Multiplying Integers: on the diverse practices of medieval Sanskrit authors

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All comments and suggestion for corrections are welcome.[1]

Abstract

We examine the diverse ways Brahmagupta (628 CE), Mahāvīra (ca. 850), Śrīdhara (ca. 750-900) and their commentators understood how a multiplication could be executed. We describe a variety of algorithms. We note how commentators give us clues to how numbers are shaped for execution, how the procedure is displayed on a working surface, etc. We attempt to evaluate in which ways resources of the decimal place value notation were used. The current historiography of elementary operations in Sanskrit sources is also revised along the way.

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1 Introduction

1.1 A variety of executions for one operation

In 1935, Datta & Singh published in Lahore a groundbreaking *History of Hindu Mathematics*[2]. In this text, they provide a homogenized point of view on medieval Sanskrit mathematics (*gaṇita*)[3]. In particular, they present a set of operations (elementary, fundamental) in arithmetics (*parikarman*) which are echoed into operations in algebra (*vidhi*). Such a set covers a bit more than what we usually call elementary operations: addition, subtraction, multiplication, division, squaring and cubing, square and cube root extraction (less the cubes for algebra)[4]:

> 'The eight fundamental operations of Hindu *gaṇita* are: (1) addition, (2) subtraction, (3) multiplication, (4) division, (5) square, (6) square-root, (7) cube and (8) cube-root.'

Datta & Singh detail the execution of elementary operations in arithmetics. They describe a variety of executions for each operation. Their reconstructions give pre-eminence to operations using resources of place value notation, since the early use of this notation is one of the important claims of the book. A same execution is studied with different texts, highlighting what is understood as the common ground of Sanskrit sources.

This homogeneous perspective has a serious basis in the corpus of medieval Sanskrit mathematical texts: there is a great continuity, over several hundred years, of classifications of mathematics involving operations (*parikarman*) and practices (*vyavahāra*)[5]. Although such classifications appear quite stable, what these classifications cover vary in number and content. Datta & Singh were well aware of this. They noted variations in classification and sometimes in execution. However, they did not attempt any serious comparison among different Sanskrit authors[6]: their overall endeavor was to explain these variations within the wider homogeneous scope of 'Hindu *gaṇita*'. Further, most variations in execution might have been perceived as trivial from the point of view of their mathematical content.

Datta & Singh's approach durably marked the historiography of mathematics in India. Thus, A. K. Bag wrote more than forty years later [7]:

> 'The eight fundamental operations of Indian arithmetic after the invention of the deci-

[2] [Datta Singh 1935].

[3] See for example [Datta Singh 1935, I, p.128sqq].

[4] [Datta Singh 1935, 128]. This point of view was probably inspired by Bhāskarācārya's (b. 1114) *Līlāvatī* and *Bījagaṇita*, whose translation by Colebrooke in 1817 ([Colebrooke 1817]) highlighted a coherent organization for operations, and an implicit theory of a set of parallel operations in between algebra and arithmetic.

[5] Together with a list of operations, mathematics could be defined by a certain number of topics, "practices" (*vyavahāra*) usually eight in number. Here also the elements of the lists, their number and contents could be subject to variation.

[6] Their comparative effort concentrates on providing parallels with what was known to them of European medieval sources.

[7] [Bag 1979, 76].

mal place value system of numeration are: addition (*saṃkalita*), subtraction (*vyavakalita* or *vyutkalita*), multiplication (*guṇana*), division (*bhāgahāra*), square (*varga*), square-root (*varga-mūla*), cube (*ghana*), cube-root (*ghana-mūla*).'

Most secondary authors of general histories of mathematics in India after them often kept to the enumeration of lists of operations, more or less taking the executions of such operations as already covered.

The unraveling of the complexity of an author's operations- what we may term each author's *practice of operation* is our greater aim[8]. Here it is narrowed to the study of practices of the executions of multiplications with integers.

Indeed, we might be tempted to consider the execution of a multiplication as an easy elementary step of mathematical practice, which does not require much discussion. However, as we try to reconstruct such executions, many questions are raised: How are numbers shaped before being multiplied ? What mathematical tools are engaged in such executions ?

Furthermore, Sanskrit medieval sources show that different executions could be carried out for multiplication. The diversity of multiplication's executions is a helpful tool to deconstruct the homogenized narratives created both by past historiography and sanskrit medieval sources themselves. The way these differences are articulated sheds light on what could have been for a given author the principles fit for either classifying different executions, or classifying the preliminary steps of an execution. In other words, executions and how they are written about illuminate how they were thought of by those who wrote mathematical texts.

Another thread followed here will be to determine how much the executions are described as resting upon resources of place value notation[9].The fact that such a notation makes for easy mechanical executions is usually the main argument used by historians to explain its popularity. In the Indian subcontinent, epigraphical proofs of the use of this notation are quite late[10]. But definitions of the place value notation existed from the 5th century, although such a notation may have been used before[11]. How widespread then was the use of resources of the decimal place value notation when executing arithmetical operations ? Among these resources which were actually mobilized in the executions ?

The corpus we have chosen to study here provides some of the earliest testimonies of multiplication execution in Sanskrit sources.

[8]What we call "practice of operation" here involves a theory of operations- a list of them, the nexus of different objects that are operated upon- whole numbers (*rūpa*), fractions (*bhinna*) (and sub-fractions), zero (*śunya*), unknowns (*avyakta*), etc.- but also how operations can be executed, with what tools, etc...

[9]This notation will also at times be called 'positional system'. Similarly, the Sanskrit word *sthāna* will be sometimes translated here as 'position' and sometimes as 'place'. In both cases, this word evokes a delineated space within a system of interacting similar delineated spaces. A *sthāna* is always in connection to other *sthāna*s.

[10][Salomon 1998].

[11][Mak 2013] and [Plofker 2009].

1.2 Sources

In the following the focus will be on three authors who wrote before Bhāskarācārya's *Līlāvatī* (12th century), concentrating on their rules (*sūtra*s) for multiplying integers. We understand these authors with the help of commentaries and of the manuscripts which contain these commentaries. The texts examined here are:

(1) Brahmagupta (628)'s mathematical section (*gaṇitādhyāya* which corresponds to chapter 12) of the *Brahmasphuṭasiddhānta* ('Treatise of the true Brahma (school)', BSS)[12]. We will read Brahmagupta through the eyes of Caturveda Pṛthūdakaśvāmin (fl. 860; PBSS), one of his earliest commentators[13].

(2) Mahāvīra's (ca. 850) mathematical text, the *Gaṇitasārasaṅgraha* ('Compendium of the Essence of Mathematics', GSS)[14]. Our understanding of Mahāvīra's texts partly rests on undated anonymous commentaries [15].

(3) Śrīdhara (ca. 750-900)[16] 's *Pāṭī-gaṇita* ('Board mathematics', PG[17]). This text was edited and published with an undated anonymous commentary (PGT) in [Shukla 1959], which will be the basis for our understanding of the PG[18].

1.3 Different multiplication names

The term *guṇana* is singled out by Datta & Singh and then A. K. Bag to name the multiplication operation. Such a word seems to have been standardly accepted as the name for this operation in later sources. For the authors and commentators examined here the term *pratyutpanna* seems to have been chosen to name the multiplication operation. It refers to the operation in all its generality; it can be applied to all sorts of objects (fractions, unknowns, different kinds of quantities). Its literal meaning is 'what is produced'. Etymologically then *pratyutpanna* designates the product. As we will see sometimes both meanings, product and multiplication, seem to be understood together.

[12] This chapter was translated in [Colebrooke 1817] and the whole of Brahmagupta's text was edited and published by [Dvivedin 1902].

[13] Pṛthūdaka's commentary is extensively translated in the footnotes of [Colebrooke 1817]. [Dvivedin 1902] includes his own Sanskrit commentary which quotes and paraphrases Pṛthūdaka. An attempt to a kind of critical edition with a mathematical analysis and Hindi commentary of the BSS, was made within a collective effort by [Sharma et alii 1966], which also rests heavily on Pṛthūdaka. A critical edition of Pṛthūdaka's commentary on the section on the Sphere (*golādhyāya* which corresponds to chapter 21) of the *Brahmasphuṭasiddhānta* was edited and translated by [Ikeyama 2003]. No edition of the whole commentary exists today, notably no edition of the mathematical section. Section B provides an edition and translation of PBSS.12.55.

[14] This text was edited and translated into English by [Rangacarya 1912]. His work was further translated into Hindi in [Jain 1963] and recently, into Kannada [Padmavathamma 2000].

[15] The part of the commentaries of interest to us here are transliterated and translated in section C. The Hindi publication of [Jain 1963] contains an Annex which exposes notes taken from palm-leaf manuscripts obtained by Dr. Hiralal Jain in 1923-24 from a Jain temple in Karanja (Maharashtra). We will refer to these as the 'Notes of Karanja'. We will also add to our sources a palm-leaf manuscript, written in old Kannada characters (GOML-13409).

[16] [Shukla 1959, Introduction, x and xxi] considers that Śrīdhara lived after Mahāvīra but this is still subject to discussion.

[17] or 'Algorithmic mathematics', since *pāṭī* can mean, either 'board' or 'algorithm'.

[18] A transliteration and translation of a portion of this commentary is provided in section D.

Other terms are used in the sources studied here. Thus Mahāvīra quotes *guṇakāra* as a synonym of *pratyutpanna*. Pṛthūdaka the commentator of Brahmagupta uses the term *guṇanā*[19], when evoking modes of multiplication (*guṇanā-prakāra*) of integers[20].

'Multiplication by parts', 'multiplication by portions', 'as it stands' , 'door-junction' are names concerning the executions of a multiplication found in our sources. Later descriptions of all of these executions are known but were not taken as part of our study, which is restricted to treatises before the 10th century[21]. Another rule exists in our corpus. Indeed, secondary sources also refer to an 'algebraical multiplication' which consists in adding or subtracting an arbitrary number to the multiplicand or multiplier and then correcting the result. Brahmagupta, in verse 56 of Chapter 12 of the BSS, thus provides a rule that is interpreted in this way by Datta & Singh[22]. We will not study it here. Other rules for multiplying integers are also documented in later texts, and not examined here.

What were the different executions of multiplications presented by Brahmagupta, Mahāvīra and Śrīdhara, as we can understand them through the commentaries studied here ? What tools were used to carry out these executions ? How much was position a central tool for such executions ? What do the rules and the way we reconstruct them tell us of how the authors and commentators understood the different algorithms for multiplication ? Has the historiography distorted the perception of such executions ? Such are the questions we will try to answer as we examine the different rules in chronological order of the treatises examined.

2 Brahmagupta and Pṛthūdaka's commentary

Brahmagupta's first verse of the mathematical chapter of the *Brahmasphuṭasiddhānta* (BSS.12), provides a definition of what a mathematician (*gaṇaka*) is: somebody who knows 'twenty operations' (*parikarma-viṃśati*)[23].

---

[19] As a feminine noun, and not in the usual neuter form of *guṇana*.

[20] A distinction seems to be established at that moment, by this commentator, between a name for a general operation operating on different types of quantities (*pratyutpanna*) and different modes of execution of a multiplication on integers (*guṇanā*). We will come back to this in section 2.5

[21] Thus we will not evoke Āryabhaṭa II's *Māhasiddhānta*, Siṃhatilaka's commentary on Śrīpati's *Gaṇita-tilaka*, or other commentaries on the *Līlāvatī* which also discuss and layout multiplications of integers. Of course it is probable that the commentaries considered for the *Gaṇitasārasaṅgraha* and the *Pāṭī-gaṇita* have been written after the 10th century. The limits of our sources are thus quite arbitrary.

[22] [Datta Singh 1935]. But Colebrooke, following Pṛthūdaka, interprets it as a rule to correct a multiplication.

[23] Indeed, BSS.12.1 runs as follows (a translation inspired by [Plofker 2007, 421] and [Colebrooke 1817]) :

> Whoever distinctly and severally knows the twenty operations beginning with addition, and the eight practices ending with shadows, is a mathematician.
> *parikarmaviṃśatiṃ saṅkalitādyāṃ pṛthag vijānāti|*
> *aṣṭau ca vyavahārān chāyāntaṃ bhavati gaṇakas saḥ||*

[Datta Singh 1935, 124] note that Brahmagupta counts 20 'operations' and remark that 'of the operations named above, the first eight have been considered to be fundamental by Mahāvīra and later writers.'

Pṛthūdaka, the commentator, lists what these operations are, and thus provides *pratyutpana* as a name for multiplication[24]:

> Addition, subtraction, multiplication, division, square, square-root, cube, cube-root, five categories [of fractions], Rule of Three, Rule of Five, Rule of Seven, Rule of Nine, Rule of Eleven, and barter and exchange are the twenty [operations].

A first *sūtra* concerning multiplication is given by Brahmagupta in verse 3 of BSS.12[25]. We will not examine it here, but note that it concerns a multiplication of fractionary quantities, an integer being possibly considered as a fraction with denominator one. In verse 55 of the same chapter, Brahmagupta comes back to multiplication and focusses on the multiplication of integers[26].

This verse runs as follows[27][28]:

> BSS.12.55. The multiplicand (*guṇya*), made into 'a zig-zag' (*go-mūtrikā*), equal in portions (*khaṇḍa*) to the multiplier (*guṇakāra*), multiplied ⟨and the partial products⟩ added is the product (*pratyutpanna*), or, ⟨the multiplicand⟩ is equal in parts (*bheda*) to the multiplier ||55||

This rule provides the gist of what we can read as two different kinds of executions of a multiplication. In Sanskrit, the rule seems symmetrical (*guṇakāra-khaṇḍa-tulyo (...) guṇkāraka-bheda-tulyo vā*), as it is organized around two subdivisions of the multiplier (*guṇakāra*): in portions (*khaṇḍa*) or in parts (*bheda*). As we will see, it is understood that the multiplicand (*guṇya*) is repeated as many times as there are parts or portions in the multiplier. Such a rule then rests on two specified operands: a multiplicand (*guṇya*) and a multiplier (*guṇakāra*) which are not treated as interchangeable, that is,

---

[24]

> *parikarmāṇi saṅkalitaṃ vyavakalitaṃ pratyutpanno bhāgahāraḥ vargo vargamūlaṃ ghano ghanamūlaṃ pañcajātyaḥ trairāśikaṃ pañcarāśikaṃ saptarāśikaṃ navarāśikaṃ ekādaśarāśikaṃ bhaṇḍapratibhāṇḍaṃ ceti viṃśatiḥ*

Although Pṛthūdaka's commentary on the mathematical section of the BSS is yet unedited and unpublished, this part has been often quoted or referred to. Indeed it can be found in translation in the footnotes of [Colebrooke 1817, ***], or paraphrased in [Dvivedin 1902, ***] and [Plofker 2009, 141].

[25]

> BSS.12.3. The integers multiplied by the denominators are added to the numerators|
> The product of the numerators divided by the product of the denominators is the multiplication (*pratyutpanna*) of two or many.||3||
> *rūpāṇi chedaguṇany aṃśayutāni dvayor bahūnāṃ vā*|
> *pratyutpanno bhavati chedavadhenohṛto 'ṃśavadhaḥ* ||*3*||

[26]This will raise a problem concerning the organization of Brahmagupta's mathematical chapter. A justification of this second rule is thus given by Pṛthūdaka at the beginning of this commentary of BSS.12.5, as seen in the first line of his commentary in section B.

[27]An edition and translation of Pṛthūdaka's commentary of this *sūtra* is given in section B. Translations of this verse can also be found in [Colebrooke 1817, 319] and [Datta Singh 1935, p. 135].

[28]

> BSS-12-55 *guṇakāra-khaṇḍa-tulyo guṇyo gomūtrikā-kṛto guṇitaḥ*|
> *sahitaḥ pratyupanno guṇkāraka-bheda-tulyo vā* ||

they do not enter the same steps in the execution procedure. What should be done to these parts in relation to the multiplicand is not precisely specified: the rule mentions a shaping of the multiplicand, a multiplication and an addition. Because of its symmetric syntax, the rule seems to be the same whatever way the multiplier is subdivided. But, we will see that in manuscripts, and probably in Pṛthūdaka's interpretation, this is not the case.

Executions of multiplications described here requires some preliminary knowledge of multiplication. Indeed, in this context, the multiplication of numbers smaller than ten has the status of tacit knowledge. The multiplication of a larger number (i.e. in case a decimal place value notation is used, a number made of a string of two or more digits) by a number smaller than ten (i.e. with a single digit) is also an implicit prerequisite. Multiplication tables were probably part of a larger elementary education. We do not know how high they went[29]. Beyond tables, it is possible to imagine that elementary executions were not always made using resources of decimal place value notation, although this would need to be substantiated with sources that we do not have[30]. If we suppose that executions did use such resources, we still do not know specifically how they were carried out. For instance, techniques concerning carry-overs are not spelled out: we do not know whether they were placed at a specific place on the working surface, and if so where, or whether they were memorized. This blind corner, which concerns also how additions were executed, will affect our reconstructions of the processes. There is little doubt that complex computations could be routinely carried out outside scholarly milieus.

Brahmagupta's rule then concentrates on the multiplication of larger numbers. The output of the algorithm is what is called 'multiplication/product': the rule retains the ambiguity of the meaning of *pratyutpanna*, and refers to the multiplication and to its result.

To evoke the subdivisions of the multiplier, Brahmagupta uses two words with a similar meaning: *khaṇḍa* ('portions') and *bheda* ('parts'). To know what they label we need a commentary. In the following, we will use Pṛthūdaka's commentary to provide an interpretation of the executions the BSS refers to. Executions will thus be reconstructed from his point of view. Previous translations, editions and analysis of BSS.12.55 will be used together with the three manuscripts which preserve Pṛthūdaka's commentary on the mathematical chapter of the BSS: Colebrooke's manuscript (I_1) which served as a basis for his 1817 translation, Dvivedi's manuscript (V_1), which served for his 1902 edition of the BSS, and a copy of Colebrooke's manuscript (I_2)[31].

Pṛthūdaka's interpretation of Brahmagupta's rule explicits partly how these multiplications should be executed. As mentioned previously, two main executions are distinguished, one in which the multiplier is subdivided in 'portions' (*khaṇḍa*) and another in which the multiplier is subdivided in 'parts'

[29] [Sarma 1997] and [Sarma 2011].

[30] Studies of numbers and computations found in inscriptions might yield some results in this respect.

[31] S. Ikeyama very generously provided the copies of V_1 used here, while copies of I_1 and I_2 were made available by the funds of the Algo ANR.

Figure 1: I_1, Colebrooke's manuscript

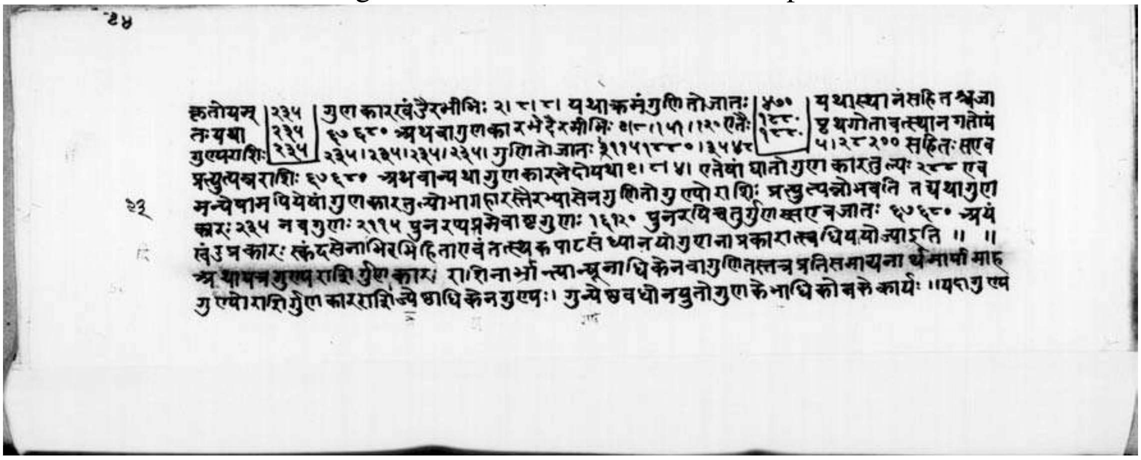


Figure 2: V_1, Dvivedi's manuscript

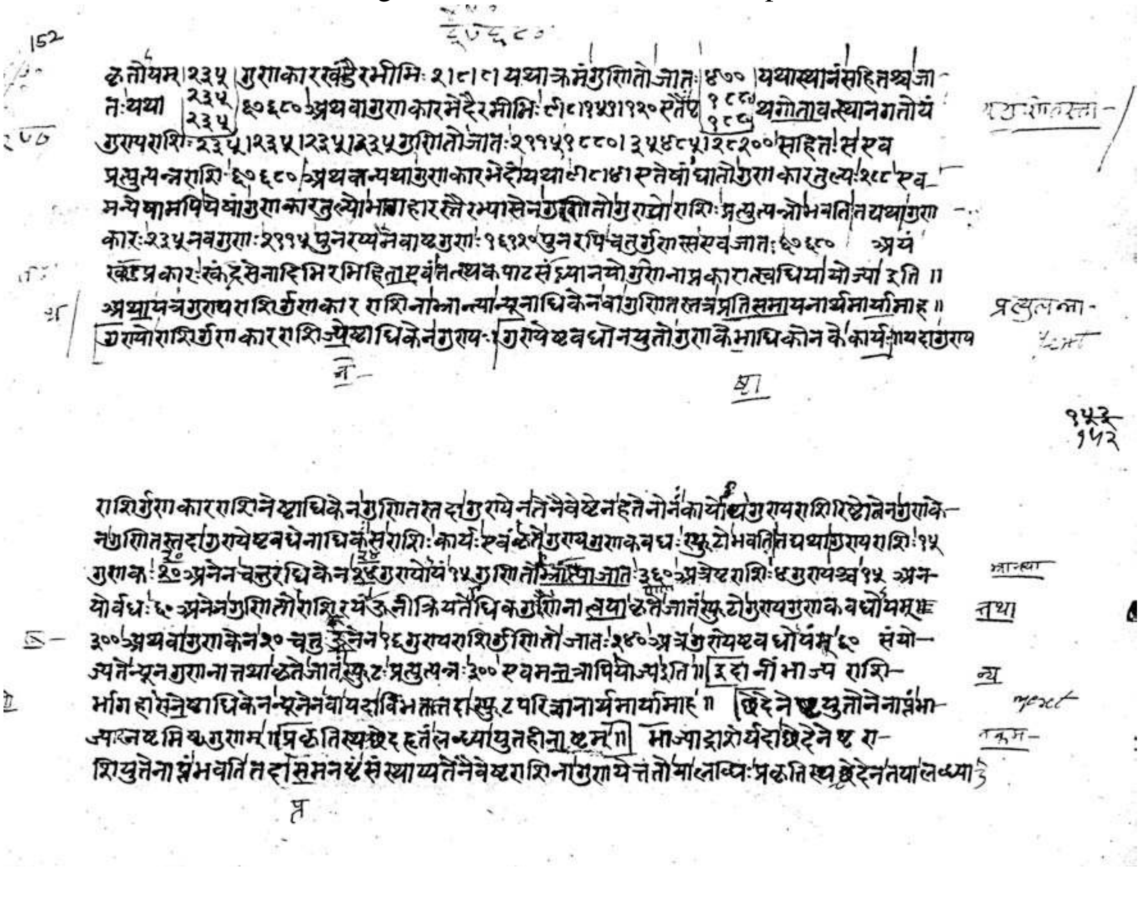
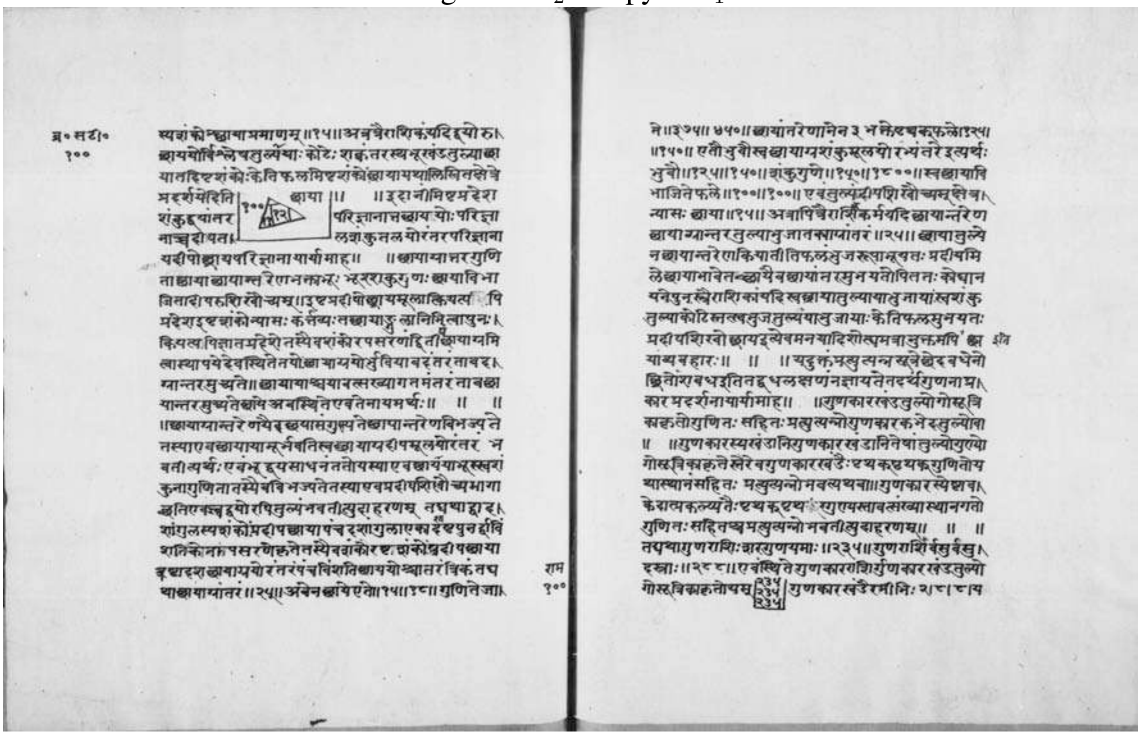


Figure 3: I_2 a copy of I_1



(*bheda*). They are discussed now.

2.1 A multiplier subdivided in 'portions' (*khaṇḍa*)

Multiplying with a multiplier subdivided in 'portions' (*khaṇḍa*) means, according to Pṛthūdaka, splitting up the multiplier according to its different powers of ten. This method then rests on the fact that counting uses base ten, and that multiplication is distributive over addition. Pṛthūdaka considers the following example: 235 is multiplied by 288. The method uses the following principle: $235 \times 288 = 235 \times (2.10^2 + 8.10^1 + 8.10^0) = (235 \times 2.10^2) + (235 \times 8.10^1) + (235 \times 8.10^0)$. Further, we will argue that different resources of the decimal place value notation are used in the execution as the commentator understands it.

2.1.1 Execution

The different steps of the execution as understood by Pṛthūdaka can be reconstructed as follows:

1. *'the multiplicand quantity, equal in portion to the multiplier, is made into a 'zig-zag''.*

   The number of digits forming the multiplier determines the number of times the multiplicand is noted in a tiered column. In the case where 235 is multiplied by 288, since 288 is made of three digits, 235 is noted three times in a tabular format.

   What exactly is meant by a 'zig-zag' remains unclear as will be discussed below. Manuscripts all agree that this refers to a display in a column:

235
235
235

Another possible interpretation, could be to understand the layout with each row written one place to the right with respect to the previous one, placing the multipliers according to the value of the respective digit of the multiplier that it will be multiplied with:

2	3	5		
	2	3	5	
		2	3	5

2. *'It is multiplied in due order by the portions of the multiplier one after the other'.*

   One by one, each digit of the multiplier multiplies one of the noted multiplicands in the column. So that with Pṛthūdaka's example, $2 \times 235 = 470$ and $8 \times 235 = 1880$:

   In manuscripts it is difficult to discern whether this layout appears as:

470
1880
1880

   Or this one:

4	7	0		
1	8	8	0	
	1	8	8	0

3. *They are 'summed according to place'.*

   The partial products are summed according to their relative places or values; providing the result of the multiplication, 67 680.

The execution as spelled out by Pṛthūdaka is carried out in four distinct steps: (a) the setting of the multiplicand, (b) the identification of the digits forming the multiplier, (c) the computation of the partial products, and then (d) their sum. This is not the order of execution, since to lay down the multiplicand (a) we need to already know how the multiplier is subdivided (b). Thus in Pṛthūdaka's commentary (b) appears as a justification of (a). The order in which the partial products are computed and summed is indifferent: these sub-steps are not made explicit in the commentary.

In the following, how the commentary and the manuscripts understand how the multiplicand and multiplier were laid out and thus how resources of place value are expressed and used is examined now. We will also see the limits of attempting to reconstruct executions when manuscripts are so far removed in time from the commentary.

2.1.2 Displaying multiplicand and multiplier

Brahmagupta's rule starts by providing a name for a specific shaping of the multiplicand, a 'zig-zag', *go-mūtrikā*. How this name should be read, understood, and translated is discussed first.

Zigzag or cow's string ? Authors do not agree on the appropriate reading of the compound describing the multiplicand's shaping. Colebrooke reads 'cow's string' (*go-sūtrikā*). This reading corresponds indeed to what is noted in all three manuscripts. Dvivedin's text, as well as Datta & Singh's interpretations suggest a different reading, *go-mūtrikā*. Datta & Singh evoke in a footnote live oral traditions through the figure of 'paṇḍits', to justify their readings[32]. The transition in manuscript from the *devanagarī* म (m) to स (s) is indeed very slight.

In the first case, the interpretation of the name remains problematic. Colebrooke evokes a string which could tie together several cows. But the existence of such strings needs to be documented.

On the other hand the expression *go-mūtrikā*[33] is known in reference to Sanskrit calligraphic poetry (*citra-kāvya*). In poetry, a *go-mūtrikā*, involves the possibility of reading twice a verse in a several lined poem- linearly or through a zig-zag between a first and a second line[34]. However, we cannot be sure that this was precisely a way of evoking the fact that the repeated multiplier was to be displayed diagonally[35]. Whatever the compound's meaning, it names a way of displaying a repeated multiplicand for execution.

A multiplicand in a column Manuscripts display the multiplicand while solving the only example given in this part of the commentary. Such manuscripts are quite late: they date from the end of the 18th century (I_1) and maybe the 19h century (I_2, V_1), while Pṛthūdaka's commentary is probably from the 9th century. All numbers in the manuscripts are written with the decimal place value notation,

[32][Datta Singh 1935, p. 147, note 4]:

> The word *gomûtrikâ* means 'similar to the course of cow's urine' hence 'zig-zag'. Colebrooke's reading *gosûtrikâ* is incorrect. The method of multiplication of astronomical quantities is called *gomûtrikâ* even unto the present day by the paṇḍits.

[33]The expression *go-mūtrikā* is a feminine noun derived from the neuter *go-mūtra* meaning 'cow's urine' with a suffix used to make tool-words. If we consider that *go-mūtrikā* should be understood as modified by a feminine word such as *rekhā* meaning 'line', then this word should be understood as: 'a cow's urine like [line]'. Several interpretations exists on how the idea of a zig-zag derive from cow's urine, most often evoking the fact that a walking cow's urine makes a zig-zag stream.

[34]This fact has been called to our attention by S. R. Sarma, may he be thanked here again. C. Minkowski, K. Preisendanz and Andrey Klebanov gave me some additional references as well. In the examples found in [Gerow 1971, 180], the emphasis is on the repetition of certain syllables, not on the diagonal display. Gerow provides an example from Daṇḍin's *Kāvyadarśa* (early 8th century). If we read in a zig-zag from the second line (*ma* second line, *da* first line, *no* on the second line, etc.), we obtain the first line of the poem.

ma	da	no	ma	di	rā	kṣī	ṇā	ma	pā	ṅgā	stro	ja	ye	da	yam
ma	de	no	ya	di	tat	kṣī	ṇā	ma	na	ṅgā	yāñ	ja	liṃ	da	de

This amounts to requiring that every other syllable be the same as the corresponding syllable in the previous quarter of verse.

[35]Additionally, one may wonder if this was the case, why authors didn't use the expression *tiryak*, which means 'diagonal'.

which is standard for texts dealing with mathematical and astral topics.

Figure 4: Multiplicand in a zigzag in I_1 for PBSS.12.55

Figure 5: Multiplicand in a zigzag in V_1 for PBSS.12.55

The manuscript's display involves noting several times the multiplicand of the example in a column, as shown in Figure 4 and 5: $\begin{array}{|c|}235\\235\\235\\\hline\end{array}$

Such notations are commonly considered a representation, within the text of a commentary, of a working surface on which the execution took place. The verticality of the layout, and the related necessity to therefore separate it from the following text by a capsule, suggests that the working surface was in a space different from the one in which the text itself was inscribed.

On the contrary, the multiplier seems to be noted within the text.

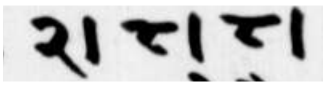


Figure 6: Multiplier by portions in I_1 for PBSS.12.55

The multiplier within the text All manuscripts also agree on the multiplier's layout, as represented in Figures 6 and 7, while executing the multiplication of 235×288. As for the multiplicand, the multiplier is usually noted using the decimal place value notation. Here however, the multiplier's different powers of ten are laid out within sentences on a horizontal row, each digit is separated by a (simple) *daṇḍa* (the Sanskrit punctuation mark: ।), in the order in which it would be noted without this separation.

On the one hand, the multiplicand seems to be shaped on a working surface separate from the text, on the other, the multiplier seems to be integrated within the text. Does this suggest that the multiplier is to be memorized and not noted ? or that it should be noted separately ? Does it indicate that for those who copied the manuscripts in the late 18th century and onwards, the working surface represented in

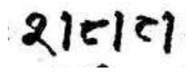


Figure 7: Multiplier by portions in V_1 for PBSS.12.55

the first execution an antiquated form and that computations were more usually integrated within the text itself ? We will come back to this topic later.

This preliminary layout raises questions on how the intermediary multiplications were carried out, and notably what place value resources were used during intermediate steps.

2.1.3 Working-out the execution

At the outset of the multiplication's execution as understood by Pṛthūdaka, the multiplier's subdivision uses the decimal place value notation: the way the multiplier is noted provides immediately the way it will be subdivided. Indeed, 288 is noted 2 | 8 | 8| and not 200 | 80 | 8 |.

Are resources of the decimal place value notation used also when executing the partial products ? When summing the partial products ? As we have seen in the reconstruction of the execution, we have not been able to decide how according to Pṛthūdaka (maybe in contrast with the manuscripts) the multiplicand and then the partial sums are laid out and computed with.

Pṛthūdaka quite clearly explicits that place (*sthāna*) is central to the different steps of the execution:

> [The multiplicand] multiplied respectively and separately by precisely those portions of the multiplier <and> added according to place (*yathā sthānaṃ*)[36] is the product.

Such places, ambiguously, could designate as much the different rows of the column where the multiplicand is repeated (in as many parts as the multiplier), as the positions in which digits are noted when noting a number with the decimal place value notation. In the first case, Pṛthūdaka's remarks concern the order in which the partial products are added; in the second he refers to their relative values.

If we consider that place here refers to the decimal place value notation, it is possible to tentatively adopt an interpretation of the columnar display which is not found in the manuscripts. Each row could have been written one place to the right with respect to the previous one, placing the multipliers according to the value of the respective digit of the multiplier that it will be multiplied with:

2	3	5		
	2	3	5	
		2	3	5

In such a 'zig-zag' , empty spaces in each row figure the product of the multiplicand by successive

[36]The expression "according to place" can be understood as referring to 'multiplied', 'added' or to both.

powers of ten: the row to be multiplied by 2 is implicitly before hand multiplied by 100, the one to be multiplied by 8, by ten. In other words, in such a reconstruction, the information on the relative value of the portions of the multiplier is captured already in the multiplicand's 'zig-zag' display. Such a layout would prepare the next step of the execution, displaying beforehand the multiplicand so that the partial products when obtained are already in the right place to be summed according to place value.

However, as seen in Figure 4 and Figure 5, manuscripts do not reproduce such a diagonal.

It is true that at this stage, before the partial products are computed, it is not essential for the execution to invest with relative values the rows of the column. It is the placement of the partial products that would be significant. Indeed, if the display involves setting the partial products according to their relative values, it should look like this:

4	7	0		
1	8	8	0	
	1	8	8	0

However, in all manuscripts, it seems that the digits of the last line are not properly placed to carry out a column by column sum as we are used to[37].

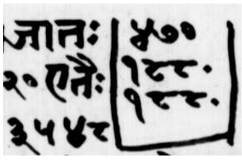


Figure 8: Partial Products in I_1 for PBSS.12.55

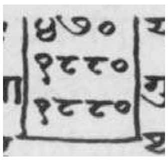


Figure 9: Partial Products in I_2 for PBSS.12.55

Figure 10: Partial Products in V_1 for PBSS.12.55

[37]The interpretation of the layouts in the manuscript is actually a bit tricky here. Although the numbers are not strictly aligned digit by digit, one can maybe read a diagonal of 'zeros' followed less clearly by a diagonal of 'seven, eight, eight'. Such diagonals seem to appear in I_1. Paleographically, strikingly enough, the zero usually a drawn circle as in the first row with 470 is followed for 1880 by what appears as not 'zeros' but simple points, which can be used in manuscripts to figure an empty space in tabular dispositions. Here however it would be difficult to understand why empty spaces would be drawn for one space and not for the following ones, or not for the 470 above...They make sense if this is a way of anticipating the lack of space to draw them out in a proper diagonal, as in the other manuscripts.. In its copy, I_2, such diagonals are less obvious and could have been ignored by the one who copied it. Maybe also the problem might be as in V_1, a lack of space.

In other words, the display in manuscripts justifies each row of number on the left in this way:

470
1880
1880

How then could such a sum "according to place" be carried out ? For the moment, no elements within the text or the manuscripts can answer this question. It is possible of course to imagine an algorithm which would take in account the relative values of each number in relation to one another (or according to their rows, which could be understood as representing from top to bottom additional decreasing powers of ten). However, such a local algorithm, for which there is so far no evidence but these columns in late manuscripts, would make little sense in what appears as an effort for generality in the algorithmic practice of operation.

If we assume that place value dictates the columnar display, then two distinct resources of the decimal place value notation would have been used in a multiplication 'with a multiplier subdivided by portions'. The first would be the decomposition in base ten of a number, noted by apposition. This is used when the multiplier 288 is subdivided by 'portions' (2|8|8) within the text of the commentary. If the repeated multiplicands are noted diagonally according to place value, then a second resource is used to develop an ephemeral computational array whose columns are decreasing power of tens[38]. This resource is precisely the one that is often drawn-out in secondary literature. We have seen however that it is possible to suppose that the columnar display may take some kind of intermediary form: using numbers noted with the decimal place value notation, it is possible to imagine that an additional value is given to such a number by the row in which it is placed, rows representing from top to bottom additional decreasing powers of ten. In this case as well, the columnar display uses position and apposition to figure value, but in this case columns and row each add value to the number. It is a 'doubly positional' display.

Let us sum up then what has already been spelled out: In the multiplication with a multiplier subdivided in 'portions', manuscripts clearly represent computations as taking place on a working surface which does not belong to the text. Spaces are delineated in the text to represent such a surface. A multiplicand is thus repeated several times in a column, this is how it is shaped as a 'zig-zag'. The subdivision of the multiplier by its digits however seems to take place not on a working surface but in a space that belongs to the text of the commentary. Such an execution rests on the decimal place value notation, at least by the subdivision of its multiplier by its digits. It is possible to interpret the shaping of the multiplicand as a 'zig-zag' as also resting on a positional configuration. Although late manuscripts and an evasive text do not enable us to reach a firm conclusion.

[38]See [Keller & Montelle, forthcoming]

Brahmagupta further distinguishes a multiplier 'subdivided by parts'.

2.2 Multiplier subdivided by 'parts' (*bheda*)

Pṛthūdaka gives two different possible understandings of the word 'part': the multiplier is either subdivided by additive or by multiplicative parts. When a multiplier is subdivided in parts, manuscripts do not shape the multiplicand in a column.

2.2.1 Additive 'parts'

In a first example, Pṛthūdaka subdivides the multiplier into additive parts. Implicitly these parts are not the powers of 10 of the numerical system. As for multiplications with 'portions', the rule rests on the distributivity of multiplication over addition. In the example given by Pṛthūdaka, 288 is subdivided into four additive parts 9, 8, 151 and 120. In other words, the underlying principle of this execution is: $235 \times 288 = 235 \times (9 + 8 + 151 + 120) = (235 \times 9) + (235 \times 8) + (235 \times 151) + (235 \times 120)$.

Execution

1. *Subdivide the multiplier in additive parts.*

   The multiplier is subdivided into four parts. In the case of Pṛthūdaka's example: 9+8+151+120=288.

2. *"the multiplicand quantity being on as many separate places <as there are parts of the multiplier>"*[39]

   Repeat the multiplicand as many times as there are parts for the multiplier. In the case of Pṛthūdaka's example, 235 is repeated four times, in the text: 235 | 235 | 235 | 235.

3. *"is multiplied by these <parts of the multiplier> one after the other"*[40].

   Each multiplicand is multiplied by one of the multiplier's parts, producing partial products. In the case of Pṛthūdaka's example, the partial products are produced within the text: 2115 | 1880 | 35485 | 28200 |.

4. *'the sum is indeed the product quantity'*[41].

   The partial products are added, yielding the result: 67680.

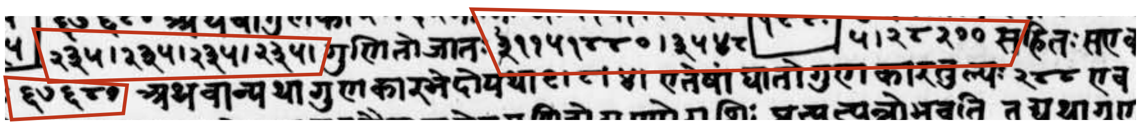


Figure 11: Multipliers having additive 'parts" in I_1 for PBSS.12.55

[39] *pṛthag etāvat sthāna-gato 'yaṃ guṇya-rāśiḥ*
[40] *taiḥ pṛthak pṛthag (...) guṇitaḥ*
[41] *sahitaḥ sa eva pratytapanna-rāśiḥ*

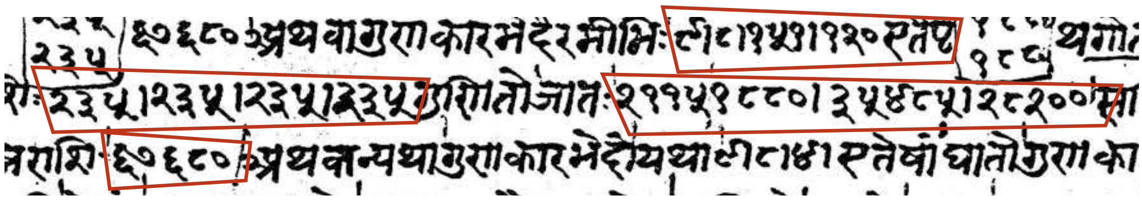


Figure 12: Multipliers having additive 'parts ' in V_1 for PBSS 12.5

Layout As previously, the decimal place value notation is used in manuscripts to note numbers during the execution of the multiplication. However, the additive parts of the multiplier are not the digits that together form the number, but quantities (some of which are digits) whose sum give the multiplier. As mentioned above, Brahmagupta's rule mentions the shaping of a multiplicand into a 'zig-zag'. This could apply to a subdivision of the multiplier by 'parts'. However, as seen in Figure 10 and 11, in all manuscripts all the numbers -the multiplicand, the multiplier and the partial products- are stated within the text, not in a column.

If we contrast the repetition of the multiplicand in a column in the first execution, and in a row integrated in the text in the second, two elements are striking: first the column requires a capsule to make sense in a space were the text is noted densely. Further, the horizontal display requires the use of the *daṇḍa*. In other words, both share a repetition, which may be what the expression *gomūtrikā* refers to. But they differ in the way they shape this repetition in relation to the text.

Are manuscripts here following Pṛthūdaka's interpretation, or doing something different ? This part of the commentary does not give us a definitive answer. Pṛthūdaka mentions in the general commentary that the multiplicand is 'on as many places for <its> numbers <as there are portions of the multiplier>' (*tāvat saṅkhyā-sthāna-gato*). During the resolution, as above, he mentions that the multiplicand is 'on as many separate places <as there are parts of the multiplier>' (*pṛthag etāvat sthāna-gato*). As previously, his use of the word place, can evoke the decimal place value notation. But it is also used standardly for cells in ephemeral computational arrays. This could indicate then that Pṛthūdaka would be referring to the column of the 'zig-zag' multiplicand.

If this was the case, why then would manuscripts represent such a different set-up from one execution to another ? In the case of a multiplication with a multiplier 'subdivided by portions', the use of a column to note the different multiplicands, might have been meaningful precisely because the repetition could take in account- in one way or another -the fact that the digits used in the method by 'portions' had implicit values. In the case of a multiplier 'subdivided by parts', the additive parts of the multiplier are quantities with no further implicit values. Manuscripts, then might be interpreting the column as indicative of a certain type of positional computation not required here.

However, the places referred to by Pṛthūdaka could also belong to a horizontal row. Additions would then be carried out, digit by digit, that is using resources of the place value, but horizontally instead of vertically. Pṛthūdaka does not specify in what space computations are carried out. No indication in the text requires that such a computational row be separated from the text, so that it is equally possible that manuscripts are actually faithful to Pṛthūdaka's original layout.

As noted in the case of a multiplier subdivided in 'portions', the text is silent on how the addition of such relatively large numbers was carried out. It makes sense to consider that the sums were made digit by digit whether it was along a line in the same space as the text or on a working surface separated from the text, either in a line or a column. But it is also possible to imagine that sums could be made without using a specific surface, in mental computation, using or not resources of place value notation.

As we see, when the text is not specific concerning layouts, the distance between the date of composition of the manuscripts and that of the original text, constitutes a blurring gap for analysis. Our ignorance of elementary executions of operations casts also its shadow here. If we subscribe to the idea that no 'zig-zag' is required from a multiplication with a multiplier subdivided by 'parts', this implies that Brahmagupta's rule is read as prescribing different steps according to how the multiplier is subdivided. This could be in a way confirmed, by the fact that a subdivision by multiplicative 'parts' does not require any sum of partial products.

2.2.2 Multiplicative 'parts'

Pṛthūdaka gives another understanding of the word 'part': the multiplier would be subdivided in aliquot parts, that is parts whose product give the multiplier. In the example given in his commentary, 288 is subdivided by three multiplicative parts, 9, 8 and 4. Such an execution then rests on the associativity of multiplication: $235 \times 288 = 235 \times (9 \times 8 \times 4) = (235 \times 9) \times 8 \times 4 = (2995 \times 8) \times 4 = 16920 \times 4 = 67860$.

Execution

1. *'Or else the parts of the multiplier are <taken> otherwise, for instance 9 | 8 | 4. Their product is equal to the multiplier 288. So with others as well which are such that <each> is a divisor <and their whole product is> equal to the multiplier'*[42].

   The multiplier is subdivided in multiplicative parts. As in the case of $288 = 9 \times 8 \times 4$.

2. *'the multiplicand quantity multiplied by these produces the product. For instance, the multiplicand 235, times nine 2995, and just as previously times eight, 16 920, this times four, once again also is precisely 67 680.'*[43]

[42] *athavānyathā guṇakāra-bhedā yathā 9|8|4 eteṣāṃ ghāto guṇakāra-tulyaḥ 288 evam anyeṣāṃ api yeṣāṃ guṇakāratulyo bhāgahāras*

[43] *tair abhyāsena guṇito guṇyo rāśiḥ pratyutpanno bhavati tadyathā guṇyakāraḥ 235 nava-guṇaḥ 2115 punar apy evāṣṭa-guṇaḥ 16920 punar api catur-guṇas sa eva jātaḥ 67680*

The multiplicand multiplies a first part: $235 \times (9 \times 8 \times 4) = 2995 \times (8 \times 4)$. The result multiplies a second part, $2995 \times 8 = 16920$. And then a third part , $16920 \times 4 = 67680$.

Layout and place value As previously, in the manuscripts, layouts are noted within sentences running horizontally, each part being separated by a double *daṇḍa*, ||, the usual sanskrit punctuation mark indicating a stop. Contrary to the previous two executions, in the case of aliquot parts there is no need to store intermediary results: the result of a product being what is then multiplied by the next part of the multiplier. When multiplying with the 'multiplier subdivided by parts', the multiplier is subdivided in parts that do not rest on the decimal place value notation.

2.3 Reconciling manuscripts and the commentary ?

Let us as way of conclusion recapture the arguments that argue in favor of a use of tabular resources of the decimal place value notation when shaping the multiplicand as a 'zig-zag': The first and foremost reason, has to do with the interpretation of the rule in the commentary and manuscripts. If we admit that manuscripts are following the commentary, then the shaping as a 'zig-zag' is not used when partitioning the multiplier with additive parts that are not powers of ten. It is thus consistent to consider that it is a feature of an execution using resources of the decimal place value notation. The second reason is that such an understanding is also consistent with the logic and ergonomy of the computation itself. Indeed, we do not know how partial sums were carried out. But it is plausible that the absolute values of these numbers are coined in one way or another in the 'zig-zag' display. A third reason would be that Pṛthūdaka specifies that the final sum is made 'according to place', which could be a reference to the use of specifically this way of coining positions as well. Finally the name of the layout, 'zig-zag', could emphasize the diagonal created when noting several times the partial products according to their different powers of ten.

In the end, as historians, to make sense of the manuscript's layout the only tools left to us at this stage are fictitious but plausible reconstructions. Colebrooke's understanding of the layout of a multiplier subdivided by portion, exposed below, is for instance a witty creation accounting for manuscripts reading and of the problems exposed in the above paragraph.

In all cases, it is clear that Pṛthūdaka understands Brahmagupta's distinction between 'portion' and 'part' as drawing a line between different kinds of methods to execute a multiplication: those that use resources of the decimal place value notation and require a specific shaping of the multiplier in a column and the others. It is not clear whether Pṛthūdaka understands the shaping of the multiplicand into a 'zig-zag' as being specific to an execution with a multiplier subdivided by 'portion'. Manuscripts opt for this reading. Pṛthūdaka's text is undecidable on this point. However, his reading of 'parts' as including both subdivision by additive and multiplicative parts, implies that he does not read Brahmagupta's rule as being symmetric in its relation to the subdivisions of the multiplier. Indeed, the use of the sub-steps of addition and multiplication are not understood in the same way in the case of a multiplier subdivided by 'parts', since multiplicative parts do not require the addition of partial sums.

Thus Brahmagupta, Pṛthūdaka and the manuscripts all give clues, sometimes contradictory or difficult to reconcile, on how to reconstruct the different executions of a multiplication. Previous scholar's have also given their interpretations of these loopholes of the sources.

2.4 Reflecting on the historiography 1

Here we will discuss how previous historians of mathematics have worked on the executions of multiplications stated by Brahmagupta.

2.4.1 How many executions are described by Brahmagupta ?

Datta & Singh count four different types of executions for a multiplication in Brahmagupta's text: [Datta Singh 1935, 135-136] 'Brahmagupta mentions four methods: (1) *gomūtrikā* (2) *khaṇḍa*, (3) *bheda* and (4) *iṣṭa*'.

The fourth execution is the one provided in verse 56. Three are listed in Datta & Singh's text then for verse 55: what is for Pṛthūdaka a way of displaying the multiplicand, is understood by Datta & Singh as an execution in itself. It is difficult to understand why they miscounted. This may be due to the fact that they overall seem to consider that most early authors considered four executions for multiplication: Śrīdhara notably[44].

The second execution, what we have translated as a 'multiplier subdivided in portions' (*khaṇḍa*) is identified by Datta & Singh as being the same as the execution called *sthāna-khaṇḍa* (that they translate as 'separation of places') defined by Bhāskarācārya in the *Līlāvatī*[45]. Such a rule is indeed characterized by the fact that the multiplier's digits separately multiply the multiplicand. After reconstructing what they deem is Pṛthūdaka's method for *go-mutrikā* (which resembles actually his multiplication by 'portions') they add[46]:

> The *sthāna-khaṇḍa* and the *go-mutrikā* methods resemble the modern plan of multiplication most closely. The *sthāna-khaṇḍa* method was employed when working on paper.

Although we do not understand why 'portions' and 'zig-zag' were separated, nonetheless Datta & Singh seem to come to the same conclusion as the one we get from adhering to Pṛthūdaka's under-

[44] This is discussed in section 4.
[45] [Datta Singh 1935, p.146].
[46] [Datta Singh 1935, p.148].

standing of Brahmagupta's rule: a 'multiplier subdivided by portions' with a multiplicand in a 'zig-zag' is used to execute a multiplication using resources of the decimal place value notation ('resemble the modern plan of multiplication most closely'). Datta & Singh also make a point that characterizes multiplications made from additive partitions of multiplier or multiplicand: these require a storage of the partial products that will later be added.Such a storage raises the question of the space of computation and the material medium and tools required for the execution ('paper').

The third execution identified by Datta & Singh '*bheda*' ('parts') corresponds to the understanding of a multiplier subdivided by parts as described in Pṛthūdaka's commentary.

It is difficult to unravel in a nutshell both Datta & Singh's subtle analysis and some of their quick regroupings. We can note however that they seem to consider that all identified executions for mathematics were known to mathematician's writing in Sanskrit after Brahmagupta. They compare such executions with those known in the historiography in medieval Europe. Such a comparison may be a way of suggesting that such methods had as origin the Indian subcontinent, or just that the latter was first. As we will see they lay an implicit emphasis on tabular resources of the decimal place value notation.

Indeed, given the evidence in the commentary and the manuscripts, the interpretation of the layouts and hence of whether the decimal place value notation is used for executions or not, presents a high degree of variation.

2.4.2 Putting the execution into tables

Colebrooke's reconstruction Colebrooke in his translation of large chunks of Pṛthūdaka's commentary (as can be seen in Figure 13) represents all executions in a two or three column table.

Figure 13: Colebrooke's Translation of Pṛthūdaka

* Example: Multiplicand two hundred and thirty-five. Multiplicator two hundred and eighty-eight.

The Multiplicator is repeated as often as there are portions in the multiplicator:

235	2
235	8
235	8

Multiplied by the portions of the multiplier in their order, there results 470 / 1880 / 1880 : which, added together according to their places, make 67680.

Or the multiplicand is repeated as often as the parts 9, 8, 151, 120; and multiplied by them

235	9	2115
235	8	1880
235	151	35485
235	120	28200

The sum is the quantity resulting from multiplication, as before, 67680.

Or the parts of the multiplier are taken otherwise: as thus 9, 8, 4; the continued multiplication of which is equal to the multiplier 288. So with others. And the multiplicand is successively multiplied by those divisors, which taken into each other equal the multiplicator. Thus the multiplicand 235, multiplied by 9, makes 2115; which, again, taken into 8, gives 16920; and this, multiplied by 4, yields 67680.

Concerning the first 'multiplication with portions' the multiplicand's column is exactly as in the manuscript he had in his possession. The multiplier on the other hand is made into a column whose diagonal represent the relative values of the multiplier's digits. The row inside the text where the multiplier was subdivided in the manuscript unfolds into a table in which multiplicand and multiplier are in the same space. The multiplier's diagonal provides a grid along which the partial products will be alined for an addition 'according to the positions', understood as the positions of the decimal place value notation.

Strikingly, the second execution of a 'multiplication with (additive) parts' is associated by Colebrooke with a three column layout: a column for (repeating) the multiplicand, a column for the parts of the multiplier and a column providing the partial products. He departs here from what was available from his manuscript. Colebrooke's tabular display presents the different steps of the algorithm, erasing none. It is possible even to imagine that the final sum is obtained below the last column.

235	9	2115
235	8	1880
235	151	35485
235	120	28200
		[67680]

He does not use such a display for 'multiplication with (multiplicative) parts'. Therefore, Colebrooke's usage of tabular displays seem justified by the need to store intermediary data (the partial products that will have to be summed later on). Indeed, when no storage is required, Colebrooke does not create a tabular display for the execution.

Datta & Singh's reconstruction also sets up multiplier and multiplicand in a two column table in what they understand of a 'zig-zag' and a 'multiplier subdivided by portions'.

Datta & Singh's reconstructions Datta & Singh's reconstruction of a zig-zag method inverts Colebrooke's proposition:

Here the multiplicand's column is the one which uses a positional grid of decreasing power of tens, while the multiplier's column spells out the grid row by row:

2	1	2	2	3		
3		1	2	2	3	
5			1	2	2	3

Such a layout represents the two resources of the decimal place value notation that we have identified in the multiplication with 'portions of the multiplier': First, the multiplier's 'portions' are enumerated by apposition of its digits in the multiplier's column. As in the manuscripts, the multiplier is subdivided by its digits, contrary to what is in the manuscripts the multiplier is displayed in a column.

Figure 14: Datta & Singh's 'zig-zag method'

essentials the same as the *sthâna-khaṇḍa* method. The following illustration is based on the commentary of Pṛthudakasvâmî.

Example. To multiply 1223 by 235

The numbers are written thus :

```
2   1223
3    1223
5     1223
```

The first line of figures is then multiplied by 2, the process beginning at the units place, thus: $2 \times 3 = 6$; 3 is rubbed out and 6 substituted in its place, and so on. After all the horizontal lines have been multiplied by the corresponding numbers on the left in the vertical line, the numbers on the *pâṭî* stand thus:

```
2 4 4 6
  3 6 6 9
    6 1 1 5
2 8 7 4 0 5
```

after being added together as in the present method.

The *sthâna-khaṇḍa* and the *gomûtrikâ* methods resemble the modern plan of multiplication most closely. The *sthâna-khaṇḍa* method was employed when working on paper.

Second, to anticipate the values of the partial products in the multiplicand's column the tabular display is used. As in Colebrooke's display, during the computation, the multiplier's column disappears, to leave place to a tabular display for the sum.

Concerning the method 'separation of places' (*sthâna-khaṇḍa*), Datta & Singh admit many different layouts. These were maybe inspired by commentaries on Bhāskarācārya's *Līlāvatī*.

Figure 15: Datta & Singh's reconstruction of a 'separation of places' (*sthâna-khaṇḍa*)

Various arrangements appear to have been employed for writing down the working. Some of these are given below:

(*i*)[1]

```
 135
 12
----
 12
  36
   60
----
1620
```

(*ii*)[2]

```
12  12  12
 1   3   5
----------
 1260
   36
 ----
 1620
```

(*iii*)[3]

```
135  135
  1    2
---------
   270
  135
  ----
  1620
```

None of these displays are associated with Brahmagupta or Pṛthūdaka; they nonetheless illustrate that the tabular resources of the decimal place value notation can take many forms.

Where should these layouts be drawn ? None of the reconstructions seen above can be found in the manuscripts.

Colebrooke's as well as Dattta and Singh's reconstructions highlight how the vertical layout makes sense especially if the execution will use the resources of the decimal place value-notation to add according to place the partial products in the multiplication by 'portions'. The use of a vertical column then exemplifies that the multiplicand was laid out on a kind of surface distinct from the running text it is embedded in. This is not the case of numbers provided in a line, which do not appear 'encapsulated' and separated from running text in manuscripts. The more general "tabulating" of executions in secondary literature, gives the illusion that place value was used through out. Datta & Singh by figuring many possible displays show that the tabular uses of place value could be various.

Manuscripts testify to the fact that several spaces for a multiplication could be used for execution, at their time: the text, a working surface, and possibly one's mind (using one's memory for instance for the products of digits). Such spaces do not necessarily entail anything concerning the use of place value. However, the shaping of the multiplicand as a 'zig-zag', might have been understood by manuscripts and maybe Pṛthūdaka as belonging to an execution resting on such a notation. It is possible that manuscripts testify to a shift in the representation of the working surface in texts: the column showing a work on an independent surface, the horizontal line that of the work on the same space as the writing of text itself[47].

2.5 Conclusion on Pṛthūdaka and Brahmagupta's executions of multiplications

As noted at the beginning of this section, Pṛthūdaka's text is silent on what he deems elementary: the multiplication of digits by digits, and the multiplication of a higher number by a number smaller than ten. His vocabulary however, as noted previously, may actually spell out distinctions. In PBSS the verb *guṇ-*, the very verbal root of *guṇanā*, is used for the multiplication executed in the intermediary steps of the execution of the multiplication operation. The term *vadha*, as well as *abhyāsa*, for product, also in the intermediary steps of the operation on integers. This would be a contrast with the use of *pratyutpanna* as the final product of a multiplication which could be carried out on different kinds of quantities (notably fractions).

Thus the execution of a multiplication has many layers, that we can but partly recover: the tools used involve tables of multiplication we have no traces of, methods for elementary multiplication that are not transmitted, diverse uses of place value some tabular some not, properties of multiplication (such as associativity or distributivity of multiplication) not requiring place value, among others.

[47]These remarks were triggered by a question raised by K. Chemla, may she be thanked here.

Pṛthūdaka ends his commentary by remarking that other execution of multiplications ('modes of multiplications') exist:

> in this way modes of multiplication (*guṇā*) such as 'as it stands' (*tat-stha*), 'door-junction' (*kapāṭa-sandhī*) or another should be used by students (*adhiyā*)[48].

Names such as 'as it stands' (*tat-stha*) and 'door-junction' (*kapāṭa-sandhī*) will be found in Mahāvīra's *Gaṇitasārasaṅgraha* to which we will now turn.

3 The *Gaṇitasārasaṅgraha* and its anonymous commentaries

Like Brahmagupta, Mahāvīra provides technical names in relation to multiplication, that we will try to understand with the help of commentaries. Are the executions considered by Brahmagupta and Mahāvīra identical ? If not, can we reconstruct their execution and determine how much they relied on place value ?

In the beginning of the *Gaṇitasārasaṅgraha* (GSS)[49], after comparing the <eight arithmetical> operations (*parikarman*) to the banks of an ocean in his well known metaphor (GSS 1.20-23), Mahāvīra provides the names of these operations (GSS 1.46-48). For him, the first operation is not addition[50] but multiplication which is called *guṇakāra* as well as *pratyutpanna*[51].

3.1 A procedural rule (GSS 2.1)

Mahāvīra's rule runs as follows[52]:

> The procedural rule (*karaṇa-sūtra*) in relation to the operation of multiplication (*pratyutpanna-parikarman*), which is the first here <among the operations> is as follows:
>
> > GSS 2.1 One should multiply the multiplicand (*guṇya*) by the multiplier (*guṇa*) after having placed <both> in the manner of the 'door-junction' (*kavāṭa-sandhi*), with <either> a portion of the quantity (*rāśi-khaṇḍa*), a portion of the value (*argha-khaṇḍa*), <or> as it stands (*tat-stha*), with the direct or reverse way (*anuloma-viloma-mārga*).

48 *evaṃ tatstha.kapāṭasandhy.ānayāḥ guṇā.prakāras tv ādhiyā.yojyā*

49 The editions used and the multiple texts relating to Mahāvīra's rule are provided in section C of the Appendix.

50 The addition (*saṅkalita*) and subtraction (*vyutkalita*) are the last two operations, and are performed on partial series of integers.

51 GSS 1.46 *ādimaṃ guṇakāro 'tra pratyutpanno 'pi tad bhavet.*

52 *tatra prathame pratyutpanna-parikarmaṇi karaṇa-sūtraṃ yathā-*

> *guṇayed guṇena guṇyaṃ kavāṭa-sandhi-krameṇa saṃsthāpya* |
> *rāśy-argha-khaṇḍa-tatsthair anuloma-viloma-mārgābhyām* || *1* ||

The statement referring to ways of multiplying is followed in Rangacarya's edition by sixteen examples. We will study the first one, since it is the only one commented in the manuscripts studied here[53] :

> GSS 2.2 Lotuses were given away in offering, eight of them to each jaina temple; how many <were given away> to a hundred and forty-four temples ?

3.2 Understanding the vocabulary

Several features of the *sūtra* make it difficult to interpret. Apart from the obvious terminology used for the multiplicand (*guṇya*) and the multiplier (*guṇa*), unless one knows what meaning is (1) behind the name 'door-junction' (*kavāṭa-sandhi*)[54], (2) how to decipher the compound *rāśy-argha-khaṇḍa-tatstha* ('quantity-value-portion-as it stands'), and (3) what are these direct or reverse ways of working (*anuloma-viloma-mārga*), the calculation cannot be executed.

Deciphering the compound raises questions : what is understood here with the word 'quantity' (*rāśi*) ? Does it mean multiplicand or multiplier ? is it restricted to one of the two ? Are 'portions' (*khaṇḍa*) of the multiplier to be understood here in the same way as in Brahmagupta's rule, viz. portions according to powers of ten ?

It appears that M. Rangacarya, in order to give a translation which made sense provided an interpretation:

> After placing (the multiplicand and the multiplier one below the other) in the manner of the hinges of a door, the multiplicand should be multiplied by the multiplier, in accordance with (either of) the two methods of normal (or) reverse working, by adopting the process of (i) dividing the multiplicand and multiplying the multiplier by a factor of the multiplicand, (ii) of dividing the multiplier and multiplying the multiplicand by a factor of the multiplier, or (iii) of using them (in the multiplication) as they are (in themselves).

As we will see, these interpretations are based on the running commentaries which can be found in manuscripts.

Further, how are 'portions of the quantity', 'portion of the value' and 'as it stands' articulated with the execution 'in the manner of a door-junction' ? In the following we will argue that they indicate a way of shaping multiplicand and multiplier before the beginning of the execution.

53

> *attāny ekaikasmai jina-bhavanāyāmbujāni tāny aṣṭau* |
> *vasatīnāṃ catur-uttara-catvāriṃśac chatāya kati* || *2* ||

54 *kavāṭa* and *kapāṭa* both mean 'a panel of a door"; the word *kavāḍa* is a prakrit one, meaning the same.

3.3 The ‘notes of Karanja’

The Hindi publication of the *Gaṇitasārasaṅgraha* ([Jain 1963]) contains in its introduction an Annex giving some notes compiled by H.L. Jain, from undated manuscripts that he found in 1923-24 in an important Jaina temple in Karanja (Maharashtra). The palm-leaf manuscript GOML-13409 (Chennai), written in old Kannada script, contains also a short running Sanskrit commentary which in many places is identical to the ‘notes of Karanja’[55].

These notes will allow us to understand how the commentators on whom Rangacarya relied interpreted the compound *rāśy-argha-khaṇḍa-tatstha* (‘portions of the quantity’, ‘portion of the value’ and ‘as it stands’).

3.3.1 ‘A portion of the quantity’ (*rāśi-khaṇḍa)*

Concerning ‘a portion of the quantity’ (*rāśi-khaṇḍa)*, the ‘notes of Karanja’ and the short commentary in GOML-13409, both start in the same way, thereby suggesting an interpretation of the expression[56]:

> When the multiplicand is divided by a quantity which is a part (*bhāga*) of it ⟨and⟩ the multiplier is multiplied by that quantity, it is an indication for the presentation (*sthāpaṇa-lakṣaṇa*) ‘portion of the quantity’ (*rāśi-khaṇḍa*).

The first numerical example given by Mahāvīra (GSS 2.2) is solved in the notes and helps to clarify what is meant[57]:

> Or else, the multiplicand and multiplier are as follows, 144 is the multiplicand; for each, the <number of> lotuses is the multiplier = 8.

2|4
<u>4 8</u>
1152 portion of the quantity (*rāśi-khaṇḍa*)

We can see that the multiplicand 144 has been divided by 3, therefore the multiplier 8 has been multiplied by 3, and the multiplication 144×8 has been replaced by 48×24. This preliminary presentation of the values to be multiplied makes use of the associative property of multiplication. As it is the multiplicand which has been divided here, we can conclude that *rāśi* in *rāśi-khaṇḍa* means ‘multiplicand’. Further, when explicating Mahāvīra’s first example numerically, the commentator calls 144 the multiplicand and 8 the multiplier. This does not correspond to our modern definitions, since

[55] All of these texts are transliterated and translated in Annex C.
[56] *yena rāśinā guṇyasya bhāgo bhavet tena guṇyaṃ bhaktvā guṇakāraṃ guṇayitvā sthāpanā-lakṣaṇo rāśi-khaṇḍaḥ.*
[57] *guṇya-guṇakāraṃ yathā va 144 guṇyaṃ = pratyeka padmāni guṇakāra iti = 8.*

if we have to calculate 8 + 8 + ... + 8 (144 times), we will take 8 as the multiplicand and 144 as the multiplier, not the reverse.

The method to be applied to perform the multiplication, once this presentation is carried out, if we rest on what is prescribed at the beginning of Mahāvīra's rule, is called a *kavāṭa-sandhi* ('door junction'). The interpretation of this execution will be presented in a later section. In this understanding therefore, a 'portion of the quantity' (*rāśi-khaṇḍa)* is not a multiplication method, but a preliminary presentation (*sthāpana*[58]) requiring a transformation of the quantities to be multiplied using a 'door-junction'method.

Let us now turn to the next part of the compound.

3.3.2 'A portion of the value' (*argha-khaṇḍa*)

The second line of the 'notes of Karanja' and the commentary in manuscript GOML-13409 both state, as an explanation[59]:

> When the multiplier is divided by a quantity (*rāśi*) which is a part of it <and> the multiplicand multiplied by that quantity, it is an indication for the presentation 'portion of the value' (*argha-khaṇḍa*).

The meaning of *argha* is 'value, price', and, according to this rule, it is associated to the multiplier.

Once again a numerical illustration is given, to solve the same problem, where the calculation of 144×8 is required[60]:

> The multiplier is 8, its part is 4, if the multiplicand is multiplied by it <we get>:

$$\begin{array}{c|ccc} 4 & 5 & 7 & 6 \\ 2 & 1/1 & 1/4 & 1/2 \end{array}$$

Here, the multiplier 8 has been divided by 4, therefore the multiplicand 144 has been multiplied by 4, and the multiplication 144×8 has been replaced by 576×2. As for the previous case, this preliminary transformation puts into play the multiplication's associativity.

The 'portion of the value' (*argha-khaṇḍa*) in this sense is not a multiplication method, but rather a preliminary transformation of the multiplicand and multiplier. The layout associated with it, which contains what seems to be a misprint, will be discussed later in section 4.1.4.

As it is the multiplier which is divided in an *argha-khaṇḍa* presentation, we can suppose that the word *argha* indicates the multiplier. It is as if a situation of multiplication was based on a model

[58]The term *sthāpaṇa* is difficult to translate here. Its usual meaning is 'statement', 'disposition', 'presentation'.

[59]*yena rāśinā guṇakārasya bhāgo bhavet tena guṇakāraṃ bhaktvā guṇyaṃ guṇayitvā sthāpanā-lakṣaṇo 'rgha-khaṇḍaḥ.*

[60]*guṇakāraṃ 8 asya bhāga 4, anena guṇyaṃ guṇita cet.* The explanation of the disposition is provided below.

reminding a Rule of Three: 'if the price/value (*argha*) of 1 object is p, what is the price/value of n objects ?'. This would explain the choice of the word *argha* for the multiplier, since it means 'price'.

3.3.3 'As it stands' (*tat-stha*)

From the same two manuscripts (GOML-13409 and the 'notes of Karanja'), we can see how the commentators understood the expression *tat-stha* ('as it stands')[61]:

> When neither the multiplicand nor the multiplier are divided, it is an indication for the presentation 'as it stands' (*tat-stha*).

Therefore, in this case, the multiplier and the multiplicand remain both unchanged. This could happen, for instance, if the numbers to be multiplied were prime or whether the multiplication is not felt to need simplification.

3.3.4 Three ways of shaping multiplicand and multiplier

According to the commentaries, the compound *rāśy-argha-khaṇḍa-tatstha* ('portions of the quantity', 'portion of the value' and 'as it stands') designates three presentations of the quantities to be multiplied, two of them requiring a preliminary transformation. This is specified in both manuscripts[62]:

> Having arranged in the manner of a 'door-junction' (*kavāṭa-saṃdhāna*) the couple of quantities which are the existing multiplicand and multiplier by means <of one among> the three ways (*tri-prakāra*) [...]

We do not know if first the couple is shaped and then set in the 'door-junction' manner or if reversely, they are first set down for a 'door-junction' and then transformed. The way they could have been set will be further discussed in section 4.1.4.

3.3.5 A 'door-junction' (*kavāṭa-sandhi*): direct and indirect way

About the two ways in which the execution can be made (*anuloma* 'direct' or *viloma* 'reverse') , in the Karanja manuscript, we can read[63]:

> Starting from the beginning of the quantity up to the end as indicated for a multiplication (*guṇana-lakṣaṇena*) in the direct way (*anuloma-mārga*); and, from the end of the quantity up to the beginning, as indicated for the multiplication in the indirect way (*viloma-*

[61] *guṇya-guṇakāro [rau] abhedayitvā sthāpanā-lakṣaṇaḥ tat-sthaḥ.*

[62] *tri-prakāraiḥ sthita-guṇya-guṇakāra-rāśi-yugalaṃ kavāṭa-saṃdhi-krameṇa vinyasya*

[63] *(...) guṇya-guṇakāra-rāśi-yugalaṃ kavāṭa-saṃdhi-krameṇa vinyasya* | *rāśer āditaḥ ārambhyānta-paryantaṃ guṇana-lakṣaṇena anuloma-mārgeṇa* | *rāśer antataḥ ārabhyādi-paryantaṃ guṇana-lakṣaṇena viloma-mārgeṇa ca guṇya-rāśiṃ guṇakāra-rāśinā guṇayet* |

mārga), one should multiply (*guṇayet*) the multiplicand quantity (*guṇya-rāśi*) by the multiplier quantity (*guṇakāra-rāśi*).

The 'beginning of the quantity' (*rāśer āditaḥ*) is the unit digit and its end (*anta*) is the digit placed at the highest order: this means that the direct order is from right to left, the reverse one from left to right[64].

In the following, we will present some reconstructions of the process on the basis of the anonymous manuscript accompanying Śrīdhara's *Pāṭī-gaṇita* given in section D, but, before we do this, let us draw some conclusions.

3.4 Reflecting on the historiography 2

The rules concerning multiplication in the GSS are different from the rules given in the BSS, in contrast with the uniformity suggested by Datta & Singh.

3.4.1 Mahāvīra versus Brahmagupta

The *tat-stha* (as it stands) and *kavāṭa-sandhi* (door-junction) methods are not given by Brahmagupta, but hinted at in the PBSS. If the name *khāṇḍa* is shared by both authors, we have seen that their meanings are different. For Brahmagupta, the multiplier (and not the multiplicand) is split according to the powers of ten and the partial multiplications are followed by additions, but for Mahāvīra, either the multiplier or the multiplicand is split as a product of aliquot parts: there is no addition involved and the property of distributivity of the multiplication over addition is not used at all. In this way, Mahāvīra's *khāṇḍa* would be closer to Brahmagupta's *bheda* (parts). So on the one hand the same name indicates two different methods for both authors, and on the other hand somewhat the same method is indicated by two different names.

Rangacarya and Colebrooke both rely on commentaries to understand the rules. However, they do not translate in the same way; Colebrooke translates literally while Rangacarya provides rather an interpretation of the rule when ambiguities arise. His translation collects the commentaries' explanations.

3.4.2 How many executions are described by Mahāvīra ?

Datta & Singh seem to consider that the 'door-junction' method is on the same level as the three other methods in Mahāvīra's work, as well as in Śrīdhara's:

[64] To proceed from the right to the left while placing the digits forming a number was the usual practice in India from the first centuries of the Christian era. See [Sarma 2009].

'Srîdhara mentions four methods of multiplication: (1) *kapâṭa-sandhi*, (2) *tastha*, (3) *rûpa-vibhâga* and (4) *sthâna-vibhâga*. Mahāvīra mentions the same four.'

This statement seems to imply that the three last elements of the above enumeration are also executions of multiplications on integers. This does not agree with what we have seen of Mahāvīra's rule and the anonymous commentaries.

Datta & Singh also seem to consider that the executions of multiplications found in the GSS are the same in number and in kind as in Śrīdhara's PG [Datta Singh 1935, 136]. Hence, let us clear what are the methods given in the PG and examine if they are the same as in the GSS ?

4 Śrīdhara and his anonymous undated commentary (PGT)

Śrīdhara states three verses on multiplication[65]. The commentary is unfortunately damaged and incomplete in this passage. It has been reconstructed for the most part by the editor, K. S. Shukla, one the basis of the only known manuscript of this text. We have not had access to this manuscript.

The first two verses describe the 'door-junction' and 'as it stands' as follows[66]:

> PG.18 Having placed the multiplicand (*guṇya*) below the multiplier quantity (*guṇa-rāśi*), according to the 'door-junction' method (*kavāṭa-sandhi-krama*), one should multiply going reversely (*viloma-gati*) or in a direct (*anuloma-mārga*) way, step by step.
> PG.19 Having shifted again and again thus should be the door-junction. This procedure (*karaṇa*) when it (the multiplier) is stationary is therefore a multiplication 'as it stands' (*tat-stha*).

Śrīdhara's rules are more detailed than the two other sūtras previously studied. Nonetheless, to reconstruct the processes evoked here, the anonymous commentary will provide much help.

4.1 A 'door-junction' (*kavāṭa-sandhi*)

The execution of a 'door junction' rests on the decimal place value notation and uses a dynamic layout. The multiplier will keep on moving, as the multiplicand's digits will be erased to be replaced by the intermediate products at each step. Thus, there will be as many steps in the execution as there are digits in the multiplicand

[65]Our translation of these verses with their commentary are provided in section D of the Appendix.
[66]*PG.18 vinyasyādho guṇyaṃ kavāṭa-sandhi-krameṇa guṇa-rāśeḥ* |
guṇayed viloma-gatyā 'nuloma-mārgeṇa vā kramaśaḥ ||*18*||
PG.19. utsāryotsārya tataḥ kavāṭa-sandhir bhaved idaṃ karaṇam|
tasmiṃs tiṣṭhati yasmāt pratyutpannas tatas tatsthaḥ ||*19*||

In the following, we reconstruct the process for the calculation of 1296 × 21 (the first example found in the anonymous manuscript of the PG). First we describe the direct way, meaning from right to left and then the indirect way. In both cases, the multiplier will glide over the multiplicand, after having multiplied its digits, one by one.

During the reconstruction we will highlight the parts that exist in the original text and that are neither Shukla's nor our own reconstructions. Bold numbers show the digits on which the computation is taking place.

4.1.1 The direct way

As 1296 contains four digits, there will be four main steps. Whenever a regrouping requires that a digit be carried over, it will be placed on a line below the multiplicand. This is justified by the only original layout found in PGT for this part of the process.

1. [67] 6 is multiplied by 21 : the first product, 6, is written below the 1, the second product, 12, below the 2, after erasing the 6 of the multiplicand. The carry-over 1 is placed below the next digit of the multiplicand, here 9.

2 **1**		**2** 1		2 1
1 2 9 **6**	6×1=6	1 2 9 **6** **6**	6×2=12	1 2 9 **2** 6
				1

[67]This step is entirely supplied by the editor Shukla.

2. 21 glides one step to the left and 9, the second digit of the multiplicand, is multiplied by 21. The first product, 9, is added to the 2 which is already there below. 11 is obtained, but as there is already the add-up 1, it is erased to become 2. The same kind of process goes on for the second product, 18, and is specified in the text[68]:

> 'when eight is added to two which stands below it, in that place: zero. One also stands below two, added to one it becomes two. '

To layout 18, 8 is written in the place of 9, and 1 below the 2 which is on the left. 8 is added to the 2, 10 is obtained. 8 is erased and replaced by 0. 1 is carried over and added to the 1 which was on the left, which becomes 2.

2 1		**2** 1		2 1
1 2 **9** 2 6	9×1=9	1 2 **9** **1** 6	9×2=18	1 2 **0** 1 6
1	9+2=11, 1+1=2	**2**	18+2=20	**2**

3. 21 glides one step to the left[69]:

> 'And then, to multiply two which is in that place, the multiplier quantity slides.'

and 2, the third digit of the multiplicand, is multiplied by 21. The first setting is the only apparent in the manuscript. The process is further described here as[70]:

> 'Now, two and twenty-one become multiplicand and multiplier. Two multiplied by one: two exactly. Below one where [zero] stands, having added two, two is produced. And two multiplied by two: four, below that, from the sum of two which stands there: six. '

2 1		**2** 1		2 1
1 **2** 0 1 6	2×1=2	1 **2** **2** 1 6	2×2=4	1 **6** 2 1 6
2	2+0=2	2	4+2=6	

4. 21 glides one step to the left and 1, the fourth digit of the multiplicand, is multiplied by 21: Each digit of the multiplicand having been multiplied by 21, the multiplier is erased : only the result, 27216 remains on the working surface[71]:

[68] *aṣṭasu ca svādhaḥ-sthita-dvika-yoge tat-sthānaṃ śūnyaṃ, rūpam api dvikādhaḥ-stithaṃ rūpeṇa yujyate dve bhavataḥ*|

[69] *tataś ca tat-sthānaṃ dvikaṃ guṇayituṃ sarpati guṇa-rāśiḥ*

[70] *idānīṃ dvayor ekaviṃśatiś ca guṇya-guṇaka-bhāvo jātaḥ, eka-guṇitau dvau dvau eva, ekādhaḥ-sthe (śūnye) dvikaṃ kṣiptvā jātau dvau, dvābhyāṃ ca dvau guṇitau catvāraḥ, svādhaḥ-sthita-dvika-yogāt ṣaṭ*

[71] *idānīm ekasyaikaviṃśatiś ca guṇya-guṇaka-bhāvo jātaḥ, tadā rūpeṇa guṇitaṃ rūpaṃ rūpam eva, ṣaṭsu kṣiptaṃ sapta, dvābhyām ekaṃ guṇitaṃ dvāv iti* | *niḥśeṣite guṇya-rāśau, guṇake nivṛtte, phalaṃ tad eva 27216*|

2 1		**2** 1		2 1
1 6 2 1 6	1×1=1	**1 7** 2 1 6	1×2=2	2 **7** 2 1 6
	1+6=7			

> 'Now, one and twenty one become multiplicand and multiplier. Then, one multiplied by one: just one, added to six is seven; 'one multiplied by two: two'. As none remains in the multiplicand quantity, <and> since the multiplier is erased (*nivṛt*), the result is just that, 27216. '

The position and regular shifting of the multiplier have an important role in this execution. Indeed, they indicate the digit on which the multiplication has to be performed. Just this step is underlined in the commentary by a fixed syntactical sentence, in which multiplicand and multiplier are named ('Now, one and twenty one become multiplicand and multiplier').

It is noteworthy that the 'door-junction' procedure can be executed in two opposite directions. This is noted both by Mahāvīra and Śrīdhara. The commentator on the PG specifies[72]:

> 'The door-junction in the indirect way is easy indeed', therefore it was mentioned first (*pūrvam uddiṣṭaḥ*) <in the verse>.

As this statement is not followed by any justification, we do not know why the reverse way of working was considered easier. Our hypothesis is that it was the only way the regrouping could be incorporated immediately in the partial results and, hence, no more than two lines were required to execute the multiplication.

4.1.2 The reverse way

The same calculations could be performed in the reverse way, from left to right. A reconstruction of the process is provided here. Two options are available in this case concerning regrouping. Either the intermediate results are incorporated step by step, or only in the end. In the process described above, the digits carried over could not be incorporated in the row of the multiplicand as it would have modified its digits and led to a wrong final result.

With the intermediary results incorporated this is how the process can be reconstructed:

1. 1, the fourth and last digit of the multiplicand, is multiplied by 21: the first product, 2, is written below 2, the second product, 1, is written below 1, after erasing the multiplicand's 1.

[72] *viloma-gatyā kavāṭa-sandhiḥ sukara iti sa eva pūrvam uddiṣṭaḥ*| This sentence is analyzed in [Hayashi 2013].

2 1		**2** 1		2 **1**
1 2 9 6	$1\times2=2$	**2** 1 2 9 6	$1\times1=1$	2 **1** 2 9 6

2. 21 glides one step to the right and 2, the third digit of the multiplicand is multiplied by 21: the first product, 4, is incorporated to 1 which is already there and erased, therefore we place 5 below 2. The second product, 2, is written below 1, after erasing the 2 which was already there.

2 1		**2** 1		2 **1**
2 1 **2** 9 6	$2\times2=4$	2 **5** 2 9 6	$2\times1=2$	2 5 **2** 9 6
	$4+1=5$			

3. 21 glides one step to the right and 9, the second digit of the multiplicand, is multiplied by 21 :

2 1		**2** 1		2 **1**
2 5 2 **9** 6	9×2=18	2 **7 0** 9 6	9×1=9	2 7 0 **9** 6
	18+2=20 ; 5+2=7			

4. 21 glides one step to the right and 6, the first digit of the multiplicand, is multiplied by 21 :

2 1		**2** 1		2 **1**
2 7 0 9 **6**	6×2=12	2 7 **2 1 6**	6×1=6	2 **7** 2 1 **6**
	12+9=21 ; 2+0=2			

The multiplier 21 cannot be pushed to the right anymore, it is erased. The result is left on the working surface.

The same process can be done with all the add-ups written on the rows below the row of the multiplicand, their sum being executed only at the end. The next to last setting would then look like this :

		2	1	
2	1	2	9	6
	4	8	2	
	1	1		

It goes without saying that such executions use tabular resources of the decimal place value notation. Contrary to what we have understood of Brahmagupta's shaping of a multiplicand in a 'zig-zag', here the table considered is made to be constantly modified both horizontally and vertically in a dynamic process. This is emphasized in the commentary by the repeated references to digits that are 'below' (*adhas*), and a multiplier gliding horizontally (*sarp-*).

Śrīdhara further evokes another name found also in Mahāvīra's treatise: *tat-stha* ('as it stands')

4.1.3 'As it stands' (*tat-stha*)

According to Śrīdhara, a *tat-stha* is like a 'door-junction', but, without moving the multiplier[73]:

[73] *vinyasyādho guṇyaṃ kavāṭa-sandhi-krameṇa guṇa-rāśeḥ |*
guṇyed viloma-gatyā 'nuloma-mārgeṇa vā kramaśaḥ ||18||
utsāryotsārya tataḥ kavāṭa-sandhir bhaved idaṃ karaṇam|
tasmiṃs tiṣṭhati yasmāt pratyutpannas tatas tatsthaḥ ||19||

PG.18 Having placed the multiplicand (*guṇya*) below the multiplier quantity (*guṇa-rāśi*), according to the 'door-junction' method, one should multiply going reversely or in a direct way, step by step.

PG.19 Having shifted again and again thus should be the door-junction. This procedure (*karaṇa*) when it (the multiplier) is stationary is therefore a multiplication 'as it stands'.

The *tat-stha* is thus here a way of executing the multiplication. It is not only a presentation of the two numbers to be multiplied as was the case in the GSS.

4.1.4 Mahāvīra's and Śrīdhara's 'door-junction'

Both Śrīdhara and Mahāvīra evoke a 'door-junction' method. The elements we have for reconstructing this dynamic execution, are very slight: the reconstruction of the 'door-junction' rests on the damaged PGT, of unknown date. Concerning the *Gaṇitasārasaṅgraha* the only elements we have are provided by the two examples in H. L. Jain's printed text of the 'Karanja manuscript'.

As we have seen the first layout appears after a transformation of the multiplicand and multiplier as a 'portion of a quantity', the set up of the multiplication is as follows:

$$\begin{array}{rl} 2|4 & \\ \underline{4\ 8} & \\ 1152 & \text{portion of the quantity } (\textit{rāśi-khaṇḍa}) \end{array}$$

We do not know what could be the exact use of the *daṇḍa* (|) between the digits 2 and 4. It evokes the device used in the manuscripts of the PBSS when the multiplier subdivided 'by portions': the *daṇḍa* then indicates the different digits of the multiplier.

Such a layout does not look like the set up of a 'door-junction' as reconstructed above. Indeed, it displays the end of the process since it states the result of the multiplication without providing its intermediary steps. It could be a representation of the result that has nothing to do with the way it was effectively executed. Such a layout then would be in contrast to the usual settings that follow the exposition of an example which display only the data before execution.

The layout for the 'portion of the value', as found in H. L. Jain's notes, does not have the same shape as any of the previous layouts so far examined.

4	|	5	7	6
2	|	1/1	1/4	1/2

The first line of the layout gives the first step, 144×4, or 576; the second line is the double of 576 obtained by writing the double of each digit. We can notice that each digit of 576 has been doubled separately, using a slash (/) to differentiate units and tens. There is an obvious error, 1/0 1/4 1/2 should be written instead of 1/1 1/4 1/2, but we do not know if this error is in the manuscript itself, or if it is a copying error due to H.L. Jain.

The correct layout then would be:

4	|	5	7	6
2	|	1/0	1/4	1/2

The final step would be to incorporate whatever has been carried-over to obtain the result 1152, by adding the digits written between two slashes. This can be done from left to right or from right to left. For instance, here we would obtain from left to right: 1, 1 (0+1), 5 (4+1), 2.

Therefore, such a layout seems to be last intermediary step of the execution: First, we can infer how the quantities were originally displayed: 144 being first noted in the same line as 4. Second, we can see the result $4 \times 144 = 576$ placed in the first line. Finally, we see how the add-ups are placed in the final line. Furthermore, the final sum is not displayed. Note, that the name 'portion of the value' is not written, contrary to the previous layout.

We do not know then how the intermediary products were computed. We can imagine that the multiplication of 4×144 was made digit by digit, and was part of the prerequisite knowledge of multiplication. The second line, which displays the add-ups recalls the 'door-junction' execution as we have reconstructed it when the add-ups are not incorporated. We can note that the add ups here are not aligned in columns.

In the preceding example, the layout includes the name of the presentation. This is not the case here. The final result is not noted. Therefore, it looks like the display of an intermediary step of the execution.

Further collation of manuscripts of the *Gaṇitasārasaṅgraha* would be necessary to elaborate any further how a 'door-junction' was understood by commentators of the GSS. A more thorough examination of all texts describing such a procedure could help us specify how standardized was this method of execution of a multiplication.

According to Śrīdhara, a *tat-stha* is like a 'door-junction', but, without moving the multiplier. In the PG, it means a lack of movement, while in the GSS it means that the two numbers to be multiplied keep their original values.

Gangadhara, a later commentator (1420) of Bhāskarācārya's *Līlāvatī*, [Patte p. 86] understands 'as it stands' (*tat-stha*) in the same way as Śrīdhara[74].

After the verses defining a 'door-junction' and 'as it stands', Śrīdhara further adds another rule[75]:

> PG.20. The procedure called 'by portions' should be two-fold depending on whether <it is a> a unit's partition or a places partition. These are four procedures when executing a multiplication.

The two procedures 'by portions' will be examined now, as, again, we meet with the word 'portion' (*khaṇḍa*). Let us see if Śrīdhara understands it the same way as Brahmagupta and Mahāvīra.

4.2 A 'units' partition' (*rūpa-vibhāga*)

The problem considered in the commentary is once again 21×1296. The commentary describes the process as follows[76]:

> And afterwards, in the 'unit's partition', having multiplied successively and separately (*pṛthak pṛthak*) by the multiplier those places which are that amount's partition (*yat-parimāṇa-vibhāga-sthānāni*), the results are to be summed, as follows: five-two-seven (725) is multiplied by twenty-one: 15225, one-seven-five (571) is multiplied by twenty-one: 11991, both are summed: 27216.

[74] He further notes that such a process makes sense when using paper (*pattra*), a medium usually not considered to have been already used before the 10th or 11th century.

[75] *rūpa-sthāna-vibhāgād dvidhā bhavet khaṇḍaṃ-saṅjñakaṃ karaṇam* |
pratyutpanna-vidhāne karaṇāny etāni catvāri ||*20*||

[76] *evaṃ rūpa-vibhāge yat-parimāna-vibhāga-sthānāni tāni pṛthak pṛthak guṇakena guṇayitvā phalānāṃ yutiḥ kāryā yathā– pañca-dvika-saptakāni eka-viṃśati-guṇāni 15225 rūpa-sapta-paṃcakāni eka-viṃśati-guṇāni 11991, yutau 27216*|

In other words, the procedure consists in replacing 1 296 by $725 + 571$. Then, $21 \times 1296 = 21 \times (725 + 571) = 15225 + 11991 = 27216$

This method, the third one indicated in the PG, seems to be similar to the multiplication by 'parts' in the BSS, as it relies on the distributivity of multiplication over addition. Nevertheless, in the BSS, the additive decomposition is for the multiplier, not the multiplicand, while it is described in PGT as a subdivision of the multiplicand.

Contrary to the previous rule, this one is stated while working within the text. No setting of the quantities on a working surface separate from the text is displayed. There is not even an allusion to such a display.

Indeed, the commentator describes this partition by explaining ' having multiplied successively and separately (*pṛthak pṛthak*) by the multiplier those places which are that amount's partition (*yat-parimāṇa-vibhāga-sthānāni*)'. He thus explicits that the different places in which the multiplier is subdivided doesn't concern the place value notation, but just the different sub-amounts (*parimāṇa*) of the multiplier considered.

The word *rūpa* refers here to an integer. Here then the expression *rūpa-vibhāga*, considers the quantity as a heap (*rāśi*) of units that one can subdivide in different ways. The term 'partition' (*vibhāga*) is used to indicate that the sub-division is such that the sum of the separate parts does give the initial quantity.

4.3 A '<place to> place partition' (*sthāna-vibhāga*)

In this fourth method, 1 296, multiplied by 21, is subdivided according to its powers of ten. The commentary describes the process as follows:

> And then, in a 'place to place partition' (*sthāna-sthāna-vibhāga*), it is as follows: One thousand multiplied by twenty-one: 21000, two hundred multiplied by twenty-one: 4200, ninety multiplied by twenty-one: 1890, six multiplied by twenty-one: 126, all are summed: 27216.

In other words, $21 \times 1296 = 21000 + 420 + 1890 + 126 = 27216$.

Such an execution uses the fact that quantities are counted in base ten, and the distributivity of multiplication over addition. The subdivisions of the multiplicand noted with place value are stated in absolute value (1000, 200, etc.). However the very name of the subdivision alludes to place value. The commentary elaborates this by contrast with the previous subdivision by naming it a 'place to place partition' (*sthāna-sthāna-vibhāga*). This could then be a way of stating in words that such a subdivision does rest on the decimal place value notation to find the partition, but not necessarily to execute the multiplication.

As previously, the multiplicand is what is subdivided here and not the multiplier as in PBSS's interpretation of a multiplication by 'portions'.

4.4 Reflecting on the historiography 3

4.4.1 Mahāvīra and Śrīdhara according to Datta & Singh

The end of PG.20 states[77]:

> These are the four procedures when executing a multiplication.

Śrīdhara gives three names for multiplication executions: a 'door-junction' (*kavaṭa-sandhi*), an 'as it stands' (*tat-stha*) and 'portions' (*khaṇḍha*). This last method is subdivided in two (*dvidhā*): units subdivision (*rūpa-vibhāga*) and <place to> place partition (*sthana-vibhāga*). So that all in all, the methods he considers are four in number.

We can now go back to Datta & Singh's statement comparing Mahāvīra and Śrīdhara: [Datta Singh 1935, 136]:

> 'Srîdhara mentions four methods of multiplication: (1) *kapâṭa-sandhi*, (2) *tastha*, (3) *rûpa-vibhâga* and (4) *sthâna-vibhâga*. Mahāvīra mentions the same four.'

We agree with the part of their statement concerning Śrīdhara but not that Mahāvīra expounds the same four methods of multiplication: we have seen that there is only one method of execution, the three other possibilities suggesting that the numbers to be multiplied could be transformed or not, before the actual execution of the multiplication. We have seen also that Mahāvīra's methods never rely on the distributivity of multiplication over addition, therefore, his methods are not identical to Śrīdhara's ones.

Let us now turn to how Shukla has understood the PG and edited the PGT.

4.4.2 Shukla translating and editing

K. S. Shukla's translation is not always literal here. In his translation of PG 20, he explicits that he believes that the partitions apply to both multiplicand and multiplier:

> The process of multiplication called *khaṇḍa* (or *khaṇḍa-guṇana*, "multiplication by parts") is of two varieties (called *rūpa-vibhāga* and *sthāna-vibhāga*), depending on whether the multiplicand or multiplier is broken up into two or more parts whose sum or product is equal to it, or the digits standing in the different notational places (*sthāna*) of the multiplicand or multiplier are taken separately.

[77] *pratyutpanna-vidhāne karaṇāny etāni catvāri*

This is not clear from the PGT. In this sense, Shukla, not unlike Rangacarya, does not hesitate to provide in his translations his interpretation of the verses.

More generally, as an editor, Shukla clearly notes the parts he is supplementing. However, this seems to have been little noted by the secondary litterature commenting on this method in the PGT[78]. K. S. Shukla's suppletions bend slightly the commentary toward explicit uses of the resources of the decimal place value notation. He thus multiplies the layouts reconstructing the intermediary steps of the multiplication. All his reconstructions are consistent with the text and probable. He also explicits the ranks of the digits of the multiplicand as the multiplier shifts: when the digit for ten and then for a hundred has to be multiplied by the multiplier, this is explicated. The commentary explicitly does so when considering the digit for a thousand. However, the repetition of both layouts and set phrases emphasizes that a detailed explicit use and understanding of the decimal place value notation can be found in the text.

5 Conclusion

Let us now turn back to some of the questions raised in our introduction on the diversity of practices of the multiplication of integers.

5.1 Brahmagupta , Mahāvīra and Śrīdhara

As we have seen, authors provide names for different elements of an execution: while some name kinds of executions others label preliminary shapings of the quantities to be multiplied. A certain number of names are shared by our authors, but they are not always undestood in the same way. The way that names refer to different subdivisions of either a multiplicand or a multiplier is summarized in Table 1.

Such decompositions however are not used in the same way in the executions of multiplications. Thus in the GSS subdivisions are preliminary to an execution with a 'door-junction', while in the BSS as understood by PBSS, the transition from preliminary transformation to execution is not clearly delimited. In the PG, the 'door-junction' with or without movement of the multiplier, in direct or reverse order, is one way of executing a multiplication, the subdivisions indicate other ways.

Two different subdivisions of multiplicand and multipliers have a same name: *khaṇḍa*. But what is called *khaṇḍa* by Mahāvīra corresponds to one kind of *bheda* in Pṛthūdaka's understanding of Brahmagupta; while Śrīdhara's *khaṇḍa* regroups for a multiplicand partly the two subdivisions of the multiplier distinguished by Brahmagupta according to Pṛthūdaka. In this case also, two different names

[78]See [Patte 2004] and [Hayashi 2013] for example.

Table 1: Names for decompositions in a Multiplication: Brahmagupta, Mahāvīra, and Śrīdhara

	BSS	GSS	PG
Additive Decomposition using powers of 10	**Multiplier** subdivided by portions (*khaṇḍa*)		**Multiplicand** with a subdivision <place to> place (*sthāna-vibhāga*). A kind of portion (*khaṇḍa*).
Any Additive Decomposition	**Multiplier** subdivided by parts (*bheda*)		**Multiplicand** with a subdivision by units (*rūpa-vibhāga*). A kind of portion (*khaṇḍa*).
Multiplicative Decomposition	**Multiplier** subdivided by parts (*bheda*)	**Multiplier**: Portions of the value (*argha-khaṇḍa*); **Multiplicand**: Portions of the quantity (*rāśi-khaṇḍa*)	
Status Quo		**Multiplier** and **Multiplicand** remain unchanged before execution (*tat-stha*)	**Multiplier** does not move (*tat-stha*) in a 'door-junction' execution

(*bheda* and *khaṇḍa*) can cover the same subdivision.

Sometimes the similarity is but partial: thus it is not clear whether PG's subdivision of the multiplicand '<place to> place' rests like the subdivision of the multiplier by portions in the BSS on the decimal place value or not, although its very name suggests this. *Tat-stha*, in the GSS indicates that no preliminary transformation of the quantities is required while in the PG it indicates an absence of movement. In both cases this name indicates a *status quo*. The object of this 'absence of change' emphasizes what is important for each author: the preliminary transformation of quantities in commentaries of the GSS; the movement for Śrīdhara. In other words, what the rules provide differs from author to author.

We have seen that some of the *sūtra*s use the distributivity of the multiplication over addition (PG), others the associativity (GSS) and sometimes both (BSS). The lack of 'distributive' multiplications in Mahāvīra's mathematics might be a distinctive mark of the executions he provides. This might be a parallel with the operation of division were factorizations were made in order to perform preliminary simplifications. If we refer to [Baptiste Méles this volume], distributive methods require some kind of short term memory to stock partial products. Associative methods on the other hand, just modify the result they have previously implemented. From an algorithmic point of view then, they are different in nature.

5.2 Describing a multiplication ?

None of the executions described here are symmetrical in respect to the multiplicand and the multiplier. This shows that the texts examined here are not making theoretical statements on multiplication and its commutativity. They are up to a certain extant on the level of execution. In this execution how multiplicand and multiplier relate to one another is important.

The question of how both multiplicand and multiplier are distinguished is raised by one of the examples in the commentaries of the *Gaṇitasārasaṅgraha*(144×8). The agent of the multiplication, the multiplier, here seems to be the repeated numerical value, while the number of times the value is repeated is understood as the multiplicand, contrary to today's definition where the repeated numerical value is the multiplicand.

All algorithms insist on the preliminary shaping of at least one of the operands. The GSS clearly distinguishes the preliminary shaping of the operands from the execution of the multiplication. In PBSS and in the PG, the preliminary shaping of the operands is included in the execution of the multiplication. The name of the executions also names the preliminary shaping of the operands, underlining how important this step was. This shaping is sometimes a numerical transformation of the operands (GSS), sometimes a preliminary layout (BSS). Sometimes it involves both.

Further, in this study we have seen that the execution of multiplications could be described in commentaries as taking place sometimes on a separate working surface, at other times in the text. Sometimes we cannot decide anything about the layout and the space where the multiplication could have taken place. We can note hypothetically, that rules that require preliminary partitions of the operands can be thought of as useful for mental computation. Those that use the resources of the decimal place value notation might require writing. The historiography then concerning working surfaces has further distinguished rules that belong to a 'dust board' type of surface (where erasing is easy), such as a 'door-junction', and those belonging to 'paper' where intermediary steps are not erased (which are not found in the corpus here)[79].

In general however, rules and even commentaries do not detail executions. An exception might be the anonymous commentator of the *Pāṭī-gaṇita* while describing the "door-junction" method. He seems to be giving different states of progression of the layout. Then, this commentary might be but highlighting what can be seen as the specificity of Śrīdhara in relation to Brahmagupta and Mahāvīra: Śrīdhara's rule for the 'door-junction' is the only one which actually describes several steps of the process of execution. By contrast, the rules of the two other authors seem rather to highlight the different modes by which a multiplication can be carried out.

5.3 Back to the 'resources' of the decimal place value notation

In contrast to a triumphant expansion of the place value notation, not all executions *explicitly* use its resources. This does not mean of course that such resources were not actually used in the executions, but shows that this was not necessarily considered their important feature.

Thus, in Pṛthūdaka's understanding of BSS.12.55 multiplications according to a multiplier subdivided in portions, or in the PGT, the subdivision of a multiplicand in units do not explicitly use such resources.

Further, when resources of the decimal place value notation are used, they are strikingly mobilized in different manners. Two aspects of this notation have been described. First such a notation readily provides a subdivision of a number in additive parts of powers of ten. One can thus 'dismember' a higher number, to subdivide the multiplication in several easier multiplications before summing the partial products. If this is clearly the case in the execution of a multiplication with a multiplier subdivided by portions in Pṛthūdaka's commentary, it is more ambiguous in the '<place to> place' subdivision of a multiplicand in the PGT. Another resource uses the fact that positions and the values they contain create tabular grids in which operations on separate digits can be made in order to obtain digit by digit the resulting number. Such grids can take many forms, correlating value and position in

[79] [Hayashi 2013].

different ways (compare the door-junction with the first execution described by Pṛthūdaka in relation to Brahmagupta's rule).

These two related but distinct mathematical properties of the notation take in fact many different avatars: this is relevant not only in the practices we have attempted to reconstruct but also in the many different ways secondary sources themselves have explored such resources.

We have seen that different understandings of the word *sthāna* (position, place) could be used in relation to operating with integers. It is not always easy to discriminate when it refers to place value, or to the subdivision of a number in several parts. Further then, the word can be used to refer to 'places' belonging outside of the text, to a working surface, or to the noting of different values within a text in separate distinguishable parts. Such ambiguous uses of *sthāna* are found both in the PBSS and in the PGT.

Nonetheless the place devoted to decimal place value resources is different in each text. Mahāvīra explicitly claims that the 'door-junction' should be used for executions. If we assume that the method used was not essentially different than the one reconstructed from Śrīdhara's rule, then for Mahāvīra all multiplications are done within a place value framework. By contrast, both Brahmagupta and Śrīdhara single out processes that rest on the decimal place value notation but also enumerate rules that do not use its resources as well. All are on the same level of co-enumerability. Nonetheless, these two authors make a clear distinction between the execution that rest on the resources of such a notation and those that rest on the partitions of one of the operands.

The decimal place value notation's popularity is often ascribed to the fact that it makes the execution of elementary operations easy. Other devices could be used with this aim as well: precisely, the preliminary transformation of the multiplier and the multiplicand could also aim at making multiplications easier. Further, being easy was not necessarily an explicit value.

5.4 Several multiplications: easy, quick or general ?

Our authors all describe different ways of multiplying. Although we can suppose that they were aiming at providing a choice enabling easier or quicker computations, strikingly numerical examples do not seem provide the best illustrations for this.

In the example solved in Pṛthūdaka's commentary, why does he choose the additive decomposition of 288=9+8+151+120 ? In the GSS why transform 144×8 into 48×24 ? In the PGT, why transform 21×1296 into $21 \times (725+571)$? Neither easiness, nor rapidity seem to be a criteria here. Maybe these decompositions aim at exploring the different difficulties that can arise when multiplying. Care seems to have been taken to avoid specific cases, thus showing that a method of executing works whatever the decomposition chosen. Not cleverly choosing a number that would make the multiplication simpler, but on the contrary, taking an arbitrary number might be a way of being general.

5.5 Historiography and the practices of Operations

We have shown that the historiography has often assumed that practices were more uniform than what sources actually reveal. In particular, when assuming that names were dealing with multiplication considered in a very global and homogenous way. Indeed, we have shown that the fact that two same names could refer to two different executions was overlooked. Further, we have seen that the preliminary shaping of the operands could be an element specifically named within a multiplication. This had not been noticed previously.

This study has attempted at bringing to light a diversity in multiplication executions. Incidentally, it has also shown a diversity in what is named in relation to the execution. When an author gives a name to a preliminary shaping of the multiplicand or the multiplier this may be a way of highlighting what is deemed an important step in the execution. This microscopic examination of an operation's execution gives us hope that we can progressively access to an author's practice of operations. W e would like to understand not only how an operation was executed, but also how the execution's mathematical properties were thought of, how different kinds of quantities were operated upon, and ultimately the epistemological values that underlined all these activities.

We thus hope that this is but a beginning of a series of studies developing a historical and philological approach to the classifications of mathematics provided by authors in Sanskrit texts[80]. Against the double aim of homogenizing this topic which exists both within the Sanskrit scholarly tradition and in the secondary literature, we hope to unravel how diverse were the executions considered by each author, but also maybe underline how within the Sanskrit mathematical scholarly world several cultures of computations could have existed.

[80]Recently variations in the classification of operations have become a theoretical itch in the secondary litterature. This is implicit for instance in the treatment of operations that can be found in [Plofker 2007] and in [Plofker 2009]. At times, the existence of a more or less homogenous structure with little historical evolution seems to be assumed for medieval Indian mathematics. At others, the fact that specific texts differ from such a scheme is underlined. [Plofker 2009, 296] actually notes among the ' few examples of fundamental questions that remain largely unanswered', at the end of her book:

> What determined the basic building-blocks or subjects of mathematics ? For example, why did Mahāvīra consider it possible in the *Gaṇitasārasaṅgraha* to dispense with addition and subtraction of numbers as canonical arithmetic operations ? How did the operations and procedures of medieval arithmetic texts originate, and how did a particular problem get assigned to a particular category ?

This paragraph shows Kim Plofker's difficulties. Although she does not state it explicitly, implicitly it seems some of the questions raised are: is there such a thing as a canonical (list of) operations and problems belonging to all medieval Indian mathematics ? Or belonging to mathematics in general ? If so how did it historically come into being ? Should one study each author's idiosyncratic point of view on operations and problems ? Are such points of view statements about the above canonical list ? What we actually believe is that before the *Līlāvatī*, there may have not existed a unique standard list of operations (or of mathematical topics). Further, when authors gave classifications, we do not know if they were providing their own classifications or if they testified of a milieu that used such classifications. Especially, it seems to us important not to postulate that at a given moment in time mathematics in the Indian subcontinent was mostly a coherent homogene field. Even more so if we consider a diachronic corpus.

A List of Abbreviations

BSS The *Brahmasphuṭasiddhānta* of Brahmagupta (628), [Dvivedin 1902];

GSS The *Gaṇitasārasaṅgraha* of Mahāvīra(ca. 850), [Rangacarya 1912];

PBSS The *Vāsanabhāṣya* of Pṛthūdaka (fl. 860), a commentary on the *Brahmasphuṭasiddhānta*;

PG The *Pāṭī-gaṇita* of Śrīdhara(ca. 950), [Shukla 1959]. PGT The anonymous and undated commentary (*ṭīkā*) on the *Pāṭī-gaṇita* of Śrīdhara,

B The *Brahmasphuṭasiddhānta* and Pṛthūdaka's commentary (PBSS)

July 24, 2014: I have included in footnotes the hesitations I had in translating some of the words and phrases used here. Some also justify the emendations I made to the Sanskrit text. I would be very happy if anybody closely reading this text while it is still a draft, would engage with me on these points

B.1 Edition

[81]

yad-uktaṃ pratyutpanna-sūtre cheda-vidhenocchrito 'ṃśavadha iti tad-vadha-lakṣaṇaṃ na jñāyate tad-arthaṃ guṇanā-prakāra-pardarśanāyāryām āha-

BSS-12-55 guṇakāra-khaṇḍa-tulyo guṇyo gomūtrikā-kṛto guṇitaḥ|
sahitaḥ pratyupanno guṇkāraka-bheda-tulyo vā ||

guṇakārasya khaṇḍāni guṇakāra-khaṇḍāni teṣāṃ tulyo guṇyo gomūtrikā-kṛtas tair eva guṇakāra-khaṇḍaiḥ pṛthak pṛthag guṇito yathā sthānāṃ sahitaḥ pratyutpanno bhavaty athavā || guṇakārasyeṣṭāv-abhedāt prakalpya taiḥ pṛthak pṛthag guṇyas tāvat saṅkhyā-sthāna-gato guṇitaḥ sahitaś ca pratyutpanno bhavatīty

udāharaṇaṃ ||

PBSS.12.55Ex.1. tadyathā guṇya-rāśiḥ śara-guṇa-yamāḥ 235 guṇa-rāśir vasu-vasu-dasrāḥ 288

evaṃ sthite guṇyakāra-rāśir guṇakāra-khaṇḍa-tulyo go-mūtrikā kṛto'yam $\left|\begin{array}{c} 235 \\ 235 \\ 235 \end{array}\right|$ *guṇakāra-kaṇḍair-abhībiḥ 2|8|8| yathā kramaṃ guṇito jātaḥ*[82]

$$\left|\begin{array}{ccccc} 4 & 7 & 0 & & \\ 1 & 8 & 8 & 0 & \\ & 1 & 8 & 8 & 0 \end{array}\right|$$

yathā-sthāna-sahitaś ca jātaḥ yathā 67680 |

[81] Editorial comments, varies and manuscript descriptions are set aside here, and will wait a fuller more proper critical edition of the commentary in a separate publication. Text in bold indicate the BSS, as well and examples which belong to the commentary.

[82] Manuscripts are ambiguous here and might be displaying $\left|\begin{array}{c} 470 \\ 1880 \\ 1880 \end{array}\right|$

athavā guṇakāra-bhedair abhībhiḥ 9|8|151|120| etaiḥ pṛthag etāvat sthāna-gato 'yaṃ guṇya-rāśiḥ 235|235|235|235| guṇite jātaḥ 2115|1880|35485|28200| sahitaḥ sa eva pratytapanna-rāśiḥ 67680|

athavānyathā guṇakāra-bhedā yathā 9|8|4 eteṣāṃ ghāto guṇakāra-tulyaḥ 288 evam anyeṣāṃ api yeṣāṃ guṇakāratulyo bhāgahāras tair abhyāsena guṇito guṇyo rāśiḥ pratyutpanno bhavati tadyathā guṇyakāraḥ 235 nava-guṇaḥ 2115 punar apy evāṣṭa-guṇaḥ 16920 punar api catur-guṇas sa eva jātaḥ 67680

ayaṃ khaṇḍa-prakāraḥ skandasenādibhir abhihita evaṃ tatstha-kapāṭasandhy-ānayano guṇaṇā-prakārāt svādhyā yojya iti|

B.2 Translation

He says an *āryā* in order to show (*pradarśana*) modes of multiplication (*guṇanā-prakāra*) on the account that when the multiplication rule (*pratyutpanna-sūtra*) is stated (BSS.12.3) 'the product of the numerators divided by the product of the denominators', a specification (*lakṣaṇa*) of such a product (*vadha*) is not explained (*jñā*):

> **BSS.12.55. The multiplicand (*guṇya*), made into 'a zig-zag' (*go-mūtrikā*), equal in portions (*khaṇḍa*) to the multiplier (*guṇakāra*), multiplied ⟨and the partial products⟩ added is the product (*pratyutpanna*), or ⟨the multiplicand⟩ is equal in parts (*bheda*) to the multiplier ||55||**

The multiplicand made into 'a zig-zag' equal to those multiplier-portions, that is, portions of the multiplier[83], **multiplied** respectively and separately by precisely those portions of the multiplier <and> added according to place (*yathā sthānaṃ*) is the product. Or else, having replaced the multiplier with optional parts (*iṣṭāvabhedāt*)[84], the multiplicand being on as many places for <its> numbers <as there are portions of the multiplier> (*tāvat saṅkhyā-sthāna-gato*)[85] multiplied respectively and separately by these <parts> and summed is the product.

An example:

For instance, the multiplicand quantity (*guṇya-rāśi*) is five-three-two 235, the multiplier quantity (*guṇakāra-rāśi*) is eight-eight-two 288[86].

[83] This sentence explicits the compound *guṇakāra-khaṇḍa-tulya*, first by showing that it is a genitive *tatpuruṣa* linking *tulya*, 'equal' to *guṇakāra-khaṇḍa* 'multiplier-portions' and then by decomposing the compound *guṇakāra-khaṇḍa* 'multiplier-portions' which is once again a genitive *tatpuruṣa*.

[84] The text is corrupt here. All manuscripts read *iṣṭāvakedāt*, Dvivedi suggests the adopted reading *iṣṭāvabhedāt*, although I have a hard time understanding the use of the ablative here, the use of the singular form, and note the unconventional use of *avabheda* instead of *bheda*. I would rather have *iṣṭābhedāni* here, which is what I have translated actually. Is there something I misunderstood in Dvivedi's readings ?

[85] This expression is echoed further down below. I'm not sure i've translated it correctly.

[86] The association of a compound enumerating digits of a number in increasing powers of ten, and their notation in decimal place value notation following, is a standard way of writing numbers and expressing values in Sanskrit mathematical texts.

Thus, when placed (*sthite*), the multiplicand quantity (*guṇyakāra-rāśi*)[87], equal in portions to the multiplier, is made into a 'zig-zag' $\left.\begin{array}{|c|}235\\235\\235\end{array}\right.$; it is multiplied in due order (*yathā krama*) by the portions of the multiplier one after the other (*abhībhiḥ*)[88] 2 | 8 | 8 , what is produced is[89]

$$\begin{array}{|cccc|}4 & 7 & 0 & \\ 1 & 8 & 8 & 0 \\ & 1 & 8 & 8 & 0\end{array}$$

and summed according to place (*yathā-sthāna-sahita*); what is produced in this way is 67680.

Or else, this multiplicand quantity, being on as many separate places <as there are parts of the multiplier> (*pṛthag etāvat sthāna-gato*), 235 | 235 | 235 | 235, is multiplied by these parts of the multiplier one after the other (*abhībhiḥ*) 9|8|151|120; what is produced is 2115 | 1880 | 35485 | 28200. The sum is indeed that product-quantity (*pratyutpanna-rāśi*) 67680.

Or else the parts of the multiplier are <taken> otherwise, for instance 9 | 8 | 4. Their product is equal to the multiplier 288. So, with others as well, which are such that <each> is a divisor <and their whole product is> equal to the multiplier, the multiplicand quantity multiplied by these produces the product (*pratyutpanna*). For instance, the multiplicand 235, times nine 2995, and just as previously times eight, 16920, this times four, once again also is precisely 67680.

This mode <of multiplication> by portions (*khaṅda-prakāra*) was mentioned by Scandasena and so forth; Similarly, the method[90] for 'as it stands' (*tatstha*) or 'door junction' (*kapāṭa-sandhī*) should be supplied from modes of multiplication with one's own reflexion (*svādhī*).[91].

C *Gaṇitasārasaṅgraha* (GSS)

For our present study, we will refer to the Sanskrit text as edited by Rangacarya in 1912, since it is the only available edition of the GSS. Rangacarya's work has been translated in Hindi by L. C. Jain in

[87]This expression is not so common. Usually the suffix *kāra* is appended to the multiplier underline its role as an 'agent' in respect to a more 'passive' multiplicand.

[88]Once again I was not sure how to understand the expression here, repeated below in an other subdivision of the divisor. It should be understood as the instrumental of an action noun derived from *abhi-I-*, which has the meaning 'to come near, approach, enter'. The adverb *abhi* on the other hand can sometimes have the meaning of 'one after the other', thus the adopted translation..

[89]Manuscripts are ambiguous here and might be displaying $\left.\begin{array}{|c|}470\\1880\\1880\end{array}\right.$

[90]The manuscripts read *ānaya* here; i've understood it by analogy with *ānayana*, which has the same verbal root *ā-Ni*, and have modified the Sanskrit in consequence. Another possible reading would be to read *anya*, as Colebrooke did, but syntactically the sentence would remain difficult to understand.

[91]This is translated as follows by Colebrooke, [Colebrooke 1817, 319] :

> This method by parts is taught by SCANDA-SÉNA and others. In like manner the other method of multiplication, as *tat-st'ha* and *capáta-sand'hi*, taught by the same authors, may be inferred by the student's own ingenuity.

1953. This publication includes short commentaries accompanying the manuscripts of Karanja, that we will call the 'notes of Karanja'. We will also rely on the short commentary found in the palm leaf manuscript GOML-13409. We will use all these sources together to better appreciate the exact content of the rule for multiplication

C.1 Mahāvīra's rule

> *tatra prathame pratyutpanna-parikarmaṇi karaṇa-sūtraṃ yathā-*
> *guṇayed guṇena guṇyaṃ kavāṭa-sandhi-krameṇa saṃsthāpya* |
> *rāśy-argha-khaṇḍa-tatsthair anuloma-viloma-mārgābhyām* || *1* ||

The translation adopted here is:

> The procedural rule (*karaṇa-sūtra*) in relation to the operation of multiplication (*pratyutpanna-parikarman*), which is the first here <among the operations> is as follows: GSS 2.1 One should multiply the multiplicand (*guṇya*) by the multiplier (*guṇa*) after having placed <both> in the manner of the "door-junction" (*kavāṭa-sandhi*), with <either> a portion of the quantity (*rāśi-khaṇḍa*), a portion of the value (*argha-khaṇḍa*), <or> as it stands (*tat-stha*), with the direct or reverse way (*anuloma-viloma-mārga*).

C.2 Rangacarya's edition

Here we present Rangacarya's translation, which differs from the one we have adopted.

> After placing (the multiplicand and the multiplier one below the other) in the manner of the hinges of a door, the multiplicand should be multiplied by the multiplier, in accordance with (either of) the two methods of normal (or) reverse working, by adopting the process of (i) dividing the multiplicand and multiplying the multiplier by a factor of the multiplicand, (ii) of dividing the multiplier and multiplying the multiplicand by a factor of the multiplier, or (iii) of using them (in the multiplication) as they are (in themselves).

C.3 Manuscript 13409

Manuscript 13409 can be found in the Government Oriental Manuscript Library (GOML) in Chennai. It is a palm leaf in kannada script, maybe from the 18th century. The rule for multiplication is on the 2nd folio (verso). The text is in the central part, a running commentary starts below, on each folio and can end on the first line of the same folio if there is no more place below, as can be seen in Figure 16.

The commentary on multiplication starts on the last line, and ends

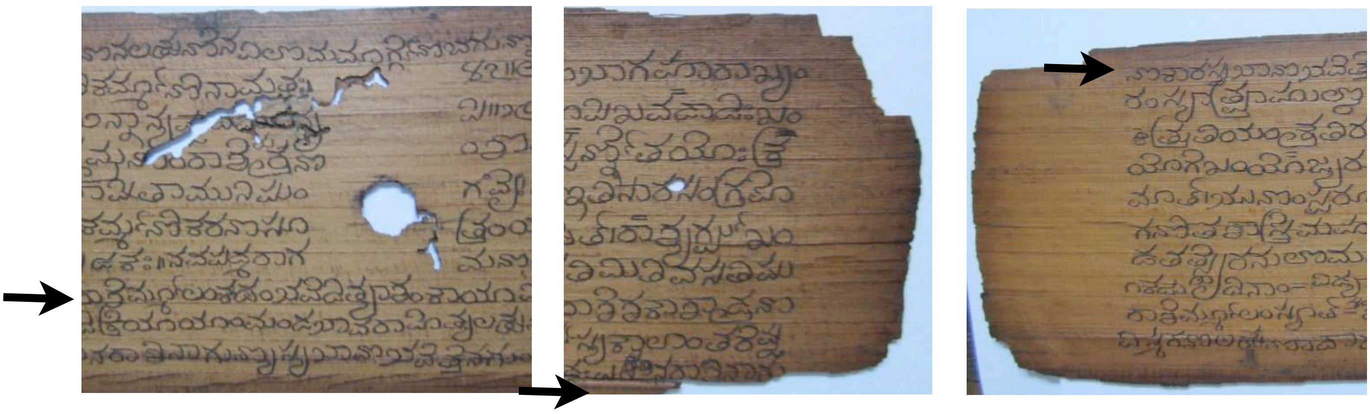


Figure 16: A detail of GOML13409: the commentary can start below and end on the first line

Running commentary :

yena rāśinā guṇyasya bhāgo bhavet tena guṃṇyaṃ bhaktvā guṇakāraṃ guṇayitvā sthāpanā-lakṣaṇo rāśi-khaṃḍḍaḥ ena rāśinā guṇakārasya bhāgo bhavet tena guṇakāraṃ bhaktvā guṇyaṃ guṇayitvā sthāpanā-lakṣaṇo 'rggha-khaṃḍaḥ guṇya-guṇakārāv abhedayitvā sthāpanā lakṣaṇaḥ tasthaḥ iti tri-prakārai(ḥ) sthita-guṇya-guṇakāra-rāśi-yugala-kavāṭa-saṃdhāna-krameṇa vinyasya rāśer āditaḥ ārabhyāṃta-paryyaṃta-guṇana-lakṣaṇena viloma-mārggena ca guṇya-rāśiṃ guṇakāra-rāśinā guṇayet

When the multiplicand is divided by a quantity which is a part (*bhāga*) of it ⟨and⟩ the multiplier is multiplied by that quantity, it is an indication for the presentation (*sthāpanā-lakṣaṇa*) 'part of the quantity' (*rāśi-khaṃḍḍaḥ*). When the multiplier is divided by a quantity which is a part of it ⟨and⟩ the multiplicand multiplied by that quantity, it is an indication for the presentation (*sthāpanā-lakṣaṇa*) "part of the value" (*arggha-khaṃḍḍaḥ*). When neither the multiplicand nor the multiplier are divided, it is an indication for the presentation "as it is" (*tastha*). That said, having arranged in the manner of a 'door-junction' (*kavāṭa-saṃdhāna*) the couple of quantities which are the existing multiplicand and multiplier by means <of one among> the three ways (*tri-prakāra*) ; having started from the beginning of the quantity up to the end, as indicated for the multiplication ⟨in the direct way⟩ and ⟨also for the multiplication⟩ in the reverse way (*viloma-mārgga*), one should multiply (*guṇayet*) the multiplicand quantity (*guṇya-rāśi*) by the multiplier quantity (*guṇakāra-rāśi*).

C.4 Karanja manuscript

The following are short notes found in manuscript A of the GSS which comes from Karanja (Maharashtra), and were copied by Dr. H.L. Jain in 1923-24. Notes are numbered.

śloka 2 – 1 A (1) yena rāśinā guṇyasya bhāgo bhavet tena guṇyaṃ bhaṅktvā guṇakāraṃ guṇayitvā

श्लोक २–१ अ (१) येन राशिना गुण्यस्य भागो भवेत् तेन गुण्यं भङ्क्त्वा गुणकारं गुणयित्वा स्थापनालक्षणो राशिखण्डः । येन राशिना गुणगुणकारस्य भागो भवेत् तेन गुणकारं भङ्क्त्वा गुण्यं गुणयित्वा स्थापनालक्षणोऽर्धखण्डः । गुण्य-गुणकारो [रौ] अभेदयित्वा स्थापनालक्षणः तत्स्थः । इति त्रिप्रकारैः स्थितगुण्य-गुणकारराशियुगलं कवाटसंधाणक्रमेण विन्यस्य । (२) राशेरादितः आरम्यान्तपर्यन्तं गुणनलक्षणेन अनुलोममार्गेण । (३) राशेरन्ततः आरभ्यादिपर्यन्तं गुणनलक्षणेन विलोममार्गेण च गुण्यराशिं गुणकार-राशिना गुणयेत् । (४) 'गुणयेत् गुणेन गुण्यं कवाटसंधिक्रमेण संस्थाप्य' इति पाठान्तर—पादद्वयम् । (५) गुण्यगुणकारं यथा व १४४ गुण्यं = प्रत्येक पद्मानि गुणकार इति = ८; २।४

४८

११५२ राशिखण्ड

(६) गुणकारं ८ अस्य भाग ४, अनेन गुण्यं गुणित चेत् ४ | ५ | ७ | ६ |
२ | १/१ | १/४ | १/२ |

sthāpanā-lakṣaṇo rāśi-khaṇḍaḥ | yena rāśinā guṇa-guṇakārasya bhāgo bhavet tena guṇakāraṃ bhaṅk-tvā guṇyaṃ guṇayitvā sthāpanā-lakṣaṇo 'rgha-khaṇḍaḥ | guṇya-guṇakāro [rau] abhedayitvā sthāpanā-lakṣaṇaḥ tat-sthaḥ | iti tri-prakāraiḥ sthita-guṇya-guṇakāra-rāśi-yugalaṃ kavāṭa-saṃdhi-krameṇa vinyasya | (2) rāśer āditaḥ ārambhyānta-paryantaṃ guṇana-lakṣaṇena anuloma-mārgeṇa | (3) rāśer antataḥ ārabhyādi-paryantaṃ guṇana-lakṣaṇena viloma-mārgeṇa ca guṇya-rāśiṃ guṇakāra-rāśinā guṇayet | (4) « guṇayet guṇena guṇyaṃ kavāṭa-saṃdhi-krameṇa saṃsthāpya » iti pāṭhantar – pāda-dvayam | (5) guṇya-guṇakāraṃ yathā va 144 guṇyaṃ = pratyeka padmāni guṇakāra iti = 8 ;

2|4

4 8

1152 portion of the quantity (*rāśi-khaṇḍa*)

(6) guṇakāraṃ 8 asya bhāga 4, anena guṇyaṃ guṇita cet

4	5	7	6
2	1/1	1/4	1/2

Verse 2 – 1 A (1) When the multiplicand is divided by a quantity which is a part (*bhāga*) of it <and> the multiplier is multiplied by that quantity, it is an indication (*lakṣaṇa*) for the presentation 'part of the quantity' (*rāśi-khaṇḍaḥ*). When the multiplier is divided by a quantity which is a part of it <and> the multiplicand multiplied by that quantity, it is an indication for the presentation 'part of the value' (*arggha-khaṃḍḍaḥ*). When neither the multiplicand nor the multiplier are divided, it is an indication for the presentation 'as it is' (*tatstha*). That said, having arranged in the manner of a 'door-junction' (*kavāṭa-saṃdhāna*) the couple of quantities which are the existing multiplicand and multiplier by means <of one among> the three ways (*tri-prakāra*) ;

(2) starting from the beginning of the quantity up to the end as indicated for a multiplication (*guṇana-lakṣaṇena*) in the direct way (*anuloma-mārga*) ;

(3) and, from the end of the quantity up to the beginning, as indicated for the multiplication in the indirect way (*viloma-mārga*), one should multiply (*guṇayet*) the multiplicand quantity (*guṇya-rāśi*) by the multiplier quantity (*guṇakāra-rāśi*).

(4) 'One should multiply the multiplicand (*guṇya*) by the multiplier (*guṇa*) after having placed (*saṃsthāpya*) ⟨both of them⟩ in the manner of the door-junction (*kavāṭa- saṃdhi-krameṇa*)' this is another reading of the two quarters of verse.

(5) *guṇya-guṇakāraṃ yathā va 144 guṇyaṃ = pratyeka padmāni guṇakāra iti = 8* ;
Or else, the multiplicand and multiplier are as follows, 144 is the multiplicand = for each, the <number of> lotuses is the multiplier = 8 ;

2|4
<u>4 8</u>
1152 portion of the quantity (*rāśi-khaṇḍa*)

(6) The multiplier is 8, its ⟨aliquot⟩ part is 4, if the multiplicand is multiplied by it ⟨we get⟩[92]

4 |	5	7	6
2 |	1/1	1/4	1/2

[92]We can see that there is a mistake, as the double of 576 should be presented as 1/0 1/4 1/2 instead of 1/1 1/4 1/2, but we do not know if this mistake is in the manuscript itself or if it is a copying error due to H.L. Jain himself, or to the printing. The name of the method (*argha-khaṇḍaḥ*) is not mentioned here.

D Śrīdhara's *Pāṭī-gaṇita* and its anonymous commentary (PGT)

D.1 Transliteration

PG.18

vinyasyādho guṇyaṃ kavāṭa-sandhi-krameṇa guṇa-rāśeḥ |
guṇyed viloma-gatyā 'nuloma-mārgeṇa vā kramaśaḥ ||18|| PG.19
utsāryotsārya tataḥ kavāṭa-sandhir bhaved idaṃ karaṇam|
tasmiṃs tiṣṭhati yasmāt pratyutpannas tatas tatsthaḥ ||19|| PG.20
rūpa-sthāna-vibhāgād dvidhā bhavet khaṇḍaṃ-saṅjñakaṃ karaṇam |
pratyutpanna-vidhāne karaṇāny etāni catvāri ||20||

[udāharaṇāni: [PGT18-20Ex1(3)]

ṣaṇ-navati-dvikam ekaṃ caikadviguṇāni ṣaṇ-navāṣṭau ca||
sapta-tri-guṇān pañcaka-ṣaṭ-khāṣṭau ca kuru ṣaṣṭi-guṇān||]

pratirūpam utpanno rāśir uddiṣṭa-rūpa-vṛndasya kiyān syād iti guṇa-guṇya[yor eka-viṃśati-ṣaṇ-ṇavaty-adhika-śata-dvādaśakayoḥ kavāṭa-sandhikrameṇa nyāsaḥ:

2 1
1 2 9 6

eka-sthāna-sthaṃ ṣaṭkaṃ rūpeṇa guṇitaṃ ṣaḍ iti ekādhaḥsthāne ṣaṭ, tataḥ dvikena guṇite ṣaṭke dvādaśa iti dvikādhaḥ sthāne dvau rūpam api navānaṇam adhaḥ jātam| nyāsaḥ

2 1
1 2 9 2 6
1

tato daśa-sthāna-sthaṃ navakaṃ guṃayituṃ sarpati guṇaraśiḥ|

2 1
nyāsaḥ 1 2 9 2 6 |
1

idānīṃ navānām eka-viṃśatiś ca guṇya-guṇaka-bhāvo jātaḥ,
rūpeṇa guṇitaṃ navakaṃ nava,
svādhaḥ-sthita-dvika-yogāt tat-sthāne rūpaṃ jayate,
rūpam api dvikādhaḥsthita-rūpeṇa yujyate dve bhavataḥ;
dvābhyāṃ guṇite navake aṣṭādaśa] pūrvavad eva tad-adho nyāsaḥ; aṣṭasu ca svādhaḥ-sthita-dvika-yoge tat-sthānaṃ śūnyaṃ, rūpam api dvikādhaḥ-stithaṃ rūpeṇa yujyate dve bhavataḥ| tataś ca tat-

sthānaṃ[93] *dvikaṃ guṇayituṃ sarpati guṇa-rāśiḥ* | *sthāpanam* $\begin{array}{ccccc} & 2 & 1 & & \\ 1 & 2 & 0 & 1 & 6 \\ & 2 & & & \end{array}$ |

idānīṃ dvayor ekaviṃśatiś ca guṇya-guṇaka-bhāvo jātaḥ, eka-guṇitau dvau dvau eva, ekādhaḥ-sthe [śūnye] dvikaṃ kṣiptvā jātau dvau, dvābhyāṃ ca dvau guṇitau catvāraḥ, svādhaḥ-sthita-dvika-yogāt ṣaṭ

[[94] $\begin{array}{ccccc} & 2 & 1 & & \\ 1 & 6 & 2 & 1 & 6 \end{array}$ *]*

tataḥ sahasra-sthāna-sthaṃ rūpaṃ guṇayituṃ sarpati guṇarāśiḥ| *nyāsaḥ [* $\begin{array}{ccccc} 2 & 1 & & & \\ 1 & 6 & 2 & 1 & 6 \end{array}$ *]*

idānīm ekasyaikaviṃśatiś ca guṇya-guṇaka-bhāvo jātaḥ, tadā rūpeṇa guṇitaṃ rūpaṃ rūpam eva, ṣaṭsu kṣiptaṃ sapta, dvābhyām ekaṃ guṇitaṃ dvāv iti|*niḥśeṣite guṇya-rāśau, guṇake nivṛtte, phalaṃ tad eva 27216*|

evaṃ rūpa-vibhāge yat-parimāna-vibhāga-sthānāni tāni pṛthak pṛthak guṇakena guṇayitvā phalānāṃ yutiḥ kāryā yathā–pañca-dvika-saptakāni eka-viṃśati-guṇāni 15225 rūpa-sapta-paṃcakāni eka-viṃśati-guṇāni 11991, yutau 27216|

evaṃ sthāna-sthāna-vibhāge, yathā–sahasram eka-viṃśati-guṇitāḥ 21000, śata-dvayam eka-viṃśati-guṇaṃ 4200, navatir eka-viṃśati-guṇitā 1890, ṣaḍ eka-viṃśati-guṇitāḥ 123, sarve yutāḥ 27296|

evaṃ ṣaṇ-ṇavāṣṭānāṃ sapta-triṃśad-guṇānāṃ tathā pañcaka-ṣaṭ-khāṣṭakānāṃ ṣaṣṭi-guṇānāṃ sthāpana-karma-phalāni darśayitavyāṇi| *viloma-gatyā kavāṭa-sandhiḥ sukara iti sa eva pūrvam uddiṣṭaḥ*|

D.2 Translation

PG.18 Having placed the multiplicand (*guṇya*) below the multiplier quantity (*guṇa-rāśi*), according to the 'door-junction' method, one should multiply going reversely or in a direct way, step by step.[95]

PG.19[96] **Having shifted again and again thus should be the door-junction. This procedure (*karaṇa*) when it (the multiplier) is stationary is therefore a multiplication**

[93]Reading as in the manuscript *tatsthāna*, rather than the *śata-sthāna*, "the place for a hundred", of the main text in Shukla's edition.

[94]This disposition is supplied by Shukla.

[95]Translated in [Shukla 1959, 7]as:

> Having placed the multiplicand below the multiplier as in the junction of two doors, multiply successively in the inverse or direct order, moving (the multiplier) each time. This process is known as *kavâṭa-sandhi* ('the door-junction method').

[96]Translated in [Shukla 1959, 7]as:

> When the multiplication is performed by keeping that (i.e., the multiplier) stationary, the process is called *tatstha* (i.e. 'multiplication at the same place') on that account

'as it stands'.

PG.20[97] The procedure called 'by portions' should be two fold depending on whether <it is a> units' partition or a places' partition. These are the four procedures when executing a multiplication.

[Examples:

PGT18-20Ex1(3) Perform, six-nine-two-one (1 296) with one-two (21) for multiplier
and six-nine-eight (896) with seven-three (37) for multiplier
and five-six-zero-eight (8 065) with sixty for multiplier ||3||]

'How much should be the quantity representing the product (*prati-rūpam utpanno raśir*) amongst the group of indicated digits (*uddiṣṭa-rūpa-vṛṇdasya*) ?

[Setting of] the multiplier and multiplicand [[98]twenty-one and six-ninety increased by twelve hundred according to the door-junction method:

```
        2   1
1   2   9   6
```

'Six which stands in the unit's place multiplied by one: six', in the place below one, six; then 'six multiplied by two: twelve'. In the place below two: two, also below nine one is produced.

Setting:

```
            2   1
1   2   9   2   6
        1
```

Then, to multiply nine which stands in the place for tens, the multiplier quantity slides.

Setting:

```
        2   1
1   2   9   2   6
        1
```

Now, nine and twenty-one become multiplicand and multiplier[99].

Nine multiplied by one: nine, by adding two which is below it, one is produced at this place, also one increased by the one which stands below the two produces two.

Nine multiplied by two: eighteen][100] just as previously, below that, there is a setting[101] and when eight is added to two which stands below it, in that place: zero. One also stands below two, added to one it becomes two.

[97]Translated in [Shukla 1959, 7]as:

> The process of multiplication called *khaṇḍa* (or *khaṇḍa-guṇana*, "multiplication by parts") is of two varieties (called *rūpa-vibhāga* and *sthāna-vibhāga*), depending on whether the multiplicand or multiplier is broken up into two or more parts whose sum or product is equal to it, or the digits standing in the different notational places (*sthāna*) of the multiplicand or multiplier are taken separately. These are four methods of multiplication.

[101]Possibly, in the original text a layout was to be represented here.

And then, to multiply two which is in that place[102] the multiplier quantity slides.

The disposition (*sthāpana*) is: $\begin{array}{ccccc} & 2 & 1 & & \\ 1 & 2 & 0 & 1 & 6 \\ & 2 & & & \end{array}$

Now, two and twenty-one become multiplicand and multiplier. Two multiplied by one: two exactly. Below one where [zero] stands, having added two, two is produced. And two multiplied by two: four, below that, from the sum of two which stands there: six.

[[103] $\begin{array}{ccccc} & 2 & 1 & & \\ 1 & 6 & 2 & 1 & 6 \end{array}$]

Then to multiply one (*rūpa*) which stands in the place for thousands, the multiplier quantity slides.

Setting: [[104] $\begin{array}{ccccc} 2 & 1 & & & \\ 1 & 6 & 2 & 1 & 6 \end{array}$]

Now, one and twenty one become multiplicand and multiplier. Then, one multiplied by one: just one, added to six is seven; 'one multiplied by two: two'. As none remains in the multiplicand quantity, <and> since the multiplier is erased (*nivṛt*), the result is just that, 27216.

And afterwards, in the 'unit's partition', having multiplied successively and separately (*pṛthak pṛthak*) by the multiplier those places which are that amount's partition (*yat-parimāṇa-vibhāga-sthānāni*), the results are to be summed, as follows: five-two-seven (725) is multiplied by twenty-one: 15225, one-seven-five (571) is multiplied by twenty-one: 11991, both are summed: 27216.

And then, in a 'place to place partition' (*sthāna-sthāna-vibhāga*), it is as follows:

One thousand multiplied by twenty-one: 21000, two hundred multiplied by twenty-one: 4200, ninety multiplied by twenty-one: 1890, six multiplied by twenty-one: 126, all are summed: 27216.

In this way the results in <any> placing method (*sthāpana-karma-phalāni*) should be shown for six-nine-eight (896) multiplied by seven-three (37), then for five-six-zero-eight (8065) multiplied by eighty.

'The 'door-junction' in the indirect way is easy indeed', thefore it was mentioned first (*pūrvam uddiṣṭaḥ*) <in the verse>.

[102] Reading as in the manuscript *tat-sthāna*, rather than the *śata-sthāna*, "the place for a hundred", of the main text in Shukla's edition.

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