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## Shedding Some Light on a Possible Origin of a Concept of Fractions in China

Division as a link between the newly discovered manuscripts and *The Gnomon of the  
Zhou [Dynasty]*

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For Menso,  
as an expression of friendship and appreciation

The earliest extant mathematical documents in Chinese, which date from the end of the third century BCE or the beginning of the second century BCE, attest to the use of fractional quantities and of algorithms allowing practitioners to carry out arithmetical operations on them. These fractions occur either standing alone, or attached to an integral quantity, which is in turn expressed with respect to a single measurement unit or a sequence of such units.<sup>2</sup> Fractions standing alone can be either a fraction of a measurement unit (e. g., “six sevenths of a *zhu*, 七分銖六 *qi fen zhu liu*,” a *zhu* being the smallest unit of weight used in these texts), or a fraction of an abstract unit (e. g., “two fifths 五分二 *wu fen er*”). In all these cases, fractions are always smaller than the smallest unit used to express the quantity (whether it is a measurement unit or an abstract unit).

A terminology was shaped to designate these fractions. Although there were variations in the terminology attested to between the end of the third century BCE and the beginning of the Common Era,<sup>3</sup> its main features remained constant, and they also characterize the

- 1 The research leading to these results received funding from the European Research Council under the European Union's Seventh Framework Programme (FP7/2007–2013) / ERC Grant agreement n. 269804 “Mathematical Sciences in the Ancient World (SAW)”. I would like to thank the members of the SAW research group, and especially Zhu Yiwen, for their questions about, and comments on, previous presentations of some of the ideas put forward here. I am also grateful to the participants in the SAW workshop about cultures of computation and quantification held between January and March 2013. The collective discussions helped me refine my initial conclusions. My heartfelt thanks to Karen Margolis for her contribution to the completion of this article.
- 2 An example can be found in the answer to the division stated in Text 1 (see translation in the appendix to this chapter).
- 3 I mention here only the general terminology for ordinary fractions, leaving aside special terms used for a small set of fractions used more often ( $1/2$ ,  $2/3$ ,  $2/3$ ). I have dealt with some of them, e. g. *ban* 半 “half,” in a glossary of technical terms used in mathematics, published in Karine Chemla and GUO Shuchun, *Les neuf chapitres. Le Classique mathématique de la Chine ancienne et ses commentaires*, Paris 2004, p. 899. Rémi Anicotte, «Nombres et expressions numériques en Chine à l'éclairage des *Ecrits sur les calculs* (début du 2e siècle avant notre ère)» (Ph. D. Thesis, INALCO (Institut National des Langues et Civilisations Orientales)), 2012 (29 September), examines systematically the variety in the formulation of fractions in one of the earliest extant mathematical books in Chinese and sketches later changes. For the oldest extant mathematical writings, the topic was touched upon in Guo Shuchun 郭書春, «*Shilun Suanshushu de lilun gongxian yu bianzuan* 試論算數書的理論貢獻與編纂 (On the theoretical achievements and the

expressions used in Chinese until the present day. The fractions  $6/7$  *zhu* and  $6/7$ , where 6 and 7 stand for any ordered pair of integers and *zhu* for any possible measurement unit, are usually referred to as 七分銖之六, *qi fen zhu zhi liu*, literally, “six (parts) of the *zhu* divided into seven parts,” and 七分之六, *qi fen zhi liu*, literally, “six (parts) of a division (of the unit) into seven parts.” This shows that the idea of cutting a unit into equal parts plays a key role in the expression of a fraction. Further, numbers like 6 and 7 are designated as, respectively, “numerator” and “denominator” (in Chinese, respectively, 子 *zi* and 母 *mu*, literally, “child” and “mother”). The algorithms that enabled practitioners to operate on fractions usually make use of these two terms to describe the elementary operations that must be applied to numerators and denominators to carry out arithmetical operations.

Until recently, the only mathematical documents in Chinese available to historians were books handed down through the written tradition. As a result, it was impossible to address the issue of the origin of this concept of a fraction. In recent decades, a significant number of new mathematical documents were found in tombs sealed around the time of the establishment of the Chinese empire or were bought on the antiquities market. Historians have also suggested dating the latter from around the early decades of the Chinese empire. The texts of some of these manuscripts have already been published. They show that division was an operation to which practitioners in early imperial China devoted much attention.<sup>4</sup> In particular, I have recently argued that these documents allow us to perceive a shift in how divisions were done between the end of the third century BCE and the first century CE. They also show the key theoretical role played by a newly established way of carrying out divisions.<sup>5</sup> The present article argues that the concept of fraction outlined above, as it can be identified through the terminology used to refer to fractions, presents striking correlations with the main ideas at play in an older way of executing division that the newly found documents enable us to reconstruct. The key point is that, throughout the procedure, this execution of division gives pride of place to changes of units, and in particular, divisions of units into smaller units. Reconstructing this older observable procedure to execute division

compilation of the *Book of Mathematical Procedures*», *Faguo hanxue* 法國漢學 (French Sinology), 6 (2002), p. 522–525, and more recently XIAO Can 肖燦, «Yuelu shuyuan Qin jian «Shu» yanjiu 嶽麓書院藏秦簡《數》研究 (Research on the Qin strips «Mathematics» kept at the Academy Yuelu)» (Ph.D. Thesis in History, Hunan University 湖南大學), 2011 p. 120–121. Karine Chemla, «Les fractions comme modèle formel en Chine ancienne», in Paul Benoit, Karine Chemla, Jim Ritter (eds.), *Histoire de fractions, fractions d'histoire* Basel 1992, p. 188–207, and Li Jimin 李繼閔, «Zhongguo gudai de fenshu lilun 中國古代的分數理論 (The theory of fractions in ancient China)», in WU Wenjun 吳文俊 (ed.), *Jiuzhang suanshu' yu Liu Hui 九章算術與劉徽 (The Nine Chapters on Mathematical Procedures and Liu Hui)* Beijing 北京 1982, p. 190–209, describe the basic features of fractional quantities and the accompanying arithmetic attested to in the earliest extant books handed down through the written tradition (see below).

- 4 Karine Chemla, 2013a, “Observing mathematical practices as a key to mining our sources and conducting conceptual history. Division in ancient China as a case study”, in Léna Soler, Sjoerd Zwart, Michael Lynch, Vincent Israël-Jost (eds.), *Science after the Practice Turn in Philosophy, History, and the Social Studies of Science*, Routledge, 2013 (forthcoming, preprint online at <http://halshs.archives-ouvertes.fr/halshs-00803425>), sketches some changes in mathematical practices involving division that can be perceived through the available documents and addresses the issue of what is at stake in these changes.
- 5 Karine Chemla, 2013b, “Working on and with division in early China” (preprint distributed at the conference held in the context of the workshop Cultures of Computation and Quantification, Paris, March 25–29, 2013) describes in detail the differences between two types of division used in early imperial China that can be reconstructed. A revised version of this preprint will be published in a forthcoming book, *Cultures of Computation and Quantification*. I do not repeat the argument here.

thus allows me to advance a hypothesis: the concept of a fraction to which the oldest extant mathematical documents in Chinese attest might have arisen in relation to, or even in the context of, this earlier mode of dividing we can observe in ancient China.<sup>6</sup>

My argument will be presented in several steps. Firstly, I outline the corpus on which I rely, and outline the reasons why I suggest there was a shift in the execution of divisions in the timespan mentioned. Secondly, I analyze a text dating probably from around 100 BCE, which in my view sheds light on the entire process of execution in this older mode of doing a division. In particular, it allows me to distinguish four phases in the division process. Thirdly, focusing on phases 2 and 3, and then on phase 1, I establish two facts. In the earliest extant documents we can find procedures to execute phase 1 or phases 2 and 3. These procedures show in which ways modifying units and, in particular, dividing units into smaller units was an essential operation in this part of the process. Finally, I show why phase 4, in which fractions are introduced as the final component of the result of a division, appears to be consistent with the other phases of execution of this type of division, from the perspective of the idea that appears to inspire the final operation. We shall also note a shift in the terminology used to introduce fractions that echoes a shift in the formulation of phase 3. This, too, reveals a strong connection between the introduction of fractions and the wider context of execution of divisions. To facilitate following the argument, I shall give the Chinese critical edition and the translation of the documents on which I rely in an appendix to this article.

#### A corpus attesting to a shift in the execution of division

Until recently, the earliest mathematical writings in Chinese available to historians were “classics” that *Li Chunfeng* 李淳風 (602–670) and his associates selected in the first half of the seventh century to compile an anthology. This collection, entitled *Ten Mathematical Classics*, was presented to the throne in 656,<sup>7</sup> and had a significant influence in shaping the set of early mathematical books that were actually handed down through the written tradition. I shall consider here the two oldest of these classics: *The Gnomon of the Zhou [Dynasty]* (*Zhou bi* 周髀) and *The Nine Chapters on Mathematical Procedures* (*Jiuzhang suanshu* 九章算術).

*The Gnomon of the Zhou [Dynasty]* deals with mathematical knowledge useful in the context of astral sciences.<sup>8</sup> Historians have expressed different theses about the completion

6 Incidentally, this result shows the benefits that can be derived from an approach such as that developed in the context of the SAW project, where our aim is to reconstruct as accurately as possible the mathematical cultures in the context of which actors practiced mathematics.

7 Two critical editions of the anthology were published in recent decades: QIAN Baocong 錢寶琮, *Suanjing shishu* 算經十書 (*Qian Baocong jiaodian* 錢寶琮校點) (*Critical punctuated edition of The Ten Classics of Mathematics*), 2 vols., Beijing 北京 1963; GUO Shuchun 郭書春 and LIU Dun 劉鈍, *Suanjing shishu* 算經十書 (*The Ten Classics of Mathematics*). GUO Shuchun, LIU Dun dianjiao 郭書春, 劉鈍 點校 (*Punctuated critical edition by Guo Shuchun and Liu Dun*), 2 vols., Shenyang 瀋陽 1998. Man-Keung Siu and Alexei Volkov, “Official Curriculum in Traditional Chinese Mathematics: How did Candidates Pass the Examinations?”, *Historia Scientiarum*, 9 (1999), p. 85–99, describe how, shortly after being presented to the throne, the anthology was used in the context of mathematical education.

8 QIAN Baocong, *The Ten Classics of Mathematics*, p. 11–80, gives a critical edition of *The Gnomon of the Zhou [Dynasty]* and some ancient commentaries on the classic: the ancient commentaries selected by *Li Chunfeng* and his associates (*Zhao Shuang*’s third century commentary and *Zhen Luan*’s sixth century commentary) as well as the subcommentary they composed in the context of the anthology. GUO Shuchun

date of the book. QIAN Baocong suggested that the book was composed around 100 BCE (QIAN, *The Ten Classics of Mathematics*, p. 3–4), whereas Cullen opposed the idea of a single date of composition (Cullen, *Astronomy and mathematics in ancient China*, p. 138–156). Instead, Cullen suggested distinguishing between various sections in *The Gnomon of the Zhou [Dynasty]*. He argued further that these sections were written at different time periods, some dating from the first century BCE, and others from the early first century CE. As for GUO Shuchun and LIU Dun, in their introduction to *The Ten Classics of Mathematics*, they suggest that the main part of the book took shape at the latest around 100 BCE, and perhaps much earlier, although it underwent minor changes later on. We shall indicate below an essential criterion for assessing these different viewpoints.

Whatever the case may be, historians usually agree on the fact that *The Gnomon of the Zhou [Dynasty]* is older than *The Nine Chapters on Mathematical Procedures* (the second book from the seventh-century mathematical anthology that I shall use and whose title I abbreviate below to *The Nine Chapters*).<sup>9</sup> Qian Baocong has suggested that *The Nine Chapters* took shape in the first century (early Eastern Han dynasty, a dynasty established in 25). I have suggested other clues supporting this conclusion. The discussion in this article incidentally provides another angle from which to address this question.<sup>10</sup>

In the last three decades, archeological discoveries have dramatically changed the state of documentary evidence attesting to mathematical activities in early imperial China. In addition to books handed down through the written tradition, historians can now rely on manuscripts found in tombs during archeological excavations or, unfortunately, obtained by illegal digs. The manuscripts on which I rely here include:

- the *Book on Mathematical Procedures* (*Suanshu shu* 算數書), which was found in a tomb sealed in around 186 BCE at Zhangjiashan 張家山 (Hubei 湖北 Province).<sup>11</sup>
- *Mathematics* (*Shu* 數), a manuscript bought on the antiquities market in 2007 and probably written at the end of the third century BCE, in the context of the Qin kingdom or the Qin dynasty (221 BCE–206 BCE), which established a Chinese Empire for the first time.<sup>12</sup>

and Liu Dun, *The Ten Classics of Mathematics*, volume 1, includes a critical edition of *The Gnomon of the Zhou [Dynasty]* and only some of its ancient commentaries. Christopher Cullen, *Astronomy and mathematics in ancient China: the Zhou bi suan jing*, *Needham Research Institute Studies* 1, ed. Christopher Cullen, Cambridge [UK], New York 1996, contains a translation of *The Gnomon of the Zhou [Dynasty]* into English.

<sup>9</sup> QIAN Baocong, *The Ten Classics of Mathematics*, p. 91–258; GUO Shuchun and LIU Dun, *The Ten Classics of Mathematics*, volume 1, contain a critical edition of *The Nine Chapters*, Liu Hui's commentary completed in 263 and included in the seventh-century anthology, as well as the subcommentary by Li Chunfeng et al. Shen Kangshen, John N. Crossley, and Anthony W.-C. Lun, *The Nine Chapters on the Mathematical Art. Companion and Commentary*, Oxford and Beijing 1999, gives an English translation of these three layers of text. Chemla and GUO, *Les Neuf Chapitres*, provide a critical edition and a French translation of *The Nine Chapters* and the previously mentioned commentaries.

<sup>10</sup> See, respectively, Qian Baocong, *The Ten Classics of Mathematics*, p. 83, and my introduction to Chapter 6, in Chemla and GUO, *Les Neuf Chapitres*, p. 475–478. In this case, too, Guo Shuchun suggests a much earlier date of composition.

<sup>11</sup> PENG Hao 彭浩, *Zhangjiashan hanjian “Suanshu shu” zhushi* 張家山漢簡《算數書》注釋 (*Commentary on the Book of Mathematical Procedures, a writing on bamboo strips dating from the Han and discovered at Zhangjiashan*), Beijing 北京 2001, provides an annotated critical edition of the book.

<sup>12</sup> ZHU Hanmin 朱漢民 and CHEN Songchang 陳松長 zhubian 主編 (eds.), *Yuelu shuyuan cang Qin jian (er)* 嶽麓書院藏秦簡 (貳) (*Qin Bamboo slips kept at the Academy Yuelu (2)*), Shanghai 上海 2011, contains a reproduction of the manuscript and an annotated critical edition. See also Xiao Can, *Research on the Qin strips “Mathematics”*. Other manuscripts have been found, but have yet to be published.

My discussion in the present article will also evoke other mathematical writings, which were likewise found in the context of illegal digs and acquired by Beijing University in 2010, in a lot that historians also date from the Qin period. I rely on the single article published on them so far.<sup>13</sup> The manuscripts found apparently contain several mathematical documents, some written on the same slips as other texts. I shall concentrate on the two pieces that *Han Wei* refers to as *Mathematical writing. Text A* (*Suan shu. Jia pian* 算書. 甲篇) and *Mathematical writing. Text B* (*Suan shu. Yi pian* 算書. 乙篇), since, in contrast to the two manuscripts mentioned above, there is no record of any title given to these writings by practitioners.

The discovery and publication of these manuscripts has revealed something very important: if we compare *The Nine Chapters* and the available manuscripts from the viewpoint of the terminology used to prescribe divisions, we observe that, aside from obvious continuity, a key change occurred.<sup>14</sup> The *Book of Mathematical Procedures* illustrates the terminology used in all the manuscripts fairly well. I shall rely on it to outline both continuities and the change mentioned. The slip number in the examples below refers to the edition of the text in PENG Hao, *Commentary on the Book of Mathematical Procedures*. In the *Book of Mathematical Procedures*, in contrast to other arithmetical operations, division is never prescribed using a verb. Rather, three sets of expression are used, and for two of them, variants are still evidenced in *The Nine Chapters*.

A first set prescribes division using expressions of the following type:

- “(...), 十而一 (...) /slip 98/  
one divides by ten,” literally *shi er yi* “ten then one.”  
This mode of prescription includes, in the *Book of Mathematical Procedures*, variants that are not found in *The Nine Chapters*, such as:
- “(...), 二尺七寸而一石 (...) /slip 146/  
dividing by two *chi* seven *cun* gives the result in *dan*,” literally *er chi qi cun er yi dan* “two *chi* seven *cun* then one *dan*.” (Here *chi* and *cun* are measurement units for volume and *dan* a measurement unit for value – features I shall not comment on, as they are not part of the present topic under consideration.)
- “(...) 三成一。 (...) /slip 115/ (...)  
one divides by three,” literally  
*san cheng yi* “three produces one”

Expressions in the second set evidenced in the *Book of Mathematical Procedures* use the common technical terms referring to the operands of division in the following ways:

13 HAN Wei 韓巍, “Beida Qin jian zhong de shuxue wenxian 北大秦簡中的数学文献 (Mathematical documents from the Qin slips of Beijing University)”, *Wenwu* 文物 (Cultural Relics), 2012, no. 6, p. 85–89, provides an initial description of these documents.

14 I summarize these changes here, referring the reader to my revised forthcoming version of Chemla, 2013b, “Working on and with division in early China,” for a more complete description, and to Chemla, 2013a, “Observing mathematical practices,” for a discussion of what is at stake in these terminological transformations. Guo Shuchun, “On the theoretical achievements,” p. 525–527, has compared the terminology for the prescription of division in the *Book of Mathematical Procedures* and *The Nine Chapters*. However, his approach to this terminology is focused on the presence or absence, in the various expressions, of the terms for the operands “dividend *shi* 實,” and “divisor *fa* 法,” and consequently misses the changes in the expressions prescribing division itself. XIAO Can, *Research on the Qin strips “Mathematics”*, p. 121–122, follows the same pattern in her description of features of this terminology in *Mathematics*.

- “實如法而一。/slip 67/  
one divides the dividend by the divisor,” literally, *shi ru fa er yi* “dividend like divisor then one.”
- “實如法得...。/slip 92/  
dividing the dividend by the divisor yields (statement of result with measurement units),” literally *shi ru fa de...* “dividend like divisor yields (...)”

Some expressions in this set are variants that do not appear in *The Nine Chapters*, such as:

- “(...) 以為法, (...) 以為實, 如法而一尺。(...) /slip 56/  
is taken as divisor (...) is taken as dividend, dividing gives (the result in the measurement unit) *chi*,” literally *ru fa er yi chi* “like divisor then one *chi*.” (Here, *chi* is a measurement unit for length.)
- “(...) 為法, (...) 為實, 實如法得尺。不盈尺者十之, 如法一寸 /slip 42/  
(...) makes the divisor, (...) makes the dividends. (Each time) the dividend is like the divisor (i. e., dividing the dividend by the divisor), this yields a *chi*. What does not fill up the *chi* (the remainder of the dividend), one ten-folds it. (Each time) it is like the divisor (i. e., dividing by the divisor yields), one *cun*” literally, for the division, *shi ru fa de chi* “dividend like divisor yields *chi*,” *ru fa yi cun* “like divisor, one *cun*” (here, *chi* and *cun* are measurement units for lengths).

*The Nine Chapters* and the manuscripts share both ways of prescribing division, which clearly manifests continuity between them. Further, despite differences between the two sets, these expressions have features in common. They all refer to a comparison between the dividend and the divisor, stating that when a part of the dividend is equal to the divisor, the operation yields a unit. Moreover, the nature of the unit thereby obtained in the quotient is often made explicit (that is, the expression prescribing the operation often involves making clear which measurement unit is attached to the units yielded). Even though this latter feature is evidenced in *The Nine Chapters*, in the latter book the expressions used more frequently refer to an abstract unit.<sup>15</sup>

These two features also characterize the third set of expressions by means of which divisions are prescribed in the *Book of Mathematical Procedures*. It is this third mode of prescription where the key difference between *The Nine Chapters* and the manuscripts occurs. This type of prescription includes expressions of the following kind:

- “除十六而得一。/slip 78/  
Dividing by sixteen yields the result,” literally *chu shiliu er de yi* “eliminate sixteen then yields one”
- “除, 如法得從一步, (...)。/slip 167/  
Dividing yields the result for the length in *bu*,” literally, *chu, ru fa de zong yi bu* “eliminate, like divisor yields, for the length, one *bu*.”

Such expressions also occur in the Qin manuscript *Mathematics*, for instance, in /slip 0789/ and /slip 0855/ (these are the excavation numbers, the editors’ numbers being 169 and 170), as well as in the numerous slips dealing with the same topic (Zhu Hanmin and Chen Songchang (eds.), *Qin Bamboo slips kept at the Academy Yuelu*, p. 22–24, 119–125). Likewise,

15 The distribution of this feature in the book, and its variations in the different ancient editions, await further research.

these expressions occur in the Qin mathematical text kept at Beijing University and entitled *Mathematical writing. Text A*, in slip 4–178 (Han Wei, “Mathematical documents from the Qin slips of Beijing University,” p. 87.) The prescriptions of division in the third set of expressions are similar to those listed in the first two sets, the main difference being the use of the verb *chu* in the sense of “eliminating,” that is “subtracting.” My claim is that here, as in all its other occurrences in the manuscripts, the verb *chu* only refers to a subtraction. This has two consequences. First, the operation of division to which these expressions refer clearly involves a subtraction, which the prescriptions make explicit. The subtraction is that of the divisor from the dividend, which yields a unit in the quotient. The idea made explicit in these prescriptions concurs with what is indicated by the other forms of prescription mentioned above: they, too, refer to a repeated subtraction of the divisor. Secondly, in *The Nine Chapters*, no such expression is ever used to prescribe a division. By contrast, this classic attests to the emergence of a new way of prescribing a division using precisely the same verb *chu*: in *The Nine Chapters*, *chu* has acquired a new meaning, since it can now be used alone to prescribe a division. This fact is easily established by observing the following expressions that occur in the parallel procedures for solving similar problems – the problems 6 and 7 of chapter 9:

- (9.6) “(...) 餘，倍出水除之，即得水深。 (...) *yu, bei chu shui chu zhi, ji de shui shen.* (...) The remainder, twice what comes out of the water divides it (*chu zhi*), thus yielding the depth of the water.”
- (9.7) “以去本自乘，令如委數而一。 *yi qu ben zicheng, ling ru wei shu er yi.* The distance from the base being multiplied by itself, one makes the division (of this) by the quantity that lies on the ground,” literally, *ling ru x er yi*, “one carries out ‘if (this) is like x then 1’.”

In these two corresponding prescriptions, the procedure following problem 9.7 makes use of a formulation attached to the first set identified above, whereas the procedure following problem 9.6 uses only the verb *chu*.<sup>16</sup> *Chu* is never used in this way in the excavated manuscripts. Note that with the change of meaning of *chu*, division can now, like all the other operations, be prescribed by a single verb. In “Observing mathematical practices,” I explore various facets of the key change to which the terminological transformation attests. In particular, I provide evidence to support the thesis that there is a new algorithm, which relies on a new number system, corresponding to the new way of prescribing the operation of division. This algorithm, I argue, is the one for the operation of division *chu*, for which the *Mathematical Classic by Master Sun* (completed around 400 CE) provides evidence. Moreover, I show that the algorithm to which *chu* “to divide” refers plays a key part in the reorganization of a set of key operations. I shall not develop any of these theses here, but shall rather restrict myself to the consideration of an older procedure for division to which, in my view, the former ways of prescribing correspond.

16 Note that the meaning of *chu* as “subtract” occurs, albeit seldom, in *The Nine Chapters*. In these rare cases, *chu* is clearly distinct from the prescription of a division. Further, it is revealing that in one of these cases, the commentator takes care to explain the meaning of “subtraction,” as if he perceived it as an archaism. By contrast, commentators never comment on the meaning of *chu* as “to divide.” See the entry *chu* “diviser, éliminer,” in my glossary of mathematical terms, Chemla and GUO, *Les Neuf Chapitres*, p. 911.



Establishing this change in the terminology, and in the practice, of the operation of division has striking consequences for the analysis of *The Gnomon of the Zhou [Dynasty]*. The observation of the prescriptions of division using *chu* in this classic shows that *The Gnomon of the Zhou [Dynasty]* uses a terminology similar to that of the recently excavated manuscripts and different from that of *The Nine Chapters*. We shall see below instances of these uses of *chu* similar to the third set of expressions analyzed in the *Book of Mathematical Procedures*. This shows in a striking way that a text like *The Gnomon of the Zhou [Dynasty]*, handed down through the written tradition, presents strong correlations with source material excavated recently from tombs sealed some decades before the date of 100 BCE, that is, the date to which *Qian Baocong* ascribes the composition of the text. This somehow supports *Qian Baocong*'s conclusion, in particular with respect to the fact that *The Gnomon of the Zhou [Dynasty]* is a text older than *The Nine Chapters*. This also shows that the process of transmission of *The Gnomon of the Zhou [Dynasty]* did not modernize the terminology. In fact, we can obtain more precise conclusions on this point. Even more striking is the following fact: two passages in the received text of *The Gnomon of the Zhou [Dynasty]* seem to contradict my conclusion. Indeed, these passages use *chu* to prescribe an operation in a similar way to *The Nine Chapters*, which is nowhere to be found in the known Qin and Han manuscripts. However, in these two places, the third century commentator *Zhao Shuang* comments in relation to the section in which these expressions occur that it does not belong to the original text of *The Gnomon of the Zhou [Dynasty]*.<sup>17</sup> Therefore there is a striking correlation between the use of a terminology I suggest regarding as “modern,” which occurs in only two passages, and the fact that the third century commentator expresses doubts as to the authenticity of the passages in question. This observation inspires us to devote particular attention to the historical information *Zhao Shuang* gives us about the text. Moreover, examination of the terminology prescribing operations appears to offer a powerful tool for dating the various sections of *The Gnomon of the Zhou [Dynasty]*. I shall not embark on this task here.

One of the key mathematical topics on which *The Gnomon of the Zhou [Dynasty]* provides abundant information is precisely the execution of divisions. The execution to which the classic attests can be correlated with evidence found in the manuscripts. We can thus rely on this batch of evidence to reconstruct an algorithm of division attached to the older terminology and probably used by the practitioners who wrote and used writings such as

17 A first occurrence of *chu* with the meaning of “to divide” occurs in *QIAN Baocong, The Ten Classics of Mathematics*, p. 77–79. *Zhao Shuang* writes about this passage “非周髀本文。蓋人問師之辭。其欲知度之所分，法術之所生。 This is not from the original text of *The Gnomon of the Zhou [Dynasty]*. Probably these are the words of someone asking a teacher. He wants to know about the division of the *du* and that from which procedures originate.” A second occurrence, related to the new use of *chu*, is in *Qian Baocong, The Ten Classics of Mathematics*, p. 27, in the context of the dialogue between *Chen zi* (Master Chen) and *Rong Fang* (the section Cullen, *Astronomy and mathematics in ancient China*, p. 175 calls B) (*QIAN Baocong, The Ten Classics of Mathematics*, p. 23–42.) Again, *Zhao Shuang* writes at the beginning of the section: “非周髀本文... This is not from the original text of *The Gnomon of the Zhou [Dynasty]*...” Cullen suggests not giving any value to the testimony provided by *Zhao Shuang*'s commentary. Contrary to the testimony, he considers section B as the “core ‘original text of the *Zhoubi*’ round which the rest of the material was assembled.” (p. 175). His idea of considering the date of *The Gnomon of the Zhou [Dynasty]* section by section is excellent. However, his conclusion about section B probably needs to be reconsidered.

our manuscripts or *The Gnomon of the Zhou [Dynasty]*.<sup>18</sup> We shall see that this algorithm presents striking correlations with the meaning of the expressions often used to prescribe division. In particular, the “units” play key parts in it. Further, the evidence in the manuscripts sheds light on how actors understood its steps. On the other hand, this algorithm is completely different from the algorithm for the division *chu* contained in the *Mathematical Classic by Master Sun*. In the following section, I examine in detail the algorithm described by *The Gnomon of the Zhou [Dynasty]*, as a preliminary to looking for parallel evidence in the manuscripts. The parallel evidence will be then used to perceive the meaning actors attached to its steps. We shall see that changes of units and, in particular, the dissection of units into smaller units is essential in them. My goal is to show that this key operation permeates the algorithm throughout, and the related idea can be correlated with the concept of fraction to which the terms described above refer.

#### An execution of division according to *The Gnomon of the Zhou [Dynasty]*

I shall explain and analyze here the division whose execution is described in detail in the passage from *The Gnomon of the Zhou [Dynasty]* translated as text 1 in the Appendix. Its purpose is to divide a length, given in the measuring unit of length *li*, by a corresponding duration, expressed in days. The quotient to be found measures the length covered in one day. The operands to which the operation is applied are thus both measured quantities, as is the quotient sought.<sup>19</sup> It will prove useful below to know that the larger measurement unit of the *li* 里 (ca. 415 m in the Han Dynasty), often used in the context of astronomical measurements, contained 300 *bu* 步 (step). More concretely, the text explains how to execute the following division:<sup>20</sup>

$$\frac{119000 \text{ li}}{182 \text{ days} + \frac{5}{8} \text{ days}}.$$

After having positioned the operands “dividend” and “divisor,” probably on a surface on which numbers were represented with counting rods,<sup>21</sup> the *first phase* of the division consists in transforming them, in correlation with each other, into integers attached to no measuring units. *The Gnomon of the Zhou [Dynasty]* prescribes this set of operations by a single verb: *tong* 通 “to make communicate.” The term is used in exactly the same way in *The Nine*

18 Note that for the moment, nothing allows us to assume that the users of the manuscripts used a single algorithm for division. We focus here on one algorithm, for which we have similar evidence in *The Gnomon of the Zhou [Dynasty]* and the manuscripts, and which is not evidenced in *The Nine Chapters*.

19 It is important to note that, in the context we explore, the decimal expansion for quantities seems to play a role different from the one we assign to it when we execute arithmetical operations.

20 For the sake of clarity, my explanations use our (place-valued decimal) number system. This does not mean that I interpret such a system as the one on which practitioners operated at the time. In what follows, I shall give information about what we know about the expression of quantities in different phases of the execution.

21 Rods were used to represent numbers from the earliest known manuscripts onwards. However, this does not mean that they were used in the same way. Chemla, 2013a, “Observing mathematical practices,” explains why I doubt that the number system used in relation to the algorithm of division described in this section was the place-valued decimal system to which *The Nine Chapters* refers.

*Chapters* in the same context of dividing between quantities of that kind.<sup>22</sup> The third century commentator on *The Nine Chapters*, *Liu Hui* 劉徽, emphasizes the complex layers of meaning he reads in this term. First, the two quantities are “brought into communication” by the fact of becoming operands in the same division. Any transformation of one must thus also be applied to the other for the communication to be maintained. Further, the operation of “bringing into communication” also applies to the integer and fraction occurring in the divisor. The integer must be brought into communication with the fraction, in a transformation that must thus also be applied to the value taken as dividend. Even though *Zhao Shuang* does not comment in detail on the meaning of *tong* 通 “to make communicate,” his explanations regarding the problem to be solved and the operations prescribed by the term are in accordance with *Liu Hui*’s. For him, the operands bring together units that are not of the same size (“homogeneous with each other”). By this expression, he clearly refers to the opposition between integral numbers of units and fractions of unit, thus accounting for a multiplication of all entities by the denominator 8 that allows them to enter into the same operations (addition and then division). These operations do correspond to those carried out in *The Gnomon of the Zhou [Dynasty]*. To my knowledge, the term *tong* 通 “to make communicate” does not occur in the excavated mathematical writings. This is one of several details that reveal differences between *The Gnomon of the Zhou [Dynasty]* and the manuscripts. However, as we see in the next section, the corresponding procedure occurs in all of them.

At the end of this phase (phase 1), the operands on which to operate have been transformed into 952000 and 1461, that is, integers expressed using a decimal expansion and attached to no specific measurement units. The *second phase* prescribes: “... MAKES THE DIVISOR. ELIMINATING (*chu*) THIS, (*Zhao Shuang*’s commentary...) (EACH TIME) THE DIVIDEND IS LIKE THE DIVISOR, THIS YIELDS ONE *li*.” In contrast to the following part of the procedure, the text gives no detail here on how the first part of the quotient, 651 *li*, is obtained.<sup>23</sup> We shall thus wait to obtain more information about the procedure. However, we can already take note of a few details.

The first part of the quotient obtained is given as an amount corresponding to a measurement unit. In fact, in contrast to the operands at this stage of the operation, the quotient will be yielded *not* as a decimal expansion, to which in the end a single measurement unit would be attached, but rather bit by bit, each bit being attached to a finer measurement unit than the preceding one. The division thus brings into play various types of numbers in a controlled way. Understanding this point will prove essential to make sense of the following part of the text. Further, we note that the measurement unit corresponding to the first bit is the one made explicit by the prescription of the division in this phase (phase 2). The same feature will recur in the subsequent iterations of phase 2 in the process. There is a correlation between the way the quotient is obtained and the way the division is prescribed in its successive phases. We also note that “eliminate this” immediately follows the determination of the divisor and will not be repeated later, when phase 2 is iterated. The prescription is, in fact, common to all iterations of phase 2. The way of proceeding to which the text testifies indicates that, once

22 See the entry *tong* 通 “faire communiquer, mettre en communication, communiquer”, in my glossary in Chemla and GUO, *Les Neuf Chapitres*, p. 994–998.

23 Clearly, the practitioner chose an initial unit from which to begin the expression of the quotient. This feature will not be examined here, since, following the purpose of this article, I shall focus instead on the process of division after this point.

the values are “brought into communication,” the divisor remains constant throughout the execution, without – in contrast to the “dividend” – undergoing any change. Below, *Zhao Shuang*’s commentary emphasizes this point, which represents a key difference between this algorithm and the one that will later be associated with the division *chu*.<sup>24</sup> The prescription of the division in this phase 2 (“Eliminating this, each time [a bit of] the dividend is like the divisor, this yields one measuring unit *li*”) seems to indicate that the execution proceeds by subtracting the constant divisor from the value of the dividend at this point, and adding a unit to the quotient, attributing to the unit obtained the name of a specified measurement unit. This hypothesis will receive some support from the following part of the execution.<sup>25</sup> The divisor somehow gives the “pattern” according to which the decimal expansion of the dividend will be transformed into the (successive) measuring units of the quotient. This process recurs in each iteration of phase 2, with a different value of the dividend and a different measurement unit. However, the divisor remains constant, and its “elimination” is not repeated, whereas the changing units obtained for the quotient by applying the pattern of the divisor are made explicit at each step.<sup>26</sup> Interestingly enough, this is precisely the meaning of the term chosen to designate the divisor in Chinese: *fa* 法 “pattern, norm.”<sup>27</sup>

Once the first occurrence of phase 2 in the execution is completed, the part of the quotient determined so far is given as 651 *li*, and 889 remains in the dividend. The expression that the text of *The Gnomon of the Zhou [Dynasty]* uses to refer to the remainder (“what [from the dividend] does not fill up the divisor”) fits well with the interpretation suggested above: The formulation does indeed evoke the understanding that one casts the magnitude in the dividend into the mold of the divisor.<sup>28</sup> The amount in *li* in the quotient was found. The next part of the execution of the division in *The Gnomon of the Zhou [Dynasty]* aims at determining the component of the quotient with respect to the next, finer measurement unit, the *bu*. Here, probably because the gap between the *li* and the *bu* is of two orders of magnitude, the text of *The Gnomon of the Zhou [Dynasty]* provides highly interesting details on the execution and the expressions used to prescribe it.

As the commentator *Zhao Shuang* writes, “one should multiply [the remainder] by three hundred” to go on with the division and yield the number of *bu*. In other words, one needs to transform the remainder in order to continue with the division. The procedure with reference to which *Zhao Shuang* wrote his commentary seems to be the execution of division that was probably widespread at the time he was writing, and that was referred to by means of the single term *chu*. However, the text of *The Gnomon of the Zhou [Dynasty]* proceeds in a different way. It suggests, first, multiplying by 3. It is in such steps that we identify a *third phase*. We shall return to these phases below. For the moment, let us continue to focus

24 The sixth century commentator reformulated the procedure that solves the problem, using the division *chu*.

25 Note that this subtraction has as operands a dividend and a divisor which at this point of the execution are both abstract decimal expansions.

26 I noticed this subtlety of the text while attempting to account for why *Zhao Shuang* had placed his commentary between “eliminating this” and “[each time] the dividend is like the divisor...”

27 See the entry *fa* 法 “diviseur”, in my glossary, Chemla and GUO, *Les Neuf Chapitres*, p. 918–919.

28 We shall see that mathematical manuscripts use another formulation here, which is revealing. The expression found at this point in *The Gnomon of the Zhou [Dynasty]* is used in *The Nine Chapters*, but only when the integral part of the quotient has been determined. At this point in the execution of division to which *The Nine Chapters* attest, the divisor is in a decimal position corresponding to its actual value. This is always the case in the division to which *The Gnomon of the Zhou [Dynasty]* refers.

on the formulation of the sequence of phase 2 in this text. After the multiplication of the remainder by 3, the elimination of the divisor, in an iteration of phase 2, is then related to the following prescription: “[each time] it [i.e., the newly computed dividend] is like the divisor, this yields a hundred *bu*.” *Zhao Shuang* interprets that as follows: To avoid “moving” the divisor (as would be the case for the execution of the division *chu*, which yields the quotient digit by digit in a decimal expansion), the text computes, in a phase 3, a dividend corresponding to a single “digit”<sup>29</sup> for the quotient, one that corresponds to the hundreds of *bu* – hence the expression he introduces to designate this dividend: “the dividend of the hundreds.” What is important for us is the fact that, in *The Gnomon of the Zhou [Dynasty]*, the usual prescription for phase 2 encountered above is now placed in a context with other similar expressions. As a result, it becomes clear that the main point of the prescription is to make explicit which measurement units correspond to the part of the quotient yielded in this phase 2. The text of *The Gnomon of the Zhou [Dynasty]* proceeds twice more in a similar fashion, using similar expressions, and is twice interpreted by *Zhao Shuang* along the same lines. Once the value of the hundreds (or the tens, respectively) of *bu* is obtained, an iteration of phase 3 multiplies the remainder by 10 each time, and the next iteration of phase 2 is prescribed as: “[each time] it [i.e., the newly computed dividend] is like the divisor, this yields ten [respectively one] *bu*.”

In conclusion, the core part in the execution of the division proceeds through an alternation of phase 2 and phase 3, modifying the dividend and then applying to the new value the same subtraction (eliminating this) by the pattern of the divisor, to yield units for the quotient whose nature the prescription explicitly specifies at each iteration. This, in my view, supports the idea that the prescription of phases 2 is not a catch phrase, as will be the case later in *The Nine Chapters*.<sup>30</sup> In *The Gnomon of the Zhou [Dynasty]*, the prescription aims rather concretely at specifying which measuring units are produced by the repeated subtraction of the divisor in different iterations of phase 2. This is the crucial question in this way of dividing. This piece of evidence from *The Gnomon of the Zhou [Dynasty]* gives additional support to the interpretation that in each phase 2, the operation carried out is a repeated subtraction of the divisor. I would like to repeat that this interpretation fits well with the fact that, in this way of practicing division, the quotient is obtained bit by bit, along a sequence of measurement units. We shall see that the manuscripts contain similar types of evidence. The only difference is that in *The Gnomon of the Zhou [Dynasty]*, emphasis is placed on a decimal expansion in the quotient, in parallel to the expansion along a sequence of measurement units.

As a result of the alternation of phases 2 and 3, the result obtained is “SIX HUNDRED AND FIFTY-ONE *li* ONE HUNDRED AND EIGHTY-TWO *bu*” with a remainder of “SEVEN HUNDRED NINETY-EIGHT.” We suggest calling the *fourth phase* the operation concluding the execution. It prescribes the transformation of the remainder (in this case, 798) into a fractional quantity, using the sentence canonically used in later texts, including *The Nine Chapters*. It reads: “WHAT DOES NOT FILL UP THE DIVISOR [i.e., the remainder of the dividend], ONE NAMES THIS WITH

29 *Zhao Shuang* creates an anachronism here, reading the successive powers of 10 as related to digits or “positions.” However, the text of *The Gnomon of the [Zhou] Dynasty* does not mention this concept.

30 *The Nine Chapters* contains no example of a sequence of prescriptions as illustrated in successive phases 2. This, together with the rather frequent lack of specification of the prescription regarding the unit produced, might indicate that the same mode of prescribing a division has taken on a new meaning.

THE DIVISOR.” We recognize, in the first part of the sentence, the expression used throughout the text in phase 3 to refer to the remainder of the dividend. However, it is not followed by a prescription to multiply the remainder, but by a statement instead. The second part of the sentence prescribes changing the “name” of the units of the remainder, substituting for it a unit (a fraction of the last one dealt with for the quotient, e. g. in our case the *bu*) whose name derives from the divisor. The etymology of the term “denominator” is no different: “naming [the remainder] with the divisor” amounts to taking the divisor 1461 as the denominator for a fraction of a *bu*. As for the remainder 798, it is taken as “numerator,” counting the number of such parts that make up the final fraction. The resulting fractional part is formulated in Chinese, literally, as “*yì qiān sì bǎi liú shí yī fēn bu zhī qī bǎi jiǔ shí ba*—一千四百六十一分步之七百九十八 seven hundred ninety-eight (parts) of the *bu* divided into one thousand four hundred sixty-one parts.” This fractional part thus completes the statement of the quotient as “SIX HUNDRED AND FIFTY-ONE *LI* ONE HUNDRED AND EIGHTY-TWO *BU* SEVEN HUNDRED NINETY-EIGHT ONE THOUSAND FOUR HUNDRED SIXTY-FIRSTS OF A *BU*.” Here too, we shall see that the manuscripts differ in a striking way from what we find in *The Gnomon of the Zhou [Dynasty]*, which in turn, in this particular respect, is closer to later mathematical texts.

To conclude, we must emphasize that *The Gnomon of the Zhou [Dynasty]* describes the execution of several divisions in similarly generous detail. All these divisions share the same features we discovered in text 1. We have thus been dealing with a paradigm, and not something marginal.

### The evidence on the execution of division in the manuscripts

I will now show that the evidence provided by the manuscripts suggests that an execution of division to which they attest was similar to that described in *The Gnomon of the Zhou [Dynasty]*. At the same time, I will show they provide information as to how actors understood the various phases of this execution of division.

### Evidence about phases 2 and 3

Let us begin with the core of the procedure: phases 2 and 3. Slips 41 and 42 in the *Book of Mathematical Procedures* (see Text 2 in the appendix) illustrate an alternation of phases 2 and 3 to execute a division which follows exactly the same pattern as in *The Gnomon of the Zhou [Dynasty]*.

The procedure in the context of which detail is given on the execution prescribes, in fact, five divisions in parallel. Let us examine the process taking the example of the last division, which divides 160 *chi* by 62 and yields 2 *chi* 5 *cun* 50/62 *cun*.<sup>31</sup> In this case, the dividend

31 PENG Hao, *Commentary on the Book of Mathematical Procedures*, p. 56, note 1, points out that the manuscript erroneously gives the result as “*yì chī wú cūn liúshí’er fēn cūn wúshí* 一尺五寸六十二分寸五十一 *chi* five *cun* fifty sixty-seconds of a *cun*.” The first “*yì* — one” was erroneously written for “*er* 二 two.” Interestingly enough, in the previous answer, which is “*chī’er cūn liúshí’er fēn cūn wúshíliú* 尺二寸六十二分寸五十六 *chi* two *cun* fifty-six sixty-seconds of a *cun*,” there is no explicit numeral before the highest measurement unit *chi*, since it is equal to one.

is a length obtained as the result of an operation and apparently expressed as a decimal expansion with respect to a single measurement unit, the *chi*. This length is divided by a number with no measurement unit. As in *The Gnomon of the Zhou [Dynasty]*, the main part of the quotient (2 *chi* 5 *cun*) is obtained as an expansion along a sequence of units of length, first the number of *chi* and then that of the unit ten times finer, the *cun*. It could have been obtained as a decimal expansion with respect to the *cun*, or as a number of *chi* followed by a fraction. But this is not the case, and this is consistently so in all the examples quoted in Texts 3 and 4.

The text shows that the quotient as an expansion along a sequence of measurement units is obtained through a process with two iterations of phase 2 (“When the dividend is like the divisor [i. e., dividing the dividend by the divisor], this yields a *chi*”, “When it [i. e., the new value for the dividend] is like the divisor [i. e., dividing by the divisor yields], one *cun*”), alternating with a phase 3 and then, in the end, a phase 4.

However, in comparison with *The Gnomon of the Zhou [Dynasty]*, the formulation of phase 3 presents a highly interesting variation, which appears in *all* the mathematical manuscripts available. Instead of the formulation in the classic, “WHAT DOES NOT FILL UP THE DIVISOR [i. e., the remainder of the dividend], ONE THREE-FOLDS THIS,” the *Book of Mathematical Procedures* reads: “What does not fill up the *chi* [i. e., the remainder of the dividend], one ten-folds it.” In this context the *chi* is the measurement unit yielded by the phase 2 executed immediately before. In Text 3 from the manuscript entitled *Mathematics*, in the extant final part of a procedure on slip 0842 (editors’ number 47), we can read details on the execution of a division for the decimal system of units of capacity measurement. The procedure alternates two iterations of phase 2 and one phase 3. The sentence introducing phase 3, after the previous phase 2 has yielded a number of *dou*, is formulated in exactly the same way (“What does not fill up the *dou* [i. e., the remainder of the dividend], one ten-folds this.”). Finally, the same feature recurs in Text 4, also from *Mathematics*, in which the quotient is an expansion along a non-decimal system of units of weight measurement. The phase 3 reads: “What does not fill up the *liang*...”.

The situation to which these expressions refer is the same in *The Gnomon of the Zhou [Dynasty]* and the manuscripts, but its description consistently uses different formulations in each of the two contexts. We note that the same opposition accounts for the distinct formulations of the opening sentence of phase 4 in *The Gnomon of the Zhou [Dynasty]* and in the manuscripts. In the former, we read “WHAT DOES NOT FILL UP THE DIVISOR” whereas in the latter we read: “What does not fill up the *cun*...” (Text 2) or “What does not fill up the *zhu*...” (Text 4). We shall return to phase 4 in our conclusion.

In fact, these expressions retrospectively shed light on how phase 2 of the execution of a division was perceived. What can we gather about actors’ perception of phase 2 from the older terminology? We have suggested above that the terminology in *The Gnomon of the Zhou [Dynasty]* (including the phrase: “what does not fill up the divisor...”) insisted on the role of the divisor as a “pattern” to cast the dividend in a new mold that transformed it into new units. The formulation “what does not fill up the divisor” invites observation of the situation from the viewpoint of the mold, i. e. the divisor. This is a viewpoint that is constant for a pair of successive phases 2 and 3 in an execution of division. By contrast, the formulation of the description of the same situation in the manuscripts reveals an observer focused throughout on the units that the pattern *fa* generates in the dividend – the *result* of the

process of molding, rather than its *operator*. The divisor appears as the standard measuring the units of the quotient in the dividend. The emphasis is placed here on whether a given measuring unit for the quotient can be generated from the dividend or not. In this formulation, the measuring unit changes for each pair of subsequent phases 2 and 3 in the execution of a division. Interestingly enough, the older formulation insists on the division as a change of unit – the change of units in the dividend that yields the units in the quotient. This is a key point, since we shall see that, although this process of division is the same in *The Gnomon of the Zhou [Dynasty]* and in the manuscripts, the way it is formulated in the manuscripts shows throughout an emphasis of the execution as a sequence of changes of units, the units in the dividend and the units produced in the quotient.

To begin with, this way of perceiving the process can be demonstrated for the operation prescribed in phase 3, thanks to a piece of evidence found in *Mathematics* (Text 4.) Here, the quotient is a weight, obtained as a quantity of *liang*, followed by a quantity of *zhu* (1 *liang* = 24 *zhu*). Once the quantity of *liang* has been determined, phase 3 reads: “What does not fill up the *liang* [i. e., the remainder of the dividend], taking one as twenty-four, one multiplies this [the remainder].” Here, the prescription of the multiplication does not directly refer to the operation, as is the case in expressions of the type “one ten-folds this.” The verb designating the operation (multiply) is preceded by the outcome expected from the multiplication: “one takes one as twenty-four.” Such sentences often occur in these mathematical texts and, as we see below, they can be interpreted as meaning: “one dissects the units (in the dividend)” into twenty-four parts.<sup>32</sup> This change of unit in the dividend allows the pattern of the divisor, in the following phase 2, to yield *zhu*. Note that the choice of how to dissect the units in the remainder derives from the next unit to be determined in the quotient.

As a result, phases 2 and 3 each effect changes of unit in the dividend, albeit different: the former transforms the dividend into the new units of the quotient, whereas the latter transforms the units of the (new value of the) dividend<sup>33</sup> as a preparation for obtaining the subsequent bit of the quotient, with respect to the following measurement unit in the system.

### Evidence about phase 1

Phase 1 transforms a dividend and a divisor expressed with respect to various units into decimal expansions with the same ratio. This phase corresponds to several procedures in the manuscripts and in *The Nine Chapters*, in particular those executing the operation “Directly sharing” or “Directly dividing into parts” (*jing fen* 徑分). Note that the same term, *fen*, is used in the expression of fractions (七分銖之六, *qi fen zhu zhi liu*, literally, “six [parts] of the *zhu* divided into seven parts”). In the name of this operation, I take it to mean “to divide

32 On the relationship between such multiplications and dissections of units into parts (in Chinese *po* 破), see footnote 40 in Karine Chemla, “Documenting a process of abstraction in the mathematics of ancient China”, in Christoph Anderl and Halvor Eifring (eds.), *Studies in Chinese Language and Culture – Festschrift in Honor of Christoph Harbsmeier on the Occasion of his 60th Birthday*, Oslo 2006, p. 185 (online at <http://halshs.archives-ouvertes.fr/halshs-00133034>, <http://www.instphi.org/Festschrift.html>). Below, we see that in the manuscripts the term *fen* also refers to this dissection.

33 All our texts systematically use assignment of variables in the formulation of algorithms. Thus the term “dividend” refers to what is in the position of the dividend at the moment when the text refers to the position.



into parts.” However, to avoid confusion with the verb “to divide,” I also translate it as “to share.” The term *fen* plays a key part in the argument.

There is abundant evidence to show that the procedures corresponding to phase 1, in the overall process of divisions, were perceived as relying on a joint modification of the units in the dividend and the divisor that transforms them conjointly into integers. Pieces of evidence for this can be found in *Mathematics*, in the *Book of Mathematical Procedures* as well as in *Liu Hui*’s third century commentary on *The Nine Chapters*.<sup>34</sup> To illustrate this point, I shall examine here a passage from *Mathematics*, which also provides other interesting information.<sup>35</sup> It is listed as Text 5 in the appendix. Explanations are required for interpreting this passage. After providing them, I shall repeat the translation below with additional information.

The context of the passage is a problem concerning the calculation of tax on a field. In this case, the tax is to be paid in “bundles” according to a norm, stating that a bundle must be paid for each amount of area of 5 *bu*. Bundles are, in fact, measured with units of weight measurement. The area of the field dealt with in the problem is 16 *bu*, and the outline of the problem states that this corresponds to 5 *jin* of bundles (1 *jin* is 16 *liang*). These data allow the reader to determine that a bundle weighs 25 *liang*. However, the outline continues, on the official certificate the amount of 5 *jin* 1 *liang* was erroneously inscribed instead of 5 *jin*. The problem requires determination of the norm corresponding to this wrong amount. In other words, it “asks to diminish the quantity of *bu*” defining the norm, the question being “for how many [*bu* should the tax] be one bundle.” The result is given as  $4 + \frac{76}{81}bu$  (*si bu bashiyi fen qishiliu* 四步八十一分七十六 “four *bu* seventy-six (parts) of the [unit] divided into eighty-one parts”<sup>36</sup>). This is clearly obtained as the result of a rule of three: 16 *bu* yielding a tax of bundles weighing 81 *liang*, what is the norm, that is, the number of *bu* yielding the weight of a bundle, namely, 25 *liang*? The reason why this must be the case derives from the following part of the text, where the “inverse computations” are addressed.

This is the passage that interests us, and it occurs after the statement of the result for the new norm. The goal of the passage is to “restore” (*fu* 復) the 81 *liang* one started with, through operating inverse computations (starting from the weight of a bundle (25 *liang*) and the field to be taxed (16 *bu*), according to the norm just computed (one bundle has to be given per  $4 + \frac{76}{81}bu$ ), one aims at restoring the weight of the tax in bundles)<sup>37</sup>.

The operations to be carried out are thus:

$$\frac{16 \text{ bu times } 25 \text{ liang}}{4 + \frac{76}{81}bu}$$

34 See, for instance, Chemla and GUO, *Les Neuf Chapitres*, p. 166–168; Chemla, “Documenting a process of abstraction”.

35 *Mathematics*, Slips 0788+0775+0984 (editors’ numbers 29-30-31), in Zhu Hanmin and Chen Songchang (eds.), *Qin Bamboo slips kept at the Academy Yuelu*, p. 6, 48.

36 Probably the measuring unit *bu* has been omitted in the expression of the fractional part.

37 The interest in “restoring” the result of a division through applying the inverse multiplication is a concern that is evidenced in our corpus, from *Mathematics* up until the third century commentary by *Liu Hui* on *The Nine Chapters*.

The corresponding text reads as follows (let us translate it before analyzing it):

“欲復之，置一束兩數以乘兌(稅)田，而令以一為八十一為[尹+貝](實)，亦【令所粟步一為八十一，不分者，從之以為】法，[尹+貝](實)如法一兩。

“If one wants to restore this (the eighty-one *liang*), one puts the quantity of *liang* of one bundle and multiplies with (this) the cropland taxed, and **makes one to be taken as eighty-one** to make the dividend;<sup>38</sup> one **likewise** takes **one as eighty-one in the *bu*** that were diminished (weakened) [KC : in the quantity  $4 + \frac{76}{81}bu$ , each unit in the “4” becomes 81; this operation does not affect the fractional part: 步 designates only the completed *bu*]; **those that are not divided into parts/shared** [KC: namely the 76 that are not cut into 81 parts] join these (4 times 81), which is taken as divisor. Dividing the dividend by the divisor yields the result in *liang*.”

This text describes in some detail the transformation of the dividend (the product 16 *bu* times 25 *liang*) and that of the divisor ( $4 + \frac{76}{81}bu$ ), before the execution of the division is carried out on operands expressed as decimal expansions. Note, as we mentioned previously, that the transformation of the dividend is prescribed as a change of unit, similar to what was analyzed in relation to phase 3 (“one makes that one is taken as eighty-one”). A similar and corresponding transformation (“one *likewise*...”) occurs in the divisor. However, it is prescribed in a highly interesting way, revealing actors’ perception of the operations carried out and the entities involved. The transformation distinguishes two parts in the quantity in the divisor: The “four *bu*,” each unit of which must be divided into eighty-one parts, and the “seventy-six eighty-first of a *bu*”. Once the “four *bu*” have been dissected, yielding 324 new units, the quantity obtained can be directly added to the 76, which were in fact already expressed with respect to the same unit (one eighty-first of a *bu*). The key point here is that these 76 are referred to as “those that are not divided into parts (*fen*).” This simple sentence contains essential information for us. First, the formulation implies that, in the eyes of the practitioners using this text, the “four *bu*” are, by contrast, “divided into parts (*fen*).” This confirms the fact that the transformation of “four *bu*” and of the dividend amounts to a change of unit, which is carried out by dissection of each unit into “eighty-one parts.” I already mentioned other pieces of evidence to establish this point above. Secondly, the transformation of the units in the dividend and the divisor carried out in phase 1 is *understood* as being of the same type as the *dissection of the unit yielding the fractions*. The same term *fen* is used in both contexts and refers in both cases to a similar kind of action on units. This remark establishes a link between phase 1 and phase 4. Thirdly, the formulation further implies that once a fractional quantity has been obtained by dissection of a unit, it is perceived as a quantity of the new unit and no longer as being divided into parts.<sup>39</sup>

38 Note that the “unit” which must be transformed into 81 is that of the product.

39 Similar expressions to the one analyzed above occur in several manuscripts. This shows the stability of this perception of the operations and the entities. See, for instance, also in the context of the inversion of a division to restore a quantity one started with, *Book of Mathematical Procedures*, slip 166 (PENG Hao, *Commentary on the Book of Mathematical Procedures*, p. 116–123).

Conclusion: The meaning of phase 4 and a hypothesis on the origin of  
a concept of fractions in China

Let us recapitulate the results obtained so far. We have shown that *The Gnomon of the Zhou [Dynasty]* and the newly found manuscripts share expressions to prescribe division that are not recorded in *The Nine Chapters*, or, to my knowledge, in later mathematical writings. Further, accordingly, *The Gnomon of the Zhou [Dynasty]* attests to a procedure for performing division which can be characterized by the fact that quotients are yielded step by step in successive bits of measured quantity. We have also provided evidence showing the existence of executions of division in the manuscripts that very probably refer to the same procedure, even though the terminology exhibits differences in some places. In other words, *The Gnomon of the Zhou [Dynasty]* and the manuscripts share ways of prescribing divisions as well as ways of executing them. As I have shown elsewhere, this procedure differs from that attached to the prescription by the verb *chu* attested to by *The Nine Chapters* and later mathematical texts, but no earlier texts. This conclusion yields striking elements that can be used in determining the date of various sections that make up *The Gnomon of the Zhou [Dynasty]*, a classic that I must emphasize has been handed down through the written tradition. The conclusion shows that in the process of transmission, the terminology of *The Gnomon of the Zhou [Dynasty]* was not modernized or rendered consistent. It also confirms the idea that *The Gnomon of the Zhou [Dynasty]* is probably a compilation of sections written down in different milieus or at different time periods. Some of these sections date from a relatively early period. Finally, the conclusion is a strong recommendation for quite serious regard for Zhao Shuang's statements made in the third century about the "original text" of *The Gnomon of the Zhou [Dynasty]*.

The comparison between *The Gnomon of the Zhou [Dynasty]* and the manuscripts would not be complete, however, if we did not take note of the promotion, in *The Gnomon of the Zhou [Dynasty]*, of the decimal expansion as playing a part in the production of the quotient similar to the expansion along a sequence of measurement units. This point awaits further research.

Establishing the parallel between *The Gnomon of the Zhou [Dynasty]* and the manuscripts has provided evidence illuminating how the practitioners using the manuscripts perceived the process of division under consideration. As a result, it appeared that the overall process of the execution of a division was understood as proceeding from end to end through changes of unit.

Phase 1 dissects the units of the dividend and divisor in a parallel way until they simultaneously become integers expressed as decimal expansions. The iterations of phase 2 transform the dividend into new units using a pattern represented by the divisor. Alternating with the successive uses of phase 2, the iterations of phase 3 dissect units in the remainder of the dividend so that, in the following phase 2, the pattern represented by the divisor yields the part of the quotient corresponding to a unit finer than a previous one. At the end of the process of execution, phase 2 is followed instead by phase 4, which is prescribed using a sentence partly similar to that prescribing a phase 3. My contention is that phase 4, which introduces the fractional component of the result, is completely consistent with the previous phases of the division under consideration. This is the basis for my suggestion that the concept of fraction evidenced from at least the manuscripts onwards might have originated from this way of executing divisions.

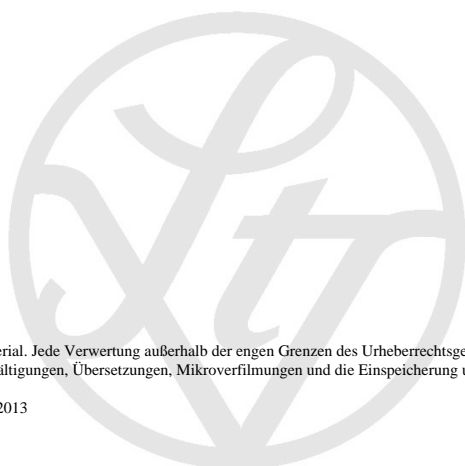
To highlight the continuity between phase 4 and the previous phases in the process, let us examine phase 3 more closely, taking Text 2 as an example. The division in Text 2 first uses the divisor as a standard to transform the units in the dividend into a number of unit *chi* (a unit of length measurement) in the quotient. Once this process is over, the division proceeds to determine how many instances of the *next* unit (*cun*, one tenth of the *chi*) the quotient contains. In other words, one must determine how many instances of the next unit can be found in the remainder of the dividend, using the divisor as a pattern. To do so, the remaining units in the dividend are dissected into 10 parts (*fen*, as we have shown). This is the meaning of the multiplication of the dividend by 10. We have noted that the coefficient ruling the dissection of the units in the dividend derives from the coefficient linking the unit in the quotient just dealt with and the finer unit to be addressed. As a result of the dissection, the same pattern of the divisor yields units ten times smaller (i. e., *cun*) in the quotient.

Now, the prescription of phase 4 begins with the same clause as phase 3 (in Text 2 “What does not fill up the *cun* (that is, the remainder of the dividend)...” and in Text 4 “What does not fill up the *zhu* (i. e., the remainder of the dividend)...”). However, either the practitioner does not want to use a more finely graduated unit for the quotient than those already obtained, or there is no such unit available. In any case, no multiplication is prescribed. On the other hand, practitioners clearly want *exact* results for division. In fact, only exact results are given for divisions throughout all the books in our corpus (whether manuscripts or texts handed down through written tradition). Incidentally, this explains why one can restore the quantity one started with by applying the inverse computations (a practitioners’ concern we remarked on above). In this context, phase 4 appears to proceed, *like phase 3*, through a dissection of the units in the remainder. However, this dissection no longer uses a general coefficient provided by a system of measurement units. Instead, it is specific to each division, using the *ad hoc* value of the divisor. In a way, the divisor now provides a pattern for the dissection. To analyze the various facets of the situation, let us call the remainder *r* and the divisor *d*. Dissecting each unit of the remainder of the dividend into *d* parts amounts to multiplying the remainder by the divisor and obtaining *rd* units in the dividend (this can be compared to a phase 3). If, as above, one applies the divisor as a pattern to this new value of the dividend, the units found for the quotient are, *from the viewpoint of their meaning*, *d* times smaller than the units of the quotient previously found: this dissection corresponds precisely to the introduction of fractions. Each such unit is a *d*-th part of the last unit obtained in the quotient. Further, *from the viewpoint of the value* of the quotient, if, as in a phase 2, one applies the pattern of the divisor to the new value of the dividend, there is no need to compute: clearly, the result is exactly *r* (result of the division of *rd* by *d*). The multiplication and the division in phase 3 and phase 2 cancel each other out. This explains why phase 4 only states the change of meaning of the remainder: *r* in the dividend is directly stated as a number of units *r* in the quotient, granted the fact that these units are *d*-th parts of the last unit computed.

In conclusion, in full continuity with the preceding phases of the process of division, phase 4 can also be interpreted as relying on a dissection of the units remaining in the dividend and a reshaping of these units with the divisor as a pattern. However, unlike the previous phases 2 and 3, phase 4 directly formulates the result of this dissection through granting a new meaning to the units in the remainder. This is precisely what *Zhao Shuang* apparently understands with respect to the unit of length *bu*, when he comments on phase 4 in Text 1 of *The Gnomon of the Zhou [Dynasty]*: “The positions (digits in the dividend) have been

exhausted up to the units of *bu*, therefore **taking the divisor to name the remaining parts (*fen*) corresponds to breaking *bu*.** This is also what the formulation of the sentences prescribing phase 4 seems to indicate. In Text 2 from the *Book of Mathematical Procedures*, Phase 4 reads: “What does not fill up the *cun* (that is, the remainder of the dividend), **one names the parts (*fen*) with the divisor.**” In Text 4 from *Mathematics*, it reads likewise: “What does not fill up the *zhu* (i. e., the remainder of the dividend), **one names the parts (*fen*) with the divisor.**” The units in the remainder are now referred to as “parts”, their name deriving from the pattern applied to the last measurement unit considered in the quotient. The term *fen* used in all these cases is the same as the one referring earlier to “division into parts” or “sharing.” Finally, the idea that, we suggest, leads to the introduction of fractions in the context of this type of execution concurs with the terminology used to designate fractions from at least the time of the manuscripts onwards: for instance, the final component of the process of division in Text 1 reads: “*yì qian sì bai liú shí yī fēn bù zhī qī bai jiǔ shí ba*—千四百六十一分步之七百九十八 seven hundred ninety-eight [parts] of the *bu* divided into one thousand four hundred sixty-one parts.”

Thus, from the viewpoint of both the terminology and the operation, the concept of fraction attested to as early as the earliest known mathematical writings fits very well with an older process of division, as we can reconstruct it. I would like to add that the reconstruction of this mathematical environment appears to present continuities with procedures that perform arithmetical operations on fractions, to which the manuscripts attest. I shall return to this point in another publication. Finally, it is worth noting that if this origin were confirmed, the fact of having carried out divisions in such a way as to yield quotients, bit by bit, as measured quantities, may have played a role in inspiring the introduction of the fractional quantities recorded in early imperial China. In the context analyzed, quotients were produced along a sequence of measurement units. The shaping of their final segment answered the question of which measurement unit should be used to complete the operation. This may have been carried out using the same ideas as those involved in the previous phases of the operation.



## Appendix

Text 1. A Division in *The Gnomon of the Zhou*.

(Qian Baocong, *The Ten Classics of Mathematics*, p. 52).

Note: as in the ancient Chinese editions, I reproduce below the text of *The Gnomon of the Zhou* [Dynasty] proper in larger characters (block capitals in the translation) and that of the commentaries in smaller characters (lower case letters in the translation). For the sake of clarity, I append the phrase “Zhao Shuang’s commentary” to the text to introduce the passages of the text of the third century commentary. The sixth century commentary is prefaced in the ancient editions by the phrase, “Your servant [Zhen] Luan says:”

(…) 其南北游，日六百五十一里一百八十二步、一千四百六十一分步之七百九十八。

術曰：置十一萬九千里爲實。以半歲一百八十二日八分日之五爲法，  
(Zhao Shuang)半歲者，從外衡去內衡以爲法。除相去之數得一日所行也。  
而通之，

(Zhao Shuang)通之者，數不合齊，常<sup>40</sup>以法等，得相通入，以八乘也。

得九十五萬二千爲實，

(Zhao Shuang)通十一萬九千里。<sup>41</sup>

所得一千四百六十一爲法，除之

(Zhao Shuang)通百八十二日、八分日之五也。

實如法得一里。不滿法者，三之，如法得百步。

(Zhao Shuang)一里三百步。當以三百乘，而言三之者，不欲轉法，便以一位爲百實。  
故從一位命爲百。

不滿法者，十之，如法得十步

(Zhao Shuang)上不用三百乘，故此十之，便以一<sup>42</sup>位爲十實。故從一位命爲十。

不滿法者，十之，如法得一步

(Zhao Shuang)復十之者，但以一位爲實。故從一位，命爲一。

不滿法者，以法命之。

(Zhao Shuang)位盡於一步，故以法命其餘分爲殘步。

臣鸞曰：求南北游法：置冬至十一萬九千里，以半歲日分母八乘之，得九十五萬二千爲實。通半歲一百八十二日、八分日之五，得一千四百六十一。以除得六百五十一里。不盡八百八十九，以三百乘之，得二十六萬六千七百。復以法除之，得一百八十二步，不盡七百九十八。卽得日南北遊日六百五十一里一百八十二步、一千四百六十一分步之七百九十八。

40 Guo Shuchun and Liu Dun, *The Ten Classics of Mathematics*, p. 41, note 132, remark that all ancient editions have *chang* here, except the Ming edition. Qian Baocong, *The Ten Classics of Mathematics*, p. 52, does not have this character. Nor does he include a note. Guo Shuchun and Liu Dun, *The Ten Classics of Mathematics*, p. 41, note 132, suggest this is the correct reading. They accordingly suggest another punctuation, which I follow here.

41 Here, as in many other places, in the editions linked to the *Siku quanshu* 四庫全書 (Complete Library of the Four Treasuries) enterprise at the end of the 18th century, Dai Zhen 戴震 includes a slightly different way of expressing the same amount: “一十一萬九千里。” The Song reading probably derives from a more ancient document. Note, however, that this feature is not systematic in Dai Zhen’s editions.

42 Guo Shuchun and Liu Dun, *The Ten Classics of Mathematics*, p. 41, note 134, indicate that this “一 yi one” was added by Dai Zhen in the editions produced in relation to the *Siku quanshu* enterprise (see Dai Zhen’s note). They adopt this suggestion.

ITS [I. E., THE SUN'S] NORTH-SOUTH DISPLACEMENT PER DAY IS SIX HUNDRED AND FIFTY-ONE *LI* ONE HUNDRED AND EIGHTY-TWO *BU* SEVEN HUNDRED NINETY-EIGHT ONE THOUSAND FOUR HUNDRED SIXTY-FIRSTS OF A *BU*.

PROCEDURE: ONE PUTS [ON THE CALCULATING SURFACE] ONE HUNDRED AND NINETEEN THOUSAND *LI* AS DIVIDEND.<sup>43</sup> ONE TAKES HALF A YEAR, ONE HUNDRED AND EIGHTY-TWO DAYS FIVE EIGHTHS OF A DAY<sup>44</sup> AS A DIVISOR,

(Zhao Shuang's commentary) The reason why one takes "half a year" is that one takes the [time elapsed for the sun to move] from the outermost *heng* to the innermost *heng* as divisor.<sup>45</sup> Dividing [by this, i. e., the time elapsed between the moments when the sun was on one and then on the other] the quantity of the distance between them yields what is travelled in one day.

AND ONE **MAKES THEM** [I. E., THE VALUES IN THE DIVIDEND AND THE DIVISOR] **COMMUNICATE**.

(Zhao Shuang's commentary) The reason why "one makes them communicate" is that these values are not homogeneous with each other, constantly making the divisor [i. e., the denominator] equal results in making them communicate with each other, [hence] one multiplies by eight.

WHICH YIELDS NINE HUNDRED FIFTY-TWO THOUSAND AS DIVIDEND,

(Zhao Shuang's commentary) this is making one hundred and nineteen thousand *li* communicate.

THE ONE THOUSAND FOUR HUNDRED SIXTY-ONE WHICH IS OBTAINED MAKES THE DIVISOR, ELIMINATING (*chu*) THIS,

(Zhao Shuang's commentary) this is making one hundred and eighty-two days five eighths of a day communicate.

(EACH TIME) THE DIVIDEND IS LIKE THE DIVISOR, THIS YIELDS ONE *LI*. WHAT DOES **NOT FILL UP THE DIVISOR** [i. e., the remainder of the dividend], ONE **THREE-FOLDS THIS**, AND [EACH TIME] IT IS LIKE THE DIVISOR, **THIS YIELDS A HUNDRED *BU***.

(Zhao Shuang's commentary) One *li* being three hundred *bu*, **one should multiply (the remainder) by three hundred**, but the reason why one says "one three-folds this" is that **one does not want to change the divisor periodically**, one then takes **a single position** [i. e., a single digit of the quotient] as corresponding to the dividend of the hundreds. Therefore the single following position [i. e., a single digit of the quotient] is named as hundred.<sup>46</sup>

WHAT DOES NOT FILL UP THE DIVISOR [I. E., THE REMAINDER OF THE DIVIDEND], **ONE TEN-FOLDS THIS**,<sup>47</sup> AND [EACH TIME] IT IS **LIKE THE DIVISOR, THIS YIELDS TEN *BU***.

(Zhao Shuang's commentary) Above, one has not used the multiplication by three hundred, therefore here one "ten-folds this." One then takes a single position [i. e., a single

43 It has been made clear in the preceding lines that this amount corresponded to the north-south displacement of the sun between the winter and the summer solstices. This passage of *The Gnomon of the Zhou [Dynasty]* (without the commentaries) is translated in Cullen, *Astronomy and mathematics in ancient China*, p. 187–188. Here I modify the translation in several places.

44 In this context, a year has  $365 + 1/4$  days.

45 I follow Cullen, *Astronomy and mathematics in ancient China*, p. 183–185, in leaving the term *heng* untranslated. In the context of *The Gnomon of the Zhou [Dynasty]*, *heng* designates the successive orbits of the sun around the celestial pole at specific moments of the year. The Sun goes from the outermost to the innermost *heng* and back during a year. Here, one is interested in the displacement per day of the orbit along a north-south axis.

46 One names the digit in the same way as the prescription of the division that formulated it: hundreds of *bu*. The term used for "naming" is the same as that used subsequently when naming the remainder with the divisor and giving the final part of the quotient as a fraction.

47 Note that this multiplication by ten is not said to be performed by a shifting of rods. Chemla, 2013b, "Working on and with division in early China," discusses the meaning of these clues.

**Your servant [Zhen] Luan says:** Method for finding the north-south displacement: One puts [on the calculating surface] the one hundred and nineteen thousand *li* [from] the winter solstice and one multiplies this by the denominator eight of the [number of] days of the half year, yielding nine hundred fifty-two thousand as dividend. One makes one hundred and eighty-two days five eighths of a day [corresponding to] the half-year communicate, which yields one thousand four hundred sixty-one. With [this] dividing yields six hundred fifty-one *li*. The eight hundred eighty-nine which are not exhausted, one multiplies them by three hundred, yielding two hundred sixty-six thousand seven hundred. Again, one divides this by the divisor, yielding one hundred and eighty-two *bu*. What is not exhausted is seven hundred ninety-eight, hence yielding for the displacement of the sun from north to south daily six hundred fifty-one *li* one hundred and eighty-two *bu* seven hundred ninety-eight one thousand four hundred sixtv-firsts of a *bu*.

Measurement units for length used here: 10 *cun* = 1 *chi*.

50 The text of the *Book of Mathematical Procedures* uses punctuation marks whose meaning differs from modern punctuation marks, and whose occurrence I reproduce in the translation.



Text 3. An execution of division in *Mathematics*.

(ZHU Hanmin and CHEN Songchang (eds.), *Qin Bamboo slips kept at the Academy Yuelu*, p. 9.)

**Slip 0757** (Editors' number 48). The slip is incomplete, showing only the end of the procedure. Perhaps it was originally placed after slip 0842 (editors' number 47, ZHU Hanmin and CHEN Songchang (eds.), *Qin Bamboo slips kept at the Academy Yuelu*, p. 8).

(...) 法，以租乘分田，如法一斗，不盈斗者，十之，如法得一升。 /Slip 0757 (editors' number 48)/

(...) divisor, one multiplies the cropland shared [?] by the tax. [Each time] it [the dividend] is like the divisor, [this yields] one *dou*. What does **not fill up the *dou*** [i. e., the remainder of the dividend], one **ten-folds** this. [Each time] it [the latter number as dividend] is like the divisor, this yields one *sheng*.

Measurement units for capacity used here: 10 *sheng* = 1 *dou*.

Text 4. An execution of division in *Mathematics*.

(ZHU Hanmin and CHEN Songchang (eds.), *Qin Bamboo slips kept at the Academy Yuelu*, p. 5.)

(...) 為[尹+貝]<sup>51</sup> (實)，(...) /slip 1743 (editors' number 17)/

(...) 為法，如法一兩，不盈兩者，以一為廿四，乘之，如法〔一朱（銖）〕，〔不〕〔盈〕朱（銖）者，以法命分。 /slip 1835+1744 (editors' number 18)/

(...) makes the dividend (...), makes the divisor. [Each time] it [the dividend] is like the divisor, [this yields] one *liang*. What does **not fill up the *liang*** [i. e., the remainder of the dividend], **taking one as twenty-four, one multiplies this** [the remainder]. [Each time] it [the latter number as dividend] is like the divisor, [this yields] one *zhu*. What does **not fill up the *zhu*** [i. e., the remainder of the dividend], one names the parts with the divisor.

Measurement units for weight used here: 24 *zhu* = 1 *liang*.

Text 5. Phase 1 in *Mathematics*.

(ZHU Hanmin and CHEN Songchang (eds.), *Qin Bamboo slips kept at the Academy Yuelu*, p. 6, 48.)

*Mathematics*, Slips 0788+0775+0984 (editors' numbers 29-30-31). Explanations for the interpretation of this passage are given in the main text for the interpretation of the passage quoted.

“(...) 欲復之，置一束兩數以乘兌(稅)田，而令以一為八十一為[尹+貝] (實)，亦【令所粟 步一為八十一，不分者，從之以為】<sup>52</sup>/slip 0775 (editors' number 30) 法，[尹+貝] (實)如法一兩。(blank space) /slip 0984 (editors' number 31)/

“If one wants to restore this, one puts the quantity of *liang* of one bundle and multiplies with [this] the cropland taxed, and **makes one to be taken as eighty-one** to make the dividend; one **likewise takes one as eighty-one in the *bu*** that were diminished

51 In the manuscript, these two characters are combined on top of one another to write the character interpreted as a variant for the character for “dividend.”

52 The right side of the lower section of the slip is damaged and only the left side of its characters can be read.

[weakened]; **those that are not divided into parts/shared** join these, which is taken as divisor. Dividing the dividend by the divisor yields the result in *liang*.”

### *Zusammenfassung*

Der Aufsatz weist darauf hin, dass die mathematische Handschriften vom dritten oder zweiten Jahrhundert v. Chr., die vor kurzem in China ausgegraben wurden, eine Terminologie für Division enthalten, die nicht in den durch die schriftliche Überlieferung übertragenen mathematischen Büchern zu finden ist. Die einzige Ausnahme davon ist das älteste mathematische Buch, das Mathematik für Astronomie vorstellt und von dem ein Teil um etwa 100 v. Chr. geschrieben wurde, *Das Zhou-Gnomon*. Die Verfasserin weist vor, dass es ein ähnliches Algorithmus für die Division in den Handschriften sowie in dem *Zhou-Gnomon* gibt, das älter als das Algorithmus *chu* ist, das später wohl verbreitet wurde. Sie stellt die Hypothese vor, dass das Begriff von Bruch, das in den ältesten mathematischen Schriften in Chinesisch bewiesen ist, mit diesem älteren Algorithmus eng verbunden ist.

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