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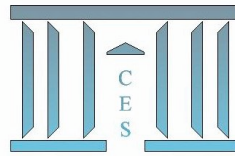
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Model with Natural Resources**

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Abstract

Upon introducing natural resources, both renewable and non-renewable, into an endogenous growth framework with R&D, this paper derives the transitional dynamics of an economy towards its long-run equilibrium. Using the Euler - Lagrange framework, this paper has successfully figured out the optimal paths of the economy. It then shows the existence and uniqueness of a balanced growth path for each type of resources. The steady state is shown to be of a saddle point stability. Along the balanced growth path, it is found that a finite size resource sector coexists with other continuously growing sectors. The paper then examines long-run responses of the economy to various changes pertaining to innovative production condition, resource sector parameters as well as rate of time preference. It also shows that positive long-run growth will be sustained regardless the type of resources used.

Keywords: R&D-based growth, natural resources, vertical innovation, transitional dynamics.

JEL classification: O13, O31, O41.

1 Introduction

There has been a growing interest in examining the pattern of economic growth when there is technological change and natural resources (e.g. Grimaud and Rouge, 2003; Lafforgue, 2008; Peretto, 2008, 2012; Peretto and Valente, 2011). This literature focuses on the interplay between economic growth resource exploitation. In particular, it studies how the adjustment of technological change induced by purposive R&D investment and natural resource stock affects economic sustainability and welfare. However, the dynamic behaviour of these

models is not well comprehended given that the existing studies mostly focus on the balanced growth path.¹ That is the main purpose of this paper. In particular, this paper sets out to analyze how the adjustment in resource exploitation and R&D investment changes the dynamic behaviour of the economy. To that end, we consider a general equilibrium model of endogenous growth with R&D and natural resources. Unlike existing studies in which only one type of resources (renewable or non-renewable) is considered, in this paper, both types of resources are examined. The economy consists of four productive sectors that are linked to each other: a research sector, an intermediate good sector, a final good sector, and a resource sector. Labour is the unique production input that is required for the production of knowledge, intermediate products and harvesting of resources. Resources (e.g. iron ore), after being extracted and processed into materials (e.g. iron), are used to produce intermediate capital goods which then serve as inputs for the production of a final consumption good. Vertical innovation targets at upgrading the quality of these intermediate products. This setting makes the decision on the allocation of labour across sector become the most important one.

Given the above setting, we first determine the optimal paths. In doing so, we characterize the time paths of labour allocations across sectors and the dynamics of the resource stock which help us to pin down the dynamics of all other variables in the economy. It should be noted that although the objective function is not concave and the constraint is not convex (due to the growth of technological knowledge), under some mild assumptions, we are able to explicitly derive the solution to this problem. More specifically, using the Euler - Lagrange equation technique, we show that as soon as the R&D sector is sufficiently productive the optimal solution can be obtained. This condition also ensures that positive long-run growth is sustained no matter what type of resources is used in production. Upon attaining these results, we move on to prove the existence and uniqueness of a balanced growth path for each type of resources. In addition, the steady state is a saddle point to which the dynamic system will converge. Although the problem is not concave, we are able to prove that the stable manifold is locally optimal.

Along the balanced growth path, while the resource sector maintains a finite size, other sectors experience continuing growth. While an improvement in productivity of research always increases growth and welfare regardless of the resource type, an improvement in that of resource production only results in the same outcome when resources are renewable. An economy endowed with renewable resources generally enjoys a higher growth rate than it would be in case of non-renewable resources.

This paper is organized as follows. Section 2 sets out the model by describing the basic structure of the economy. Section 3 offers equilibrium concepts for this economy. Section 4 studies transitional dynamics, the existence and stability

¹Previous studies often consider natural resources, R&D, and growth separately; either between resource abundance and economic growth (e.g. Sachs and Warner, 1995; Lederman and Maloney, 2007) or between R&D-based innovation and economic performance (e.g. Romer, 1990; Grossman and Helpman, 1991; Aghion and Howitt, 1992).

of a balanced growth path. Section 4 considers key properties of this balanced growth path. Conclusions are summarized in Section 5.

2 The economy

2.1 *The final goods sector*

This sector is assumed to be competitive with a large number of identical firms producing an homogeneous consumption good Y according to the following technology:

$$Y_t = \int_0^1 A_{it} x_{it}^\alpha di, \quad \alpha \in (0, 1) \quad (1)$$

where x_{it} is the amount of intermediate good of vintage i (indexed on a unit interval), and A_{it} is a productivity parameter attached to the latest version of that intermediate good.

The final good is taken as a *numeraire* ($P_Y = 1$). The final good producers' profit function is:

$$\pi = Y_t - \int_0^1 p_{xit} x_{it} di$$

where p_{xit} denotes the price of intermediate good i at time t . Profit maximization gives the (inverse) demand function for each intermediate good:

$$p_{xit} = \alpha A_{it} x_{it}^{\alpha-1}, \quad \forall i \in [0, 1] \quad (2)$$

2.2 *The intermediate goods sector*

This sector is assumed to be monopolistically competitive. Each intermediate producer faces the following production technology:

$$x_{it} = \frac{M_{it}^\beta L_{it}^{1-\beta}}{A_{it}}, \quad \beta \in [0, 1], \quad \forall i \in [0, 1] \quad (3)$$

Here, L_{it} is labour employment in industry i at time t and M_{it} is the use of processed natural resource materials. The appearance of A_{it} in the denominator is aimed to capture the fact that products of higher degree of complexity require more resources to produce.

Profit function for the representative monopolist i is:

$$\pi_{xit} = p_{xit} x_{it} - p_{mt} M_{it} - w_t L_{it}$$

The monopolist's objective is to maximize this profit function subject to the demand equation (2) and production technology equation (3). In terms of notation, p_{mt} is the unit price of processed material and w_t is the cost of hiring one unit of labour. After taking the first order conditions with respect to M_{it} and L_{it} and then rearranging and summing over i , we obtain:

$$M_t = \frac{\alpha^2 \beta Y_t}{p_{mt}} \quad (4)$$

$$L_{xt} = \frac{\alpha^2 (1 - \beta) Y_t}{w_t} \quad (5)$$

where $M_t = \int_0^1 M_{it} di$ is the aggregate stock of materials used and $L_{xt} = \int_0^1 L_{it} di$ is the total labour employment employed for producing intermediate goods. Plugging these results into equation (3) yields $x_{it} = x_t = \left(\frac{M_t^\beta L_{xt}^{1-\beta}}{Y_t}\right)^{\frac{1}{1-\alpha}}$, $\forall i$. Plugging this result into the production function in (1) gives:

$$Y_t = A_t^{1-\alpha} (M_t^\beta L_{xt}^{1-\beta})^\alpha \quad (6)$$

where $A_t = \int_0^1 A_{it} di$ is the economy wide aggregate stock of knowledge.²

2.3 The research sector

This sector is assumed to be competitive with no entry restrictions. There is only one type of innovation aiming at improving the quality of existing intermediate products (vertical innovation). Each time, when an innovation is successful, a new (better) vintage of an intermediate product is introduced and replaces its older version in the final good production. Assume that designs or blue prints are protected by the patent law so that each successful innovator can charge a monopoly price over their product until the next successful innovator occurs in that industry.

With access to the stock of knowledge, research firms use labour to develop new blueprints with a Poisson arrival rate $\lambda > 0$. A successful innovation lifts up the knowledge level by a factor $\mu > 1$. Because the prospective payoff is the same in each industry, a same amount will be spent on vertical R&D in each industry. If L_{rt} is the total amount of labour devoted to doing research then the evolution of A_t can be shown as:

$$\dot{A}_t = \lambda(\mu - 1)L_{rt}A_t \quad (7)$$

With free entry, in equilibrium, marginal cost of an extra unit of labour is equal to its expected marginal benefit:

$$\lambda V_t = w_t \quad (8)$$

Here, V_t is the value of a vertical innovation such that:

$$V_t = \int_t^\infty \pi_{xt\tau} e^{-\int_t^\tau (r_s + I_s) ds} d\tau \quad (9)$$

²Because the number of intermediate industries is indexed on a unit interval, A_t coincides with the economy's average technology level.

where r_s is the instantaneous interest rate at date s , $I_s = \lambda L_{rs}$ is the rate of successful innovation arrival at date s , and $\pi_{xt\tau}$ is the flow of operating profit at date τ to any firm in the sector whose technology is of vintage t . In other words, as the market for design is competitive, the value of vertical innovation is equal to the expected present value of future operating profits to be earned by the incumbent intermediate monopolist until being replaced by the next innovator in the industry.

Following Caballero and Jaffe (1993) and Howitt and Aghion (1998), assume that the leading-edge technology parameter $A_t^{\max} \equiv \max\{A_{it}, \forall i \in [0, 1]\}$ is available to any successful innovator and its growth is due to knowledge spillovers produced by innovations. As shown in Howitt and Aghion (1998), the ratio of the leading-edge technology A_t^{\max} to the average technology A_t is constant. As innovation occurs at rate $I_t = \lambda L_{rt}$ per product and the average change across innovating sectors is $A_t^{\max} - A_t$ so:

$$\frac{\dot{A}_t}{A_t} = \lambda L_{rt} \left(\frac{A_t^{\max}}{A_t} - 1 \right)$$

Comparing this together with (7) yields $\frac{A_t^{\max}}{A_t} = \mu$.

2.4 The primary or resource sector

Assume that the resources are owned by households. Following Schaefer (1957), the amount of materials extracted depends on the amount of labour input used, L_{mt} , and the availability of the stock of resources, R_t :

$$M_t = BL_{mt}R_t \quad (10)$$

In this formulation, B is the productivity of resource production. The dynamics of the stock of resources are as follows:

$$\dot{R}_t = f(R_t) - M_t \quad (11)$$

Here, $f(R_t)$ is the natural growth of the resources that takes the following logistic growth form:

$$f(R_t) = \eta R_t \left(1 - \frac{R_t}{\bar{R}} \right), \quad \eta \geq 0 \quad (12)$$

where \bar{R} is the carrying capacity of the environment and η represents the intrinsic growth rate of resources. When $\eta > 0$, the natural resources are renewable and when $\eta = 0$, they are non-renewable. Combining (10), (11), and (12) delivers:

$$\dot{R}_t = \eta R_t \left(1 - \frac{R_t}{\bar{R}} \right) - BL_{mt}R_t \quad (13)$$

3 Equilibrium

Assume constant population and normalize the size of population to 1 ($L = 1$) for simplicity. Hence, under the assumption of full employment, the labour market equilibrium requires that:

$$L_{xt} + L_{rt} + L_{mt} = 1 \quad (14)$$

And the goods market equilibrium dictates that:

$$C_t = Y_t$$

where Y_t is given by equation (6).

The program of the social planner is to maximize the utility:

$$U = \int_0^{\infty} \log(Y_t) \cdot e^{-\rho t} dt$$

subject to the dynamic equations of technology and natural resources:

$$\frac{\dot{A}_t}{A_t} = \lambda(\mu - 1)L_{rt} \quad (15)$$

$$\dot{R}_t = \eta R_t \left(1 - \frac{R_t}{\bar{R}}\right) - B L_{mt} R_t \quad (16)$$

We define our equilibrium in this economy as follows:

Definition 1. *An equilibrium of this centralized economy is an infinite sequence of quantity allocations $\{C_t, Y_t, A_t, R_t, L_{xt}, L_{mt}, L_{rt}\}_{t=0}^{\infty}$ such that consumers' welfare is maximized subject to intertemporal constraints facing the social planner.*

Definition 2. *A balanced growth path (BGP) is an equilibrium path where all variables grow at a constant rate and the allocations of labour across the intermediate goods, resource, and the R&D sectors are also constant.*

Specifically, along this BGP, L_{xt} , L_{mt} , L_{rt} are all constant; C_t , Y_t , A_t , R_t grow at constant rates g_C , g_Y , g_A , and g_R respectively. In this paper, we will analyze transitional dynamics to and local stability around these BGPs. After that, we examine comparative statics along these BGPs.

4 Characterization of the optimal path and local stability of the BGP

4.1 Transitional dynamics of the optimal paths

In this centralized economy, the key dynamic equations are given by those describing the evolution of technical knowledge and the dynamics of the stock of natural resources given in (15) and (16) respectively. From these equations, we will derive optimal time paths of the economy, the balanced growth path, and work out conditions for achieving the convergence to the steady state.

We will assume that L_{rt} and L_{mt} are continuous. For any t , L_{rt} and L_{mt} belong to the interval $[0, 1]$. Any solution R_t to (16) is continuously differentiable. Observe that when $R(t) \geq \bar{R}$, we have $\dot{R}_t < 0$, $\forall t$. Predicting that $R(t) \leq \bar{R}$, we can state the following:

Lemma 1 *Assume $R_0 < \bar{R}$. Then $R(t) \leq \bar{R}$ for all t . And, hence, $\log(R_t) \geq \log(R_0) - Bt$.*

Proof. See Appendix.

We now summarize our first key results in the proposition below:

Proposition 1 *To simplify notation, define $\varphi = \frac{(1-\alpha)\lambda(\mu-1)}{\alpha(1-\beta)}$. Assume $\eta = 0$ and $\frac{\rho}{\varphi} + \frac{\beta\rho}{\varphi(1-\beta)+B\beta} < 1$. Then the solution to the social planner's maximization problem is an optimal BGP that is uniquely defined as follows:*

$$\begin{aligned}\hat{L}_x &= \frac{\rho}{\varphi} \\ \hat{L}_m &= \frac{\beta\rho}{\varphi(1-\beta)+B\beta} \\ \hat{L}_r &= 1 - \hat{L}_x - \hat{L}_m \\ \hat{R}_t &= R_0 e^{-B\hat{L}_m t} \\ \hat{A}_t &= A_0 e^{\lambda(\mu-1)(1-\hat{L}_m-\hat{L}_x)t}\end{aligned}$$

Proof. See Appendix.

Lemma 2 *Let L_{xt}^* and L_{mt}^* be solutions to the social planner's maximization problem. Then L_{xt}^* and L_{mt}^* satisfy the following differential equations:*

$$\frac{\dot{L}_{xt}}{L_{xt}} = \varphi L_{xt} - \rho \quad (17)$$

$$\rho\alpha\beta + \frac{\alpha\beta\dot{L}_{mt}}{L_{mt}} + \frac{\alpha\beta\eta R_t}{\bar{R}} - [\varphi\alpha(1-\beta) + B\alpha\beta] L_{mt} = \frac{\alpha(1-\beta)\eta}{L_{xt}\bar{R}} L_{mt} R_t \quad (18)$$

where $\varphi = \frac{(1-\alpha)\lambda(\mu-1)}{\alpha(1-\beta)}$.

Proof. See Appendix.

Lemma 3 *The solutions to the social planner's maximization problem, L_{xt}^* and L_{mt}^* , take the following forms:*

$$L_{xt}^* = \frac{1}{\frac{\varphi}{\rho} + c_x e^{\rho t}} \quad (19)$$

$$L_{mt}^* = \frac{1}{e^{\int_0^t b(u)du} \left(c_1 - \int_0^t h(x) e^{-\int_0^x b(u)du} dx \right)} \quad (20)$$

where

$$\begin{aligned}b(u) &= \rho + \eta \frac{R_u^*}{\bar{R}} \\ h(x) &= \frac{\varphi(1-\beta)}{\beta} + B + \frac{\eta(1-\beta)}{\beta L_{xt}^*} \times \frac{R_x^*}{\bar{R}} \\ \varphi &= \frac{(1-\alpha)\lambda(\mu-1)}{\alpha(1-\beta)} \\ c_1 &> 0, c_x \geq 0\end{aligned}$$

Proof. See Appendix.

Lemma 4 Let $v(\cdot)$ be a continuous function, then the following applies:

$$\int_0^t v(x) e^{-\int_0^x v(u) du} dx = 1 - e^{-\int_0^t v(u) du}$$

Proof. See Appendix.

Lemma 5 As soon as $c_x = 0$, the following condition holds:

$$\frac{1 - \beta}{L_{xt}^*} < \frac{\beta}{L_{mt}^*}$$

Proof. See Appendix.

Now let $z_t = \log(A_t)$ and $w_t = \log(R_t)$. From (26), one can define that:

$$\begin{aligned} \mathcal{M}(z_t, \dot{z}_t, w_t, \dot{w}_t) &= (1 - \alpha)z_t + \alpha\beta \left[w_t + \log \left(\eta \left(1 - \frac{e^w}{\bar{R}} \right) - \dot{w}_t \right) \right] \\ &+ \alpha(1 - \beta) \log \left[1 - \frac{\dot{z}_t}{\lambda(\mu - 1)} - \frac{\eta}{B} \left(1 - \frac{e^{w_t}}{\bar{R}} \right) + \frac{\dot{w}_t}{B} \right] \end{aligned}$$

To simplify notations, we define the function:

$$G(\dot{z}_t, w_t, \dot{w}_t) = \left[1 - \frac{\dot{z}_t}{\lambda(\mu - 1)} - \frac{\eta}{B} \left(1 - \frac{e^{w_t}}{\bar{R}} \right) + \frac{\dot{w}_t}{B} \right]$$

The following lemma is the corrolary of Lemma 5:

Lemma 6 When $c_x = 0$, we have:

$$-\frac{\alpha\beta}{\eta \left(1 - \frac{e^{w_t}}{\bar{R}} \right) - \dot{w}_t} + \frac{\alpha(1 - \beta)}{G(\dot{z}_t, w_t, \dot{w}_t)} \times \frac{1}{B} < 0$$

Proof. See Appendix.

Lemmas 4, 5, 6, and their corrolary are crucial for our next main results below:

Proposition 2 Assume $R_0 < \bar{R}$ and $\frac{\rho}{\varphi} < 1 - \beta$. Then the social planner's optimal solutions are

$$L_{xt}^* = \frac{\rho}{\varphi} \tag{21}$$

$$L_{mt}^* = \frac{1}{e^{\int_0^t b(u) du} \left(c_1 - \int_0^t h(x) e^{-\int_0^x b(u) du} dx \right)} \tag{22}$$

$$\frac{\dot{A}_t^*}{A_t^*} = \lambda(\mu - 1)(1 - L_{xt}^* - L_{mt}^*)$$

$$\frac{\dot{R}_t^*}{R_t^*} = \eta \left(1 - \frac{R_t^*}{R}\right) - BL_{mt}^*$$

R_0, A_0 are given and

$$\varphi = \frac{(1-\alpha)\lambda(\mu-1)}{\alpha(1-\beta)}$$

$$b(u) = \rho + \eta \frac{R_t^*}{R}$$

$$h(x) = \frac{(1-\alpha)\lambda(\mu-1)}{\beta\alpha} + B + \frac{\eta(1-\beta)}{\beta L_{xt}^*} \times \frac{R_t^*}{R}$$

$$c_1 = \int_0^\infty h(x) e^{-\int_0^x b(u) du} dx$$

Proof. See Appendix.

Remark 1 If $\eta = 0$, then L_{mt}^* takes the value of the BGP derived in Proposition 1.

4.2 Long-run properties of the optimal paths: convergence to the BGP

In this subsection, we will first prove the existence and uniqueness of the BGP. We will then show that the optimal paths obtained in the previous subsection will converge to the BGP.

Proposition 3 Assume

$$\frac{2B\rho + (1-\beta)\varphi\eta - \sqrt{\Delta}}{2B(1-\beta)\varphi} \leq 1$$

where $\Delta = 4B^2\beta^2\rho^2 + (1-\beta)^2\varphi^2\eta^2$ and $\varphi = \frac{(1-\alpha)\lambda(\mu-1)}{\alpha(1-\beta)}$ then there exists a unique BGP that is described by

$$\hat{L}_x = \frac{\rho}{\varphi}$$

$$\hat{L}_m = \frac{2B\beta\rho + (1-\beta)\varphi\eta - \sqrt{\Delta}}{2B(1-\beta)\varphi}$$

Proof. See Appendix.

We next prove the convergence of the optimal paths to the BGP:

Proposition 4 Assume parameters are such that the BGP exists. Then the dynamic system is saddle point convergent.

Proof. See Appendix.

5 Properties of the BGP: a comparative statics study

Having known that the economy will converge to the BGP in the long-run, it will be interesting to discuss the properties of this BGP. In other words, we can do the comparative statics at this long-run equilibrium and analyze possible impacts on output growth and welfare. This section is devoted to that task.

Proposition 5 *Other things equal, along the BGP for each type of resources, output growth and welfare are increasing in parameters characterizing productivity of the R&D sector (λ and μ) but decreasing in the rate of time preference (ρ).*

Proof. See Appendix.

The results are quite intuitive. When λ or μ increases, it becomes more socially efficient to invest in the R&D sector (relatively to other sectors) so the social planner will choose a higher level of \hat{L}_r which then enhances growth of technological knowledge and output. An increase in ρ means households value current consumption relatively more than future consumption. In order to produce more output to meet higher consumption demand today, the social planner will direct more labour to work in the resource sector (\hat{L}_m increases) and the intermediate goods sector (\hat{L}_x increases). As a result, there will be a fall in \hat{L}_r meaning lower growth of technology and output. Consumption growth will also be lower because consumers increase current consumption relatively to future consumption.

Because an increase in either λ or μ raises output and consumption so welfare rises. However, an increase in ρ reduces welfare as it makes the whole path of utility fall below the one before the shock.

Proposition 6 *Other things equal, along the BGP, an improvement in the productivity of the resource sector (an increase in B):*

- *increases both welfare and output growth if resources are renewable.*
- *increases welfare but has no impact on output growth if resources are non-renewable.*

Proof. See Appendix.

The results can be explained as follows. An increase in B makes it more productive to extract natural resources. Equivalently, less labour is needed for producing resource material to meet the existing market demand. Hence, the social planner will allocate less labour to the resource sector (\hat{L}_m decreases) and more into the R&D activities (\hat{L}_r increases).³ This change will increase welfare as there is more output and consumption created. It will also increase

³Another way of looking at this is that as the social planner always knows the optimal level of natural resources to be $R = \bar{R}(1 - \frac{B L_m}{\eta})$, he will reduce L_m in accordance with the amount of increase in B .

output growth for the case of renewable resources because the growth rate of technology is higher. However, it does not affect output growth under non-renewable resources. The reason is that an increase in B , on the one hand, increases \hat{L}_r and, hence, technological change, will also exhaust resources at a faster rate on the other (the fall in \hat{L}_m is less than the increase in B). These two opposing effects cancel out each other at the optimum.

Proposition 7 *Assume parameters are such that there exists a BGP for each type of resources then output growth is generally higher under renewable resources than under non-renewable resources.*

Proof. See Appendix.

In this economy, output growth comes from two different sources: the evolution of technological knowledge and the natural resource dynamics. Because natural resources cannot grow without bound, the best trajectory that the social planner can choose is to reach the optimal level of resources at which the rate of resource extraction is equal to the rate of natural growth. However, this policy is only achievable in case of renewable resources. With non-renewable resources, the rate of resource extraction always softens the rate of growth of output as output needs to increase to make up for the amount of natural resources that has been depleted. Given that technological progress is the key driver of the economy, it in turn requires the evolution of technological knowledge be strong enough to lift the economy up out of the stagnation trap.

6 Concluding remarks

This paper has introduced a resource sector into an endogenous growth model with R&D investment. We have shown how the dynamic equilibrium could be represented by a dynamic system characterized by the sectoral allocation of labour, the evolution of technology and the dynamics of the resource stock. We show that under plausible assumptions, the social planner can achieve a stable transitional dynamics to an optimal balanced growth path. The model results in an equilibrium in which the stock of resources remains in finite size while other sectors carry continuous growth.

The balanced growth path equilibrium has the following features. Equilibrium growth rate in an economy endowed with renewable resources is higher than it would be in the case of non-renewable resources. As soon as the research sector is highly productive, positive growth is always sustained.

We have also examined the long-run reaction of the economy to a number of changes regarding innovative production capacity, rate of time preference, and resource sector productivity. While an improvement in productivity of the R&D is always growth and welfare enhancing, that of the resource sector is subject to the type of resources that is considered. As for future research agenda, it would be interesting to see the extent to which empirical evidence is consistent with these theoretical predictions.

Appendix

Proof of Lemma 1

We will prove this lemma by method of contradiction. To that end, assume there exists t_0 such that $R(t_0) > \bar{R}$. For any $\varepsilon > 0$ which satisfies $\bar{R} + \varepsilon < R(t_0)$, there exists $t \in (0, t_0)$ such that $R(t) = \bar{R} + \varepsilon$. Now define:

$$I_\varepsilon = \{t \in [0, t_0] : R(t) = \bar{R} + \varepsilon\}$$

Since R_t is continuous, the set I_ε is compact. Let $t_1 = \max\{t : t \in I_\varepsilon\}$ then $t_1 < t_0$. Evaluating the dynamics of resources at time t_1 , we have:

$$\left(\frac{\dot{R}(t)}{R(t)}\right)_{t_1} = -\frac{\eta\varepsilon}{\bar{R}} - BL_m < 0$$

Hence, for $t' \in (t_1, t_0)$ that is close enough to t_1 , we have $R(t') < R(t_1) = \bar{R} + \varepsilon < R(t_0)$. In this case, there must be $t_2 \in (t', t_0)$ such that $R(t_2) = \bar{R} + \varepsilon$. This implies $t_2 \in I_\varepsilon$ and $t_2 \leq t_1 < t'$ which is a contradiction. Therefore, $R(t) \leq \bar{R}$ for any t .

From equation (16), we have:

$$\frac{d \log R}{dt} \geq -BL_{mt} \geq -B$$

since $L_{mt} \leq 1$. By integrating this inequality we get $\log(R_t) \geq \log(R_0) - Bt$.

■

Proof of Proposition 1

We will prove this proposition in two parts. In the first part, we prove that there exists a unique BGP that solves the social planner's maximization problem. In the second part, we show that this BGP is optimal.

Using (6) and (10), the utility function can be rewritten as:

$$U = \int_0^\infty \log(A_t^{1-\alpha} B^{\alpha\beta} L_{mt}^{\alpha\beta} R_t^{\alpha\beta} L_{xt}^{\alpha(1-\beta)}) \cdot e^{-\rho t} dt$$

On the BGP, we now have $R_t = R_0 e^{-tBL_m}$ where R_0 is the initial stock of natural resources. With a note that $\int_0^\infty t e^{-\rho t} dt = \frac{1}{\rho^2}$ and $\int_0^\infty e^{-\rho t} dt = \frac{1}{\rho}$, the utility function on the BGP is:

$$\begin{aligned} \rho U &= (1-\alpha) \log(A_0) + \alpha\beta \log(B) + \alpha\beta \log(L_m) + \alpha\beta \log(R_0) \\ &+ \alpha(1-\beta) \log(L_x) - \frac{\alpha\beta BL_m}{\rho} + \frac{(1-\alpha)\lambda(\mu-1)(1-L_x-L_m)}{\rho} \end{aligned}$$

L_x and L_m will be chosen to maximize this utility functions. The first order conditions give:

$$\hat{L}_x = \frac{\alpha\rho(1-\beta)}{\lambda(\mu-1)(1-\alpha)} = \frac{\rho}{\varphi} \quad (23)$$

$$\hat{L}_m = \frac{\alpha\beta\rho}{\lambda(\mu-1)(1-\alpha) + B\alpha\beta} = \frac{\beta\rho}{\varphi(1-\beta) + B\beta} \quad (24)$$

where $\varphi = \frac{(1-\alpha)\lambda(\mu-1)}{\alpha(1-\beta)}$. Clearly, $\hat{L}_x > 0$ and $\hat{L}_m > 0$. Hence, the value of \hat{L}_r is:

$$\hat{L}_r = 1 - \left[\frac{\rho}{\varphi} + \frac{\beta\rho}{\varphi(1-\beta) + B\beta} \right] \quad (25)$$

When $\hat{L}_r > 0$ or $\frac{\rho}{\varphi} + \frac{\beta\rho}{\varphi(1-\beta) + B\beta} < 1$ we automatically have $0 < \hat{L}_x, \hat{L}_m, \hat{L}_r < 1$. With these obtained results, we can calculate the growth rates of technology and natural resources along the BGP as follows:

$$\begin{aligned} \hat{R}_t &= R_0 e^{-B\hat{L}_m t} \\ \hat{A}_t &= A_0 e^{\lambda(\mu-1)(1-\hat{L}_m-\hat{L}_x)t} \end{aligned}$$

To prove that this solution is optimal, we compute the following:

$$\lim_{T \rightarrow \infty} \left[\int_0^T \log \left(\hat{A}_t^{1-\alpha} B^{\alpha\beta} \hat{L}_{mt}^{\alpha\beta} \hat{R}_t^{\alpha\beta} \hat{L}_{xt}^{\alpha(1-\beta)} \right) . e^{-\rho t} dt - \int_0^T \log \left(A_t^{1-\alpha} B^{\alpha\beta} L_{mt}^{\alpha\beta} R_t^{\alpha\beta} L_{xt}^{\alpha(1-\beta)} \right) . e^{-\rho t} dt \right]$$

Using integration by parts we have:

$$\int_0^T \log \left(\hat{A}_t^{1-\alpha} \right) . e^{-\rho t} dt = \left[-\frac{1}{\rho} e^{-\rho t} \log \left(\hat{A}_t^{1-\alpha} \right) \right]_0^T + \int_0^T \frac{1}{\rho} e^{-\rho t} (1-\alpha) \frac{\dot{\hat{A}}_t}{\hat{A}_t} dt$$

$$\int_0^T \log \left(\hat{R}_t^{\alpha\beta} \right) . e^{-\rho t} dt = \left[-\frac{1}{\rho} e^{-\rho t} \log \left(\hat{R}_t^{\alpha\beta} \right) \right]_0^T + \int_0^T \frac{1}{\rho} e^{-\rho t} \alpha\beta \frac{\dot{\hat{R}}_t}{\hat{R}_t} dt$$

$$\int_0^T \log \left(A_t^{1-\alpha} \right) . e^{-\rho t} dt = \left[-\frac{1}{\rho} e^{-\rho t} \log \left(A_t^{1-\alpha} \right) \right]_0^T + \int_0^T \frac{1}{\rho} e^{-\rho t} (1-\alpha) \frac{\dot{A}_t}{A_t} dt$$

$$\int_0^T \log \left(R_t^{\alpha\beta} \right) . e^{-\rho t} dt = \left[-\frac{1}{\rho} e^{-\rho t} \log \left(R_t^{\alpha\beta} \right) \right]_0^T + \int_0^T \frac{1}{\rho} e^{-\rho t} \alpha\beta \frac{\dot{R}_t}{R_t} dt$$

In addition, applying the inequality $x \geq \log(1+x)$ we have:

$$\log\left(\frac{L_{mt}}{\hat{L}_{mt}}\right) = \log\left(1 + \frac{L_{mt} - \hat{L}_{mt}}{\hat{L}_{mt}}\right) \leq \frac{L_{mt} - \hat{L}_{mt}}{\hat{L}_{mt}}$$

Hence

$$-\alpha\beta \log\left(\frac{L_{mt}}{\hat{L}_{mt}}\right) \geq -\alpha\beta \cdot \frac{L_{mt} - \hat{L}_{mt}}{\hat{L}_{mt}}$$

or

$$\alpha\beta \left[\log(\hat{L}_{mt}) - \log(L_{mt}) \right] \geq \alpha\beta \cdot \frac{\hat{L}_{mt} - L_{mt}}{\hat{L}_{mt}}$$

Similarly, the following holds:

$$\alpha(1 - \beta) \left[\log(\hat{L}_{xt}) - \log(L_{xt}) \right] \geq \alpha(1 - \beta) \cdot \frac{\dot{L}_{xt} - L_{xt}}{\hat{L}_{xt}}$$

Inserting these results into the equation for ΔU and observing that:

$$\begin{aligned} \left[-\frac{1}{\rho} e^{-\rho t} \log(\hat{A}_t^{1-\alpha}) \right]_0^{+\infty} &= \left[-\frac{1}{\rho} e^{-\rho t} \log(A_t^{1-\alpha}) \right]_0^{+\infty} = \frac{1}{\rho} \log(A_0^{1-\alpha}) \\ \left[-\frac{1}{\rho} e^{-\rho t} \log(\hat{R}_t^{\alpha\beta}) \right]_0^T &= \left[-\frac{1}{\rho} e^{-\rho t} \log(R_t^{\alpha\beta}) \right]_0^T = \frac{1}{\rho} \log(R_0^{\alpha\beta}) \end{aligned}$$

we get:

$$\begin{aligned} \Delta U \geq & \int_0^{+\infty} \frac{1}{\rho} e^{-\rho t} (1 - \alpha) \frac{\dot{A}_t}{A_t} dt + \int_0^{+\infty} \frac{1}{\rho} e^{-\rho t} \alpha\beta \frac{\dot{R}_t}{R_t} dt + \int_0^{+\infty} e^{-\rho t} \alpha\beta \cdot \frac{\dot{L}_{mt} - L_{mt}}{\hat{L}_{mt}} dt + \\ & \int_0^{+\infty} e^{-\rho t} \alpha(1 - \beta) \cdot \frac{\dot{L}_{xt} - L_{xt}}{\hat{L}_{xt}} dt - \int_0^{+\infty} \frac{1}{\rho} e^{-\rho t} (1 - \alpha) \frac{\dot{A}_t}{A_t} dt - \int_0^{+\infty} \frac{1}{\rho} e^{-\rho t} \alpha\beta \frac{\dot{R}_t}{R_t} dt \end{aligned}$$

Using (23) and (24) then:

$$\alpha\beta \cdot \frac{\dot{L}_{mt} - L_{mt}}{\hat{L}_{mt}} = \frac{\lambda(\mu-1)(1-\alpha) + B\alpha\beta}{\rho} \cdot (\hat{L}_{mt} - L_{mt})$$

$$\alpha(1 - \beta) \cdot \frac{\dot{L}_{xt} - L_{xt}}{\hat{L}_{xt}} = \frac{\lambda(\mu-1)(1-\alpha) + B\alpha\beta}{\rho} \cdot (\hat{L}_{xt} - L_{xt})$$

Plugging these results in, we can figure out that $\Delta U \geq 0$.

■

Proof Lemma 2

The maximization problem for the social planner of this economy is to solve:

$$\max \int_0^{\infty} \mathcal{L}(A_t, \dot{A}_t, R_t, \dot{R}_t) e^{-\rho t} dt$$

where

$$\begin{aligned} \mathcal{L}(A_t, \dot{A}_t, R_t, \dot{R}_t) &= (1 - \alpha) \log(A_t) + \alpha\beta \log \left[\eta R_t \left(1 - \frac{R_t}{\bar{R}} \right) - \dot{R}_t \right] \\ &+ \alpha(1 - \beta) \log \left[1 - \frac{\dot{A}_t}{A_t \lambda(\mu - 1)} - \frac{\eta}{B} \left(1 - \frac{R_t}{\bar{R}} \right) + \frac{\dot{R}_t}{B R_t} \right] \end{aligned} \quad (26)$$

Considering interior solutions, we have the following Euler-Lagrange equations:

$$\frac{d}{dt} \left[\frac{\partial \mathcal{L}}{\partial \dot{A}_t} e^{-\rho t} \right] = \frac{\partial \mathcal{L}}{\partial A_t} e^{-\rho t} \quad (27)$$

$$\frac{d}{dt} \left[\frac{\partial \mathcal{L}}{\partial \dot{R}_t} e^{-\rho t} \right] = \frac{\partial \mathcal{L}}{\partial R_t} e^{-\rho t} \quad (28)$$

The LHS of (27) is given by:

$$\rho e^{-\rho t} \frac{\alpha(1-\beta)}{\lambda(\mu-1)} \frac{1}{A_t L_{xt}} + \frac{e^{-\rho t}}{A_t L_{xt}} \alpha(1-\beta) \left[L_{rt} + \frac{\dot{L}_{xt}}{\lambda(\mu-1)L_{xt}} \right]$$

while its RHS is equal to:

$$e^{-\rho t} \frac{1-\alpha}{A_t} + \frac{\alpha(1-\beta)}{L_{xt}} \frac{L_{rt}}{A_t} e^{-\rho t}$$

Equating the LHS with the RHS gives:

$$\rho \frac{\alpha(1-\beta)}{\lambda(\mu-1)} \frac{1}{L_{xt}} + \frac{1}{L_{xt}} \alpha(1-\beta) \frac{\dot{L}_{xt}}{\lambda(\mu-1)L_{xt}} = 1-\alpha$$

or

$$\rho \alpha(1-\beta) + \alpha(1-\beta) \frac{\dot{L}_{xt}}{L_{xt}} = (1-\alpha)\lambda(\mu-1)L_{xt}$$

Further simplifying leads to:

$$\rho + \frac{\dot{L}_{xt}}{L_{xt}} = \frac{(1-\alpha)\lambda(\mu-1)}{\alpha(1-\beta)} L_{xt}$$

Using the notation that is previously defined $\frac{(1-\alpha)\lambda(\mu-1)}{\alpha(1-\beta)} = \varphi$ then:

$$\frac{\dot{L}_{xt}}{L_{xt}} = \varphi L_{xt} - \rho$$

Similarly, the LHS of (28) is:

$$\begin{aligned} & -\rho e^{-\rho t} \left[-\frac{\alpha\beta}{BR_t L_{mt}} + \frac{\alpha(1-\beta)}{BR_t L_{xt}} \right] \\ & + e^{-\rho t} \left[\frac{\alpha\beta}{B} \left(\frac{\dot{L}_{mt}}{R_t L_{mt}^2} + \frac{1}{L_{mt} R_t^2} \dot{R}_t \right) - \frac{\alpha(1-\beta)\dot{L}_{xt}}{BR_t L_{xt}^2} - \frac{\alpha(1-\beta)}{BR_t^2 L_{xt}} \dot{R}_t \right] \end{aligned}$$

and its RHS is:

$$e^{-\rho t} \left[\frac{\alpha\beta}{BL_{mt} R_t} \eta \left(1 - \frac{2R_t}{\bar{R}} \right) + \frac{\alpha(1-\beta)}{L_{xt}} \left(\frac{\eta}{B\bar{R}} - \frac{\dot{R}_t}{BR_t^2} \right) \right]$$

Now equating LHS with RHS noting $\frac{(1-\alpha)\lambda(\mu-1)}{\alpha(1-\beta)} = \varphi$ to get:

$$\rho\alpha\beta + \frac{\alpha\beta\dot{L}_{mt}}{L_{mt}} + \frac{\alpha\beta\eta R_t}{\bar{R}} - [\varphi\alpha(1-\beta) + B\alpha\beta] L_{mt} = \frac{\alpha(1-\beta)\eta}{L_{xt}\bar{R}} L_{mt} R_t$$

■

Proof of Lemma 3

Consider equation (17), we write $\Upsilon = \frac{1}{L_{xt}}$ to obtain $\dot{\Upsilon} = \rho\Upsilon - \varphi$. After sloving this simplified differential equation we get:

$$L_{xt}^* = \frac{1}{\frac{\varphi}{\rho} + c_x e^{\rho t}}$$

with $c_x \geq 0$ being a positive constant.

To get the functional form of L_{mt}^* we transform equation (18) to obtain:

$$\dot{L}_{mt} + \left(\rho + \frac{\eta R_t}{R} \right) L_{mt} = \left(\frac{\varphi(1-\beta)}{\beta} + B + \frac{\eta(1-\beta)R_t}{\beta R L_{xt}} \right) L_{mt}^2$$

To simplify notations, define $b(t) = \rho + \frac{\eta R_t}{R}$ and $h(t) = \frac{\varphi(1-\beta)}{\beta} + B + \frac{\eta(1-\beta)R_t}{\beta R L_{xt}}$. The above equation now becomes:

$$\dot{L}_{mt} + b(t)L_{mt} = h(t)L_{mt}^2$$

Let $z_t = \frac{1}{L_{mt}}$ (noting that $L_{mt} \neq 0$) then $\dot{z}_t = -\frac{\dot{L}_{mt}}{L_{mt}^2}$. This is equivalent to $\dot{L}_{mt} = -\dot{z}_t L_{mt}^2$. Substituting results into the above equation and simplifying gives:

$$\dot{z}_t - b(t)z_t = -h(t)$$

The homogeneous equation takes the form:

$$\dot{z}_t - b(t)z_t = 0$$

This equation yields the homogeneous solution as (noting that $z_t \neq 0$)

$$z_h = c_1 e^{\int_0^t b(u) du}$$

Returning to the original non-homogeneous equation given above ($h(t) \neq 0$), assume that a particular solution exists and takes the following form:

$$z_p = v(t) \cdot e^{\int_0^t b(u) du}$$

where $v(t)$ will need to be determined. Substituting this into the LHS of the non-homogeneous equation yields:

$$\dot{z}_p - b(t)z_p = v'(t)e^{\int_0^t b(u) du} + v(t)e^{\int_0^t b(u) du} b(t) - b(t)v(t)e^{\int_0^t b(u) du} = v'(t)e^{\int_0^t b(u) du}$$

The function $v(t)$ must be chosen so that:

$$v'(t)e^{\int_0^t b(u) du} = -h(t)$$

or equivalently

$$v'(t) = \frac{-h(t)}{e^{\int_0^t b(u) du}}$$

Upon taking integral we get:

$$v(t) = -\int_0^t \frac{h(x)}{e^{\int_0^x b(u) du}} dx + c_2$$

For simplicity, set $c_2 = 0$ then $z_p = -e^{\int_0^t b(u) du} \int_0^t \frac{h(x)}{e^{\int_0^x b(u) du}} dx = -e^{\int_0^t b(t) dt} \int_0^t h(x) e^{-\int_0^x b(u) du} dx$. The general solution to the non-homogeneous equation will be:

$$z_t = z_h + z_p = c_1 e^{\int_0^t b(u) du} - e^{\int_0^t b(u) du} \int_0^t h(x) e^{-\int_0^x b(u) du} dx$$

Thus,

$$L_{mt}^* = \frac{1}{e^{\int_0^t b(u) du} (c_1 - \int_0^t h(x) e^{-\int_0^x b(u) du} dx)}$$

where $c_1 > 0$ is a constant. It can be shown that in the special case when $\eta = 0$, we have:

$$L_{mt}^* = \frac{1}{\left(\frac{\varphi(1-\beta)}{\beta} + B\right) \cdot \frac{1}{\rho} + c_m e^{\rho t}}$$

where $c_m = c_1 - \left(\frac{\varphi(1-\beta)}{\beta} + B\right) \cdot \frac{1}{\rho}$ is a positive constant.

■

Proof of Lemma 4

We have:

$$v(x) e^{-\int_0^x v(u) du} = -\frac{d}{dx} \left(e^{-\int_0^x v(u) du} \right)$$

Hence, it follows that:

$$\begin{aligned} \int_0^t v(x) e^{-\int_0^x v(u) du} dx &= - \left[e^{-\int_0^x v(u) du} \right]_{x=0}^{x=t} \\ &= 1 - e^{-\int_0^t v(u) du} \end{aligned}$$

■

Proof of Lemma 5

Let $c_x = 0$. First, since $L_{mt}^* > 0, \forall t$ we must have $c_1 \geq \int_0^\infty h(x) e^{-\int_0^x b(u) du} dx$. Let $\Delta = \frac{1-\beta}{L_{xt}^*} - \frac{\beta}{L_{mt}^*}$, we have $\frac{1-\beta}{L_{xt}^*} = (1-\beta) \frac{\rho}{\beta}$. Tedious computations lead to:

$$h(t) e^{-\int_0^t b(u) du} = \left[\frac{\varphi(1-\beta)}{\beta} + B + \frac{\eta(1-\beta)\varphi R(t)}{\beta\rho\bar{R}} \right] e^{-\rho t - \int_0^t \frac{\eta R(x)}{\bar{R}} dx}$$

Using Lemma 4 we obtain:

$$\int_0^t h(x) e^{-\int_0^x b(u) du} dx = B \int_0^t e^{-\int_0^x b(u) du} dx + \frac{\varphi(1-\beta)}{\rho\beta} \left(1 - e^{-\int_0^t b(u) du} \right)$$

so that

$$\begin{aligned} \frac{\beta}{L_{mt}^*} = \\ \beta c_1 e^{\int_0^t b(u) du} - \beta B e^{\int_0^t b(u) du} \int_0^t e^{-\int_0^x b(u) du} dx - \frac{\varphi(1-\beta)}{\rho} e^{\int_0^t b(u) du} + \frac{\varphi(1-\beta)}{\rho} \end{aligned}$$

Therefore

$$\begin{aligned}\Delta &= -\beta c_1 e^{\int_0^t b(u)du} + \beta B e^{\int_0^t b(u)du} \int_0^t e^{-\int_0^x b(u)du} dx + \frac{\varphi(1-\beta)}{\rho} e^{\int_0^t b(u)du} \\ &= -\beta e^{\int_0^t b(u)du} \left(c_1 - B \int_0^t e^{-\int_0^x b(u)du} dx - \frac{\varphi(1-\beta)}{\rho\beta} \right)\end{aligned}$$

Since

$$\begin{aligned}c_1 &\geq \int_0^\infty h(x) e^{-\int_0^x b(u)du} dx \\ &= B \int_0^\infty e^{-\int_0^x b(u)du} dx + \frac{\varphi(1-\beta)}{\rho\beta} \left[1 - e^{-\int_0^\infty b(u)du} \right]_\infty\end{aligned}$$

we get

$$\begin{aligned}&c_1 - B \int_0^t e^{-\int_0^x b(u)du} dx - \frac{\varphi(1-\beta)}{\rho\beta} \\ &\geq B \int_t^\infty e^{-\int_0^x b(u)du} dx - \frac{\varphi(1-\beta)}{\rho\beta} e^{-\int_0^\infty b(u)du} \\ &\geq B \int_t^\infty e^{-\int_0^x b(u)du} dx > 0\end{aligned}$$

since $e^{-\int_0^\infty b(u)du} = 0$. Thus, $\Delta < 0$.

■

Proof of Lemma 6

We have $\eta(1 - \frac{e^{w_t}}{R}) - \dot{w}_t = BL_{mt}$ and $G(\dot{z}_t, w_t, \dot{w}_t) = L_{xt}$. Hence

$$\begin{aligned}-\frac{\alpha\beta}{\eta(1 - \frac{e^{w_t}}{R}) - \dot{w}_t} + \frac{\alpha(1-\beta)}{G(\dot{z}_t, w_t, \dot{w}_t)} \times \frac{1}{B} &= \frac{-\alpha\beta}{BL_{mt}} + \frac{\alpha(1-\beta)}{BL_{xt}} \\ &= \frac{\alpha}{B} \left(\frac{1-\beta}{L_{xt}} - \frac{\beta}{L_{mt}} \right) < 0\end{aligned}$$

where the result obtained in the second line comes from Lemma 5.

■

Proof of Proposition 2

Part of the proof for this proposition has been done in Lemma 3. Here we only need to show that the solutions to the social planner's maximization problem are optimal. We have:

$$\begin{aligned}c_1 &= B \int_0^\infty e^{-\int_0^x b(u)du} dx + \frac{\varphi(1-\beta)}{\rho\beta} \left[1 - e^{-\int_0^\infty b(u)du} \right] \\ &= B \int_0^\infty e^{-\int_0^x b(u)du} dx + \frac{\varphi(1-\beta)}{\rho\beta}\end{aligned}$$

since $e^{-\int_0^\infty b(u)du} = 0$. Hence, it follows that

$$\begin{aligned} c_1 - \int_0^t h(x) e^{-\int_0^x b(u)du} dx &= c_1 - B \int_0^t e^{-\int_0^x b(u)du} dx - \frac{\varphi(1-\beta)}{\rho\beta} + \frac{\varphi(1-\beta)}{\rho\beta} e^{-\int_0^t b(u)du} \\ &= B \int_t^\infty e^{-\int_0^x b(u)du} dx + \frac{\varphi(1-\beta)}{\rho\beta} e^{-\int_0^t b(u)du} \end{aligned}$$

Therefore

$$L_{mt}^* = \frac{1}{B e^{\int_0^t b(u)du} \int_t^\infty e^{-\int_0^x b(u)du} dx + \frac{\varphi(1-\beta)}{\rho\beta}}$$

Obviously, $L_{mt}^* \leq \frac{\rho\beta}{\varphi(1-\beta)}$. Now observe that

$$\begin{aligned} e^{\int_0^t b(u)du} \int_t^\infty e^{-\int_0^x b(u)du} dx &= \int_t^\infty e^{-\int_t^x b(u)du} dx \\ &\leq \int_t^\infty e^{-\rho(x-t)} dx = \frac{1}{\rho} \end{aligned}$$

Thus, $L_{mt}^* \geq \frac{1}{\frac{B}{\rho} + \frac{\varphi(1-\beta)}{\rho\beta}} = \frac{\rho\beta}{B\beta + \varphi(1-\beta)}$. Since $L_{xt}^* + L_{mt}^* < 1$, we need to impose that:

$$\frac{\rho}{\varphi} + \frac{\rho\beta}{\varphi(1-\beta)} < 1$$

or equivalently

$$\frac{\rho}{\varphi} < 1 - \beta$$

Again, let $z_t = \log(A_t)$ and $w_t = \log(R_t)$

$$\begin{aligned} \mathcal{M}(z, \dot{z}, w, \dot{w}) &= (1-\alpha)z + \alpha\beta \left[w + \log\left(\eta\left(1 - \frac{e^w}{R}\right)\right) - \dot{w} \right] \\ + \alpha(1-\beta) \log\left(1 - \frac{\dot{z}}{\lambda(\mu-1)} - \frac{\eta}{B}\left(1 - \frac{e^w}{R}\right) + \frac{\dot{w}}{B}\right) \end{aligned} \quad (29)$$

The maximization problem is to solve:

$$\max \int_0^\infty \mathcal{M}(z_t, \dot{z}_t, w_t, \dot{w}_t) e^{-\rho t} dt$$

Considering interior solutions, we have the following Euler-Lagrange equations:

$$\frac{d}{dt} \left[\frac{\partial \mathcal{M}}{\partial \dot{z}_t} e^{-\rho t} \right] = \frac{\partial \mathcal{M}}{\partial z_t} e^{-\rho t} \quad (30)$$

$$\frac{d}{dt} \left[\frac{\partial \mathcal{M}}{\partial \dot{w}_t} e^{-\rho t} \right] = \frac{\partial \mathcal{M}}{\partial w_t} e^{-\rho t} \quad (31)$$

Let

$$G(\dot{z}, w, \dot{w}) = \left(1 - \frac{\dot{z}}{\lambda(\mu-1)} - \frac{\eta}{B} \left(1 - \frac{e^w}{R} \right) + \frac{\dot{w}}{B} \right)$$

One can check that (30) can be written as

$$\frac{d}{dt} \left(-\frac{\alpha(1-\beta)}{\lambda(\mu-1)} \times \frac{1}{G(\dot{z}, w, \dot{w})} e^{-\rho t} \right) = (1-\alpha)e^{-\rho t} \quad (32)$$

while (31) can be written as:

$$\begin{aligned} & \frac{d}{dt} \left[\left(-\frac{\alpha\beta}{\eta(1-\frac{e^w}{R})-\dot{w}} + \frac{\alpha(1-\beta)}{G(\dot{z}, w, \dot{w})} \times \frac{1}{B} \right) e^{-\rho t} \right] \\ &= \left(\alpha\beta - \frac{\alpha\beta}{\eta(1-\frac{e^w}{R})-\dot{w}} \times \frac{\eta}{R} e^w + \frac{\alpha(1-\beta)}{G(\dot{z}, w, \dot{w})} \times \frac{\eta}{RB} e^w \right) e^{-\rho t} \end{aligned} \quad (33)$$

We will show that

$$\lim_{T \rightarrow +\infty} \int_0^{\infty} [\mathcal{M}(z^*, \dot{z}^*, w^*, \dot{w}^*) - \mathcal{M}(z, \dot{z}, w, \dot{w})] e^{-\rho t} dt \geq 0$$

where $z_t = \log(A_t^*)$, $w_t = \log(R_t^*)$, $\dot{z}_t = \log(A_t)$, $\dot{w}_t = \log(R_t)$. The variables A_t^* , R_t^* are given in the statement of Proposition 2. The variables A_t , R_t satisfy the dynamic equations (15) and (16).

We have

$$\begin{aligned} & \mathcal{M}(z^*, \dot{z}^*, w^*, \dot{w}^*) - \mathcal{M}(z, \dot{z}, w, \dot{w}) \geq \\ & (1-\alpha)(z_t^* - z_t) + \alpha\beta(w_t^* - w_t) + \frac{\alpha\beta}{\eta(1-\frac{e^{w^*}}{R})-\dot{w}^*} \left[-\frac{\eta}{R}(e^{w^*} - e^w) - (\dot{w}^* - \dot{w}) \right] \\ & + \frac{\alpha(1-\beta)}{G(\dot{z}^*, w^*, \dot{w}^*)} \left[-\frac{1}{\lambda(\mu-1)}(\dot{z}^* - \dot{z}) + \frac{\eta}{BR}(e^{w^*} - e^w) + \frac{\dot{w}^* - \dot{w}}{B} \right] - \frac{\alpha\beta}{\eta(1-\frac{e^{w^*}}{R})-\dot{w}^*} (\dot{w}_t^* - \dot{w}_t) \\ & = (1-\alpha)(z_t^* - z_t) + \alpha\beta(w_t^* - w_t) + \left(\frac{\alpha\beta}{\eta(1-\frac{e^{w^*}}{R})-\dot{w}^*} - \frac{\alpha(1-\beta)}{G(\dot{z}^*, w^*, \dot{w}^*)B} \right) \frac{\eta}{R} (-e^{w^*} + e^w) \\ & - \frac{\alpha(1-\beta)}{G(\dot{z}^*, w^*, \dot{w}^*)} \times \frac{1}{\lambda(\mu-1)} (\dot{z}^* - \dot{z}) + \frac{\alpha(1-\beta)}{G(\dot{z}^*, w^*, \dot{w}^*)B} (\dot{w}^* - \dot{w}) - \frac{\alpha\beta}{\eta(1-\frac{e^{w^*}}{R})-\dot{w}^*} (\dot{w}_t^* - \dot{w}_t) \end{aligned}$$

On the one hand, we have:

$$-e^{w^*} + e^w \geq -e^{w^*} (w^* - w)$$

On the other hand, from Lemma 6, we have:

$$\left(\frac{\alpha\beta}{\eta(1-\frac{e^{w^*}}{R})-\dot{w}^*} - \frac{\alpha(1-\beta)}{G(\dot{z}^*, w^*, \dot{w}^*)B} \right) > 0$$

Hence

$$\left(\frac{\alpha\beta}{\eta(1-\frac{e^{w^*}}{R})-\dot{w}^*} - \frac{\alpha(1-\beta)}{G(\dot{z}^*, w^*, \dot{w}^*)B} \right) \times \frac{\eta}{R} (-e^{w^*} + e^w)$$

$$> - \left(\frac{\alpha\beta}{\eta(1 - \frac{e^{w^*}}{R}) - \dot{w}^*} - \frac{\alpha(1-\beta)}{G(\dot{z}^*, w^*, \dot{w}^*)B} \right) \times \frac{\eta}{R} e^{w^*} (w^* - w)$$

and

$$\begin{aligned} & \mathcal{M}(z^*, \dot{z}^*, w^*, \dot{w}^*) - \mathcal{M}(z, \dot{z}, w, \dot{w}) \geq \\ & \left[(1-\alpha)(z_t^* - z_t) - \frac{\alpha(1-\beta)}{\lambda(\mu-1)} \frac{1}{G(\dot{z}^*, w^*, \dot{w}^*)} (\dot{z}^* - \dot{z}) \right] \\ & + \alpha\beta(w_t^* - w_t) - \left(\frac{\alpha\beta}{\eta(1 - \frac{e^{w^*}}{R}) - \dot{w}^*} - \frac{\alpha(1-\beta)}{G(\dot{z}^*, w^*, \dot{w}^*)B} \right) \times \frac{\eta}{R} e^{w^*} (w^* - w) + \frac{\alpha(1-\beta)}{G(\dot{z}^*, w^*, \dot{w}^*)B} (\dot{w}^* - \dot{w}) \\ & - \frac{\alpha\beta}{\eta(1 - \frac{e^{w^*}}{R}) - \dot{w}^*} (\dot{w}_t^* - \dot{w}_t) \end{aligned}$$

Let

$$\Delta_T = \int_0^T [\mathcal{M}(z^*, \dot{z}^*, w^*, \dot{w}^*) - \mathcal{M}(z, \dot{z}, w, \dot{w})] e^{-\rho t} dt \geq I_T + J_T$$

where

$$\begin{aligned} I_T &= \int_0^T \left[(1-\alpha)(z_t^* - z_t) - \frac{\alpha(1-\beta)}{\lambda(\mu-1)} \frac{1}{G(\dot{z}^*, w^*, \dot{w}^*)} (\dot{z}^* - \dot{z}) \right] e^{-\rho t} dt \\ J_T &= \int_0^T \left[\alpha\beta(w_t^* - w_t) - \left(\frac{\alpha\beta}{\eta(1 - \frac{e^{w^*}}{R}) - \dot{w}^*} - \frac{\alpha(1-\beta)}{G(\dot{z}^*, w^*, \dot{w}^*)B} \right) \times \frac{\eta}{R} e^{w^*} (w^* - w) \right] e^{-\rho t} dt \\ &+ \int_0^T \frac{\alpha(1-\beta)}{G(\dot{z}^*, w^*, \dot{w}^*)B} (\dot{w}^* - \dot{w}) e^{-\rho t} dt - \int_0^T \frac{\alpha\beta}{\eta(1 - \frac{e^{w^*}}{R}) - \dot{w}^*} (\dot{w}_t^* - \dot{w}_t) e^{-\rho t} dt \end{aligned}$$

Computing I_T , we get:

$$\begin{aligned} I_T &= \int_0^T \left[(1-\alpha)e^{-\rho t} + \frac{\alpha(1-\beta)}{\lambda(\mu-1)} \frac{d}{dt} \left(\frac{1}{G(\dot{z}^*, w^*, \dot{w}^*)} e^{-\rho t} \right) \right] (z_t^* - z_t) dt \\ &\quad - \frac{\alpha(1-\beta)}{\lambda(\mu-1)} \left[\frac{1}{G(\dot{z}^*, w^*, \dot{w}^*)} (z_t^* - z_t) e^{-\rho t} \right]_0^T \end{aligned}$$

Using (32), we have:

$$I_T = - \frac{\alpha(1-\beta)}{\lambda(\mu-1)} \left[\frac{1}{G(\dot{z}^*, w^*, \dot{w}^*)} (z_t^* - z_t) e^{-\rho t} \right]_0^T = - \frac{\alpha(1-\beta)}{\lambda(\mu-1)} \frac{z_T^* - z_T}{G(\dot{z}_T^*, w_T^*, \dot{w}_T^*)} e^{-\rho T}$$

since $z_0^* = z_0 = \log(A_0)$ and $G(\dot{z}_0^*, w_0^*, \dot{w}_0^*) = L_{x0}^* = \frac{\rho}{\varphi}$. Observe that we have:

$$\begin{aligned} G(\dot{z}_T^*, w_T^*, \dot{w}_T^*) &= L_{xT}^* = \frac{\rho}{\varphi} \\ \log(A_0) &\leq z_T^* \leq \log(A_0) + \lambda(\mu-1)T \end{aligned}$$

and

$$\log(A_0) \leq z_T \leq \log(A_0) + \lambda(\mu - 1)T$$

Therefore, we obtain:

$$\lim_{T \rightarrow +\infty} I_T = 0$$

Now consider J_T . We have:

$$\begin{aligned} J_T &= \int_0^T \alpha\beta(w_t^* - w_t)e^{-\rho t} dt + \frac{\alpha(1-\beta)}{B} \left\{ \left[\frac{(w_t^* - w_t)e^{-\rho t}}{G(\dot{z}^*, w^*, \dot{w}^*)} \right]_0^T - \int_0^T (w_t^* - w_t) \frac{d}{dt} \left(\frac{e^{-\rho t}}{G(\dot{z}^*, w^*, \dot{w}^*)} \right) dt \right\} \\ &\quad - \int_0^T \left(\frac{\alpha\beta}{\eta(1 - \frac{e^{w^*}}{R}) - \dot{w}^*} - \frac{\alpha(1-\beta)}{G(\dot{z}^*, w^*, \dot{w}^*)B} \right) \times \frac{\eta}{R} e^{w^*} (w^* - w) \\ &\quad - \left[\frac{\alpha\beta}{\eta(1 - \frac{e^{w^*}}{R}) - \dot{w}^*} (w_t^* - w_t) e^{-\rho t} \right]_0^T + \int_0^T \alpha\beta(w_t^* - w_t) \frac{d}{dt} \left(\frac{e^{-\rho t}}{\eta(1 - \frac{e^{w^*}}{R}) - \dot{w}^*} \right) dt \end{aligned}$$

Now using (33), and observing that $G(\dot{z}^*, w^*, \dot{w}^*) = L_x^*$, $\eta(1 - \frac{e^{w^*}}{R}) - \dot{w}^* = BL_m^*$, and $w = w_0$ we get

$$J_T = \left[\frac{\alpha(1-\beta)}{BL_x^*} - \frac{\alpha\beta}{BL_m^*} \right] (w_T^* - w_T) e^{-\rho T}$$

We have:

$$\begin{aligned} \log R_0 - bT &\leq \log R_T^* = w_T^* \leq \log \bar{R} \\ \log R_0 - bT &\leq \log R_T = w_T \leq \log \bar{R} \end{aligned}$$

which implies

$$\begin{aligned} (\log R_0 - bT) e^{-\rho T} &\leq w_T^* e^{-\rho T} \leq \log \bar{R} e^{-\rho T} \\ (\log R_0 - bT) e^{-\rho T} &\leq w_T e^{-\rho T} \leq \log \bar{R} e^{-\rho T} \end{aligned}$$

Hence, we can induce that:

$$\lim_{T \rightarrow \infty} w_T^* e^{-\rho T} = \lim_{T \rightarrow \infty} w_T e^{-\rho T} = 0$$

These imply:

$$\lim_{T \rightarrow +\infty} J_T = 0$$

And finally, we obtain:

$$\lim_{T \rightarrow +\infty} \Delta_T \geq 0$$

That is the end of the proof.

■

Proof of Proposition 3

On the BGP, we have $A_t = A_0 e^{tg_A}$ where A_0 is the initial level of technology and g_A is the constant rate of growth of technology. Given that the rate of growth of resources $g_R = \eta \left(1 - \frac{\hat{R}_t}{R}\right) - B\hat{L}_m$ is constant on the BGP then $\hat{R} = \bar{R} \left(1 - \frac{B\hat{L}_m}{\eta}\right)$ is also constant. Note that the condition $0 \leq \hat{L}_m \leq \frac{\eta}{B}$ must hold for $R \geq 0$.

The utility on the BGP is:

$$\begin{aligned} U &= (1 - \alpha) \int_0^\infty [\log(A_0) + tg_A] e^{-\rho t} dt + \alpha\beta \int_0^\infty \log(B) e^{-\rho t} dt \\ &+ \alpha\beta \int_0^\infty \log(\hat{L}_m) e^{-\rho t} dt + \alpha\beta \int_0^\infty \log(\hat{R}) e^{-\rho t} dt \\ &+ \alpha(1 - \beta) \int_0^\infty \log(\hat{L}_x) e^{-\rho t} dt \end{aligned}$$

Using $\int_0^\infty t e^{-\rho t} dt = \frac{1}{\rho^2}$, $\int_0^\infty e^{-\rho t} dt = \frac{1}{\rho}$ and noting $g_A = \lambda(\mu - 1)\hat{L}_r = \lambda(\mu - 1)(1 - \hat{L}_x - \hat{L}_m)$ the above utility function becomes:

$$\begin{aligned} \rho U &= (1 - \alpha) \log(A_0) + \alpha\beta \log(B) + \alpha\beta \log(\hat{L}_m) \\ &+ \alpha\beta \log(\bar{R}) + \alpha\beta \log\left(1 - \frac{B\hat{L}_m}{\eta}\right) + \alpha(1 - \beta) \log(\hat{L}_x) \\ &+ \frac{(1 - \alpha)\lambda(\mu - 1)(1 - \hat{L}_x - \hat{L}_m)}{\rho} \end{aligned}$$

We now maximize U with respect to \hat{L}_x and \hat{L}_m . The first order conditions with respect to these choice variables give:

$$\frac{B}{\eta - B\hat{L}_m} = \frac{1}{\hat{L}_m} - \frac{(1 - \alpha)\lambda(\mu - 1)}{\alpha\beta\rho} \quad (34)$$

and

$$\hat{L}_x = \frac{\alpha\rho(1 - \beta)}{(1 - \alpha)\lambda(\mu - 1)} = \frac{\rho}{\varphi} \quad (35)$$

The left hand side (LHS) of equation (34) is increasing in \hat{L}_m , equal $\frac{B}{\eta}$ when $\hat{L}_m = 0$ and approaching $+\infty$ when $\hat{L}_m \rightarrow \frac{\eta}{B}$. The right hand side (RHS) is, in contrast, decreasing in \hat{L}_m , approaching $+\infty$ when $\hat{L}_m \rightarrow 0$ and equal $\frac{B}{\eta} - \frac{(1 - \alpha)\lambda(\mu - 1)}{\alpha\beta\rho}$ when $\hat{L}_m = \frac{\eta}{B}$. Hence, this equation has a unique solution $\hat{L}_m \in (0, \frac{\eta}{B})$.

Actually, equation (34) can be solved explicitly to get the expression of \hat{L}_m . Indeed, rearranging (34) gives a quadratic equation of \hat{L}_m :

$$B\lambda(\mu - 1)(1 - \alpha)\hat{L}_m^2 - [2B\alpha\beta\rho + \lambda(\mu - 1)(1 - \alpha)\eta]\hat{L}_m + \alpha\beta\rho\eta = 0$$

Noting that $\varphi = \frac{(1-\alpha)\lambda(\mu-1)}{\alpha(1-\beta)}$, this equation can be rewritten as:

$$B(1-\beta)\varphi\hat{L}_m^2 - [2B\beta\rho + (1-\beta)\varphi\eta]\hat{L}_m + \beta\rho\eta = 0$$

This equation has two distinct real roots but one of them has to be ruled out due to violating $\hat{L}_m < \frac{\eta}{B}$.⁴ The accepted root is $\hat{L}_m = \frac{2B\beta\rho + (1-\beta)\varphi\eta - \sqrt{\Delta}}{2B(1-\beta)\varphi}$ where $\Delta = 4B^2\alpha^2\beta^2\rho^2 + (1-\beta)^2\varphi^2\eta^2$.

Having obtained \hat{L}_x and \hat{L}_m , we can calculate $\hat{L}_r = 1 - \hat{L}_x - \hat{L}_m = 1 - \frac{2B\rho + (1-\beta)\varphi\eta - \sqrt{\Delta}}{2B(1-\beta)\varphi}$. Since $\hat{L}_x \geq 0$, $\hat{L}_m \geq 0$, the condition $\hat{L}_r \geq 0$ or $\frac{2B\rho + (1-\beta)\varphi\eta - \sqrt{\Delta}}{2B(1-\beta)\varphi} \leq 1$ is sufficient for $L_x, L_m, L_r \leq 1$. Using these results, the growth rates of technology, natural resources, output, and consumption are calculated as follows:

$$g_A = \lambda(\mu - 1)\hat{L}_r$$

$$g_R = 0$$

$$g_Y = g_C = (1 - \alpha)g_A = (1 - \alpha)\lambda(\mu - 1)\hat{L}_r$$

■

Proof of Proposition 4

We consider dynamic equations (16) and (18). We can write them as follows

$$\begin{aligned} \frac{\dot{R}_t}{R_t} &= G(L_{mt}, R_t) \\ \frac{\dot{L}_{mt}}{L_{mt}} &= H(L_{mt}, R_t) \end{aligned}$$

where

$$G(L_{mt}, R_t) = \eta\left(1 - \frac{R_t}{R}\right) - BL_{mt}$$

$$H(L_{mt}, R_t) = \frac{\varphi\alpha(1-\beta) + B\alpha\beta}{\alpha\beta}L_{mt} + \frac{\alpha(1-\beta)\eta}{\alpha\beta L_{xt}R}L_{mt}R_t - \rho - \frac{\eta R_t}{R}$$

The BGP (\hat{L}_m, \hat{R}) are solutions to the system:

$$G(L_{mt}, R_t) = 0, H(L_{mt}, R_t) = 0$$

However, we can introduce the variables $w_t = \log(R_t)$, $v_t = \log(L_{mt})$. To simplify notations, we drop the time subscript t unless there is a confusion. The dynamic system becomes:

$$\dot{w} = \eta\left(1 - \frac{e^w}{R}\right) - Be^v$$

⁴The ruled out root is $L_m = \frac{2B\alpha\beta\rho + \lambda(\mu-1)(1-\alpha)\eta + \sqrt{\Delta}}{2B\lambda(\mu-1)(1-\alpha)}$.

$$\dot{v} = \left[\frac{\varphi(1-\beta)+B\beta}{\beta} \right] e^v + \frac{(1-\beta)\eta}{\beta L_x R} e^v e^w - \rho - \frac{\eta}{R} e^w$$

The BGP satisfies:

$$\eta \left(1 - \frac{e^w}{R}\right) - B e^v = 0$$

$$\left[\frac{\varphi(1-\beta)+B\beta}{\beta} \right] e^v + \frac{(1-\beta)\eta}{\beta L_x R} e^v e^w - \rho - \frac{\eta}{R} e^w = 0$$

The Jacobian matrix of the system is:

$$J = \begin{bmatrix} -\frac{\eta e^w}{R} & -B e^v \\ \frac{(1-\beta)\eta}{\beta L_x R} e^v e^w - \frac{\eta e^w}{R} & \left[\frac{\varphi(1-\beta)+B\beta}{\beta} \right] e^v + \frac{(1-\beta)\eta}{\beta L_x R} e^v e^w \end{bmatrix}$$

Let λ_1 and λ_2 be eigen values of vector J then they will be solutions to the following equation:

$$\lambda^2 - tr(J)\lambda + \det(J) = 0$$

It can be seen that λ_1 and λ_2 will satisfy:

$$\lambda_1 + \lambda_2 = -tr(J)$$

$$\lambda_1 \lambda_2 = \det(J)$$

Evaluating at BGP value, we obtain:

$$tr(J) = \rho$$

$$\det(J) = \frac{\eta e^w}{R} \left[B e^v \left(\frac{(1-\beta)L_m^*}{\beta L_x^*} - 1 \right) - \rho - \frac{\eta e^w}{R} \right]$$

Because $\left(\frac{(1-\beta)L_m^*}{\beta L_x^*} - 1 \right) < 0$ according to Lemma 5, $\det(J) < 0$. This means that λ_1 and λ_2 take real value with $\lambda_1 > 0$ and $\lambda_2 < 0$. The BGP is saddle-point. Since the problem is not concave, we will prove that the stable manifold is actually optimal.

Now take a feasible path on the stable manifold and close to the BGP. Denoting it by (z^*, w^*) , we have:

$$L_{mt}^* = \frac{1}{e^{\int_0^t b(u) du} \left(c_1 - \int_0^t h(x) e^{-\int_0^x b(u) du} dx \right)}$$

in which $c_1 \geq \int_0^\infty h(x) e^{-\int_0^x b(u) du} dx$ and

$$\begin{aligned} h(x) &= \frac{(1-\alpha)\lambda(\mu-1)}{\beta\alpha} + B + \frac{\eta(1-\beta)}{\beta L_{xt}^*} \times \frac{R_x^*}{R} \\ &= \frac{(1-\alpha)\lambda(\mu-1)}{\beta\alpha} + B + \frac{\eta(1-\beta)\varphi}{\beta\rho} \times \frac{R_x^*}{R} \end{aligned}$$

We claim that $c_1 = \int_0^\infty h(x)e^{-\int_0^x b(u)du} dx$. Indeed, we have

$$\int_0^\infty h(x)e^{-\int_0^x b(u)du} dx = B \int_0^\infty e^{-\int_0^x b(u)du} dx + \frac{\varphi(1-\beta)}{\rho\beta}$$

. Assume $c_1 = a + B \int_0^\infty e^{-\int_0^x b(u)du} dx + \frac{\varphi(1-\beta)}{\rho\beta}$ with $a > 0$ then

$$\begin{aligned} e^{\int_0^t b(u)du} \left(c_1 - \int_0^t h(x)e^{-\int_0^x b(u)du} dx \right) &= ae^{\int_0^t b(u)du} + Be^{\int_0^t b(u)du} \int_t^\infty e^{-\int_0^x b(u)du} dx + \frac{\varphi(1-\beta)}{\rho\beta} \\ &\geq ae^{\int_0^t b(u)du} dx \rightarrow +\infty \text{ when } t \rightarrow +\infty \end{aligned}$$

Therefore $L_{mt}^* \rightarrow 0$ when $t \rightarrow +\infty$. Because we are on the stable manifold at which L_{mt}^* converges to a strictly positive value, we have a contradiction. Thus $c_1 = \int_0^\infty h(x)e^{-\int_0^x b(u)du} dx$ and

$$L_{mt}^* = \frac{1}{Be^{\int_0^t b(u)du} \int_t^\infty e^{-\int_0^x b(u)du} dx + \frac{\varphi(1-\beta)}{\rho\beta}}$$

We compute again

$$\Delta_T = \int_0^T [\mathcal{M}(z^*, \dot{z}^*, w^*, \dot{w}^*) - \mathcal{M}(z, \dot{z}, w, \dot{w})] e^{-\rho t} dt$$

where (z, w) is a feasible path. The technique in the proof of Proposition 2 can be used again to obtain:

$$\Delta_T \geq I_T + J_T$$

where

$$I_T = -\frac{\alpha(1-\beta)}{\lambda(\mu-1)} \left[\frac{1}{G(\dot{z}^*, w^*, \dot{w}^*)} (z_t^* - z_t) e^{-\rho t} \right]_0^T = -\frac{\alpha(1-\beta)}{\lambda(\mu-1)} \frac{z_T^* - z_T}{L_{xt}^*} e^{-\rho T}$$

and

$$J_T = \left[\frac{\alpha(1-\beta)}{BL_x^*} - \frac{\alpha\beta}{BL_m^*} \right] (w_T^* - w_T) e^{-\rho T}$$

We still have:

$$z_0^* = z_0 = \log(A_0)$$

$$L_{x0}^* = \frac{\rho}{\varphi}$$

and

$$\lim_{t \rightarrow \infty} I_T = 0$$

We also have:

$$\begin{aligned} \log R_0 - bT &\leq \log R_T^* = w_T^* \leq \log \bar{R} \\ \log R_0 - bT &\leq \log R_T = w_T \leq \log \bar{R} \end{aligned}$$

which implies

$$\begin{aligned} (\log R_0 - bT)e^{-\rho T} &\leq w_T^* e^{-\rho T} \leq \log \bar{R} e^{-\rho T} \\ (\log R_0 - bT)e^{-\rho T} &\leq w_T e^{-\rho T} \leq \log \bar{R} e^{-\rho T} \end{aligned}$$

Because L_{mt}^* converges to a steady state which is strictly positive, we get:

$$\lim_{T \rightarrow +\infty} J_T = 0$$

This means that the optimal path locally converges to the BGP.

■

Proof of Proposition 5

(i) When resources are renewable:

When λ (or μ) increases, the RHS of (34) decreases while its LHS does not change. The graph of the RHS shifts down implying that \hat{L}_m decreases. It is obvious that in this case \hat{L}_x decreases. Hence, \hat{L}_r increases since $\hat{L}_r = 1 - \hat{L}_x - \hat{L}_m$. As a result, g_Y increases.

When ρ increases, \hat{L}_x increases as per (35). The graph of the RHS of (34) shifts up while the LHS does not change implying an increase in \hat{L}_m . Therefore, \hat{L}_r decreases and, thus, g_Y decreases as well.

As for the welfare effects, we have:

$$\begin{aligned} \frac{\partial U}{\partial \lambda} &= \frac{1}{\rho} \cdot \frac{\partial \hat{L}_m}{\partial \lambda} \left[\frac{\alpha\beta}{\hat{L}_m} - \frac{B\alpha\beta}{\eta - B\hat{L}_m} - \frac{(1-\alpha)\lambda(\mu-1)}{\rho} \right] \\ &+ \frac{1}{\rho} \cdot \frac{\partial \hat{L}_x}{\partial \lambda} \left[\frac{\alpha(1-\beta)}{\hat{L}_x} - \frac{(1-\alpha)\lambda(\mu-1)}{\rho} \right] \\ &+ \frac{(1-\alpha)(\mu-1)(1-\hat{L}_x-\hat{L}_m)}{\rho^2} \end{aligned}$$

Observe that the first two terms are equal to zero along the optimal BGP as per (34) and (35). Hence, $\frac{\partial U}{\partial \lambda} = \frac{(1-\alpha)(\mu-1)(1-\hat{L}_x-\hat{L}_m)}{\rho^2} > 0$. Similarly, we have

$$\frac{\partial U}{\partial \mu} = \frac{(1-\alpha)\lambda(1-\hat{L}_x-\hat{L}_m)}{\rho^2} > 0 \text{ and } \frac{\partial U}{\partial \rho} = -\frac{1}{\rho}U - \frac{(1-\alpha)(\mu-1)(1-\hat{L}_x-\hat{L}_m)}{\rho^3} < 0.$$

(ii) When resources are non-renewable:

It is obvious from (23) and (24) that when λ (or μ) increases, \hat{L}_x and \hat{L}_m both decrease meaning \hat{L}_r increases and g_Y increases (because $g_Y = (1-\alpha)\lambda(\mu-1)\hat{L}_r - \alpha\beta B\hat{L}_m$). By contrast, when ρ increases, \hat{L}_x and \hat{L}_m both increase implying \hat{L}_r decreases and g_Y decreases.

Regarding the welfare effect, we have:

$$\begin{aligned}
\frac{\partial U}{\partial \lambda} &= \frac{1}{\rho} \cdot \frac{\partial \hat{L}_m}{\partial \lambda} \left[\frac{\alpha\beta}{\hat{L}_m} - \frac{B\alpha\beta}{\rho} - \frac{(1-\alpha)\lambda(\mu-1)}{\rho} \right] \\
&+ \frac{1}{\rho} \cdot \frac{\partial \hat{L}_x}{\partial \lambda} \left[\frac{\alpha(1-\beta)}{\hat{L}_x} - \frac{(1-\alpha)\lambda(\mu-1)}{\rho} \right] \\
&+ \frac{(1-\alpha)(\mu-1)(1-\hat{L}_x-\hat{L}_m)}{\rho^2}
\end{aligned}$$

With a note that the first two terms are zero according to (23) and (24) then $\frac{\partial U}{\partial \lambda} = \frac{(1-\alpha)(\mu-1)(1-\hat{L}_x-\hat{L}_m)}{\rho^2} > 0$. A similar result applies for μ as $\frac{\partial U}{\partial \mu} = \frac{(1-\alpha)\lambda(1-\hat{L}_x-\hat{L}_m)}{\rho^2} > 0$. However, $\frac{\partial U}{\partial \rho} = -\frac{1}{\rho}U - \frac{\alpha\beta B\hat{L}_m}{\rho^3} - \frac{(1-\alpha)(\mu-1)(1-\hat{L}_x-\hat{L}_m)}{\rho^3} < 0$.

■

Proof of Proposition 6

(i) When resources are renewable:

From (34) and (35), it can be seen that an increase in B does not affect \hat{L}_x . However, it reduces \hat{L}_m as the graph of the LHS of (34), which is increasing in \hat{L}_m , shifts up while the RHS of that equation, which is decreasing in \hat{L}_m , stays the same. Hence, \hat{L}_r rises and so does g_Y .

With respect to welfare, using (34), we have $\frac{\partial U}{\partial B} = \frac{\alpha\beta\hat{L}_m}{\rho B} \left[\frac{1}{\hat{L}_m} - \frac{B}{\eta - B\hat{L}_m} \right] = \frac{(1-\alpha)\lambda(\mu-1)\hat{L}_m}{\rho^2 B} > 0$ meaning welfare rises with B .

(ii) When resources are non-renewable:

Plugging the values of \hat{L}_m and \hat{L}_r from (24) and (25) into the equation for the growth rate of output along the BGP we obtain:

$$g_Y = \lambda(\mu-1)(1-\alpha) - \alpha\rho$$

It can be seen that B does not appear in the result for g_Y . Hence, an increase in B does not have any impact on long-run output growth.

As for the welfare, we have $\frac{\partial U}{\partial B} = \frac{\alpha\beta}{\rho} \left[\frac{1}{B} - \frac{\hat{L}_m}{\rho} \right]$. From (24), it can be figured out that $\hat{L}_m < \frac{\rho}{B}$. Therefore, $\frac{\partial U}{\partial B} > 0$ or U is increasing in B .

■

Proof of Proposition 7

Under renewable resources, output growth is:

$$g_Y = \alpha(1-\beta)\varphi - \alpha\rho + \frac{\alpha\sqrt{\Delta} - \alpha(1-\beta)\varphi\eta}{2B}$$

where $\Delta = 4B^2\alpha^2\beta^2\rho^2 + (1-\beta)^2\varphi^2\eta^2$. Under non-renewable resources, output growth is:

$$g_Y = \alpha(1-\beta)\varphi - \alpha\rho$$

Clearly, $\sqrt{\Delta} - (1 - \beta)\varphi\eta \geq 0$ implying a generally higher output growth for renewable resource case. The two rates are equal only when $\sqrt{\Delta} - (1 - \beta)\varphi\eta = 0$ or $\beta = 0$ meaning there is absolutely no utilization of natural resources in intermediate good production.

Obviously, along the optimal BGP for renewable resources, output growth is non-negative. Along the optimal BGP for non-renewable resources, output growth may be negative if:

$$(1 - \beta)\varphi = \frac{(1-\alpha)\lambda(\mu-1)}{\alpha} < \rho$$

However, this condition violates what stated in Lemma 5 so negative growth will not occur. Hence, we always have $\lambda(\mu - 1) > \frac{\alpha\rho}{1-\alpha}$ implying that as soon as the R&D sector being sufficiently productive, positive growth will be sustained no matter what type of resources is employed for production.

■

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