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Utilitarianism with Prior Heterogeneity

Antoine BILLOT, Vassili VERGOPOULOS

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Utilitarianism with Prior Heterogeneity

Antoine Billot* and Vassili Vergopoulos†

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Abstract

Harsanyi's axiomatic justification of utilitarianism is extended to a framework with subjective and heterogeneous priors. Contrary to the existing literature on aggregation of preferences under uncertainty, society is here allowed to formulate probability judgements, not on the actual state of the world as individuals do, but rather on the opinion they each have on the actual state. An extended Pareto condition is then proposed that characterizes the social utility function as a convex combination of individual ones and the social prior as the independent product of individual ones.

Keywords: utilitarianism, prior heterogeneity, Pareto condition.

JEL classification: D71, D81.

1 Introduction

Harsanyi (1955) provides an axiomatic justification of utilitarianism that is based on two principles: Bayesian rationality and the Pareto condition. While the former requires each alternative to be evaluated at the social level solely in terms of the consequences it is likely to induce, the latter rather requires each alternative to be evaluated solely in terms of individual evaluations. As shown by Harsanyi, these two principles imply that the utility function of society is a convex combination of individual ones. However, a tension emerges between these two requirements in a framework *à la* Savage (1954) where probabilities are subjective and thus depend upon each individual. In this setting, Diamond (1967), Hylland and Zeckhauser (1979), Mongin (1995) and Chambers and Hayashi (2006) show that a society trying to aggregate individual preferences into social ones faces an impossibility: in case of prior heterogeneity, it has to reject at least one of Savage's version of Bayesian rationality, i.e. the Subjective Expected Utility (SEU) model, or the Pareto condition. (See Mongin and Pivato, 2014, for a recent and more general formulation of this impossibility result.) Unfortunately, calling any of these requirements into question does not lead to satisfactory

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solutions. On the one hand, weakening the Savage model might give rise to some sort of inconsistency in social decisions (Al-Najjar and Weinstein, 2009). On the other hand, weakening the Pareto condition may lead to puzzling situations where society is allowed to contradict unanimity (Gilboa, Samet and Schmeidler, 2004).

Gilboa, Samet and Schmeidler, henceforth GSS, develop the following example to motivate weakening the Pareto condition: two gentlemen are optimistic enough about their respective probability of success in a potential duel between them and thus unanimously prefer to fight. While Bayesian rationality leads society to forbid the duel, the Pareto condition rather leads society to allow the duel, a solution that might hurt common sense. Hence, the duel example makes it especially natural to restrict the domain of application of the Pareto condition. However, what makes the GSS approach so natural seems to be the optimism of individuals facing potential losses. Consider an opposite situation of pessimism in face of potential gains: a father of two children, Alice and Bob, is wondering how to finance their higher education. Due to budget constraints and academic supply, he only has two available strategies: either he funds a three-year BA degree for each child or funds a eight-year PHD to only one of them and leaves nothing to the other one. In the latter case, he waits for the next school test to determine who gets the PHD opportunity. Assume additionally that the children are pessimistic enough about their own probability to get the best grade. As a result, they unanimously prefer the BA solution. This time, while the Pareto condition requires the father to fund two BA degrees, Bayesian rationality rather requires him to fund a PHD degree only, a solution that appears just as counterintuitive as the choice of a duel in GSS example.

This work elaborates an axiomatic justification of utilitarianism that is somehow dual to the GSS approach: while the latter develop an appropriate weakening of the Pareto condition to deal with the duel example, this paper develops an appropriate version of Bayesian rationality to solve the father example. To understand the key insight of this paper, recall that the literature on preference aggregation implicitly assumes some principle of anthropomorphism according to which social preferences are of the same nature as individual preferences. In contrast, it is argued here that there is a deep difference in the two sorts of preferences. Individual preferences rank, as usual, alternatives defined as distributions of outcomes across states of the world while social preferences are here considered to compare alternatives defined as distributions of outcomes across *states of opinion* about the actual state of the world, where a state of opinion only encodes, by definition, what each individual foresees as the actual state of the world. In this context, the main result presented in this paper shows that Bayesian rationality over states of opinion and an appropriate extension of the Pareto condition produce together utilitarianism.

The extension of the Pareto condition that is proposed thereafter involves social preferences to be (1) defined in terms of states of opinion and (2) conditional upon potentially heterogeneous information. Supposing that formal results may suggest some normative devices, the consistency of this version of the Pareto condition with Bayesian rationality is a stimulating result. Actually, the impossibility to aggregate preferences under heterogeneous beliefs leads to a puzzling conclusion, as already noticed by GSS: any democratic institution, president or ruling party, could rationally consider that its role is not to rep-

resent the society as a whole, i.e. all the individuals, but rather the only ones who base its majority. However, *our moral intuition on this is issue (...) demands that a majority should not disregard opinions of minorities* (GSS, 2004, p. 935). Finally, our extended Pareto condition makes it possible to overcome the preference aggregation impossibility in explaining how the opinion and information of minorities, as well as majorities, could be taken into account in the social decision process.

Section II presents the framework, the extended Pareto axiom and the main result and the Appendix displays the proof.

2 The main result

2.1 The framework

Let (S, Σ) be a σ -measurable space where S is a set of *individual states* (of nature) and Σ is a σ -algebra of *events*. Denote by X a set of *outcomes* endowed with a σ -algebra. The set $F(S) = \{f \mid f : S \rightarrow X, f \text{ is } \Sigma\text{-measurable}\}$ is the set of *individual acts*. Society is a set of individuals $N = \{1, \dots, n\}$. Individual $i \in N$ has preferences $\succsim_i \subseteq F(S) \times F(S)$. For $i \in N$, the relations \sim_i and \succ_i are defined as the symmetric and asymmetric parts of \succsim_i . Individual preferences are assumed to be represented by expected utility maximization; that is, there are a measurable and bounded utility function $u_i : X \rightarrow \mathbb{R}$ and a probability measure λ_i on Σ such that, for any $f, g \in F(S)$, $f \succsim_i g$ iff $\int_S u_i(f(s))d\lambda_i \geq \int_S u_i(g(s))d\lambda_i$. Moreover, we assume that, for any $i \in N$, λ_i is countably additive and nonatomic and that u_i is not constant.

Let $\Omega = S^N$ be the set of *social states* and \mathcal{F} be the product σ -algebra on Ω . Thus, a social state $\omega \in \Omega$ is a *state of opinion* in the sense that it encodes the opinion of each individual: at state ω , individual i 's opinion is that $\omega_i \in S$ will occur. In the introductory example, *(Alice gets the best grade, Alice gets the best grade)* is the social state at which Alice as well as Bob form the opinion that Alice will get the PhD opportunity. The set X^N of *social outcomes* is endowed with the product σ -algebra. A social outcome $(x_i)_{i \in N} \in X^N$ thus encodes an outcome $x_i \in X$ for each individual $i \in N$. The set $F(\Omega) = \{F \mid F : \Omega \rightarrow X^N, F \text{ is } \mathcal{F}\text{-measurable}\}$ is the set of *social acts*. Thus, given a social act, the outcome received by an individual depends not only on his own opinion but also on the opinions of others. Society has preferences $\succsim_N \subseteq F(\Omega) \times F(\Omega)$. The relations \sim_N and \succ_N are defined as the symmetric and asymmetric parts of \succsim_N . Social preferences are also assumed to be represented by expected utility maximization; that is, there are a measurable and bounded utility function $u_N : X^N \rightarrow \mathbb{R}$ and a probability measure λ_N on \mathcal{F} such that, for any $F, G \in F(\Omega)$, $F \succsim_N G$ iff $\int_\Omega u_N(F(\omega))d\lambda_N \geq \int_\Omega u_N(G(\omega))d\lambda_N$. As above, λ_N is countably additive and nonatomic, and u_N is not constant. At last, it is assumed that there exists $x^0, x^1 \in X$ such that, for any $i \in N$, $u_N(x^0, \dots, x^0) = u_i(x^0) = 0$ and $u_N(x^1, \dots, x^1) = u_i(x^1) = 1$.

2.2 The axiom

Additional notation is required for the formulation of the extended Pareto axiom. For any $i \in N$ and $E \in \Sigma$, let \succsim_i^E be the preference relation on $F(S)$ defined, for any $f, g \in F(S)$, by $f \succsim_i^E g$ iff there exists $h \in F(S)$ such that $f_E h \succsim_i g_E h$. Note that, if E is λ_i -nonnull (i.e. $\lambda_i(E) > 0$), then \succsim_i^E is represented by expected utility maximization with respect to u_i and $\lambda_i(\cdot|E)$. If E is λ_i -null (i.e. $\lambda_i(E) = 0$), then \succsim_i^E is trivial: any two acts $f, g \in F(S)$ satisfy $f \sim_i^E g$. Similarly, for any $\mathcal{E} \in \mathcal{F}$, let $\succsim_N^\mathcal{E}$ be the preference relation on $F(\Omega)$ defined, for any $F, G \in F(\Omega)$, by $F \succsim_N^\mathcal{E} G$ iff there exists $H \in F(\Omega)$ such that $F_\mathcal{E} H \succsim_N G_\mathcal{E} H$. Naturally, λ_N -nullity and nonnullity are defined as above.

Finally, for any $i \in N$ and $\omega \in \Omega$, ω_i denotes the i -th component of ω and ω_{-i} denotes the other components. For a family $(f_i)_{i \in N} \in F(S)^N$, define a social act according to $(f_1 \otimes \dots \otimes f_n)(\omega) = (f_1(\omega_1), \dots, f_n(\omega_n))$ for any $\omega \in \Omega$. Let $\tilde{F}(\Omega)$ stand for the set of such acts. It is nothing but the set of all social acts that induce, to any individual, an outcome that is independent of the opinion of others.

THE EXTENDED PARETO CONDITION. Let $F, G \in F(\Omega)$ and $\tilde{F}, \tilde{G} \in \tilde{F}(\Omega)$. Let $(E_i)_{i \in N} \in \Sigma^N$ and $\mathcal{E} = E_1 \times \dots \times E_n \in \mathcal{F}$.

- (1) If, for any $i \in N$ and $\omega \in \Omega$, $F_i(\cdot, \omega_{-i}) \succsim_i^{E_i} G_i(\cdot, \omega_{-i})$, then $F \succsim_N^\mathcal{E} G$.
- (2i) Let \mathcal{E} be λ_N -nonnull. If, for any $i \in N$ and $\omega \in \Omega$, $\tilde{F}_i(\cdot, \omega_{-i}) \succsim_i^{E_i} \tilde{G}_i(\cdot, \omega_{-i})$ and there exists $i \in N$ such that, for any $\omega \in \Omega$, $\tilde{F}_i(\cdot, \omega_{-i}) \succ_i^{E_i} \tilde{G}_i(\cdot, \omega_{-i})$, then $\tilde{F} \succ_N^\mathcal{E} \tilde{G}$.
- (2ii) Let \mathcal{E} be λ_N -null. Then, there exists $i \in N$ such that E_i is λ_i -null.

This axiom provides a natural translation of the Pareto condition to the present extended framework. More precisely, consider an individual $i \in N$ and two social acts $F, G \in F(\Omega)$. Note that, for any $\omega_{-i} \in \Omega_{-i} = \prod_{j \neq i} S$, $F_i(\cdot, \omega_{-i})$ and $G_i(\cdot, \omega_{-i})$ are individual acts and consequently comparable through $\succsim_i^{E_i}$ for any event $E_i \subseteq S$. Then, individual i prefers social act F to social act G conditional upon E_i whenever $F_i(\cdot, \omega_{-i}) \succsim_i^{E_i} G_i(\cdot, \omega_{-i})$ for any given state of opinion ω_{-i} . In this context, part (1) of the axiom requires society to prefer weakly a social act that is individually unanimously preferred in the latter sense and part (2) adapts the strict version of the Pareto condition. However, the axiom departs from a naive translation of the Pareto condition in three different ways. First, it involves preferences conditional upon information and applies eventwise. Social preferences conditional upon each individual i 's having his opinion in E_i must be consistent in the Pareto sense with individual preferences conditional upon E_i . Second, part (2i) is restricted to $\tilde{F}(\Omega)$. This restriction makes it sure that social preferences are affected by a strict individual preference only in case of social acts inducing individual outcomes that are independent of the opinion of others. Third, part (2ii) is a Pareto requirement for nonnull events: if every individual $i \in N$ considers E_i to be possible, then society has to consider that the event \mathcal{E} at which every $i \in N$ has his opinion in E_i is possible as well.

2.3 The theorem

Social probability λ_N is said to be the independent product of $(\lambda_i)_{i \in N}$ if it is the unique probability measure on (Ω, \mathcal{F}) satisfying $\lambda_N(E_1 \times \dots \times E_n) = \lambda_1(E_1) \times \dots \times \lambda_n(E_n)$, for

any $(E_i)_{i \in N} \in \Sigma^N$. Social utility u_N is said to be a convex combination of $(u_i)_{i \in N}$ with positive coefficients if there exists $(\alpha_i)_{i \in N} \in [0, 1]$ with $\alpha_1 + \alpha_2 + \dots + \alpha_n = 1$ such that $u_N(x_1, \dots, x_n) = \alpha_1 u(x_1) + \dots + \alpha_n u(x_n)$, for any $(x_i)_{i \in N} \in X^N$.

THEOREM 1. The extended Pareto condition holds iff λ_N is the independent product of $(\lambda_i)_{i \in N}$ and u_N is a convex combination of $(u_i)_{i \in N}$ with positive coefficients.

According to this result, if society adheres to the extended Pareto condition, then it conforms to Harsanyi's utilitarianism and aggregates beliefs in a multiplicative way. Social preferences are thus fully determined by individual preferences. Additionally, this multiplicative aggregation of beliefs makes it possible to retrieve individual probabilities from social probability. No information about individual beliefs is lost through the aggregation process. Thus, Theorem 1 shows that the state space extension provides an axiomatic justification of Harsanyi's utilitarianism that respects maximally individuals and their heterogeneous beliefs while preserving the Pareto condition.

More practically, consider that society must choose between f and g , for any $f, g \in F(S)$. Then, the present modeling approach suggests that society should choose f iff social act $f \otimes \dots \otimes f$ is socially preferred to social act $g \otimes \dots \otimes g$. Formally, in any given decision problem, society chooses a feasible act $f \in F(S)$ by maximizing the following quantity:

$$\mathbb{E}_{\lambda_N} u_N(f \otimes \dots \otimes f).$$

Given the aggregation process characterized in Theorem 1, the latter quantity can be rewritten in the following way:

$$\mathbb{E}_{\lambda_N} u_N(f \otimes \dots \otimes f) = \sum_{i \in N} \alpha_i \mathbb{E}_{\lambda_i} u_i(f).$$

First, the latter equality clearly reveals that a convex combination of expected utilities over S can always be rewritten as an expected utility over $\Omega = S^N$. Consequently, the appropriate space for social aggregation of a set of n individual preferences over S is the Cartesian product $\Omega = S^N$. In addition, in case of prior homogeneity, the latter decision rule is consistent with Harsanyi's standard aggregation procedure: it is indeed equivalent to expected utility maximization with respect to this common prior and a convex combination of utilities.

Finally, the introductory example is now used to illustrate Theorem 1. There are two individual states $S = \{A(\text{lice}), B(\text{ob})\}$, each encoding who gets the best grade. Denote by f the individual act that represents the intuitively fair alternative that gives a three-year BA opportunity to each child and g the individual act that represents the unfair alternative giving an eight-year PHD opportunity to one child only and nothing to the other one. The utilities induced by the two alternatives, as well as individual probabilities λ_A and λ_B over S , are given in the table below.

	<i>A</i>	<i>B</i>
λ_A	1/3	2/3
λ_B	2/3	1/3
<i>f</i>	(3, 3)	(3, 3)
<i>g</i>	(8, 0)	(0, 8)

If, just like his children, the father uses S as the state space for his preferences, then, for any possible probability $\lambda = (p, 1 - p)$ on S and a utility function u given by a (.5, .5)-convex combination of the children's utilities, he necessarily chooses g in spite of their common preference for f . Hence a violation of the Pareto condition. Indeed, the father's values for f and g are as follows:

$$\begin{aligned}\mathbb{E}_\lambda u(f) &= (3 \times .5 + 3 \times .5)p + (3 \times .5 + 3 \times .5)(1 - p) = 3, \\ \mathbb{E}_\lambda u(g) &= (8 \times .5)p + (8 \times .5)(1 - p) = 4.\end{aligned}$$

Assume now that the father sticks to the aggregation rule of Theorem 1 and thus uses the four social states $\Omega = \{(A, A), (A, B), (B, A), (B, B)\}$, each encoding the opinion of each child about who gets the best grade. For instance, at state (B, A) , each child thinks that the other one gets the best grade while, at state (A, B) , each one thinks that he gets himself the best grade. The table below provides the father's probability λ_N computed as the independent product over Ω of λ_A and λ_B together with the individual utilities associated with the social acts induced by f and g . For instance, since each child thinks at state (B, A) that the other one gets the best grade, both Alice and Bob necessarily think that the choice of f will result in a level of utility equal to 0.

	(<i>A, A</i>)	(<i>A, B</i>)	(<i>B, A</i>)	(<i>B, B</i>)
λ_N	2/9	1/9	4/9	2/9
$f \otimes f$	(3, 3)	(3, 3)	(3, 3)	(3, 3)
$g \otimes g$	(8, 0)	(8, 8)	(0, 0)	(0, 8)

Assume that the father's utility u_N is a (.5, .5)-convex combination of individual utilities u_A and u_B : for any $x, y \in X$, $u_N(x, y) = .5u_A(x) + .5u_B(y)$. To make a choice between f and g , the father compares $f \otimes f$ to $g \otimes g$:

$$\begin{aligned}\mathbb{E}_{\lambda_N} u_N(f \otimes f) &= (3 \times .5 + 3 \times .5)\left(\frac{2}{9} + \frac{1}{9} + \frac{4}{9} + \frac{2}{9}\right) = 3, \\ \mathbb{E}_{\lambda_N} u_N(g \otimes g) &= (8 \times .5)\frac{2}{9} + (8 \times .5 + 8 \times .5)\frac{1}{9} + 0\frac{1}{9} + (8 \times .5)\frac{2}{9} = \frac{8}{3}.\end{aligned}$$

Hence, the father chooses the BA option consistently with the children's preferences. Intuitively, recall that each of the children is pessimistic in the sense of assigning a high probability to the other one getting the best grade. Such prior heterogeneity over S is reflected within Ω by a high probability on state (B, A) at which each child thinks that he gets a utility of 0 if the PHD option is chosen. Thus, the latter loses much of its value, which explains the father's choice of the BA option. As a conclusion, the state space extension from S to Ω makes it possible for the father to respect both expected utility maximization and the Pareto condition.

Appendix

Proof of Theorem 1

Necessity of the axiom: Assume that λ_N is the independent product of $(\lambda_i)_{i \in N}$ (which makes part (2ii) of the extended Pareto condition straightforward) and that u_N is a convex combination of $(u_i)_{i \in N}$: $\forall (x_i)_{i \in N} \in X^N$, $u_N(x_1, \dots, x_n) = \alpha_1 u_1(x_1) + \dots + \alpha_n u_n(x_n)$, where $\alpha_i > 0$ for any $i \in N$ and $\alpha_1 + \dots + \alpha_n = 1$. Additionally, for any $i \in N$, let λ_{-i} be the probability on $\Omega_{-i} = S^{n-1}$ defined as the independent product of $(\lambda_j)_{j \neq i}$. Assume that, for any $i \in N$ and $\omega \in \Omega$, $F_i(\cdot, \omega_{-i}) \succsim_i^{E_i} G_i(\cdot, \omega_{-i})$. Let us compare the social values $V_N^\mathcal{E}(F)$ and $V_N^\mathcal{E}(G)$ of F and G conditional upon \mathcal{E} : $V_N^\mathcal{E}(F) = \int_\Omega u_N(F(\omega)) d\lambda_N(\cdot | \mathcal{E}) = \sum_{i \in N} \alpha_i \int_\Omega u_i(F_i(\omega)) d\lambda_N(\cdot | \mathcal{E})$. Now, the Fubini theorem implies, for any individual $i \in N$, $\int_\Omega u_i(F_i(\omega)) d\lambda_N(\cdot | \mathcal{E}) = \int_{\Omega_{-i}} (\int_S u_i(F_i(\omega_i, \omega_{-i})) d\lambda_N^i(\cdot | \mathcal{E})) d\lambda_{-i}^i(\cdot | \mathcal{E})$ where $\lambda_N^i(\cdot | \mathcal{E})$ and $\lambda_{-i}^i(\cdot | \mathcal{E})$ are the i -marginal and the $-i$ -marginal of $\lambda_N(\cdot | \mathcal{E})$. It is easy to see that $\lambda_N^i(\cdot | \mathcal{E}) = \lambda_i(\cdot | E_i)$. Therefore, for any $i \in N$, $\int_\Omega u_i(F_i(\omega)) d\lambda_N(\cdot | \mathcal{E}) \geq \int_\Omega u_i(G_i(\omega)) d\lambda_N(\cdot | \mathcal{E})$ and, finally, $V_N^\mathcal{E}(F) \geq V_N^\mathcal{E}(G)$. If in addition \tilde{F} and \tilde{G} are elements of $\tilde{F}(\Omega)$ and if there is $i \in N$ such that for any $\omega \in \Omega$, $\tilde{F}_i(\cdot, \omega_{-i}) \succ_i^{E_i} \tilde{G}_i(\cdot, \omega_{-i})$, then $\int_\Omega u_i(\tilde{F}_i(\omega)) d\lambda_N(\cdot | \mathcal{E}) = \int_S u_i(\tilde{F}_i(\omega_i, \omega_{-i})) d\lambda_i(\cdot | E_i)$, which is independent of ω_{-i} . Then, $\int_\Omega u_i(\tilde{F}_i(\omega)) d\lambda_N(\cdot | \mathcal{E}) > \int_\Omega u_i(\tilde{G}_i(\omega)) d\lambda_N(\cdot | \mathcal{E})$ and, since $\alpha_i > 0$, that $V_N^\mathcal{E}(F) > V_N^\mathcal{E}(G)$.

Sufficiency of the axiom: Let us introduce additional notations. For any $x, y \in X$ and $i \in N$, $(x, y_{-i}) \in X^N$ is defined as $(y, \dots, y, x, y, \dots, y)$ where x lies in the i -th component. For any partition $(\mathcal{E}_j)_{j=1}^m$ over Ω and $(F_j)_{j=1}^m \in F(\Omega)^m$, the social act $F_{1\mathcal{E}_1} F_{2\mathcal{E}_2} \dots F_{m-1\mathcal{E}_{m-1}} F_m \in F(\Omega)$ is equal to F_j on \mathcal{E}_j for any $j \in [1, m]$. Finally, a uniform partition over Ω is defined as a partition $(\mathcal{E}_i)_{i \in N}$ made of equiprobable cells.

Consider the two following conditions for social preferences.

c1. For any $y \in X$, $(x_i)_{i \in N} \in X^N$, uniform partition $(\mathcal{E}_i)_{i \in N}$ over Ω and $i \in N$, $(x_1, y_{-1})_{\mathcal{E}_1} (x_2, y_{-2})_{\mathcal{E}_2} \dots (x_{n-1}, y_{-(n-1)})_{\mathcal{E}_{n-1}} (x_n, y_{-n}) \sim_N (x_1, \dots, x_n)_{\mathcal{E}_i} (y, \dots, y)$.

c2. For any $(E_i)_{i \in N} \in \Sigma^N$, $(f_i)_{i \in N}, (g_i)_{i \in N} \in F(S)^N$, if $f_i \succsim_i^{E_i} g_i$ for all $i \in N$, then $f_1 \otimes \dots \otimes f_n \succsim_N^\mathcal{E} g_1 \otimes \dots \otimes g_n$ where $\mathcal{E} = E_1 \times \dots \times E_n$. If, in addition, $\lambda_N(\mathcal{E}) > 0$ and there exists $i \in N$ such that $f_i \succ_i^{E_i} g_i$, then $f_1 \otimes \dots \otimes f_n \succ_N^\mathcal{E} g_1 \otimes \dots \otimes g_n$.

CLAIM 1. The extended Pareto condition implies both c1 and c2.

Proof. (C1) Define $F = (x_1, y_{-1})_{\mathcal{E}_1} (x_2, y_{-2})_{\mathcal{E}_2} \dots (x_{n-1}, y_{-(n-1)})_{\mathcal{E}_{n-1}} (x_n, y_{-n})$ and $G = (x_1, \dots, x_n)_{\mathcal{E}_i} (y, \dots, y)$. It is sufficient to show that, for any $j \in N$ and $\omega \in \Omega$, $F_j(\cdot, \omega_{-j}) \sim_j G_j(\cdot, \omega_{-j})$. Now, one has $F_j(\cdot, \omega_{-j}) = x_{j\mathcal{E}_j} y = G_j(\cdot, \omega_{-j})$. Hence the result.

(C2) Note that c2 is a direct consequence of the extended Pareto condition. However, for expositional clarity, a proof of c2 is given that only relies on the unconditional version of part (1) of the extended Pareto condition. Define $F = f_1 \otimes \dots \otimes f_n$ and $G = g_1 \otimes \dots \otimes g_n$. By assumption, for any $i \in N$, $f_i \succsim_i^{E_i} g_i$; that is, there exists $h_i \in F(S)$ such that $f_i \succ_{iE_i} h_i \succsim_i g_i \succ_{iE_i} h_i$. One has to show $F \succsim_N^\mathcal{E} G$; that is, there exists $H \in F(\Omega)$ such that $F_\mathcal{E} H \succsim_N G_\mathcal{E} H$. Define $H = h_1 \otimes \dots \otimes h_n$. Given the extended Pareto condition, it is sufficient to show that, for any $i \in N$ and $\omega \in \Omega$, $(F_\mathcal{E} H)_i(\cdot, \omega_{-i}) \succsim_i (G_\mathcal{E} H)_i(\cdot, \omega_{-i})$. So fix $i \in N$ and $\omega \in \Omega$.

Case 1: $\omega_{-i} \in \prod_{j \neq i} E_j$. Then, $(F_{\mathcal{E}}H)_i(\cdot, \omega_{-i}) = f_{iE_i} h_i$ and $(G_{\mathcal{E}}H)_i(\cdot, \omega_{-i}) = g_{iE_i} h_i$ so that $(F_{\mathcal{E}}H)_i(\cdot, \omega_{-i}) \succsim_i (G_{\mathcal{E}}H)_i(\cdot, \omega_{-i})$.

Case 2: $\omega_{-i} \notin \prod_{j \neq i} E_j$. Then, $(F_{\mathcal{E}}H)_i(\cdot, \omega_{-i}) = h_i$ and $(G_{\mathcal{E}}H)_i(\cdot, \omega_{-i}) = h_i$ so that again $(F_{\mathcal{E}}H)_i(\cdot, \omega_{-i}) \succsim_i (G_{\mathcal{E}}H)_i(\cdot, \omega_{-i})$.

Thus, $F \succsim_N^{\mathcal{E}} G$. Assume additionally that there exists $i \in N$ such that $f_i \succ_i^{E_i} g_i$. Apply part (2i) of the extended Pareto condition to conclude. Q.E.D.

Consider the two following conditions for social preferences.

D1. For any $(y_i)_{i \in N} \in X^N$ and $i \in N$, there exists $\alpha_i(y_1, \dots, y_n) > 0$ and $\beta_i(y_1, \dots, y_n)$ such that, for all $x \in X$, $u_N(y_1, \dots, y_{i-1}, x, y_{i+1}, \dots, y_n) = \alpha_i(y_1, \dots, y_n)u_i(x) + \beta_i(y_1, \dots, y_n)$.

D2. For any $(E_i)_{i \in N} \in \Sigma^N$ and $i \in N$, if $\lambda_i(E_i) > 0$ and $\lambda_N(\mathcal{E}) > 0$, then $\lambda_N^i(\cdot | \mathcal{E}) = \lambda_i(\cdot | E_i)$, where $\lambda_N^i(\cdot | \mathcal{E})$ is the i -marginal of $\lambda_N(\cdot | \mathcal{E})$ and $\mathcal{E} = E_1 \times \dots \times E_n$.

CLAIM 2. Condition C2 implies both conditions D1 and D2.

Proof. For convenience, denote (f_i, y_{-i}) the element of $\tilde{F}(\Omega)$ defined by $(f_i, y_{-i}) = y_1 \otimes \dots \otimes y_{i-1} \otimes f_i \otimes y_{i+1} \otimes \dots \otimes y_n$. By C2, for any $f_i, g_i \in F(S)$, one has: $f_i \succsim_i^{E_i} g_i \iff (f_i, y_{-i}) \succsim_N^{\mathcal{E}} (g_i, y_{-i})$. Now $\succsim_i^{E_i}$ is of the expected utility type with respect to utility u_i and probability $\lambda_i(\cdot | E_i)$. On the other hand, the social value of (f_i, y_{-i}) at \mathcal{E} is given by: $\int_{\Omega} u_N((f_i, y_{-i})(\omega)) d\lambda_N(\cdot | \mathcal{E}) = \int_S v_i(f_i(s)) d\lambda_N^i(\cdot | \mathcal{E})$ where $v_i : X \rightarrow \mathbb{R}$ is defined by $v_i(x) = u_N(x, y_{-i})$ for all $x \in X$. Finally, D1 follows from the uniqueness (up to a positive affine transformation) of utility while D2 follows from the uniqueness of probability (Savage, 1954; Arrow, 1965). Q.E.D.

CLAIM 3. Under C2 and D2, λ_N is the independent product of $(\lambda_i)_{i \in N}$.

Proof. Fix $(E_i)_{i \in N} \in \Sigma^N$ to show $\lambda_N(E_1 \times \dots \times E_n) = \lambda_1(E_1) \times \dots \times \lambda_n(E_n)$.

Case 1: assume $\lambda_N(E_1 \times \dots \times E_n) > 0$. Then, $\lambda_N(E_1 \times \dots \times E_{n-1} \times S) > 0$ and, by D2:

$$\lambda_n(E_n) = \lambda_N[E_1 \times \dots \times E_n | E_1 \times \dots \times E_{n-1} \times S].$$

Similarly, $\lambda_N(E_1 \times \dots \times E_{n-2} \times S \times S) > 0$ and, by D2: $\lambda_{n-1}(E_{n-1}) = \lambda_N[E_1 \times \dots \times E_{n-1} \times S | E_1 \times \dots \times E_{n-2} \times S \times S]$. Combining the two latter equalities, one obtains:

$$\lambda_{n-1}(E_{n-1})\lambda_n(E_n) = \lambda_N(E_1 \times \dots \times E_n) / \lambda_N(E_1 \times \dots \times E_{n-2} \times S \times S).$$

Moreover, proceeding as above, $\lambda_N(E_1 \times \dots \times E_{n-3} \times S \times S \times S) > 0$ and, by D2: $\lambda_{n-2}(E_{n-2}) = \lambda_N[E_1 \times \dots \times E_{n-2} \times S \times S | E_1 \times \dots \times E_{n-3} \times S \times S \times S]$. Again, combining the two latter equalities, one obtains:

$$\lambda_{n-2}(E_{n-2})\lambda_{n-1}(E_{n-1})\lambda_n(E_n) = \lambda_N(E_1 \times \dots \times E_n) / \lambda_N(E_1 \times \dots \times E_{n-3} \times S \times S \times S).$$

Finally, repeating this process delivers:

$$\lambda_1(E_1) \times \dots \times \lambda_n(E_n) = \lambda_N(E_1 \times \dots \times E_n) / \lambda_N(\Omega) = \lambda_N(E_1 \times \dots \times E_n).$$

Case 2: assume $\lambda_N(E_1 \times \dots \times E_n) = 0$. Then, by (2ii) in the extended Pareto condition, there exists $i \in N$ such that $\lambda_i(E_i) = 0$ and the result is straightforward. Q.E.D.

CLAIM 4. Conditions C1 and D1 imply that u_N is a convex combination of $(u_i)_{i \in N}$ with positive coefficients.

Proof. By D1, one has, for any $x_1, \dots, x_n \in X$, $u_N(x^o, x_2, \dots, x_n) = \beta_1(x^o, \dots, x_n) = \beta_1(x_1, \dots, x_n)$ and $u_N(x_1, \dots, x_n) = \alpha_1(x_1, \dots, x_n)u_1(x_1) + \beta_1(x_1, \dots, x_n)$. Combining these two equations yields:

$$u_N(x_1, \dots, x_n) = \alpha_1(x_1, \dots, x_n)u_1(x_1) + u_N(x^o, x_2, \dots, x_n). \quad (1)$$

Let us compute in the same way $u_N(x^o, x_2, \dots, x_n)$. By D1, one has $u_N(x^o, x^o, x_3, \dots, x_n) = \beta_2(x^o, x^o, x_3, \dots, x_n) = \beta_2(x^o, x_2, x_3, \dots, x_n)$ and $u_N(x^o, x_2, \dots, x_n) = \alpha_2(x^o, x_2, \dots, x_n)u_2(x_2) + \beta_2(x^o, x_2, \dots, x_n)$. Combining the two latter equalities and using (1) yields:

$$u_N(x_1, \dots, x_n) = \alpha_1(x_1, \dots, x_n)u_1(x_1) + \alpha_2(x^o, x_2, \dots, x_n)u_2(x_2) + u_N(x^o, x^o, x_3, \dots, x_n).$$

Repeating this process for any $i = 3 \dots n$ delivers:

$$u_N(x_1, \dots, x_n) = \sum_{i \in N} \alpha_i(x^o, \dots, x^o, x_{i+1}, \dots, x_n)u_i(x_i).$$

Then, in particular, one has $u_N(x^o, \dots, x^o, x_n) = \alpha_n(x^o, \dots, x^o)u_n(x_n)$, and, by proceeding similarly in various orders, one even obtains $u_N(x^o, \dots, x^o, x_i, x^o, \dots, x^o) = \alpha_i(x^o, \dots, x^o)u_i(x_i)$, for any $i \in N$. Now by C1:

$$u_N(x_1, \dots, x_n) = \sum_{i \in N} u_N(x^o, \dots, x^o, x_i, x^o, \dots, x^o) = \sum_{i \in N} \alpha_i(x^o, \dots, x^o)u_i(x_i).$$

Define $\alpha_i = \alpha_i(x^o, \dots, x^o)$ which is positive by D1. Moreover, note that $1 = u(x^1, \dots, x^1) = \sum_{i \in N} \alpha_i(x^o, \dots, x^o)u_i(x^1) = \sum_{i \in N} \alpha_i$. Q.E.D.

End of sufficiency part of proof: By Claim 1, the extended Pareto Condition implies both C1 and C2 while, by Claim 2, C2 implies both D1 and D2. The result then follows by Claims 3 and 4.

References

- Al-Najjar, Nabil I., and Jonathan L. Weinstein. 2009. "The Ambiguity Aversion Literature: A Critical Assessment." *Economics and Philosophy* 25: 249-84.
- Arrow, Kenneth J. 1965. *Aspects of the Theory of Risk-Bearing*. Helsinki: Yrjö Jahnsson Säätiö. Reprinted in *Essays in the Theory of Risk-Bearing*. Amsterdam: North-Holland, 1971.
- Chambers, Christopher P., and Takashi Hayashi. 2006. "Preference Aggregation under Uncertainty: Savage vs. Pareto." *G.E.B.* 54: 430-440.
- Diamond, Peter A. 1967. "Cardinal Welfare, Individualistic Ethics, and Interpersonal Comparison of Utility: Comment." *J.P.E.* 75: 765.
- Gilboa, Itzhak, Dov Samet, and David Schmeidler. 2004. "Utilitarian Aggregation of Beliefs and Tastes." *J.P.E.* 112: 932-38.
- Harsanyi, John C. 1955. "Cardinal Welfare, Individualistic Ethics, and Interpersonal Comparison of Utility." *J.P.E.* 63: 309-21.
- Hylland, Aanund, and Richard J. Zeckhauser. 1979. "The Impossibility of Bayesian Group Decision Making with Separate Aggregation of Beliefs and Values." *Econometrica* 47: 1321-36.
- Mongin, Philippe. 1995. "Consistent Bayesian Aggregation." *J. Econ. Theory* 66: 313-51.
- Mongin, Philippe, and Marcus Pivato. 2014. "Ranking Multidimensional Alternatives and Uncertain Prospects." *MPRA working paper n°42515*.
- Savage, Leonard J. 1954. *The Foundations of Statistics*. New York: Wiley. 2d ed. New York: Dover, 1972.