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GROUPE D'ANALYSE ET DE THÉORIE ÉCONOMIQUE LYON - ST ÉTIENNE

WP 1440

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December 2014

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The Sequential Equal Surplus Division for Rooted Forest Games and an Application to Sharing a River with Bifurcations[1]

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June 9, 2014

Abstract

We introduce a new allocation rule, called the sequential equal surplus division for rooted forest TU-games. We provide two axiomatic characterizations for this allocation rule. The first one uses the classical property of component efficiency plus an edge deletion property. The second characterization uses standardness, an edge deletion property applied to specific rooted trees, a consistency property, and an amalgamation property. We also provide an extension of the sequential equal surplus division applied to the problem of sharing a river with bifurcations.

Keywords: Amalgamation – Consistency – Fairness – Rooted forest – Sequential equal surplus division – Water allocation.

1 Introduction

A situation in which a set of players can obtain certain payoffs by cooperating can be described by a cooperative game with transferable utility (henceforth a TU-game). In many economic applications, agents are part of a relational structure which possibly affects the cooperation possibilities. Examples are, *e.g.* one machine sequencing games of Curiel *et al.* (1989) where a finite number of agents, each having one job, are queued in front of a machine waiting for their jobs; the hierarchical production games of van den Brink (2008) where a set of workers and managers are partially ordered according to some hierarchical/permission structure affecting the potential production possibilities of the workers in the firm; the river games of Ambec and Sprumont (2002) where the riparian agents are totally ordered by their location on a river flowing from upstream to downstream. In all of these examples, some agents are not able to cooperate fully without the presence of other agents.

In this article, we follow in the footsteps of Khmelnitskaya (2010) by considering situations where the players of a TU-game are located at the nodes of a digraph. We restrict ourselves to the class of digraphs where each component of a digraph is a rooted tree, *i.e.* each digraph is a rooted forest. The article is divided into three parts.

In the first part of the article, we introduce a new allocation rule, the sequential equal surplus division as we call it, for rooted forest TU-games. The construction of this allocation rule is done sequentially by following the direction of the edges of the rooted tree of each component of the rooted forest. The player located at the root of this rooted tree has possibly several successors. The coalition formed by a successor and all the players connected with this successor through a directed path constitutes a set of subordinates for the player located at the root. Together with each set of subordinates, one for each successor, the player located at the root achieves some surplus (positive or negative depending on the properties of the TU-game), measured by

[1]For helpful comments received, the authors want to thank six anonymous referees, Antoinette Baujard, Gustavo Bergantiños, and the participants of the fourth World Congress of the Game Theory Society and 13th SAET conference. Financial support by the National Agency for Research (ANR) – research program "DynaMITE: Dynamic Matching and Interactions: Theory and Experiments", contract ANR-13-BSHS1-0010 – and the "Mathématiques de la décision pour l'ingénierie physique et sociale" (MODMAD) project is gratefully acknowledged.

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the difference between the worth achieved by the component and the sum of the worths achieved separately by the player located at the root and each set of his or her subordinates. In order to distribute this surplus, the player located at the root as well as the coalitions formed by these sets of subordinates are considered as single entities. The participation of each entity is necessary for the production of this surplus. Therefore, it seems natural to give to each entity an equal share of the surplus in addition to the worth it can secure in the absence of cooperation. This is equivalent to reward each entity by the well-known equal surplus division for TU-games. For each coalition formed by a set of subordinates, the obtained total payoff is what it remains to be shared among its members. The sequential equal surplus division then consists of applying repeatedly the above step to all players located at the root of the subtree induced by a set of subordinates of a player. The distribution of the surplus of cooperation between two or more coalitions is also at the heart of the construction of many of the solutions proposed in the literature. Either this surplus is fully allocated to only one player or it is shared on a more egalitarian basis (see, *e.g* van den Brink *et al.* 2007, Khmelnitskaya 2010). We will compare the sequential equal surplus division with existing solutions constructed on rooted trees.

In the second part of the article, we provide two characterizations of the sequential equal surplus division. The first one uses the classical property of component efficiency and an edge deletion property. The second characterization mobilizes four properties. The first one is a property of fairness stating that in a two-player situation the allocation rule reduces to the standard solution. The second one is an edge deletion property applied to specific rooted trees. The third one is a consistency property, and the last one is an amalgamation property. Edge deletion properties, consistency properties and amalgamation properties are customary in ressource allocation problems. Here, we construct specific properties which take into account the characteristics of the relational structure.

In the third part of the article, we apply the sequential equal surplus division solution to the problem of sharing a river with bifurcations as modeled by Ambec and Sprumont (2002) and Khmelnitskaya (2010). More specifically, we suggest a natural extension of the sequential equal surplus division that accounts for the specific features of the river sharing problems. It relies on a more realistic distribution of the surplus generated by the cooperation of several agents along a river with bifurcations. Based on one of the characterizations of the sequential equal surplus division, we are able to provide a characterization of this allocation rule on the class of river TU-games with bifurcations.

The organization of the article is as follows. In Section 2, basic definitions and notations are introduced. Section 3 is devoted to the construction of the sequential equal surplus division and to its basic properties. A comparison with the existing allocation rules is also proposed. Section 4 presents the properties used for the characterizations of the sequential surplus division rule. Section 5 contains the two characterizations. Section 6 addresses the problem of sharing a river with bifurcations. Section 7 concludes.

2 Preliminiaries

2.1 Cooperative TU-games

Notations For a finite set A, the notation $|A|$ stands for the number of elements of A. Weak set inclusion is denoted by $\subseteq$, whereas proper set inclusion is denoted by $\subset$.

A *cooperative game* with transferable utility (henceforth called a TU-game) is a pair (N, v) consisting of a finite player set $N \subseteq \mathbb{N}$ of size n and a coalition function $v : 2^N \longrightarrow \mathbb{R}$ satisfying $v(\emptyset) = 0$. An element S of 2^N is a coalition, and $v(S)$ is the maximal worth that the members of S can obtain by cooperating. Denote by $\mathcal{C}$ the set of all TU-games. For each nonempty coalition $S \in 2^N$, the *subgame* of (N, v) induced by S is the TU-game $(S, v_S) \in \mathcal{C}$ such that for any $T \in 2^S$, $v_S(T) = v(T)$. A TU-game (N, v) is *superadditive* if $v(S \cup T) \geq v(S) + v(T)$ for any pair of disjoint coalitions S and T.

In the TU-game (N, v), each agent $i \in N$ may receive a payoff $z_i \in \mathbb{R}$. A *payoff vector* $z = (z_i)_{i \in N} \in \mathbb{R}^n$ lists a payoff z_i for each $i \in N$. For any nonempty coalition $S \in 2^N$ the notation z_S stands for $\sum_{i \in S} z_i$. An *allocation rule* Φ on TU-games $\mathcal{C}$ is a map that assigns to each TU-game $(N, v) \in \mathcal{C}$ a payoff vector $\Phi(N, v) \in \mathbb{R}^n$.

2.2 Digraph TU-games

In several social situations there is an underlying ordering of the players, which describes some social, technical, or communicational structure. In this article, we assume that the social structure is represented by a digraph on the player set representing some dominance relation between these players.

Precisely, a digraph TU-game is a triple (N, v, D) where $(N, v) \in \mathcal{C}$ and (N, D) is a directed graph. A *directed graph or digraph* is a pair (N, D), where N is a finite set of nodes (representing the players) and $D \subseteq N \times N$ is a binary relation on N. An ordered pair of elements $(i, j) \in D$ represents a *directed edge* from i to j. We assume the digraph to be irreflexive, i.e., $(i, i) \notin D$ for all $i \in N$. Let $E_i \subseteq D$ be the set of directed edges to which $i \in N$ is incident, i.e., directed edges of the form $(i, j) \in D$ or $(j, i) \in D$. For any subset $C \subseteq N$, the *subdigraph* induced by C is the pair (C, D_C) where $D_C = \{(i, j) \in D : i, j \in C\}$. For $i \in N$, the nodes in $S_D(i) = \{j \in N : (i, j) \in D\}$ are called the *successors* of i, and the nodes in $P_D(i) = \{j \in N : (j, i) \in D\}$ are called the *predecessors* of i in (N, D). A *directed path* from i to j in N is a sequence of distinct nodes $(i_1, \ldots, i_p)$, $p \geq 2$, such that $i_1 = i$, $i_{q+1} \in S_D(i_q)$ for $q = 1, \ldots, p-1$, and $i_p = j$. The number $p-1$ is the *length* of the path. Given two nodes $i, j \in N$, j is a *subordinate* of i in (N, D) if there is a directed path from i to j. The set of i's subordinates is denoted by $\hat{S}_D(i)$, and we will use the notation $\hat{S}_D[i]$ to represent the union of $\hat{S}_D(i)$ and $\{i\}$. Note that for any $i \in N$, $S_D(i) \subseteq \hat{S}_D(i)$. We refer to the players in $\hat{P}_D(i) = \{j \in N : i \in \hat{S}_D(j)\}$ as the *superiors* of i in (N, D). We have $P_D(i) \subseteq \hat{P}_D(i)$.

A *quasi-path* is a sequence of distinct nodes $(i_1, \ldots, i_p)$, $p \geq 2$, such that $i_1 = i$, $i_{q+1} \in S_D(i_q)$ or $i_{q+1} \in P_D(i_q)$ for $q = 1, \ldots, p-1$, and $i_p = j$. A coalition S is *connected* in a directed digraph (N, D) if either it is a singleton or for any pair of distinct nodes i and j in S there is a quasi-path from i to j. A coalition S which is not connected admits a unique partition into maximally (with respect to set inclusion) connected parts, called its *components*.

A directed path $(i_1, \ldots, i_p)$, $p \geq 2$, in (N, D) is a *directed cycle* in D if $(i_p, i_1) \in D$. We call digraph (N, D) *acyclic* if it does not contain any directed cycle. Note that acyclicity of digraph (N, D) implies that D has at least one node that does not have a predecessor. A digraph (N, D) is a *rooted tree* if (a) (N, D) is acyclic, (b) there is exactly one node, denoted by $r \in N$ and called the *root* of the tree, such that $P_D(r) = \emptyset$, (c) for each $i \in N \setminus \{r\}$, there is exactly one directed path from r to i. If (N, D) is a tree rooted at $r \in N$, $p_D(i)$ refers to the unique predecessor of $i \in N \setminus \{r\}$. The *depth* of a node $i \in N$ in a rooted tree (N, D) is the length of the unique directed path from r to i, with the convention that the depth of r is set to 0. The *depth of a rooted tree* (N, D) is the depth of its deepest nodes. A digraph (N, D) is a *rooted forest* if it exists a partition $\{C_1, \ldots, C_p\}$ of N such for each $q = 1, \ldots, p$, the subdigraph (C_q, D_{C_q}) is a rooted tree. Each C_q is a component of the rooted forest (N, D). Denote by $\mathcal{D}$ the set of rooted forests. If the digraph $(N, D) \in \mathcal{D}$ possesses several components, we use the notation r_C for the root in (C, D_C).

In this article, we restrict our attention to the digraph TU-games (N, v, D) such that $(N, v) \in \mathcal{C}$ and $(N, D) \in \mathcal{D}$. Denote this set of digraph TU-games by $\mathcal{CD}$. An *allocation rule* Φ on $\mathcal{CD}$ is a map that assigns to each TU-game $(N, v, D) \in \mathcal{CD}$ a payoff vector $\Phi(N, v, D) \in \mathbb{R}^n$.

Example 1 *Let us illustrate these notions with the rooted tree (N, D) depicted in Figure 1. Its only component is N. The set of successors of node 1, the root, is $S_D(1) = \{2, 3\}$, the set of its subordinates including itself is $\hat{S}_1[1] = \{1, 2, 3, 4\}$, whereas the set of successors of node 3 is $S_D(3) = \{4\}$, $\hat{S}_D[3] = S_D[3]$, and $E_3 = \{(3, 4), (1, 3)\}$. Node 1 has no predecessor, and the unique predecessor $p_D(3)$ of node 3 is node 1. The set of superiors of node 4 is $\hat{P}_D(4) = \{1, 3\}$.* □

3 The sequential equal surplus division

3.1 A motivating example

Pick any $(N, v, D) \in \mathcal{CD}$ such that $N = \{1, 2\}$ and $D = \{(1, 2)\}$. The standard solution assigns first to a player its stand-alone worth and then distributes an equal share of the left surplus created by the cooperation. That

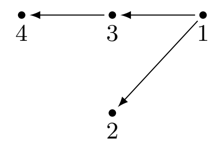


Figure 1: A rooted tree (N, D).

is, player $i \in \{1,2\}$ gets:

$$v(\{i\}) + \frac{v(\{1,2\}) - v(\{2\}) - v(\{1\})}{2}.$$

Many allocation rules for (undirected) graph TU-games coincide with the standard solution for the two-player one-edge case. This is true, among others, for the Myerson value (Myerson, 1977) and the average tree solution (Herings *et al.*, 2008). We would like to generalize this principle from the two-player case to the situation where n players are located on the nodes of a rooted forest. If we assume that players are responsible of the structure of the digraph (N, D), the ethics of responsibility usually conveys the idea that the "social planner" should let the agents exercise their responsibility and bear the consequences of such exercise, without trying to distort their outcomes in a particular way and with particular incentives. This means that if (N, D) represents a structure of control over the players for which the latter are responsible, they should be held responsible for the outcomes of their own choices. Thus, when 1 is the superior of 2, the standard solution is disputable, in particular when the surplus resulting from the cooperation is positive. But, if we assume that players are not responsible of the structure of the digraph (N, D), they should not be held responsible for the unchosen circumstances in which they make choices. An egalitarian principle may require that the effect of differential non-responsible characteristics to be counter-balanced. A handicap in non-responsible characteristics deserves a compensation (see, e.g., Fleurbaey, 1994).

So, assume that a set of players N of size $n > 2$ are located at the nodes of a rooted forest (N, D). Without loss of generality, assume that N is the only component of (N, D). Pick any player $i \in N$, and focus on the worth generated by this player and all his or her subordinates. The worth $v(\hat{S}_D[i])$ is jointly created by i and the existing coalitions $\hat{S}_D[j]$ for each $j \in S_D(i)$. The reason is obvious: on the one hand, player i can threaten to not cooperate; but, on the other hand, player i and each coalition of players $\hat{S}_D[j]$ for each $j \in S_D(i)$ have to cooperate in order to produce the total worth $v(\hat{S}_D[i])$. In the absence of cooperation, the total worth achieved by these players reduces to:

$$v(\{i\}) + \sum_{j \in S_D(i)} v(\hat{S}_D[j]).$$

If one considers each coalition $\hat{S}_D[j]$ as a single entity in the problem of distributing the total welfare $v(\hat{S}_D[i])$, then we can apply the well-known equal surplus division to such a $|S_D[i]|$-agent situation. We have:

$$v(\{i\}) + \frac{1}{|S_D[i]|}\left(v(\hat{S}_D[i]) - v(\{i\}) - \sum_{j \in S_D(i)} v(\hat{S}_D[j])\right)$$

to player i and, for each coalition $\hat{S}_D[j]$, $j \in S_D(i)$,

$$v(\hat{S}_D[j]) + \frac{1}{|S_D[i]|}\left(v(\hat{S}_D[i]) - v(\{i\}) - \sum_{j \in S_D(i)} v(\hat{S}_D[j])\right).$$

The allocation rule that we construct consists in a sequential application of the equal surplus division principle from upstream to downstream, which means that for player i and his or her subordinates, it might remain more than $v(\hat{S}_D[i])$ to share. Before defining formally this allocation rule, we provide an example.

Example 2 *Consider again the digraph (N, D) in Figure 1. Initially the four players cooperate with each other and the worth $v(\{1,2,3,4\})$ is to be distributed. The players receive their payoff sequentially according to their distance to the root. We compute the payoff of player 1 first on the basis of what would happen if he or she refuses to cooperate. In such a case, the grand coalition would partition into three coalitions: $\{1\}$, $\{2\}$ and $\{3,4\}$, where $\{3,4\}$ acts as a single entity. How much should player 1 get from the worth $v(\{1,2,3,4\})$ as it threatens to refuse the cooperation? Intuitively these three entities should first get the worth they can secure without the cooperation of the other players, i.e. $v(\{1\})$, $v(\{2\})$ and $v(\{3,4\})$, respectively. As for the surplus $v(\{1,2,3,4\}) - v(\{1\}) - v(\{2\}) - v(\{3,4\})$, one can argue that since player 1 is now negotiating with coalition $\{2\}$ and with coalition $\{3,4\}$ as a whole, and the surplus is jointly created by these three parties, the equal surplus division for 3-player TU-games can be applied so that each party should get one third of the joint surplus. Consequently, player 1 obtains its individual worth plus one third of the joint surplus:*

$$v(\{1\}) + \frac{1}{3}\Big(v(\{1,2,3,4\}) - v(\{1\}) - v(\{2\}) - v(\{3,4\})\Big).$$

The worths remaining for coalitions $\{2\}$ and $\{3,4\}$, which we call the egalitarian remainders for $\{2\}$ and $\{3,4\}$, are given respectively by:

$$v(\{2\}) + \frac{1}{3}\Big(v(\{1,2,3,4\}) - v(\{1\}) - v(\{2\}) - v(\{3,4\})\Big) \tag{1}$$

and

$$v(\{3,4\}) + \frac{1}{3}\Big(v(\{1,2,3,4\}) - v(\{1\}) - v(\{2\}) - v(\{3,4\})\Big).$$

Then, the above procedure has to be repeated on each branch of the rooted tree. That is, both remaining coalitions $\{2\}$ and $\{3,4\}$ behave independently on their own branch of the rooted tree in order to negotiate the allocation of their respective remainder. On the one hand, player 2 is the unique player on its branch so that the final payoff of 2 is given by (1). On the other hand, player 3 can now be considered as the root of the branch on which it is located. As above, the egalitarian remainder for player 4 will be:

$$v(\{4\}) + \frac{1}{2}\Big(v(\{3,4\}) + \frac{1}{3}\Big(v(\{1,2,3,4\}) - v(\{1\}) - v(\{2\}) - v(\{3,4\})\Big) - v(\{3\}) - v(\{4\})\Big),$$

while player 3 will get:

$$v(\{3\}) + \frac{1}{2}\Big(v(\{3,4\}) + \frac{1}{3}\Big(v(\{1,2,3,4\}) - v(\{1\}) - v(\{2\}) - v(\{3,4\})\Big) - v(\{3\}) - v(\{4\})\Big).$$

□

3.2 Definition and first properties

Extending the argument developped in Example 2 to each (N, v, D), we obtain a solution on $\mathcal{CD}$, called the *sequential equal surplus division*. The sequential equal surplus division on $\mathcal{CD}$, denoted by Φ^e, is defined recursively as follows. Pick any $(N, v, D) \in \mathcal{CD}$, any component C of (N, D), any $i \in C$, and define recursively the *egalitarian remainder* α for coalition $\hat{S}_D[i]$ as:

$$\alpha(\hat{S}_D[i]) = \begin{cases} v(C) & \text{if } i = r_C, \\ v(\hat{S}_D[i]) + \dfrac{1}{|S_D[p_D(i)]|}\Big(\alpha(\hat{S}_D[p_D(i)]) - \displaystyle\sum_{j \in S_D(p_D(i))} v(\hat{S}_D[j]) - v(\{p_D(i)\})\Big) & \text{otherwise.} \end{cases} \tag{2}$$

Note that, by definition, $i \in S_D(p_D(i))$ for each $i \neq r_C$. The sequential equal surplus division assigns to each $i \in N$, the payoff given by:

$$\Phi^e_i(N, v, D) = v(\{i\}) + \frac{1}{|S_D[i]|}\left(\alpha(\hat{S}_D[i]) - \sum_{j \in S_D(i)} v(\hat{S}_D[j]) - v(\{i\})\right). \tag{3}$$

The allocation rule Φ^e generalizes the individual standardized remainder vectors proposed by Ju, Borm and Ruys (2007) for the class of all TU-games. Each individual standardized remainder vector is constructed from a bijection over the player set. Because a bijection on N induces a total order on the player set, it is isomorphic to a directed path. In this special case, the resulting individual standardized remainder vector coincides with (2). The major difference with our allocation rule is that more than two coalitions may negotiate to share a surplus when the rooted tree separates into different branches.

We show below in Proposition 1 that for each coalition formed by a player and his or her subordinates, the egalitarian remainder is fully redistributed among its members, *i.e.* for each $i \in N$, it holds that:

$$\Phi^e_{\hat{S}_D[i]}(N, v, D) = \alpha(\hat{S}_D[i]).$$

Because $\hat{S}_D[r_C] = C$, $\alpha(C) = v(C)$ by (2), we have: $\Phi^e_C(N, v, D) = v(C)$. This means that Φ^e satisfies the well-known axiom of component efficiency introduced by Myerson (1977) in the context of undirected graph TU-games. The sequential equal surplus division satisfies two other interesting properties whenever the underlying TU-game (N, v) is superadditive: for each $i \in C$, it holds that:

$$\Phi^e_{\hat{S}_D[i]}(N, v, D) \geq v(\hat{S}_D[i]), \quad \text{and} \quad \Phi^e_i(N, v, D) \geq v(\{i\}).$$

The first property indicates that a coalition consisting of any player and all his or her subordinates obtains a total payoff that is at least as large as the worth its members can achieve without the cooperation of the other players. The second property indicates that each player obtains at least his or her stand-alone worth. These properties cannot be extended to all coalitions to show that $\Phi^e_i(N, v, D)$ belongs to the core of (N, v) when its core is nonempty.

Example 3 *Assume that* $N = \{1, 2, 3, 4\}$, $D = \{(1, 2), (2, 3), (3, 4)\}$. *Assume further that* $v(S) = 16$ *for each* $S \in \{\{1, 2\}, \{1, 2, 3\}, \{1, 2, 4\}, N\}$, *and* $v(S) = 0$ *for each other coalition* S. *The TU-game* (N, v) *is superadditive, and its core is nonempty. For instance, the payoff vector* $z = (8, 8, 0, 0)$ *lies in the core, i.e.* $z_S \geq v(S)$ *for each* $S \in 2^N$, *and* $z_N = v(N)$. *On the other hand,* $\Phi^e(N, v, D) = (8, 4, 2, 2)$ *does not belong to the core of* (N, v) *since* $\Phi^e_1(N, v, D) + \Phi^e_2(N, v, D) < v(\{1, 2\})$. □

It is important to point out extra properties when the underlying TU-game is superadditive because it is the case in most of the applications. These properties are summarized in the following proposition and stated for all rooted forest TU-games.

Proposition 1 *For each* $(N, v, D) \in \mathcal{CD}$, *and each* $i \in N$, *it holds that:*
(i) $\Phi^e_{\hat{S}_D[i]}(N, v, D) = \alpha(\hat{S}_D[i])$;
If (N, v) *is superadditive, then it holds that:*
(ii) $\alpha(\hat{S}_D[i]) \geq v(\hat{S}_D[i])$;
(iii) $\Phi^e_i(N, v, D) \geq v(\{i\})$.

Proof. Pick any $(N, v, D) \in \mathcal{CD}$. The proof of part (i) is by induction on the number of subordinates of i.
Initial step: Assume that i has no subordinates, *i.e.* $\hat{S}_D(i) = \emptyset$. Then, (3) implies:

$$\Phi^e_{\hat{S}_D[i]}(N, v, D) = \Phi^e_i(N, v, D) = \alpha(\{i\}).$$

Induction hypothesis: Assume that the assertion holds when $\hat{S}_D(i)$ contains at most q elements.

INDUCTION STEP: Assume that $\hat{S}_D(i)$ contains $q+1$ elements. Then:

$$\Phi^e_{\hat{S}_D[i]}(N,v,D) = \Phi^e_i(N,v) + \sum_{j \in S_D(i)} \Phi^e_{\hat{S}_D[j]}(N,v,D).$$

Since $j \in S_D(i)$, each $\hat{S}_D(j)$ contains at most q elements. By the induction hypothesis, the right-hand side of the above equality is equivalent to:

$$\Phi^e_i(N,v,D) + \sum_{j \in S_D(i)} \alpha(\hat{S}_D[j]).$$

Using the definition (2) of the egalitarian remainder, the fact that, for each $j \in S_D(i)$, $p_D(j) = i$, we have:

$$\begin{aligned}
\Phi^e_{\hat{S}_D[i]}(N,v,D) &= \Phi^e_i(N,v,D) + \sum_{j \in S_D(i)} \alpha(\hat{S}_D[j]) \\
&= v(\{i\}) + \frac{1}{|S_D[i]|}\Big(\alpha(\hat{S}_D[i]) - \sum_{j \in S_D(i)} v(\hat{S}_D[j]) - v(\{i\})\Big) \\
&\quad + \sum_{j \in S_D(i)} \bigg(v(\hat{S}_D[j]) + \frac{1}{|S_D[i]|}\Big(\alpha(\hat{S}_D[i]) - \sum_{k \in S_D(i)} v(\hat{S}_D[k]) - v(\{i\})\Big)\bigg) \\
&= v(\{i\}) + \frac{1}{|S_D[i]|}\Big(\alpha(\hat{S}_D[i]) - \sum_{j \in S_D(i)} v(\hat{S}_D[j]) - v(\{i\})\Big) \\
&\quad + \sum_{j \in S_D(i)} v(\hat{S}_D[j]) + \frac{|S_D(i)|}{|S_D[i]|}\Big(\alpha(\hat{S}_D[i]) - \sum_{k \in S_D(i)} v(\hat{S}_D[k]) - v(\{i\})\Big) \\
&= \alpha(\hat{S}_D[i]),
\end{aligned}$$

as desired.

The proof of part (ii) is by induction on the depth q of a player i in the rooted tree induced by the component C he or she belongs to.

INITIAL STEP: Assume that $q = 0$ so that i is the root r_C of the rooted tree on C, *i.e.* $\hat{S}_D(r_C) = C$. By (2), we get $\alpha(C) = v(C)$, as desired.

INDUCTION HYPOTHESIS: Assume that the assertion is true for each player $i \in C$ such that his or her depth is inferior or equal to q.

INDUCTION STEP: Pick any $i \in N$ such that his or her depth is $q+1$. It follows that the depth of number of $p_D(i)$ is equal to q. By the induction hypothesis and the superadditivity of (N,v), we get:

$$\begin{aligned}
\alpha(\hat{S}_D[i]) &= v(\hat{S}_D[i]) + \frac{1}{|S_D[p_D(i)]|}\Big(\alpha(\hat{S}_D[p_D(i)]) - \sum_{j \in S_D(p_D(i))} v(\hat{S}_D[j]) - v(\{p_D(i)\})\Big) \\
&\geq v(\hat{S}_D[i]) + \frac{1}{|S_D[p_D(i)]|}\Big(v(\hat{S}_D[p_D(i)]) - \sum_{j \in S_D(p_D(i))} v(\hat{S}_D[j]) - v(\{p_D(i)\})\Big) \\
&\geq v(\hat{S}_D[i]),
\end{aligned}$$

as desired.

The proof of part (iii) follows from (ii), the definition (3) of Φ^e and the superadditivity of (N,v). It suffices to note that the collection of coalitions $\hat{S}_D[j]$, $j \in S_D(i)$, and the singleton $\{i\}$ constitute a partition of $\hat{S}_D[i]$. ■

It is interesting to compare our solution to the existing ones for TU-games in which players are assumed to be internally organized according to some social structure. For some of the articles cited below, the social structure is represented by an undirected graph (N, D), *i.e.* D is symmetric, $(i, j) \in D$ implies $(j, i) \in D$. It is only when the solution is designed that players are hierarchically ordered according to some asymmetric relation. For the sake of presentation, and without loss of generality, we will not insist on this aspect below. A notable exception is the socially structured games introduced by Herings et al. (2007). For this class of TU-games, the social structure is defined by a power function assigning to each nonempty coalition a power vector of its members. This power vector represents the strength of the position of every player within the underlying social structure of the coalition.

The solution provided by Ambec and Sprumont (2002) for river TU-games and reinterpreted in van den Brink *et al.* (2007) in terms of rooted forest TU-games is defined for the subclass of rooted forest TU-games (N, v, D) where each component of the rooted forest induces a unique directed path. Khmelnitskaya (2010) extends this solution to all rooted forests. The solution suggested by Khmelnitskaya (2010) coincides with the so-called hierarchical outcome introduced by Demange (2004) in another context, and defined by:

$$\forall i \in N, \quad h_i(N, v, D) = v(\hat{S}_D[i]) - \sum_{j \in S_D(i)} v(S_D[j]). \tag{4}$$

Each player's payoff is equal to his marginal contribution to all his subordinates when he cooperates with them. In case the rooted tree (C, D_C) induced by the component C, say equal to $\{1, \ldots, c\}$, is the directed path $\{(i, i+1) : i \in C \setminus \{c\}\}$, each player $i \in C$ other than player c has exactly one successor $i+1$, so that $\hat{S}_D[i] = \{i, i+1, \ldots, c\}$ and $\hat{S}_D[i+1] = \{i+1, i+2, \ldots, c\}$ for $\{i+1\} = S_D(i)$. The hierarchical outcome reduces to:

$$h_c(N, v, D) = v(\{c\}) \text{ and for } i \in C \setminus \{c\},\ h_i(N, v, D) = v(\{i, \ldots, c\}) - v(\{i+1, \ldots, c\}). \tag{5}$$

As shown by Demange (2004), van den Brink *et al.* (2007) and Khmelnitskaya (2010), this solution possesses several advantages. Firstly, it is efficient by component, *i.e.* $h_C(N, v, D) = v(C)$, and satisfies also point (ii) in Proposition 1 not only for coalitions formed by a player and all its subordinates but for all connected coalitions. Secondly, it also gives an incentive for a player to cooperate with his or her subordinates in the sense that all the surplus resulting from this cooperation is allocated to this player. Indeed, this surplus is given by:

$$v(\hat{S}_D[i]) - \sum_{j \in S_D(i)} v(S_D[j]) - v(\{i\})$$

and i's payoff in (4) is exactly the same as:

$$v(\{i\}) + v(\hat{S}_D[i]) - \sum_{j \in S_D(i)} v(\hat{S}_D[j]) - v(\{i\}).$$

This solution is the opposite of the solution suggested by Ambec and Sprumont (2002) and van den Brink *et al.* (2007), which distributes all the surplus resulting from the cooperation of a player with its superiors:

$$\text{the root } r = 1 \text{ gets } v(\{1\}) \text{ and each other player } i \text{ gets } v(\{1, 2, \ldots, i\}) - v(\{1, 2, \ldots, i-1\}). \tag{6}$$

Herings *et al.* (2007) show that the solutions given by (5) and (6) can be obtained as unique socially stable core elements, where the solution selected depends on whether the superior players or the subordinate players in the directed path are the more powerful.

Another way to compare (4) with our solution is to use point (i) of Proposition 1. Indeed, for each $i \in C$, we have:

$$\Phi^e_i(N, v, D) = \Phi^e_{\hat{S}_D[i]}(N, v, D) - \sum_{j \in S_D(i)} \Phi^e_{\hat{S}_D[j]}(N, v, D) = \alpha(\hat{S}_D[i]) - \sum_{j \in S_D(i)} \alpha(\hat{S}_D[j]).$$

Thus, each player's payoff is equal to its contribution to the egalitarian remainder when it cooperates with all his or her subordinates.

Béal *et al.* (2010, 2012b) and van den Brink *et al.* (2012) envisage another route: their solution assigns a weighted average of different hierarchical outcomes, computed from various other rooted trees than the natural ones depicted by (N, D). These solutions generalize the so-called average tree solution introduced by Herings *et. al.* (2008). Instead of averaging between different payoff vectors, our solution is constructed recursively from the root of each component of the rooted forest.

Games with a permission structure constitute another strand of the literature initiated by Gilles, Owen and van den Brink (1992). Besides a cooperative game, a game with permission structure is endowed with a permission structure specifying that a coalition is feasible if for any of its members, some (disjunctive approach) or all (conjunctive approach) players preceding him in the structure also belong to the coalition. In particular, a permission structure can be constructed from a digraph by requiring that a coalition is feasible if any (or all) superiors of each of its members also belong to the coalition. The Shapley value of a modified game is studied by van den Brink and Gilles (1996) and van den Brink (1997) under the conjunctive and disjunctive approaches respectively. Like for the Myerson value, the construction of this kind of solutions is not really similar to ours.

4 Properties for Φ^e

In the section, we introduce a list of properties for allocation rules, which will be used in the next section to provide two characterizations of Φ^e. These properties express desirable principles of consistency, efficiency, fairness, and amalgamation/collusion of players.

The first property states that the allocation rule coincides with the standard solution if the directed graph induced by a component involves only two players connected by one directed edge.

Standardness[2] For each $(N, v, D) \in \mathcal{CD}$, each component C such that $C = \{1, 2\}$ and $D_C = \{(1, 2)\}$, it holds that:

$$\forall i \in \{1, 2\}, \quad \Phi_i(N, v, D) = v(\{i\}) + \frac{v(\{1, 2\}) - v(\{1\}) - v(\{2\})}{2}.$$

The second property states that the total sum of payoffs in a component equals the worth of this component. This property, introduced by Myerson (1977) for undirected graph TU-games, is satisfied by many allocation rules including those mentioned in the previous section.

Component efficiency For each $(N, v, D) \in \mathcal{CD}$, each component C of (N, D), it holds that:

$$\Phi_C(N, v, D) = v(C).$$

Many allocation rules for directed or undirected graph TU-games are based on the principle of component efficiency, and an edge deletion property which incorporates a principle of fairness by indicating how the payoffs of certain players evolve when a subset of edges are deleted. For instance, the Myerson value (1977) is characterized by component efficiency and fairness on the class of all undirected graph TU-games. Fairness says that deleting an edge between two players yields for both players the same change in payoff. The average tree solution (Herings *et al.*, 2008) is characterized by component efficiency and component fairness on the class of all undirected acyclic graph TU-games. Component fairness says that deleting an edge between two players yields for both resulting components the same average change in payoff, where the average is taken over the players in the component. In addition to component efficiency, van den Brink *et al.* (2007) characterize the solutions defined as (5) and (6) by lower equivalence and upper equivalence respectively. Lower equivalence indicates the payoff of player on a directed path does not depend on the presence of the upward directed edges, while upper equivalence indicates that the payoff of player on a directed path does not depend on the presence of the downward directed edges.

We introduce two new edge deletion properties. The first one, called subordinate fairness, says that deleting the directed edges between a player i and all her neighbors, *i.e.*, deleting the set of directed edges E_i, yields the

[2]Standardness is the usual name given to this axiom even if it is not satisfied by several solutions on $\mathcal{CD}$.

same change in total payoff for the resulting components containing his or her subordinates in (N, D) including herself.

Subordinate fairness For each $(N, v, D) \in \mathcal{CD}$, each component C with at least two players, and each $i \in C$ such that $S_D(i) \neq \emptyset$, it holds that:

$$\forall j \in S_D(i), \quad \Phi_i(N, v, D) - \Phi_i(N, v, D \setminus E_i) = \Phi_{\hat{S}_D[j]}(N, v, D) - \Phi_{\hat{S}_D[j]}(N, v, D \setminus E_i).$$

In particular, subordinate fairness implies:

$$\forall j, k \in S_D(i), \quad \Phi_{\hat{S}_D[j]}(N, v, D) - \Phi_{\hat{S}_D[j]}(N, v, D \setminus E_i) = \Phi_{\hat{S}_D[k]}(N, v, D) - \Phi_{\hat{S}_D[k]}(N, v, D \setminus E_i).$$

Because $\mathcal{D}$ is closed under the edge deletion operation, $(N, v, D \setminus E_i)$ belongs to $\mathcal{CD}$.

The second edge deletion property, called star fairness, applies to components where the root is incident to each directed edge in this component. The rooted tree induced by this component is called an outward pointing star. Figure 2 depicts such a situation. Star fairness indicates that the deletion of a directed edge between the root and one of his or her successors yield for all other players including the root the same change in payoff.

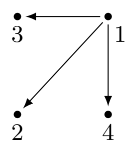


Figure 2: An outward pointing star.

Star fairness For each $(N, v, D) \in \mathcal{CD}$, each component C such that the rooted tree is an outward pointing star with root $r_C \in C$, and $|C| \geq 3$, the following holds. For each $i \in S_D(r_C)$, we have:

$$\forall j, k \in C \backslash \{i\}, \quad \Phi_j(N, v, D) - \Phi_j(N, v, D \setminus \{(r_C, i)\}) = \Phi_k(N, v, D) - \Phi_k(N, v, D \setminus \{(r_C, i)\}).$$

Star fairness has the same flavor as the property of fairness for neighbors invoked in Béal *et al.* (2012a), which requires, for arbitrary (undirected) graph games, an equal payoff variation between a player and each of his remaining neighbors in the graph obtained by deleting one of his links. More specifically, the two properties become equivalent when considered link belongs to a star graph. As for fairness for neighbors, star fairness can be justified by the fact that it makes sense to apply an equal gain/loss principle to pairs of players who continue to be linked in the graph resulting from the removal of a link. By contrast, Myerson's (1977) fairness applies the same equal gain/loss principle but to the unique pair of players who become disconnected.

The next property incorporates a consistency principle. Informally, a consistency principle states the following. Fix a solution for a class TU-games. Assume that some players leave the game with their payoffs, and examine the reduced problem that the remaining agents face. The solution is consistent if for this reduced game, there is no need to re-evaluate the payoffs of the remaining agents. As noted by Aumann (2008) and Thomson (2013), the consistency principle has been examined in the context of a great variety of concrete problems of resource allocation. In one form or another, it is common to almost all solutions and often plays a key role in axiomatic characterizations of the solutions. In the context of rooted forest TU-games, we design a consistency property which takes into account the orientation of the edges.

Pick any $(N, v, D) \in \mathcal{CD}$, any component C with root $r_C \in C$ and any $i \in C$. Assume that the payoffs have been distributed according to the payoff vector $z \in \mathbb{R}^n$ and that the members $C \backslash \hat{S}_D[i]$ leave the game with their component of the vector z. Let us re-evaluate the situation of $\hat{S}_D[i]$ at this point. To do this, we define the reduced game they face. The worths of coalitions in the reduced games can depend on (i) the worths that these coalitions could earn on their own in the original game, (ii) what these coalitions could earn with the leaving players, and (iii) the payoff with which the leaving players left the game. Our *reduced TU-game* $(\hat{S}_D[i], v_{z,i}, D_{\hat{S}_D[i]})$ induced by $\hat{S}_D[i]$ and z is defined as follows: $v_{z,i}(\hat{S}_D[i]) = v(C) - z_{C\setminus \hat{S}_D[i]}$ and for each other coalition $S \subset \hat{S}_D[i]$, $v_{z,i}(S) = v(S)$. Thus, $v_{z,i}(\hat{S}_D[i])$ is the total worth left for the remaining players who interact according to $(\hat{S}_D[i], v_{z,i}, D_{\hat{S}_D[i]})$ on the induced subtree on $\hat{S}_D[i]$.[3]

Subordinate consistency For each $(N, v, D) \in \mathcal{CD}$, each $i \in N$, it holds that:

$$\forall j \in \hat{S}_D[i], \quad \Phi_j(N, v, D) = \Phi_j(\hat{S}_D[i], v_{\Phi,i}, D_{\hat{S}_D[i]}).$$

Subordinate consistency is a robustness requirement guaranteeing that the coalition formed by the subordinates of a player respects the recommendations made by Φ when the other players in the component have already received their payoffs according to the solution Φ.

The last property incorporates an amalgamation principle. This principle says something about the changes in payoffs when two or more players are amalgamated to act as if they were a single player. It states that if some players are amalgamated into one entity, just because they have colluded, then the payoff of these players in the new game coincides with the sum of the payoffs of the amalgamated player in the original game. This principle is imilar to those in Lehrer (1988) and Haller (1994), even if Lehrer only imposes a weak inequality and if Haller keeps a fixed player set. Amalgamation principles have been used by Lehrer (1988), Albizuri (2001) and Albizuri, Aurrekoetxea (2006), among others, in order to characterize power indexes in voting games. Recently, Ju and Park (2014), and van den Brink (2012) have used a similar principle to characterize the hierarchical outcome (Demange, 2004) and the average tree solution (Herings *et al.* 2008).

Pick any $(N, v, D) \in \mathcal{CD}$, any component C with root $r_C \in C$, and any $i \in C$. For each $j \in S_D(i)$, the members of $\hat{S}_D[j]$ collude and act as a single entity so that they are amalgamated into a new single entity denoted by $\hat{S}^*_D[j]$. From this amalgamation, we define a companion forest directed TU-game $(N^i, v^i, D^i) \in \mathcal{CD}$ as follows. The new player set N^i is given by:

$$N^i = \Big[N \setminus \Big(\bigcup_{j\in S_D(i)} \hat{S}_D[j]\Big)\Big] \bigcup_{j\in S_D(i)} \Big\{\hat{S}^*_D[j]\Big\}.$$

The rooted tree on

$$\Big[C \setminus \Big(\bigcup_{j\in S_D(i)} \hat{S}_D[j]\Big)\Big] \bigcup_{j\in S_D(i)} \Big\{\hat{S}^*_D[j]\Big\}$$

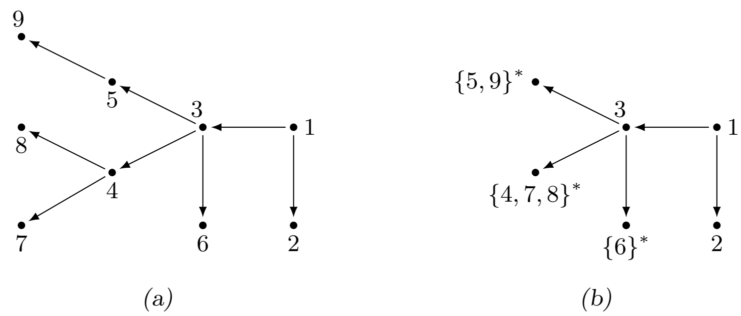
contains the directed edges $(i, \hat{S}^*_D[j])$ for each $j \in S_D(i)$ plus all the original directed edges between pairs of players that belong to $C \setminus \big(\bigcup_{j\in S_D(i)} \hat{S}_D[j]\big)$. The other components are left unchanged.

Example 4 *The amalgamation process is illustrated by the rooted tree in Figure 3. Part (b) represents the rooted tree obtained after amalgamation of the players located on each branch downstream of player 3. Therefore, the coalitions* $\{5, 9\}$, $\{4, 7, 8\}$ *and* $\{6\}$ *have been transformed into single entities* $\{4, 7, 8\}^*$, $\{5, 9\}^*$ *and* $\{6\}^*$ *respectively. Note that the amalgamated entity* $\{6\}^*$ *is formally identical to the original player 6. Player 3 and his or her superiors are not affected. Regarding the directed edges of the new rooted tree, the directed edges* $(5, 9)$, $(4, 7)$ *and* $(4, 8)$ *within the former branches completely disappear. The former directed edges* $(3, 4)$, $(3, 5)$ *and* $(3, 6)$ *are somehow replaced by directed edges* $(3, \{4, 7, 8\}^*)$, $(3, \{5, 9\}^*)$ *and* $(3, \{6\}^*)$ *respectively. The other directed edges are not affected.* □

Since the members of $\hat{S}_D[j]$ behave as a single entity, the coalitions contained in $\hat{S}_D[j]$ as well as their directed edges are not taken into account in the description of the new coalition function. Therefore, we define $(N^i, v^i) \in \mathcal{C}$ as follows: for each $S \in 2^{N^i}$,

$$v^i(S) = \begin{cases} v(S) & \text{if } S \cap \hat{S}^*_D[j] = \emptyset \text{ and } j \in S_D(i), \\ v\Big(\big(S \setminus \{\hat{S}^*_D[j] : j \in S_D(i)\}\big) \bigcup_{\{j\in S_D(i):\hat{S}^*_D[j]\in S\}} \hat{S}_D[j]\Big) & \text{otherwise.} \end{cases}$$

[3]This reduced TU-game, also called the projected reduced game, is customary used in cooperative game theory to construct consistency axioms (see Thomson, 2013).



(a) (b)

Figure 3: An operation of amalgamation on (N, D).

In particular, for each player $S^*_D[j]$, we have $v^i(S^*_D[j]) = v(S_D[j])$. Of course, if $S_D(i)$ is empty, then (N^i, v^i, D^i) coincides with (N, v, D).

Subordinate amalgamation For each $(N, v, D) \in \mathcal{CD}$ and each $i \in N$ such that $S_D(i) \neq \emptyset$, it holds that:

$$\forall j \in S_D(i), \quad \Phi_{\hat{S}_D[j]}(N, v, D) = \Phi_{\hat{S}^*_D[j]}(N^i, v^i, D^i).$$

5 Two characterizations of Φ^e

This section contains the main results of this article. We start by showing that Φ^e satisfies all the properties listed in the previous section.

Proposition 2 *The allocation rule Φ^e defined in (3) satisfies standardness, component efficiency, subordinate fairness, star fairness, subordinate consistency, and subordinate amalgamation on $\mathcal{CD}$.*

Proof. Recall that the allocation rule Φ^e is computed component by component. From (2) and (3), it is easily verified that Φ^e satisfies standardness. The fact that Φ^e satisfies component efficiency follows directly from part (i) in Proposition 1 and (2). To check that Φ^e satisfies subordinate fairness, pick any $(N, v, D) \in \mathcal{CD}$, any component C and any $i \in C$ such that $S_D(i) \neq \emptyset$. Deleting the directed edges E_i from D has the following effect on i's subordinates: $\hat{S}_D[i]$ splits into new components $\{i\}$ and $\hat{S}_D[j]$ for each $j \in S_D(i)$. Thus, for each $j \in S_D(i)$, we have:

$$\begin{aligned}
\Phi^e_{\hat{S}_D[j]}(N, v, D) - \Phi^e_{\hat{S}_D[j]}(N, v, D \setminus E_i) &= \Phi^e_{\hat{S}_D[j]}(N, v, D) - v(\hat{S}_D[j]) \\
&= \alpha(\hat{S}_D[j]) - v(\hat{S}_D[j]) \\
&= \frac{1}{|S_D[i]|}\Bigg(\alpha(\hat{S}_D[i]) - \sum_{k \in S_D(i)} v(\hat{S}_D[k]) - v(\{i\})\Bigg) \\
&= \Phi^e_i(N, v, D) - v(\{i\}) \\
&= \Phi^e_{\{i\}}(N, v, D) - \Phi^e_{\{i\}}(N, v, D \setminus E_i),
\end{aligned}$$

where the first equality follows from the fact that Φ^e satisfies component efficiency. The second equality follows from part (i) in Proposition 1. The third and fourth equalities come from $p_D(j) = i$, (2)–(3), and component efficiency respectively. Therefore, Φ^e satisfies subordinate fairness.

Next, consider any $(N, v, D) \in \mathcal{CD}$ such that there is a component C, $|C| \geq 3$, that induces an outward pointing star. Pick any directed edge (r_C, i). In order to verify that Φ^e satisfies star fairness, we first compute $\Phi^e_j(N, v, D)$ for each $j \in C$. By construction and definition of the egalitarian remainder, we have: $\hat{S}_D[r_C] = C$, $\alpha(C) = v(C)$ and, for each $j \in N \backslash \{r_C\}$, $p_D(j) = r_C$ and also $\hat{S}_D[j] = \{j\}$. Consequently, from (2) and (3) we get:

$$\forall j \in C, \quad \Phi^e_j(N,v,D) = v(\{j\}) + \frac{1}{|C|}\Bigg(v(C) - \sum_{l\in C} v(\{l\})\Bigg). \tag{7}$$

In $(N,v,D\setminus\{(r_C,i)\}) \in \mathcal{CD}$, the component C is replaced by the two new components $C\setminus\{i\}$ and $\{i\}$. By a similar computation on $(N,v,D\setminus\{(r_C,i)\}) \in \mathcal{CD}$, we get that for each $j \in C\setminus\{i\}$:

$$\Phi^e_j(N,v,D\setminus\{(r_C,i)\}) = v(\{j\}) + \frac{1}{|C|-1}\Bigg(v(C\setminus\{i\}) - \sum_{l\in C\setminus\{i\}} v_{C\setminus\{i\}}(\{l\})\Bigg). \tag{8}$$

From (7)–(8), we get:

$$\forall j,k \in C\backslash\{i\}, \quad \Phi^e_j(N,v,D) - \Phi^e_j(N,v,D\setminus\{(r_C,i)\}) = \Phi^e_k(N,v,D) - \Phi^e_k(N,v,D\setminus\{(r_C,i)\}),$$

which proves that Φ^e satisfies star fairness.

In order to verify that Φ^e satisfies subordinate consistency, we first compare the payoffs $\Phi^e_i(N,v,D)$ and $\Phi^e_i(\hat{S}_D[i], v_{\Phi^e,i}, D_{\hat{S}_D[i]})$ for any $(N,v,D) \in \mathcal{CD}$ and any $i \in C$ for some component C. By definition of $(\hat{S}_D[i], v_{\Phi^e,i}, D_{\hat{S}_D[i]})$, we have $v_{\Phi^e,i}(\hat{S}_D[i]) = v(C) - \Phi^e_{C\setminus\hat{S}_D[i]}(N,v,D)$. Because Φ^e satisfies component efficiency and by part (i) of Proposition 1, we get:

$$v_{\Phi^e,i}(\hat{S}_D[i]) = \Phi^e_{\hat{S}_D[i]}(N,v,D) = \alpha(\hat{S}_D[i]).$$

Using the definition of $(\hat{S}_D[i], v_{\Phi^e,i}, D_{\hat{S}_D[i]})$, it follows that:

$$\begin{aligned}
\Phi^e_i(\hat{S}_D[i], v_{\Phi^e,i}, D_{\hat{S}_D[i]}) &= v_{\Phi^e,i}(\{i\}) + \frac{1}{|S_D[i]|}\Bigg(v_{\Phi^e,i}(\hat{S}_D[i]) - \sum_{j\in S_D(i)} v_{\Phi^e,i}(\hat{S}_D[j]) - v_{\Phi^e,i}(\{i\})\Bigg)\\
&= v(\{i\}) + \frac{1}{|S_D[i]|}\Bigg(\alpha(\hat{S}_D[i]) - \sum_{j\in S_D(i)} v(\hat{S}_D[j]) - v(\{i\})\Bigg)\\
&= \Phi^e_i(N,v,D),
\end{aligned}$$

as desired for player i. It follows that the egalitarian remainder for each $\hat{S}_D[j]$, $j \in S_D(i)$, is the same in $(\hat{S}_D[i], v_{\Phi^e,i}, D_{\hat{S}_D[i]})$ and (N,v,D). So, using the definition of $(\hat{S}_D[i], v_{\Phi^e,i}, D_{\hat{S}_D[i]})$, we also have $\Phi^e_j(\hat{S}_D[i], v_{\Phi^e,i}, D_{\hat{S}_D[i]}) = \Phi^e_j(N,v,D)$ for each $j \in S_D(i)$. Continuing in this fashion for each subordinate of i, we reach the desired conclusion.

Finally, consider any $(N,v,D) \in \mathcal{CD}$, any $i \in C$ for some component C such that $S_D(i) \neq \emptyset$. Pick any $j \in S_D(i)$. On the one hand, by part (i) of Proposition 1, we have $\Phi^e_{\hat{S}_D[j]}(N,v,D) = \alpha(\hat{S}_D[j])$. On the other hand, from (3) and definition of $(N^i, v^i, D^i) \in \mathcal{CD}$, we deduce that $\Phi^e_i(N^i,v^i,D^i) = \Phi^e_i(N,v,D)$. To understand this equality, note that the egalitarian remainder for $\hat{S}_D[i]$ in (N,v,D) do not rely on proper coalitions of $\hat{S}_D[j]$, $j \in S_D(i)$ so that the computations from v give the same results as the computations from v^i for $\hat{S}_{D^i}[i]$. It is also true for each $j \in S_D(i)$: the egalitarian remainder for $\hat{S}_D[j]$ in (N,v,D) coincides with the egalitarian remainder $\alpha^i(\hat{S}^*_D[j])$ for the amalgamated player $\hat{S}^*_D[j]$ in (N^i,v^i,D^i). Indeed, take into account the following six facts: the previous fact on the egalitarian remainder for $\hat{S}_{D^i}[i]$; $p_{D^i}(\hat{S}^*_D[j]) = i$; $|S_D[i]| = |S_{D^i}[i]|$; for each $j \in S_D(i)$, $v^i(\{\hat{S}^*_D[j]\}) = v(\hat{S}_D[j])$, $\hat{S}_{D^i}[\hat{S}^*_D[j]] = \{\hat{S}^*_D[j]\}$, and $v^i(\{i\}) = v(\{i\})$. Then, we get:

$$\begin{aligned}
\alpha^i(\hat{S}^*_D[j]) &= v^i(\{\hat{S}^*_D[j]\}) + \frac{1}{|S_{D^i}[p_{D^i}(\hat{S}^*_D[j])]|}\Bigg(\alpha^i(\hat{S}_{D^i}[p_{D^i}(\hat{S}^*_D[j])]) - \sum_{k\in S_{D^i}(p_{D^i}(\hat{S}^*_D[j]))} v^i(\hat{S}_{D^i}[k]) - v^i(\{p^i_D(\hat{S}^*_D[j])\})\Bigg)\\
&= v(\hat{S}_D[j]) + \frac{1}{|S_D[i]|}\Bigg(\alpha(\hat{S}_D[i]) - \sum_{k\in S_D(i)} v(\hat{S}_D[k]) - v(\{i\})\Bigg)\\
&= \alpha(\hat{S}_D[j]).
\end{aligned} \tag{9}$$

Because each amalgamated player $\hat{S}^*_D[j]$, $j \in S_D(i)$, has no subordinate in the resulting rooted tree, we easily conclude that $\Phi^e_{\hat{S}^*_D[j]}(N^i, v^i, D^i) = \alpha^i(\hat{S}^*_D[j]) = \alpha(\hat{S}_D[j]) = \Phi^e_{\hat{S}_D[j]}(N, v, D)$. This proves that Φ^e satisfies subordinate amalgamation. ■

Combining component efficiency and subordinate fairness we obtain a characterization of the sequential equal surplus division on $\mathcal{CD}$.

Proposition 3 *The sequential equal surplus division Φ^e is the unique allocation rule that satisfies component efficiency and subordinate fairness on $\mathcal{CD}$.*

Proof. By Proposition 2, Φ^e satisfies component efficiency and subordinate fairness. Next, consider an allocation rule Φ that satisfies component efficiency and subordinate fairness on $\mathcal{CD}$. To show: Φ is uniquely determined on $\mathcal{CD}$. Pick any $(N, v, D) \in \mathcal{CD}$. First note that, for each $i \in N$, we have:

$$\Phi_i(N, v, D) = \Phi_{\hat{S}_D[i]}(N, v, D) - \sum_{j \in S_D(i)} \Phi_{\hat{S}_D[j]}(N, v, D).$$

It remains to prove that, for each $i \in N$, $\Phi_{\hat{S}_D[i]}(N, v, D)$ is uniquely determined by component efficiency and subordinate fairness.Pick any component C of (N, D). We proceed by induction on the depth of number q of the players in C.
Initial step: If $q = 0$, then player $i = r_C$. By component efficiency, we get:

$$\Phi_{\hat{S}_D[r_C]}(N, v, D) = \Phi_C(N, v, D) = v(C).$$

Thus, $\Phi_{\hat{S}_D[r_C]}(N, v, D)$ is uniquely determined.
Induction hypothesis: Assume that the assertion is true for each player $i \in C$ whose depth is less or equal to q.
Induction step: Pick any player $i \in C$ whose depth is equal to $q+1$. By subordinate fairness and component efficiency of Φ, we have:

$$\Phi_{p_D(i)}(N, v, D) - v(\{p_D(i)\}) = \Phi_{\hat{S}_D[i]}(N, v, D) - v(\hat{S}_D[i]).$$

Denote this quantity by δ. By subordinate fairness and component efficiency of Φ, we also obtain:

$$\forall j \in S_D(p_D(i)), \quad \Phi_{\hat{S}_D[j]}(N, v, D) = v(\hat{S}_D[j]) + \delta.$$

On the other hand, we have $\Phi_{p_D(i)}(N, v, D) = v(\{p_D(i)\}) + \delta$. Summing all these equalities, we get:

$$\Phi_{\hat{S}_D[p_D(i)]}(N, v, D) = \sum_{j \in S_D(p_D(i))} v(\hat{S}_D[j]) + v(\{p_D(i)\}) + |S_D[p_D(i)]|\delta.$$

By the induction hypothesis, $\Phi_{\hat{S}_D[p_D(i)]}(N, v, D)$ is uniquely determined. As a consequence, the parameter δ is uniquely determined. This gives the result for i since $i \in S_D(p_D(i))$ and so $\Phi_{\hat{S}_D[i]}(N, v, D) = v(\hat{S}_D[i]) + \delta$. Because (N, v, D) and C have been chosen arbitrarily, conclude that Φ is uniquely determined on $\mathcal{CD}$. ■

The next proposition provides an alternative characterization of the sequential equal surplus division.

Proposition 4 *The sequential equal surplus division Φ^e is the unique allocation rule that satisfies standardness, star fairness, subordinate consistency and subordinate amalgamation on $\mathcal{CD}$.*

In order to prove the above statement, we need two intermediary results. The first one states that standardness, subordinate fairness and component efficiency are not logically independent on $\mathcal{CD}$. The second one establishes that if an allocation rule satisfies component efficiency, standardness and star fairness, then, for each $(N, v, D) \in \mathcal{CD}$ containing a component C inducing an outward pointing star, it holds that $\Phi_i(N, v, D) = \Phi^e_i(N, v, D)$ for each $i \in C$.

Lemma 1 *If an allocation rule Φ satisfies standardness and subordinate consistency on $\mathcal{CD}$, then it also satisfies component efficiency on $\mathcal{CD}$.*

Proof. Consider any allocation rule Φ satisfying standardness and subordinate consistency on $\mathcal{CD}$. Pick any $(N, v, D) \in \mathcal{CD}$, and any component C of (N, D). We distinguish three cases according to the size of C.
Case 1 $|C| = 1$. We introduce two subcases.

Case 1.1 Assume that $N = \{i\}$ so that $D = \emptyset$ and $C = N$. From $(\{i\}, v, \emptyset)$ construct the forest directed graph TU-games $(\{i, j\}, w, \{(j, i)\})$, where $j \in \mathbb{N} \setminus \{i\}$, the worth $w(\{j\})$ is chosen arbitrarily in $\mathbb{R}$, $w(\{i\}) = v(\{i\})$, and $w(\{i, j\}) = w(\{i\}) + w(\{j\})$. Applying standardness, we get:

$$\Phi_i(\{i, j\}, w, \{(j, i)\}) = v(\{i\}) \text{ and } \Phi_j(\{i, j\}, w, \{(j, i)\}) = w(\{j\}).$$

Because i has no successor in $(\{i, j\}, \{(j, i)\})$, the reduced TU-game constructed from i's subordinates including herself is given by $(\{i\}, w_{\Phi,i}, \emptyset)$, where

$$w_{\Phi,i}(\{i\}) = w(\{j, i\}) - \Phi_j(\{i, j\}, w, \{(j, i)\}) = v(\{i\}).$$

Therefore, $(\{i\}, w_{\Phi,i}, \emptyset)$ and $(\{i\}, v, \emptyset)$ coincide. By subordinate consistency, we obtain the desired result:

$$\Phi_i(\{i\}, v, \emptyset) = \Phi_i(\{i\}, w_{\Phi,i}, \emptyset) = \Phi_i(\{i, j\}, w, \{(j, i)\}) = w(\{i\}) = v(\{i\}).$$

Case 1.2 Assume that N contains at least two players. First note the following fact: for any component of a forest digraph, it is allowed to construct the reduced TU-game on C by picking the root of the subtree induced by this component. Such a reduced TU-game represents a situation where all the players except the members of C leave the game with their payoffs, while all the players in C continue to play the game. So, consider again $C = \{i\}$, and construct the reduced TU-game $(\{i\}, v_{\Phi,i}, \emptyset)$. By definition of $v_{\Phi,i}$, we have$v_{\Phi,i}(\{i\}) = v(\{i\})$. Thus, by subordinate consistency and **Case 1.1**, we obtain:

$$\Phi_i(N, v, D) = \Phi_i(\{i\}, v_{\Phi,i}, \emptyset) = \Phi_i(\{i\}, v, \emptyset) = v(\{i\}),$$

as desired.
Case 2 $|C| = 2$. It suffices to apply standardness.

Case 3 C contains more than two elements. Pick any $i \in C$ such that $S_D(i) = \emptyset$. Such a player exists in a rooted tree. The reduced TU-game $(\hat{S}_D[i], v_{\Phi,i}, D_{\hat{S}_D[i]})$ is such that:

$$\hat{S}_D[i] = \{i\}, \quad D_{\hat{S}_D[i]} = \emptyset, \text{ and } v_{\Phi,i}(\hat{S}_D[i]) = v(\hat{S}_D[i]) - \Phi_{C \setminus \hat{S}_D[i]}(N, v, D).$$

Thus, by subordinate consistency, we have:

$$\Phi_i(N, v, D) = \Phi_i(\{i\}, v_{\Phi,i}, \emptyset).$$

By **Case 1.1**, we know that if the player set of a forest digraph TU-game is a singleton, then Φ is component efficient. Applying this result to $(\{i\}, v_{\Phi,i}, \emptyset)$ and using the definition of $v_{\Phi,i}$, we get:

$$\Phi_i(N, v, D) = \Phi_i(\{i\}, v_{\Phi,i}, \emptyset) = v_{\Phi,i}(\{i\}) = v(C) - \Phi_{C \setminus \{i\}}(N, v, D).$$

Therefore, Φ is component efficient. This completes the proof of the lemma. ■

Lemma 2 *If an allocation rule Φ satisfies component efficiency, standardness and star fairness on $\mathcal{CD}$, then for each $(N, v, D) \in \mathcal{CD}$ containing a component C inducing an outward pointing star, it holds that $\Phi_i(N, v, D) = \Phi_i^e(N, v, D)$ for each $i \in C$.*

Proof. By Proposition 2, Φ^e satisfies component efficiency, standardness and star fairness on $\mathcal{CD}$. Consider any allocation rule Φ that satisfies component efficiency, standardness and star fairness on $\mathcal{CD}$ and pick any $(N,v,D) \in \mathcal{CD}$. We proceed by induction on the number of elements of component C.

Initial step: If $C = \{j\}$ for some $j \in N$, then by component efficiency, $\Phi_j(N,v,D) = \Phi^e_j(N,v,D)$. If $N = \{i,j\}$, then by standardness, $\Phi_i(N,v,D) = \Phi^e_i(N,v,D)$ and $\Phi_j(N,v,D) = \Phi^e_j(N,v,D)$.

Induction hypothesis: Assume that the statement is true for C with at most $q \in \mathbb{N}$ players.

Induction step: Assume that C contains $q+1$ players, that r_C is the root of this outward pointing star, and delete the directed edge (r_C, i) for some $i \in C \setminus \{r_C\}$. The component C splits into two new components: $C \setminus \{i\}$ and $\{i\}$. Thus, $C \setminus \{i\}$ contains exactly q elements. By star fairness and the induction hypothesis, we have:

$$\begin{aligned}\forall j,k \in C \setminus \{i\}, \quad \Phi_j(N,v,D) - \Phi_k(N,v,D) &= \Phi_j(N,v,D \setminus \{(r_C,i)\}) - \Phi_k(N,v,D \setminus \{(r_C,i)\})\\ &= \Phi^e_j(N,v,D \setminus \{(r_C,i)\}) - \Phi^e_k(N,v,D \setminus \{(r_C,i)\})\\ &= \Phi^e_j(N,v,D) - \Phi^e_k(N,v,D).\end{aligned}$$

Thus, there is a constant $\delta \in \mathbb{R}$ such that:

$$\forall j \in C \setminus \{i\}, \quad \Phi_j(N,v,D) - \Phi^e_j(N,v,D) = \delta.$$

Because these equalities remain valid whatever the chosen directed edge, we have:

$$\forall j \in C, \quad \Phi_j(N,v,D) - \Phi^e_j(N,v,D) = \delta.$$

By component efficiency, $\Phi_C(N,v,D) = \Phi^e_C(N,v,D) = v(C)$ and so $\delta = 0$. Thus, for each $j \in C$, $\Phi_j(N,v,D) = \Phi^e_j(N,v,D)$. ■

Proof. (of Proposition 4). By Proposition 2, Φ^e satisfies standardness, star fairness, subordinate consistency and subordinate amalgamation on $\mathcal{CD}$. Next, pick any allocation rule Φ that satisfies standardness, star fairness, subordinate consistency and subordinate amalgamation on $\mathcal{CD}$. Consider any $(N,v,D) \in \mathcal{CD}$ and any component C. To show: for each $i \in C$, $\Phi_i(N,v)$ is uniquely determined. We proceed by induction on the depth of the rooted tree induced by C.
Initial step: Assume that the depth of the rooted tree is equal to zero or one. In such case, the rooted tree is a star. By Lemma 1, we know that Φ also satisfies component efficiency. As a consequence, Lemma 2 implies that the payoffs $(\Phi_i(N,v,D))_{i \in N}$ are uniquely determined.
Induction hypothesis: Assume that the payoffs $(\Phi_i(N,v,D))_{i \in N}$ are uniquely determined for each rooted tree whose depth is q.
Induction step: Assume that the depth of the rooted tree is $q+1 > 1$. From (N,v,D) and the root r_C, construct the companion rooted forest TU-game $(N^{r_C}, v^{r_C}, D^{r_C})$ where r_C's subordinates are amalgamated. By construction, r_C is now the root of an outward pointing star, *i.e.* we have $(r_C, \hat{S}^*_D[j]) \in D^{r_C}$ for each $j \in S_D(r_C)$. Since by assumption there is at least one player with no successor whose depth is strictly greater than one, $(N^{r_C}, v^{r_C}, D^{r_C})$ does not coincide with (N,v,D). By Lemma 2, for each $j \in S_D(r_C)$, the payoff $\Phi_{\hat{S}^*_D[j]}(N^{r_C}, v^{r_C}, D^{r_C})$ is uniquely determined. By subordinate amalgamation, we get:

$$\forall j \in S_D(r_C), \quad \Phi_{\hat{S}^*_D[j]}(N^{r_C}, v^{r_C}, D^{r_C}) = \Phi_{S_D[j]}(N,v,D),$$

which means that the total payoff of each coalition $\hat{S}_D[j]$, $j \in S_D(r_C)$, is uniquely determined in the original game (N,v,D). Using component efficiency, we obtain that player r_C's payoff is uniquely determined in (N,v,D):

$$\Phi_{r_C}(N,v,D) = v(C) - \sum_{j \in S_D(r_C)} \Phi_{\hat{S}_D[j]}(N,v).$$

It remains to show that the payoff of each other player than r_C in C is uniquely determined. Consider any $j \in S_D(r_C)$ and construct the reduced TU-game $(\hat{S}_D[j], v_{\Phi,j}, D_{\hat{S}_D[j]}) \in \mathcal{CD}$. By definition of the reduced TU-game, we have:

$$\forall S \subset \hat{S}_D[j], \quad v_{\Phi,j}(S) = v(S) \text{ and } v_{\Phi,j}(\hat{S}_D[j]) = v(C) - \Phi_{C \setminus \hat{S}_D[j]}(N, v, D).$$

Note that $\Phi_{C \setminus \hat{S}_D[j]}(N, v, D)$ is uniquely determined by the previous step so that $(\hat{S}_D[j], v_{\Phi,j}, D_{\hat{S}_D[j]})$ is well defined. By construction, the depth of each subtree $(\hat{S}_D[j], D_{\hat{S}_D[j]})$, $j \in S_D(r_C)$, is at most q. By the induction hypothesis, for each $j \in S_D(r_C)$ and $i \in \hat{S}_D[j]$, $\Phi_i(\hat{S}_D[j], v_{\Phi,j}, D_{\hat{S}_D[j]})$ is uniquely determined. By subordinate consistency we get:

$$\forall j \in S_D(r_C), \forall i \in \hat{S}_D[j], \quad \Phi_i(\hat{S}_D[j], v_{\Phi,j}, D_{\hat{S}_D[j]}) = \Phi_i(N, v, D),$$

which means that the payoff of each subordinate of r_C is uniquely determined in (N, v, D). This completes the proof. ■

Remark Note that on the subclass of rooted forest TU-games where the rooted forests are such that the components are directed paths, Φ^e is characterized by standardness, subordinate amalgamation, and subordinate consistency.

Logical independence of the properties of Proposition 3 and Proposition 4 are studied in Appendix 8.1.

6 Sharing a river with bifurcations

Ambec and Sprumont (2002) model the problem of water allocation in which a group of agents is located along a river from upstream to downstream as a cooperative TU-game. van den Brink *et al.* (2007) show this river TU-game can be naturally embedded into the framework of directed graph TU-games where the directed graph is a directed path. Khmelnistaya (2010) extends the model to the cases where the river is either a rooted tree (the river possesses one source and some bifurcations) or a sink tree (the river has multiple sources and one sink). The model is as follows.

Let N be a finite set of agents, located along a river. At the location of each $i \in N$, rainfall and inflow from tributaries increase the total river flow by $e_i \in \mathbb{R}_+$. The most upstream agent along the river is called its spring. The river may have many bifurcations, *i.e.* it is shaped like a tree (N, D) rooted at its spring. Figure 4 provides an example with inflows and root 1.

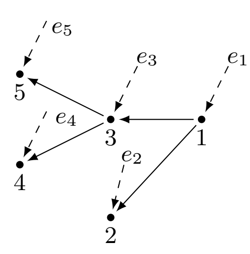


Figure 4: A river with bifurcations

The agents value both water and money and are endowed with a quasi-linear utility function $u_i : \mathbb{R}_+ \times \mathbb{R} \longrightarrow \mathbb{R}$ that assigns to each pair $(x_i, t_i) \in \mathbb{R}_+ \times \mathbb{R}$ the utility $u_i(x_i, t_i) = b_i(x_i) + t_i$, where $t_i \in \mathbb{R}$ is a monetary transfer. A welfare distribution is a pair (x, t) where $x \in \mathbb{R}^n_+$ is a consumption plan and $t = (t_i)_{i \in N} \in \mathbb{R}^n$ is a compensation scheme satisfying the budget constraint $\sum_{i \in N} t_i \leq 0$. The welfare distribution (x^*, t) is Pareto optimal if and only if x^* is the optimal consumption plan and the budget constraint is balanced, *i.e.* if transfers add to zero. Each agent i receives the payoff $z_i = u_i(x_i^*, t_i)$, and the sum of these payoffs is equal to the optimal social welfare $\sum_{i\in N} b_i(x_i^*)$. The problem is to find a fair distribution of this social welfare. Ambec and Sprumont (2002) formulate the following assumption about the benefit functions.

Assumption 1. The water inflow e_1 is strictly positive. Each benefit function b_i is strictly increasing, strictly concave, and differentiable on $\mathbb{R}_{++}$, $b_i(0) = 0$, and the first derivative $b_i^{(1)}(x_i)$ tends to infinity as x_i vanishes.

Define as $B_D(i)$ the, possibly empty, set of agent i's brothers in (N, D), *i.e.* the other agents in i's component which have the same predecessor as i. Formally, $B_D(i) = \{j \in N \setminus \{i\} : p_D(j) = p_D(i)\}$. A consumption plan $x = (x_i)_{i\in N} \in \mathbb{R}^n_+$ satisfies the following constraints:

$$\forall k \in N, \sum_{j\in \hat{P}_D[k]} x_j \leq \sum_{j\in \hat{P}_D[k]} e_j, \text{ and } \sum_{j\in \hat{P}_D[k]\cup B_D(k)} x_j \leq \sum_{j\in \hat{P}_D[k]\cup B_D(k)} e_j. \tag{10}$$

The first constraints in (10) indicate that each agent $k \in N$ consumes at most the sum of the water inflow at its location and the water inflows not consumed by its upstream agents (its superiors in the rooted tree). The second constraints in (10) indicate that the cumulated water consumption along the river does not exceed at any location the total available inflow. In particular, these conditions specify that if an agent k decides to let pass some water downstream, then this amount of water is split among the several continuing branches of the river indicated by country k's brothers. Under Assumption 1 there is a unique optimal consumption plan.

The worth of each connected coalition S is given by the social welfare that it can secure for itself without the cooperation of the other agents. Therefore, for any connected coalition S, the worth $v(S)$ is given by:

$$v(S) = \sum_{k\in S} b_k(x_k^S),$$

where $x^S = (x_i^S)_{i\in S}$ solves

$$\max \sum_{k\in S} b_k(x_k) \text{ s.t.}$$

$$\forall k \in S, \sum_{j\in \hat{P}_D[k]\cap S} x_j \leq \sum_{j\in \hat{P}_D[k]\cap S} e_j, \quad \sum_{j\in \hat{P}_D[k]\cup B_D(k)\cap S} x_j \leq \sum_{j\in \hat{P}_D[k]\cup B_r(k)\cap S} e_j. \tag{11}$$

Assumption 1 ensures that this program has a unique solution. Recall that a disconnected coalition S admits a unique partition into components. By definition of a TU-game, the agents belonging to $N\backslash S$ do not cooperate with S and act non-cooperatively. Since the benefit functions are strictly increasing in the water consumed, each agent located between two components of S will consume all the water inflow entering its location. Therefore, a component of S will never receive the water left over by another component located upstream. It follows that the worth of a non-connected coalition S is the sum of the worths of its components. Note also that a river TU-game is superadditive. By allocating a payoff $\Phi_i(N, v, D)$ to each $i \in N$, we determine a compensation scheme t as follows: for each agent $i \in N$, $t_i = \Phi_i(N, v, D) - b_i(x_i^*)$. If $\sum_{i\in N} \Phi_i(N, v, D) = v(N)$, then the optimal social welfare is integrally allocated among the agents and t is budget balanced. The difficulty is thus to find a fair agreement on the allocation of the social welfare resulting from an optimal consumption plan, which in turn determines monetary compensations. One can slightly extend this definition of the river TU-game by assuming a collection of independent rivers so that N splits into several components. It follows that the class of river TU-games (N, v, D) is a subset of $\mathcal{CD}$ where (N, v) is superadditive.

Khmelnistaya (2010) applies the hierarchical outcome given by (4) to the river TU-games. She characterizes this solution on $\mathcal{CD}$ by using component efficiency and the following link deletion property.

Successor equivalence For each $(N, v, D) \in \mathcal{CD}$, each $(i, j) \in D$, it holds that:

$$\forall k \in \hat{S}_D[j], \quad \Phi_k(N, v, D \setminus \{(i, j)\}) = \Phi_k(N, v, D).$$

When (i, j) is deleted from D, the component C containing i and j splits into two new components. Successor equivalence states that the payoffs of j and all his or her subordinates do not depend on the presence the directed edge from i to j. This characterization also holds on the subclass of river TU-games since if (N, v, D) is a river TU-game, then $(N, v, D\setminus\{(i, j)\})$ is a river TU-game. Successor equivalence is a generalization to rooted forests of the lower equivalence property introduced by van den Brink *et al.* (2007) in order to characterize solution (5) when the river is depicted as collection of directed paths.

Substituting successor equivalence by subordinate fairness in the characterization provided by Khmelnistaya (2010), we get the sequential equal surplus division rule by Proposition 3. Note also that this characterization remains valid on the class of river TU-games because if (N, v, D) is a river TU-game, then $(N, v, D \setminus E_i)$ is a river TU-game as well. This is no longer the case when we apply Proposition 4. The reason is that we need to construct associated rooted forest TU-games in order to apply subordinate consistency or subordinate amalgamation, which are not necessarily river TU-games. This fact is illustrated in Appendix 8.2.

We conclude this section by suggesting a natural extension of the sequential equal surplus division that accounts for the specific features of the river sharing problems. In fact, it is perhaps possible to elaborate a more realistic distribution of the surplus generated by the cooperation of several agents. Consider again a river represented by the rooted tree given in Figure 1 and the distribution of the surplus

$$v(\{1, 2, 3, 4\}) - v(\{1\}) - v(\{2\}) - v(\{3, 4\})$$

created by the three negotiating coalitions $\{1\}$, $\{2\}$ and $\{3, 4\}$. Further, suppose that the benefit functions yield an optimal consumption plan x^* such that $x^*_3 + x^*_4 = e_3 + e_4$. This extreme situation raises the question of which (asymmetrical) shares of the surplus should be distributed. In fact, the equality $x^*_3 + x^*_4 = e_3 + e_4$ means that the optimal total water consumption of the branch of the river on which agents 3 and 4 are located coincides with its total inflow. In other words, the achievement of the optimal welfare does not require the spring of the river to let pass any water to its downstream agent 3. As such, the branch of the river on which agents 3 and 4 are located does not contribute any unit to the surplus. Therefore, it makes sense to deprive coalition $\{3, 4\}$ of any share of the surplus, leaving this coalition with a remainder equal to $v(\{3, 4\})$. This example is rather extreme but can be easily generalized to any situation. It is enough to require, at each step, that each negotiating coalition gets a share of the surplus which is proportional to the extra amount of water it consumes in order to reach the optimal consumption plan. This intuitive aspect of the river sharing problem can be formalized as follows.

In the definition (3) of Φ^e_i for agent i, recall that the surplus

$$\alpha(\hat{S}_D[i]) - \sum_{j\in S_D(i)} v(\hat{S}_D[j]) - v(\{i\})$$

is equally redistributed among i and all coalitions $\hat{S}_D[j]$, $j \in S_D(i)$, formed by i's downstream agents, *i.e.* is equally redistributed among $|S_D[i]|$ negotiating coalitions. Now, for each downstream neighbor $j \in S_D(i)$, denote by $e^*_{i,j}$ the amount of water that i lets pass to agent j with respect to the optimal consumption plan $x^* = (x^*_i)_{i\in N}$. Furthermore, define $e^*_i \geq e_i \geq 0$ the total amount of water which is available at i's location with respect to x^*. More specifically:

$$e^*_i = x^*_i + \sum_{j\in S_D(i)} e^*_{i,j} > 0,$$

where the strict inequality comes from the fact that x^*_i is strictly positive under Assumption 1. The remainder of $\hat{S}_D[r_C]$ remains $v(C)$ for the spring r_C of component C. Next, if each branch of the river $\hat{S}_D[j]$, $j \in S_D(i)$, obtains a share of the surplus proportional to the quantity $e^*_{i,j}$, then its weighted remainder becomes:

$$\alpha^w(\hat{S}_D[j]) = v(\hat{S}_D[j]) + \frac{e^*_{i,j}}{e^*_i}\left(\alpha^w(\hat{S}_D[i]) - \sum_{k\in S_D(i)} v(\hat{S}_D[k]) - v(\{i\})\right)$$

so that the share of the surplus which goes to agent i is

$$\frac{x_i^*}{e_i^*}\Bigg(\alpha^w(\hat{S}_D[i]) - \sum_{k\in S_D(i)} v(\hat{S}_D[k]) - v(\{i\})\Bigg).$$

The new allocation rule $\Phi^{w,e}$ constructed from these weighted remainders assigns to every $i \in N$ in each river TU-game (N, v, D) the payoff:

$$\Phi_i^{w,e}(N, v, D) = v(\{i\}) + \frac{x_i^*}{e_i^*}\Bigg(\alpha^w(\hat{S}_D[i]) - \sum_{j\in S_D(i)} v(\hat{S}_D[j]) - v(\{i\})\Bigg).$$

Proceeding as in the proof of Proposition 1, we conclude that $\Phi^{w,e}$ satisfies points (i), (ii) and (iii) of Proposition 1 and so is component efficient on the class of all river TU-games. While $\Phi^{w,e}$ does not satisfy subordinate fairness, it satisfies the following weighted version of this axiom: for each river TU-game (N, v, D) and each component C with at least two agents and each $i \in C$ such that $S_D(i) \neq \emptyset$, it holds that:

$$\forall j \in S_D(i), \quad e_{i,j}^*\Bigg(\Phi_i(N, v, D) - \Phi_i(N, v, D \setminus E_i)\Bigg) = x_i^*\Bigg(\Phi_{\hat{S}_D[j]}(N, v, D) - \Phi_{\hat{S}_D[j]}(N, v, D \setminus E_i)\Bigg).$$

In particular, if $e_{i,j}^* = 0$, then this property implies that $\Phi_{\hat{S}_D[j]}(N, v, D) = \Phi_{\hat{S}_D[j]}(N, v, D \setminus E_i)$ because $x_i^* > 0$. It is indeed true for $\Phi^{w,e}$ since, in such a case, $\alpha^w(\hat{S}(D[j])) = v(\hat{S}_D[j])$ so that $\Phi^{w,e}_{\hat{S}_D[j]}(N, v, \setminus E_i) = v(\hat{S}_D[j]) = \Phi^{w,e}_{\hat{S}_D[j]}(N, v, D)$ by point (i) of Proposition 1. This feature is consistent with the extreme example considered above. Straightforward modifications of the proof of Proposition 3 yield the following result.

Proposition 5 *$\Phi^{w,e}$ is the unique allocation rule that satisfies component efficiency and the weighted version of subordinate fairness on the class of all river TU-games.*

7 Conclusion

In this article we consider rooted forest TU-games and provide two characterizations of a new allocation rule, called the sequential surplus division rule. This rule is constructed sequentially from the root of the tree on each component of the rooted forest. We provide two characterizations of this allocation rule and one extension to the problem of sharing a river with bifurcations. Khmelnitskaya and Talman (2014) study the more general class of acyclic directed graph TU-games, and introduce axiomatically new allocation rules, called web values. Each of these allocation rules are constructed with respect to a chosen coalition of players that is assumed to be an anti-chain in the directed graph and is considered as a management team. In case the graph is a rooted tree and the management team is formed by the player located at the root, the web value reduces to the hierarchical outcome (Demange, 2004, Khmelnitskaya, 2010). It would be a challenging issue to examine the way in which the sequential equal surplus division can be extended to the class of acyclic directed graph TU-games.

Our solution can also be seen as the generalization of the individual standardized remainder vectors proposed by Ju, Borm and Ruys (2007) for the class of all TU-games. A next step in the line of research examined by our article would be to consider games associated with undirected rooted forests, and to compute the average, over all its induced rooted forests, of our solution. The resulting solution would parallel, for the individual standardized remainder vectors, what did Herings, van der Laan and Talman (2008) for the hierarchical vectors in Demange (2004). However, characterizing such a solution would be tedious since it violates most of the axioms invoked in our article.

8 Appendix

8.1 Logical independence

The logical independence of the properties used in Proposition 3 is illustrated as follows.

- Define the allocation rule Φ as follows. For each $(N, v, D) \in \mathcal{CD}$ and each component C of (N, D), Φ coincides with the equal division rule ED on (C, v_C, D_C):

$$\forall i \in C, \quad \Phi_i(N, v, D) = \text{ED}_i(C, v_C, D_C) = \frac{v(C)}{|C|}.$$

  This allocation rule satisfies component efficiency but violates subordinate fairness.

- Consider the allocation rule Φ which assigns to each player in each (N, v, D) her Shapley value (Shapley, 1953) in (N, v):

$$\forall i \in N, \quad \Phi_i(N, v, D) = \sum_{S \in 2^N : S \ni i} \frac{(|N| - |S|)!(|S| - 1)!}{|N|!}(v(S) - v(S \setminus \{i\})).$$

  It is well known that this allocation rule is efficient, and so can not be component efficient. Nonetheless, it trivially satisfies subordinate fairness.

The logical independence of the properties used in Proposition 4 is illustrated as follows.

- Define the allocation rule Φ as follows. For each $(N, v, D) \in \mathcal{CD}$ and each component C of (N, D), Φ coincides with the equal surplus division rule ESD on (C, v_C, D_C):

$$\forall i \in C, \quad \Phi_i(N, v, D) = \text{ESD}_i(C, v_C, D_C) = v(\{i\}) + \frac{v(C) - \sum_{j \in C} v(\{j\})}{|C|}.$$

  This allocation rule satisfies standardness, star fairness, subordinate consistency but violates subordinate amalgamation.

- Consider the allocation rule Φ constructed as follows on $\mathcal{CD}$: Φ distributes to the root and each successor of the root in a component the payoff resulting of the application of the equal surplus division rule in the game where the subordinates of the root are amalgamated. The other members of the component receive a null payoff. Precisely, for each $(N, v, D) \in \mathcal{CD}$ and each component C of (N, D), we have:

$$\forall i \in S_D[r_C], \quad \Phi_i(N, v, D) = \text{ESD}_i(N^{r_C}, v^{r_C}, D^{r_C}) \text{ and } \forall i \in C \setminus S_D[r_C], \quad \Phi_i(N, v, D) = 0.$$

  This allocation rule satisfies standardness, star fairness, subordinate amalgamation but violates subordinate consistency.

- Consider now the allocation rule Φ defined as follows. Pick any $(N, v, D) \in \mathcal{CD}$, any component C of (N, D), any $i \in C$, and define the *equal remainder* β for coalition $\hat{S}_D[i]$ as:

$$\beta(\hat{S}_D[i]) = \begin{cases} v(C) & \text{if } i = r_C, \\ \frac{1}{|S_D[p_D(i)]|}\beta(\hat{S}_D[p_D(i)]) & \text{otherwise.} \end{cases}$$

  The allocation rule Φ is given by:

$$\Phi_i(N, v, D) = \frac{1}{|S_D[i]|}\beta(\hat{S}_D[i]).$$

  It satisfies star fairness, subordinate amalgamation, subordinate consistency but violates standardness.

- Pick any $(N,v,D) \in \mathcal{CD}$, any component C of (N,D), any $i \in C$, and define the *root-advantaged remainder* γ for coalition $\hat{S}_D[i]$ as:

$$\gamma(\hat{S}_D[i]) = \begin{cases} v(C) & \text{if } i = r_C, \\ v(\hat{S}_D[i]) + \dfrac{1}{2|S_D(p_D(i))|}\Big(\gamma(\hat{S}_D[p_D(i)]) - \sum_{j \in S_D(p_D(i))} v(\hat{S}_D[j]) - v(\{p_D(i)\})\Big) & \text{otherwise.} \end{cases}$$

The allocation rule Φ given by

$$\Phi_i(N,v,D) = v(\{i\}) + \frac{1}{2}\Big(\gamma(\hat{S}_D[i]) - \sum_{j \in S_D(i)} v(\hat{S}_D[j]) - v(\{i\})\Big),$$

satisfies standardness, subordinate amalgamation, subordinate consistency but violates star fairness.

8.2 The river TU-games

We proceed in two steps. In a first step, we show that the class of river TU-games does not coincide with the class of rooted forest TU-games with positive and superadditive underlying TU-games. In a second step, we show that the characterization provided in Proposition 4 does not hold on the class of river TU-games.

Step 1 Pick any $(N,v,D) \in \mathcal{CD}$ such that $N = \{1,2,3\}$, $D = \{(1,2),(2,3)\}$, and v is such that $v(\{i\}) = 1$ for each $i \in N$, $v(\{1,2\}) = v(\{2,3\}) = v(\{1,3\}) = 2$. Note that the TU-game (N,v) is superadditive if and only if $v(\{1,2,3\}) \geq 3$. It is sufficient to show that $(N,v,D) \in \mathcal{CD}$ is a river TU-game if and only if $v(\{1,2,3\}) = 3$. So, assume that (N,v,D) is a river TU-game. At each location $i \in N$, there is an inflow of water $e_i \geq 0$, where $e_1 > 0$. By Assumption 1 and definition of $v(\{i\})$, $b_i(e_i) = v(\{i\}) = 1$ for each $i \in N$. By the constraints (11), for the optimal consumption plan (x_i, x_{i+1}) for coalition $\{i, i{+}1\}$, $i \in \{1,2\}$, we have $x_i \leq e_i$ and $x_i{+}x_{i+1} \leq e_i{+}e_{i+1}$. Because $v(\{i,i+1\}) = 2$, we get $b_i(x_i) + b_{i+1}(x_{i+1}) = v(\{i,i+1\}) = 2 = b_i(e_i) + b_{i+1}(e_{i+1})$, $i \in \{1,2\}$. Because there is a unique optimal plan under Assumption 1, we have $x_i = e_i$ and $x_{i+1} = e_{i+1}$. This implies (see Ambec and Sprumont, 2002, Béal *et al.*, 2013) that:

$$\forall i \in \{1,2\}, \quad b_i^{(1)}(e_i) \geq b_{i+1}^{(1)}(e_{i+1}). \tag{12}$$

Consider now the unique optimal consumption plan (x_1,x_2,x_3) for coalition N. By Assumption 1, $x_1 \leq e_1$, $x_1 + x_2 \leq e_1 + e_2$ and $x_1 + x_2 + x_3 \leq e_1 + e_2 + e_3$. We show that $x_2 = e_2$.

(i) If (x_1,x_2,x_3) is the optimal consumption plan for coalition N, then $x_2 \geq e_2$.

Assume that (x_1,x_2,x_3) is the optimal consumption plan for coalition N. For the sake of contradiction, assume that $x_2 < e_2$. Two cases arise. If $x_2{+}x_3 \leq e_2{+}e_3$, then we easily conclude by the previous argument on coalition $\{2,3\}$ that the consumption plan (x_1,e_2,e_3) is feasible and optimal, a contradiction. If $x_2{+}x_3 > e_2{+}e_3$, then $x_3 > e_3 + \delta$ where $\delta = e_2 - x_2 > 0$. Consider the feasible consumption plan $(x_1, e_2, x_3 - \delta)$. We have:

$$b_2(e_2) - b_2(x_2) = b_2(e_2) - b_2(e_2 - \delta) > b_3(e_3 + \delta) - b_3(e_3) > b_3(x_3) - b_3(x_3 - \delta),$$

where the first equality follows from the definition of δ; the first inequality follows from the fact that (e_2,e_3) is the unique optimal consumption plan for coalition $\{2,3\}$; the second inequality follows from $x_3 > e_3 + \delta$ and the strict concavity of b_3. It follows that:

$$b_1(x_1) + b_2(e_2) + b(x_3 - \delta) > b_1(x_1) + b_2(x_2) + b_3(x_3),$$

which contradicts that (x_1,x_2,x_3) is the optimal consumption plan for coalition N. Therefore, if (x_1,x_2,x_3) is the optimal consumption plan for coalition N, then we necessarily have $x_2 \geq e_2$.

(ii) If (x_1, x_2, x_3) is the optimal consumption plan for coalition N, then $x_2 \leq e_2$.

Assume that (x_1, x_2, x_3) is the optimal consumption plan for coalition N. For the sake of contradiction, assume that $x_2 > e_2$. Set $\delta = x_2 - e_2 > 0$, and consider the consumption plan $(x_1 + \delta, e_2, x_3)$. It is feasible since, under Assumption 1, $x_1 + x_2 \leq e_1 + e_2$, which in turn implies $x_1 + \delta \leq e_1$. We have:

$$b_2(x_2) - b_2(e_2) = b_2(e_2 + \delta) - b_2(e_2) < b_1(e_1) - b_1(e_1 - \delta) < b_1(x_1 + \delta) - b_1(x_1),$$

where the first equality follows from the definition of δ; the first inequality follows from the fact that (e_1, e_2) is the unique consumption plan for coalition $\{1, 2\}$; the second inequality follows from $x_1 + \delta \leq e_1$ and the strict concavity of b_1.

$$b_1(x_1 + \delta) + b_2(e_2) + b(x_3) > b_1(x_1) + b_2(x_2) + b_3(x_3),$$

a contradiction. Therefore, if (x_1, x_2, x_3) is the optimal consumption plan for coalition N, then we necessarily have $x_2 \leq e_2$.

From (i) and (ii), deduce that the unique optimal consumption plan for N is of the form (x_1, e_2, x_3). It remains to prove that agent 1 has no incentive to pass water to agent 3, *i.e.* (e_1, e_2, e_3) is the optimal consumption plan for N. By (12), we have $b_1^{(1)}(e_1) \geq b_3^{(1)}(e_3)$. Because b_3 is increasing in x_3, player will consume at least e_3. By strict concavity of b_1, we have:

$$\forall \delta \in]0, e_1[, \quad b_1(e_1) - b_1(e_1 - \delta) = \int_{e_1-\delta}^{e_1} b_1^{(1)}(t)dt > \int_{e_1-\delta}^{e_1} b_1^{(1)}(e_1)dt = b_1^{(1)}(e_1)\delta.$$

In a similar way, we have:

$$\forall \delta > 0, \quad b_3(e_3 + \delta) - b_3(e_3) = \int_{e_3}^{e_3+\delta} b_3^{(1)}(t)dt < \int_{e_3}^{e_3+\delta} b_3^{(1)}(e_3)dt = b_3^{(1)}(e_3)\delta.$$

From these strings of inequalities and the fact that $b_1^{(1)}(e_1) \geq b_3^{(1)}(e_3)$, we deduce:

$$\forall \delta \in]0, e_1[, \quad b_3(e_3 + \delta) + b_2(e_2) + b_1(e_1 - \delta) < b_1(e_1) + b_2(e_2) + b_3(e_3).$$

Conclude that agent 1 has no incentive to pass water to agent 3 so that (e_1, e_2, e_3) is the optimal consumption plan for N. Therefore, $v(\{1, 2, 3\}) = b_1(e_1) + b_2(e_2) + b_3(e_3) = 3$, as desired.

Step 2 Consider the river sharing problem given by $N = \{0, 1, 2, 3\}$, $D = \{(0, 1), (1, 2), (2, 3)\}$, $e_i = 16$ and $b_i(x_i) = \sqrt{x_i}/4$ for each $i \in N \backslash \{1\}$, and $e_1 = 4$ and $b_1(x_1) = \sqrt{x_1}/2$. In the associated river TU-game, it is easy to compute that $v(\{i\}) = 1$ for each $i \in N$, that $v(\{0, 1\}) = 2.5$, and that for each other $S \in 2^N$, we have:

$$v(S) = \begin{cases} \displaystyle\sum_{i \in S} v(\{i\}) & \text{if } \{0, 1\} \not\subseteq S, \\ \displaystyle v(\{0, 1\}) + \sum_{i \in S \backslash \{0,1\}} v(\{i\}) & \text{if } \{0, 1\} \subseteq S. \end{cases}$$

Consider any allocation rule Φ satisfying standardness, subordinate consistency and subordinate amalgamation. To begin with, from (N, v, D), construct the rooted forest TU-game (N^1, v^1, D^1) in which coalition $\{1, 2, 3\}$ is amalgamated into the new player $\{1, 2, 3\}^*$. Since $N^1 = \{0, \{1, 2, 3\}^*\}$ and $D^1 = \{(0, \{1, 2, 3\}^*)\}$, standardness and subordinate amalgamation imply that:

$$\begin{aligned} \Phi_{\{1,2,3\}}(N, v, D) &= \Phi_{\{1,2,3\}^*}(N^1, v^1, D^1) \\ &= v^1(\{1, 2, 3\}^*) + \frac{1}{2}\Big(v^1(\{0, \{1, 2, 3\}^*\} - v^1(\{0\}) - v^1(\{1, 2, 3\}^*)\Big) \\ &= v(\{1, 2, 3\}) + \frac{1}{2}\Big(v(N) - v(\{0\}) - v(\{1, 2, 3\})\Big). \end{aligned}$$

By Lemma 1, Φ also satisfies component efficiency, which in turn implies that:

$$\Phi_0(N,v,D) = v(\{0\}) + \frac{v(N) - v(\{0\}) - v(\{1,2,3\})}{2}.$$

By definition of v, we obtain $\Phi_0(N,v,D) = 1.25$. Knowing this payoff, we can construct the reduced game induced by Φ and $\{1,2,3\}$. More specifically, this reduced game is the rooted forest TU-game $(\{1,2,3\}, v_{\Phi,1}, D_{\{1,2,3\}})$, where $D_{\{1,2,3\}} = \{(1,2),(2,3)\}$ and function $v_{\Phi,1}$ is described as follows:

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v_{\Phi,1}(S)$	1	1	1	2	2	2	3.25

Since $v_{\Phi,1}(\{1,2,3\}) > 3$, this positive and superadditive rooted forest TU-game is not a river TU-game as shown in **Step 1**.

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