāṇa*). Many variations of the number and contents of both operations and practices for arithmetics- usually rules of proportions including the Rule of Three (*trairāśika*) are included as operations-, and types of equations for algebra, are known. However, such an ordered structure, that is found in all the mathematical texts handed to us could have been that of the progressive learning of mathematics. As we will see in the section on higher education in mathematics, algebra may have been seen as a more advanced topic in mathematics. Variations then of structures could account for the fact that although belonging to a common Sanskrit mathematical culture, each school, each family had its own particularities.

The question of the nature of practices (*vyavahāra*), and how they relate for instance to vocational training, is still very much open to research. Hayashi 2001 provides elements showing that accountants and calculators, as scribes, seem to have been regularly needed for all sorts of administrations. One can thus imagine that they needed to have a solid mathematical formation. However these worldly mathematics do not provide any rules for accounting. Indeed, each “practice” was a scholarly topic. Bhāskara (628 C.E) thus evokes a series of scholars who before him had worked, composed and compiled treatises, as Sanskrit scholarly knowledge self proclaims itself:

«Or else, the scope of mathematics is vast, there are eight practices Mixtures, Series, Figures, Excavations, Piles of brick, Sawings, Mounds of grain, and Shadows. (...) For each, rules and books were made and compiled by master Maskari, Pūraṇa, Mudgala, and others.»

In all cases, whether explicitly or implicitly these topics seemed to have been at least partially mastered for one who was to know astronomy.

Thus at the outset of the mathematical chapter (*gaṇitādhyāya*) of his astronomical treatise the *Brahmasphuṭasiddhānta*, Brahmagupta (629 C.E) claims:

«BSS.12.1 Whoever knows separately the twenty operations (*parikarman*) beginning with addition, and the eight practices (*vyavahāra*) ending with shadows, he is a calculator (*gaṇaka*)»¹.

Prthudakasvamin, a IXth century commentator specifies: «He has the rank (*adhikārin*) to learn the Sphere». In other words, to learn mathematical astronomy - and specifically the movement of planets which is the topic called *gola* (the Sphere)- one had first to master the different operations of «board mathematics».

Rules and problems given in mathematical texts can give us an idea of what was required of a good mathematician. That is, the skills that one learning mathematics should acquire.

Riddle culture.

The Sanskrit elite culture no doubt valued playfulness. This could have also been part of the mathematical pedagogy. Texts of «board mathematics» often contain versified mathematical problems, that can be seen as mathematical riddles. Their authorship is often doubtful: similar type problems are known to have travelled from text to text. Further, all mathematical commentaries of astronomical treatises contain list of versified examples, some of which echo those of «board mathematics» texts. They are often use the vocative and encourage rapidity and wit. Let us take for instance a problem of computations of the purity of gold melting given in the *Līlavātī*.

L.102. Example: Parcels of gold weighing severally ten, four, two and four *māṣas*, and of fineness thirteen, twelve, eleven and ten respectively, being melted together, tell me quickly, merchant who art conversant with the computation of gold, what is the fineness of the mass? If the twenty *māṣas* of gold be reduced to sixteen by refining, tell me instantly the touch of the purified mass. Or, if its purity when refined be of sixteen, prithee, what is the number to which the twenty *māṣas* are reduced ²?

Statement:	Touch	13	12	11	10
	Weight	10	4	2	4

¹ Note that Brahmagupta counts more operations than Bhāskarācārya, he adds the Rules of Three, Five, Nine and Eleven, the inverse Rule of Three, Five rules to reduce fractions, Barter and Exchange, and Rules to sell living beings.

² If x_i is the fineness (or “touch”) of the i th piece of gold, and y_i its purity, x and y their respective value for the melted mass of gold, then $xy = \sum x_i y_i$. Therefore, $x = \sum y_i$, and $y = \sum x_i$.

Answer: After melting,	fineness	12
	Weight	20

After refining, the weight being sixteen *māṣas*; the touch is 15. The touch being sixteen, the weight is 15³.

Such gold melting problems can be found in all known «board mathematics» texts, including those mentioned above, the *Bakhshālī Manuscript*, the *Gaṇitasārasaṃgraha* and the *Pāṭīgaṇita*, the *Gaṇitasārakaumudī*, and the Patan manuscript.

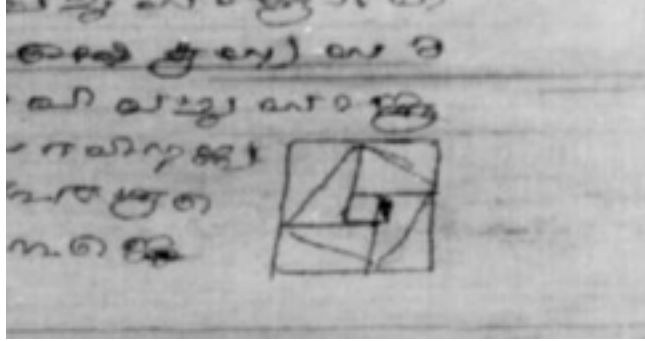
The playfulness of the problem can be seen in the ever ceasing variations of the weight or the fineness of the gold. Indeed, the problem seems rather to deal with an algorithm to sum and divide rather than with the formation of the apprentice of a jeweler or of a super-intendant of coins (as described with the rules for alligation in the *Arthaśāstra*). Problems, as we see, address a reader or a listener, evoking his or her qualities, the most common one, being the training in being quick when the rule that is exemplified in the problem is known.

The problem is set down in a sort of numerical table on a working surface: First products are taken within the pairs forming each column. Then, columns are summed, before dividing respectively either by the intended fineness or weight. Such graphic dispositions may have been carried out on a working surface, such as a dust board or a slate with chalk. They could have also served the purpose of representing mentally computations to be carried out.

Indeed, if problems evoke the witty quick one who needs not write computations, commentaries sometimes hit on the dull minded one who is too slow and for which an extended explanation is needed. Bhāskara thus describes a diagram- probably to illustrate the «Pythagorean Theorem»- that he will draw for the dull minded.

When one has sketched an equi-quadrilateral field and divided <it> in eight, one should form four rectangles whose breadth and length are three and four and whose diagonals are five. There, in the same way, stands in the middle a field whose sides are the diagonals of the <four> rectangles which were the selected

³ $13 \cdot 10 + 12 \cdot 4 + 11 \cdot 2 + 10 \cdot 4 = 240 = xy$. Therefore in the first case $x = 240/20 = 12$. In the second case, $y = 240/16 = 15$. In the third case $x = 240/15 = 16$.



quadrilaterals <and> which is an equi-quadrilateral field. And the square of the diagonal of a rectangular field there, is the area in the interior equi-quadrilateral field. (...) And a field is sketched in order to convince a dull-minded one (*duhvidha*):

Mss Burnell 517 British Library

However, mathematical texts do describe tools for drawing diagrams (pairs of compasses, ropes and chalk) or solid objects made of clay to explain computations involving solids. They often evoke the explanation of an expert, to actually “show” (teach, explain, prove) the rule.

Knowing how to apply the general rule

Riddles, solved problems and commentaries are all directed towards an hermeneutic act, the proper interpretation of a rule. Bhāskara, the seventh century commentator we have seen drawing for the dull one, sees the rule provided by the treatise he comments as a seed (*bīja*) to be grown. In other words, the rule was a general statement that could englobe many different rules. This no doubt was part of a more general scholarly conception of how a rule should be coined. Looking at this endeavor from the point of view of acquired skills, shows that one had to know how to apply such general rules to many different cases. Thus if we turn for instance to rules for summing (*saṅkalita*), such as the one given in the *Brahmasphuṭhasiddhānta*:

BG. 12.2 Of two quantities the denominators and the numerators when multiplied by the opposite denominators have the same divisor. In addition the numerators are added together; in subtraction the difference of the numerators is to be computed.

We immediately read this rule as concerning fractions. Pṛthudhakasvamin, the IXth century commentator specifies that integer numbers can be seen as having a denominator equal to one. So that such a rule can thus also be applied to integers. He further extends the rule to series, echoing other the rules “for addition” given in worldly mathematical texts. Indeed, through a solved example, he provides an interpretation of the rule as a procedure to sum arithmetic progressions.

Higher Education in mathematics and mathematical astronomy

In the mutation from the Vedic period to the Classical and middle ages, mutations in the organization of state and local royalties show the development of learned settlements in association with religious complexes (*maṭhas*), giving rise to what may have been the oldest universities of the ancient world. Such universities were known to have cultivated the study of vernacular languages not restricting themselves to scholarly Sanskrit. Chinese travelers and buddhist monks have left us testimonies of great buddhist centers of learning, such as Nalanda, in the Eastern Gangetic plain, but they do not mention the teaching of mathematics or astronomy in this context. Thus Fa-Hien (fl. 399 CE) describes Pāṭaliputra and its hundreds of monks. The Vth century astronomer Āryabhaṭa mentions that he has learned astronomy in Kusumpura, and presents this as a title of glory. His VIth century commentator Bhāskara, will gloss this name with Pāṭaliputra, while others will recognize here the name of Kurukṣetra, another centre of higher learning.

Astral science and within it mathematical astronomy was understood as a scholarly topic with divine origin and a long stream of forefathers, from gods (often Brahma) to seers (men with supernatural powers), to great scholars and gurus.

Varāhamihira in his *Bṛhatsamhitā* has recorded the names of a large number of astronomers: among which we can mention Garga who will often be recalled and a certain number of foreigners such as the recognisable Romaka (Roman?).

Sūryadeva Yajvan a XIIth century commentator of Āryabhaṭa describes this process of transmission. The field of astral science is first seen or intuited (*drś-*) by Brahma. He then founds the discipline, or composes a treatise. Which he teaches to a great scholar, who in turn synthetises it, writes a new treatises, and teaches it to his followers and so forth. Compiling, reducing/synthetising and transmitting then are the movement from which a student becomes a teacher, a pupil somebody who knows.

We have seen that a minimum of mathematical knowledge seems to have been required to study mathematical astronomy. But, on the other hand, mathematics is often brought up as the ultimate step in mastering astronomy, as it enables one to ground astronomical rules. For instance, after having evoked what was necessary to make a calculator, or a mathematician, on the outset of his algebraical chapter, Brahmagupta states:

BSS.18.2 « [One becomes] a master (*ācārya*) among experts (*vidyā*) of the doctrine (*tantra*) by knowing the pulverizer, zero, negatives, positives, unknowns, elimination of the middle [term], [reduction to one] colour/unknown, *bhāvita* (multiplication) and the square nature.

Indeed, for many late medieval authors, algebra is a tool to ground arithmetics. With the Rule of Three and the Pythagorean procedure it serves as the basis of another skill, that of understanding and then producing explanations for Rules.

Criticizing

In the formation of a doctor one had to hear debates, and finally participate in them. Indeed some of the wry irony of treatises and commentaries support the idea that debates could be quite virulent in astral science and mathematics as well.

Thus, the ultimate scholar, not really a student anymore, most probably was one who could engage in debates with his teacher, but not criticize him. This was certainly not but a premodern privilege. In Bhāskara's, medieval commentary of the *Āryabhaṭīya*, the author is compared as he starts his treatise, to a warrior on the battle, raising his sword. As Bhāskara later comments a compound (*samavṛttiparidhi*, the circumference of an evenly circular [figure]) that one can understand as either referring to a disk or a circle. He notes about the first meaning, nonsensical in the context:

This very analysis [of the compound] has been taught by master Prabhākara. Because he is a *guru*, we are not blaming him.

Meaning here that however such an explanation of the compound is wrong.

The early *Arthaśāstra* states thus:

5.5A science imparts discipline to one, whose intellect has (the qualities of) desire to learn, listening, learning, retention, thorough understanding, reflection, rejection and intentness on truth, not to any other person.

If it thus defines the ideals scholars by the qualities a good student needs, including an eventual rejection of bad rules. It also adds

5.6 But training and discipline in the sciences (are acquired) by (accepting) the authoritativeness of the teachers in the respective sciences.

This tension of criticising and respecting forefathers will be carried out to the premodern period.

Premodern India (XIIIth-XVIIth century)

By the beginning of the early modern period, information on what the secondary literature calls astronomical «schools» can be gathered. Most Sanskrit mathematical manuscripts handed to us by the tradition, were copied at the end of this period and after. Through their history and those of family libraries, family lineages running through centuries in astral science (including mathematical astronomy) are revealed. Thus the XIIth century astronomer and mathematician Bhāskarācārya be-

longed to a family of astronomers. His sons and nephews were also well known court astrologers and composed astronomical texts.

The most famous school was initiated by Mādhava (c. 1340-1425) near the town today known as Kochi. Nīlakaṇṭha (1445-1545) was from a Nambuttiri family in Trikkantiyur on the coastal part of South Kerala. He thus travelled to learn his mathematics from Parameśvara (fl.ca.1430)'s son Dāmodara (fl.ca.1460) at Ālattūr (Aśvatthagrāma), Kerala. The “Kerala school” produced astronomers that were also specialists of other fields of knowledge. This can be seen as a “premodern” features of scholarly Sanskrit India, where traditional boundaries are breaking down, and scholars writing in vernacular languages seek new structures of knowledge. Nīlakaṇṭha, known also as a philosopher, had many pupils as Śaṅkara (fl. 1550) who produced an important commentary on both the *Līlāvati* and Nīlakaṇṭha's works: although renewing deeply the mathematical tools and theories, authors positioned themselves in a continuous tradition.

The main turn in premodern mathematics seems to be the emphasis on providing proofs (*upapatti*) of canonical author's rules. The “Kerala school” is indeed known for its endeavor at correcting and grounding Āryabhaṭa's parameters, leading to its development of Taylor series. Commentators of Bhāskarācārya wrote books with algebraical “proofs” of his arithmetical rules. Higher students no doubt had first to learn these explanations, before creating their own groundings of mathematical algorithms.

For the premodern period we know of rival schools and families. Thus, Divākara (fl.ca.1530) a Gujarati astronomer of the XVIth century went to study next to Benares with Gaṇeśa (b. 1507) famous among other texts for his commentary of the *Līlāvati* which includes many proofs. Divākara 's descendants became important astrologers in this city, although they came under attack by a rival family of astrologers, notably for their acceptance of Islamic astronomical theories.

Indeed, the premodern period also saw the arrival and installation in most of North India, of muslims from central Asia, Afghanistan and Persia. They arrived with an arabic and persian culture in mathematics and astronomy. However, more even than the mathematical education of Hindu, Buddhist and Jains, the mathematical education of Indian muslims is, to my knowledge, mainly uncharted territory. For muslims of the Indian subcontinent, most probably as elsewhere in muslim countries, elementary mathematical education were developed in Madrassas. Under the reign of Sultan Fīrūz Shāh Tughluq (1305-1388), who commissioned many scientific texts and translations, it is known that a number of Madrassas were opened to encourage literacy and numeracy. Arabic and Persian mathematics flourished in the Indian subcontinent: many manuscripts of such texts can be found in Indian Libraries, notably in those of the Asiatic Society of Mumbai and Kolkatta. Certain-

ly, centers of learning in the Indian subcontinent were devoted to studying these texts. Further, during this period astrolabes and table-texts influenced by Arabic and Persian literature were translated into Sanskrit. Reversely many Sanskrit texts were translated into Persian. Court patronages were available for both Hindu and Muslim astrologers cum mathematicians: a close study of the courts of such Moghol rulers would probably yield much information on how mathematics and astral sciences were taught. Note that Jain monks were known to have been active players enabling the cross fertilization of both mathematical and astronomical traditions. Jain mathematicians then from the late Vedic period to the premodern seemed to have been crucial actors of mathematical activity in the Indian sub-continent. Investigating how mathematics could have been taught to them is yet another venue for further research.

Conclusion

All texts want to impart an information: one can read them as having the intention to instruct, and therefore, often to quickly, reconstruct from them a class room icon. Our knowledge is for all uncertain, but maybe one should imagine an assembly where pupils, teachers and scholars at different stage of learning were working together, and this representation should extend from the Vedic period to the late modern period. This is a sort of dream, a fantasy such as the mirage of Jai Singh's XVIIIth century court in which we would like to imagine Jesuit priests, Hindu pandits, Arabic and Persian scholars translating Euclid together. But as the story goes⁴, it is but a mirage, an incentive for further research.

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⁴ A translation into Sanskrit was made from al-Tūsi's Persian version but obviously Euclid's text was not understood by the pandits who undertook this work.

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