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Different Cultures of Computation in Seventh Century China from the Viewpoint of Square Root Extraction

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Abstract

The aim of this paper is to bring to light a previously unknown geometrical method for extracting the square root in seventh century China. In order to achieve this goal, a seventh century commentary by the scholar Jia Gongyan, 賈公彥, on a Confucian canon, the *Rites of Zhou Dynasty* [*Zhouli* 周禮], is analysed. This is compared with the commentary by his contemporary Li Chunfeng, 李淳風, which is referred to in another mathematical book, the *Mathematical Procedures of the Five Canons*, [*Wujing Suanshu* 五經算術]. Although these two scholars probably knew each other, they used very different methods to solve the same problem in relation to square root extraction. It is argued that the differences mainly lie in two aspects: firstly, Jia Gongyan mostly made use of geometry while Li Chunfeng used counting rods; secondly, the two methods had different geometrical interpretations. Given the fact that the method of square root extraction Jia Gongyan uses is one among many other methods he employed in mathematics, and it has the same features as the others; moreover, other commentators on the Confucian Canons use similar mathematical methods, this paper closes with a general discussion on mathematical cultures. It is suggested that there were three elements to mathematical practice in seventh century China: geometry, counting rods, and written texts. The interplay and structure between the three elements is seen to influence mathematical practices.

Key words

Cultures of computation, Jia Gongyan, Li Chunfeng, square root extraction, seventh century, Tang dynasty, mathematical practices, counting rods, geometrical diagrams, mathematical cultures, *Rites of Zhou Dynasty*, *Mathematical Procedures of the Five Canons*

1. Introduction

The *Nine Chapters on Mathematical Procedures* [Jiuzhang Suanshu 九章算術, hereafter the *Nine Chapters*] includes a procedure, called Kai Fang Shu¹ [開方術]. This procedure, using counting rods, is very significant because it presents the first complete method in the history of mathematics in China for what in modern mathematics is known as square root extraction.² Historians have published many articles,³ not only on this procedure in the *Nine Chapters*, but also on other procedures for root extraction, including square roots and cube roots, in the *Ten Classic Books of Mathematics*, [Suanjing Shishu 算經十書 hereafter the *Ten Mathematical Books*]⁴. Thanks to this meticulous work, it is known that despite minor differences, all the procedures in these mathematical canons, written by different scholars, have the same basic features.⁵ The various procedures in the *Ten Mathematical Books* share the same general characteristics:⁶ the instruments, counting rods⁷, used for carrying out the procedures are the same; the terminology used is very similar; and, there is a common geometrical basis underlying the procedures, which was first presented by Liu Hui⁸ 劉徽 in the third century, and was subsequently confirmed by Li Chunfeng⁹ 李淳風 in the seventh century.¹⁰

However, this topic is the subject of much debate and some unsolved problems remain. For example, the details of the procedures carried out with counting rods, and the minor differences between the procedures. Moreover, the role of geometry

¹ This procedure is to extract the square root in modern mathematics. Literally speaking, Kai means to open/setup, Fang means square, and Shu means procedure. Therefore, the literal interpretation will be to open/ set up a square.

² In the 19th century, after mathematics was introduced from the west into China, Qing scholars borrowed the ancient term “Kai Fang” to designate the operation of square root extraction.

³ All works are divided into two categories: books in which scholars discussed this issue in one or more chapters, and independent papers on this issue. I just mention some of them here: Qian Baocong, 1964, 46-51. Li Yan, 1937, 63-68. Li Jimin, 1990, 389-396. Guo Shuchun, 1992, 29-33. Liu Dun, 1993, 191-198. Joseph Needham, 1959, 62-67. Karine Chemla, 1994. Jean-Claude Martzloff, 2006, 221-228.

⁴ During the seventh century, the *Ten Classic Books of Mathematics* includes: the *Mathematical Canon on Gnomon of Zhou* [Zhoubi Suanjing 周髀算經], the *Nine Chapters on Mathematical Procedures* [Jiuzhang Suanshu 九章算術], the *Mathematical Canon on Sea Island* [Haidao Suanjing 海島算經], the *Mathematical Canon of Sunzi* [Sunzi Suanjing 孫子算經], the *Mathematical Canon of Zhang Qiujian* [Zhang Qiujian Suanjing 張丘建算經], the *Mathematical Canon of Xia Houyang* [Xia Houyang Suanjing 夏侯陽算經], the *Joint Mathematical Procedures* [Zhuishu 綴術], the *Mathematical Canon on Five Bureaus* [Wucaosuanjing 五曹算經], the *Mathematical Procedures of the Five Canons* [Wujing Suanshu 五經算術], and the *Mathematical Canon on Continuation of Ancient Mathematics* [Jigu Suanjing 緝古算經]. In 263 CE, Liu Hui commented on the *Nine Chapters on Mathematical Procedures*; and in 656 CE, Li Chunfeng and his colleagues edited, commented and made canonical versions for these ten mathematical books, which were used in Imperial University [國子監]. Later, when the book of the *Joint Mathematical Procedures* was lost, Bao Hanzhi [鮑漸之 13th century] replaced it by another mathematical book that was the *Notes on Traditions of Numerological Procedures* [Shushu Jiye 數術記遺]. 1963, Qian Baocong produced critical editions for these books. Guo Shuchun and Liu Dun produced critical editions recently [1998].

⁵ Here what I stress is that most scholars agree to call the procedures “square roots extraction using counting rods”[籌算開方術], although the procedures evolved over time.

⁶ Jean-Claude Martzloff, 2006, 222.

⁷ The use of counting rods for square root extraction lasted until the 16th century and the rise in adoption of the abacus.

⁸ Liu Hui is the earliest known commentator of the *Nine Chapters* that was handed down through the written tradition. His commentary on the *Nine Chapters* dates from 263 CE.

⁹ Li Chunfeng was a 7th century scholar. He and his colleagues edited, and commented on the *Nine Chapters* in 656 CE.

¹⁰ Liu Hui first explained the geometrical basis. The fact that “Fang” in the term “Kai Fang” means square also implies its correlation to geometry.

needs further clarification.¹¹ The problems seem to stem from different elements of mathematical practice. In the case of square root extraction, the procedures were written in ancient Chinese characters on slips of paper or bamboo, while the procedures were carried out in their entirety with counting rods on a computational surface. These elements suggest different facets of the same topic, but raise the question as to how these different elements inter-relate.

The question is difficult to answer. Recently discovered sources may provide new insights and thus be useful in answering part of the question. It has become apparent, at least for seventh century China, that the historical picture is incomplete. Other procedures for square root extraction existed besides those historians have addressed so far. The new source is the commentaries on the Confucian Canons by seventh century scholars.

1.1 Aims

On the basis of previous studies, various procedures have been seen to refer to the same operation, that is, square root extraction. A geometrical interpretation can also be seen, which seems parallel to the arithmetical procedures and refers to the same operation. That is, there are different elements of mathematical practice related to the execution of the same operation. What were the practitioners' intentions in extracting the square root? What did the operation actually mean during those times?¹² The aim of this paper is to present a new historical perspective on square root extraction and on mathematics from Confucian Canons in seventh century China.

1.2 Analysis

Key to this paper are the writings of Jia Gongyan, [賈公彥, seventh century]. He wrote his commentary on the *Rites of Zhou Dynasty* [Zhouli 周禮]¹³ during 650-655 C.E. These precise dates come from one of Jia Gongyan's official titles, an Erudite of the National University, [Taixue Boshi 太學博士], as recorded on the first page of the *Commentary on the Rites of Zhou Dynasty* and the *Commentary on Ceremonies and Rites*¹⁴. Also, from Jia Gongyan's biography in the *Old Book of Tang Dynasty*¹⁵,

¹¹ Karine Chemla discusses the use of diagrams, and how they changed, in Chinese mathematical writings, see Karine Chemla, 2010a. To me, many questions still need to be solved concerning the geometrical facets of the history of mathematics in China.

¹² One key hint for answering such a question lies in the understanding of Kai Fang in the *Nine Chapters*, which literally means "to open" or "to set up" a square. This term was not only used for the procedure in the *Nine Chapters*, but also for the same operation elsewhere. Moreover, this term is usually used simultaneously with another term Chu 除, which means "to divide" or "to subtract". Therefore, Kai Fang or Kai Fang Chu shows the relation between procedures and the operation, and the relation between arithmetic and geometry, raising the question of what the practitioners actually meant when they said Kai Fang or Kai Fang Chu.

¹³ The *Rites of Zhou Dynasty*, one of the *Three Rites*, was a classic ritual book. This book was produced through a continuous process of writing, editing and compiling in which different scholars, in different eras, took part. Its basic definite version was produced in the Warring States period. It contained six parts. The sixth part was lost at some point during the Han dynasty. Liu Xin[劉歆], a scholar of the Xin dynasty, replaced it with the *Record Inspecting Public Works* [Kaogong Ji 考工記]. The *Record Inspecting Public Works* was to record the standards for construction and manufacturing). It was produced in the Warring States period [475-220 BC].

¹⁴ The *Commentary on the Rites of Zhou Dynasty* [Zhouli Zhushu 周禮注疏] and the *Commentary on Ceremonies and Rites* [Yili Zhushu 儀禮注疏] were two classic ritual books.

¹⁵ Liu Xu, *Jiu Tangshu*, 1975, 4950.

we know in the Yonghui period [650-655], he was an Erudite of the National University. His commentary shows approach to square root extraction and another way of doing the operation that was different from the procedure using counting rods, both of which are related to geometry in a specific way and not seen before. These sources will help us answer the questions raised above.

This paper is in four sections. The first two will give a translation and analysis of two excerpts from Jia Gongyan's commentary, both concerning square root extraction. In the third part, Li Chunfeng, who was Jia Gongyan's colleague in the Tang court, is introduced,¹⁶ along with his mathematical method of square root extraction. A comparison will be made between the two methods, with a general discussion on the different cultures of computation from the point of view of root extraction.¹⁷

2. Jia Gongyan's procedure

In two excerpts of commentaries from the *Rites of Zhou Dynasty*¹⁸, there are two examples of the execution of square root extraction. The procedures, which the two examples present, are very similar. Both examples relate to the construction of a carriage. The first analysis focuses on curved canopy ribs. The same problem is dealt with in one of the *Ten Mathematical Books*, that is *Mathematical Procedures of the Five Canons* edited by Li Chunfeng and his colleagues.

2.1 The problem of "one trisects the length of rib"

The structure of the text has three layers: the text of the Classic¹⁹, Zheng Xuan's 鄭玄²⁰ [127-200 C.E.] commentary, and Jia Gongyan's further commentary. It explains how a rib is articulated so as to obtain the curve of the canopy. According to the *Rites of Zhou Dynasty*, each canopy has twenty-eight ribs. Here the general idea of the problem is given, followed by an analysis of the layers of the text.

Looking from above, the twenty-eight ribs supporting the canopy look like a claw and are called "the claw" in this text, diagram 1. Picture 1 is of the carriage

¹⁶ We don't have direct evidence that Li Chunfeng and Jia Gongyan knew each other. One important piece of indirect evidence that led me to make this statement is that Wang Zhenru 王真儒 who worked with Jia Gongyan on Confucian canons (Ouyang Xiu, Song Qi, *Xintangshu*, 1975, 1428. *Quantangwen*, 1983, 1375.), also worked with Li Chunfeng on mathematical canons. (Liu Xu, *Jiu Tangshu*, 2719. Ouyang Xiu, Song Qi, *Xin Tangshu*, 5798.).

¹⁷ The concept of "cultures of computation" used in the title comes from seminars organized by Karine Chemla, Christine Proust, and Agathe Keller. Moreover, Karine Chemla describes Chinese mathematical culture mainly from the viewpoint of counting rods in many articles. Karine Chemla, 2010b. I hope this paper will help to enrich this concept of "cultures of computation" and mathematical cultures in the history of China.

¹⁸ In this paper, I quote *Rites of Zhou Dynasty* and Jia Gongyan's commentary from Ruan Yuan's edition, that is in the *Shisanjing zhushu* 十三經注疏 (*Commentary on Thirteen Canons*). Ruan Yuan 阮元 (ed.). Photocopied by Zhonghua Book Company, Beijing, 1979, 631-940.

¹⁹ That is the original text of the *Rites of Zhou Dynasty*.

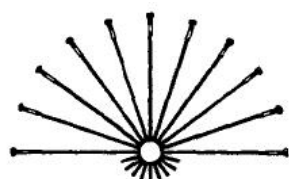
²⁰ Zheng Xuan was an official and scholar in the Eastern Han dynasty [25-220 C.E.]. He was a very important scholar for the study of Confucian canons. He commented on the *Three Rites* during 169-183 C.E. In the *Commentary on the Rites of Zhou Dynasty* handed down through the written tradition, Zheng Xuan and Jia Gongyan are the only two commentators on this book whose commentaries exist today.

excavated along with the Terracotta Army²¹. Its canopy is similar to the one discussed here.²²

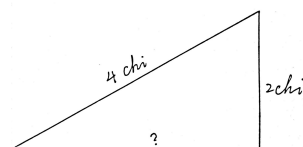
According to the text, each rib is 6 *chi* long and articulated. The articulation, point **c** in diagram 2, is at $\frac{1}{3}$ of the length of the rib from the centre point of the claw, point **a**. As a result, each rib has two parts: 2 *chi*, [**ac** in diagram 2], and 4 *chi*, [**bc** in diagram 2]. The 2 *chi* part is horizontal, and the 4 *chi* part slopes downward. The difference in height between the highest and lowest part of the 4 *chi* part is exactly equal to $\frac{1}{3}$ the length of the rib, which is 2 *chi*, [**cd** in diagram 2]. In the text corresponding to diagram 2, point **a** is the centre of the canopy; point **b** is the outer extremity of the claw; point **c** is the articulation point; the line segment **ac** is the horizontal part of the rib near the centre of the canopy; the line segment **bc** is the sloping outer part of the rib; and, the line segment **cd** is the height. The handle of the canopy is 10 *chi* long and divided into two parts: the 2 *chi* part “da chang” 達常 and the 8 *chi* “gai jiang” 蓋杠, diagram 2.

The line segment **bd** is sought. Therefore, the problem involves the computation of the length of one side of a right angle triangle when the length of the hypotenuse and the other side [in this case, the short side] are known, diagram 3. The reason this problem is raised is that one wants to know the scale of the canopy and how much area the canopy can cover. We shall see the canopy is required to cover the axletree of the carriage.

Diagrams 1 to 3

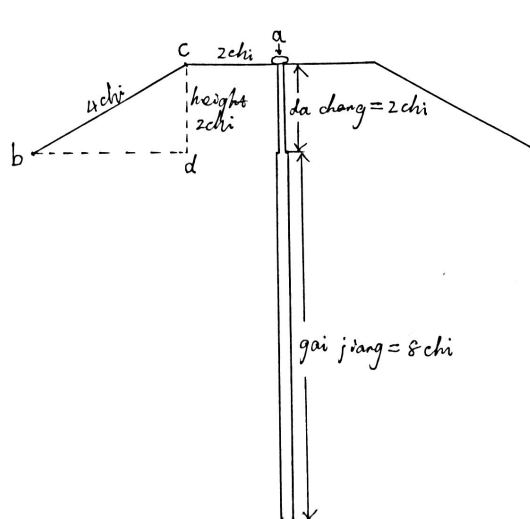


1: the claw



²¹ The Terracotta Army is a collection of terracotta sculptures depicting the armies of the first emperor of China.

²² According to the excavation report, the total length of this carriage and horse is 2.25 m; the total height of the carriage is 1.56 m. The carriage is not exactly the same as the one described in the *Rites of Zhou Dynasty*. However, it is usually used to compare the excavated carriage to the records in texts handed down by the written tradition. I add this carriage because it illustrates the type described in the source using an archeological artifact. See: Qinshihuang bingmayong bowuguan 秦始皇兵馬俑博物館 (Museum of Qin Shi Huang's Terracotta Warriors), 1998.



2:

\widehat{acb} : the rib

a: the centre of the canopy

b: the outer extremity of the claw

c: the articulation point of the rib

ac: the horizontal part [of the rib] connected to the centre of the canopy

bc: the sloping outer part [of the rib]

cd: the height

bd: to be looked for

3: Two *chi* being taken as the short side, four *chi* being taken as the hypotenuse, one looks for the corresponding long side.

Picture 1: The excavated carriage of the Terracotta Army



Source: Qinshihuang bingmayong bowuguan 秦始皇兵馬俑博物館 (Museum of Qin Shi Huang's Terracotta Warriors), 1998. *Qinshihuangling tongchema fajue baogao* 秦始皇陵銅車馬發掘報告 (Excavation Report for the Copper Carriages in Qin Shi Huang's Tomb). Culture Relic Press, Beijing. Colorplate 6.

The text will be translated sentence by sentence with the original Chinese text given underneath. Each sentence will be given an ordinal number for easier identification and analysis. The measurement units for length are: 1 *zhang* = 10 *chi*, 1 *chi* = 10 *cun*, 1 *cun* = 10 *fen*. 1 *chi* is approximately equal to 23.1 *cm* in the Han dynasty, and 29.6 *cm* in the Tang dynasty.

The text of the Classic says²³:

²³ *Shisanjing zhushu*, 1979, 910.

1. One trisects the length of the rib, taking one of the thirds as the *Zun*.²⁴

參分弓長，以其一爲之尊。

Zheng Xuan's commentary says:

2. *Zun* means height.

注：尊，高也。

3. For a rib of six *chi*, the higher horizontal part near the centre of the canopy is two *chi*, hence one extremity of the claw is lower than the centre of the canopy by two *chi*.

六尺之弓，上近部平者二尺，爪末下於部二尺。

4. Two *chi* being taken as the short side, four *chi* being taken as the hypotenuse, one looks for the corresponding long side.

二尺爲勾，四尺爲弦。求其股。

5. The [square of the] long side is twelve [*chi*].

股十二。

6. One divides it²⁵, which makes the sides [of the square of its long side] three *chi* and nearly half [a *chi*].

除之，面三尺幾半也。

The text of the Classic is simple, only giving the general idea of the articulated rib. In 2 Zheng Xuan explains the meaning of *zun*. In 3 he shows how the rib was articulated. Sentence 4 reformulates the question as a mathematical problem with a right-angle triangle. In 5 Zheng Xuan gives the square of the long side. In 6, the final sentence, he mentions the execution of a kind of division, and gets an approximate result, which is 3 *chi* and nearly a half, that is nearly 3½ *chi*. However, Zheng Xuan does not give any details about how he computed the square root. Jia Gongyan's commentary addresses precisely this point.

Jia Gongyan's commentary says:

7. When [the *Rites of Zhou Dynasty*] says "taking one of the thirds as *zun*", this means [the articulation in the rib is], two *chi* from the centre of the canopy, with respect to the extremity, the four *chi* being made to go downwards, [the articulation] is two *chi* higher.

云「以其一爲之尊」者，正謂近部二尺者，對末頭四尺者爲下，以二尺者爲高。

8. When [Zheng Xuan] says "the extremity of the claw is lower by two *chi* than the centre of the canopy", this means that, the height of the lower part, [the *gai jiang*], and the higher part, [the *da chang*], of the canopy handle being one *zhang*, and [from the extremity of the claw to the extremity of the lower part] being eight *chi*, thus the ribs on the four directions are /articulated so as to hang down by two *chi*.

云「爪末下於部二尺」者，正謂蓋杠并達常高一丈，八尺，故四面宇曲，垂二尺也。

9. When he says "two *chi* being taken as the short side, four *chi* being taken as the hypotenuse, one looks for the corresponding long side", Zheng Xuan wants to explain the decrease due to the articulation in the ribs, which means the width the decreased canopy was able to cover the axletree, [called *Zhi*], less than the net width [that is, the width of the un-articulated canopy].

²⁴ It's clear that the commentator Zheng Xuan perceives the term "*Zun*" requires a gloss.

²⁵ According to *Mathematical Procedures* by Xia Houyang, square root extraction is a kind of division.

云「二尺爲句，四尺爲弦，求其股」者，鄭欲解字曲之減，減蓋之寬覆軹，不及幹之意。

Jia Gongyan makes a much longer commentary. In **7**, he comments on **1**, **2** and **3**. In **8** he clarifies the reason why the rib is articulated downward by 2 *chi* in relation to the structure of the canopy handle. In **9** Jia Gongyan explains that Zheng Xuan's commentary is to compute the area the canopy covers. In the following, Jia Gongyan's procedure can be divided into two parts, one of which is to compute the square of the long side.

Jia Gongyan continues by saying:

10. For all these computation procedures²⁶: one takes two *chi*, by which the extremity of the claw is lower [than the centre of the canopy], that is one takes the two *chi* by which it is lower, as the short side.

凡算法：以蚤低二尺，即以低二尺者爲句。

11. One further takes the length of four *chi* of the supporting part [of the rib] as the hypotenuse, and further takes the length of the straight and flat [line segment] from the extremity of the claw [to the point directly below the articulation in the rib, as the long side.

又以持長四尺爲弦，又蚤末直平者爲股。

12. The [length of the] hypotenuse is four *chi*, four times four is sixteen, making one *zhang* six *chi*.

弦者四尺，四四十六，爲丈六尺。

13. The [length of] the short side is two *chi*, two times two is four, making four *chi*.

句者二尺，二二而四，爲四尺。

14. One wants to look for the length of the corresponding long side, which is the straight and flat [line segment] [from the extremity of the claw to the point just below the articulation in the rib].

欲求其股之直平者。

15. Computation procedure: removing the [square of the] short side from the [square of the] hypotenuse, what remains makes the [square of the] long side.

算法：以句除弦，餘爲股。

16. When one removes the four *chi* of the [square of] short side from the [square of the] hypotenuse, which is to remove four *chi* from one *zhang* six *chi*, there still remains one *zhang* two *chi*.

將句之四尺除弦，丈六尺中除四尺，仍有丈二尺在。

In **10** and **11** Jia Gongyan repeats that the hypotenuse is 6 *chi* long, and the short side is 4 *chi* long. In **12** and **13** he respectively squares the hypotenuse and the short side. In **12** he shows: using “four times four is sixteen”, one gets 4 *chi* x 4 *chi* = 1 *zhang* 6 *chi*, the meaning will soon be made clear by the following procedure, diagrams **4** and **5**. That is, Jia Gongyan interprets 1 *zhang* 6 *chi* as a rectangle whose

²⁶ The referee suggests the literal meaning of *Suanfa* is “procedure using counting rods”. The literal translation is correct, but here “computation procedure” is a more correct translation for two reasons: 1. Jia Gongyan didn't use counting rods to execute his procedure. He used geometrical figures as the tool. 2. Jia Gongyan said in the *Rites of Zhou Dynasty*, “Computation is *Suanfa*, where names of multiplication and division come from.” 計者算法，乘除之名處於此也。 *Shisanjing zhushu*, 1979, 656. This shows Jia Gongyan built a relation between *suanfa* and computation, which is different from Li Chunfeng's.

width is 1 *chi* and length is 1 *zhang* 6 *chi*. $4\ chi \times 4\ chi$ refers to a square whose sides are 4 *chi*, diagrams 4 and 5. It will be shown that Jia Gongyan's calculation relies on the relation between a quantity and its geometrical meaning.

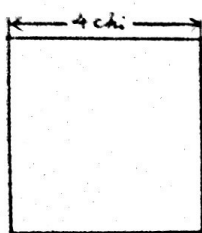


Diagram 4

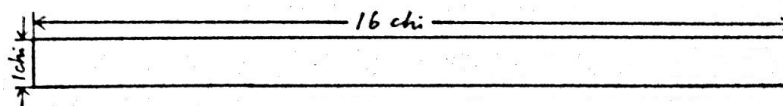
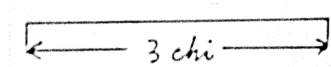
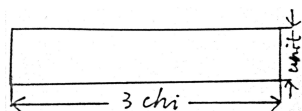


Diagram 5

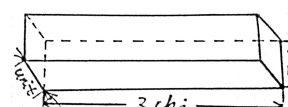
The key principle needs some explanation. This principle was first observed and analysed by Li Jimin²⁷, that is to say, the ancients used a linear quantity to express length, area and volume. When the linear quantity is used for length, it just means a given quantity of length units. When used for area, it means a given quantity of length units and a unit for the width. When it is for volume, it refers to a given quantity of length units, and a cross section whose sides are a unit. Diagrams 6 to 8 show the use of the same quantity, 3 *chi*, to express length, area, and volume respectively. However, whether the quantity refers to the length, area or volume, depends on the context in which the quantity was used.²⁸



3 *chi* for length



3 *chi* for area



3 *chi* for volume

Diagrams 6-8

Thus, Jia Gongyan views the calculation $4\ chi \times 4\ chi$ as a geometrical problem in which one knows the sides of a square and one looks for the length of a rectangle. Having the same area [as the aforementioned square] but whose width is 1 *chi*. “Four times four is sixteen” comes from a multiplication table, namely the Nine-Nine table, in which there are a series of multiplication calculations starting from nine times nine.²⁹ It gives the length of the rectangle. Perhaps with another sentence “*chi* times *chi* is *chi*”, one could find the unit of the result.³⁰ One further uses “1 *zhang* = 10 *chi*” to get the result, which is 1 *zhang* 6 *chi*. In 13, by the same method, Jia Gongyan gives “ $2\ chi \times 2\ chi = 4\ chi$ ”. This 4 *chi* also refers to a rectangle whose length is 4 *chi* and width is 1 *chi*. In 14 to 16 Jia Gongyan subtracts the square of the short side from the square of the hypotenuse, and finds the square of the long side, making $16\ chi - 4\ chi = 12\ chi$.

²⁷ Li Jimin, 1990, 768-778.

²⁸ This is the point I develop Li Jimin's viewpoint.

²⁹ The Nine-Nine table was used as early as the Warring States period in China. It is named after and starts with its the highest value, “nine times nine”.

³⁰ In the first mathematical bamboo slips excavated from an ancient tomb dated 186 BCE, namely the *Book of Mathematics* [Suan Shu Shu 算数書] we find this sentence, which relates to the multiplication of units.

Now Jia Gongyan had to compute the length of the short side, and thus had to extract a square root. He continues by saying:

17. Then one divides³¹ it by the procedure of computation.

然後以算法約之。

18. [The rectangle is] one *chi* wide and one *zhang* two *chi* long; make a square of it.
廣一尺，長丈二尺，方之。

Sentences **17** and **18** appear essential, since Jia Gongyan reveals how he perceives the meaning of the operation of square root extraction. In **18** he states that in his view, the previous result of 12 *chi* refers to a rectangle whose length is 12 *chi* and width is 1 *chi*. Then he says, in transforming this rectangle into a square of the same area, he has to calculate the sides of the square. Diagrams **9** and **10** illustrate this procedure.

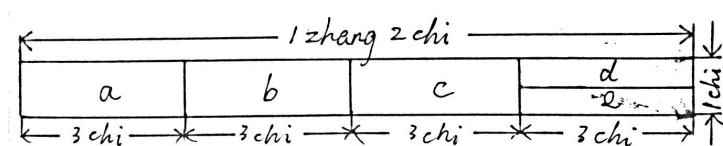


Diagram 9

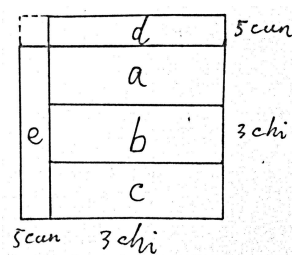


Diagram 10

19. One takes nine *chi* from one *zhang* two *chi*;

丈二尺，取九尺，

20. Each three *chi*, one cuts once.

三尺一截，

21. Complementing them with each other yields a square with sides of three *chi*.

相裨得方三尺。

Jia Gongyan takes a rectangle whose length is 9 *chi* and width is 1 *chi* from the original 12 *chi* x 1 *chi* rectangle. He then cuts this rectangle into three pieces, [rectangles **a**, **b**, and **c** in diagram 9], each of which is 3 *chi* long and 1 *chi* wide. In **21** he reorganizes the three pieces, [**a**, **b** and **c**], to make a square whose sides are 3 *chi*. Jia Gongyan takes 9 *chi* as $3\text{chi} \times 3\text{chi} = 9\text{chi}$. Hence, the geometrical operation is based on computation. However, Jia Gongyan doesn't show how he does the computation. Specifically, he doesn't state why he takes a 9 *chi* long rectangle and not another rectangle from the original rectangle; he doesn't explain why he cuts this rectangle into three pieces. These two questions are related and there are two possible reasons. One is that since there is already the sentence "three times three is nine", Jia Gongyan just needs to look at the multiplication table to find the biggest square number smaller than 12. Another possibility is that, since Zheng Xuan

³¹ The character 約 *yue* means to simplify or to make simple, usually in relation to the simplification of fractions. Karine Chemla, Guo Shuchun, 2004, 1028. In *the Mathematical Canon of Xia Houyang*, the operation of reduction of fractions was named *yue chu* 約除, literally to divide by simplifying. Moreover, the operation of *yue chu* was one of the five divisions. Jia Gongyan uses this term to comment on Zheng xuan's *chu*. Therefore, this character has a similar meaning to *chu* [that is to divide]. In this context, the operation of *yue* refers to square root extraction. On the other hand, *chu* can be also used for describing the square root extraction. This fact further shows the close relation between *yue* and *chu*.

mentions “three *chi* and near half [a *chi*]”, Jia Gongyan aims to account for this amount, therefore the integer, 3 *chi*, is obtained first. Perhaps these two possibilities could both be true. In **21** Jia Gongyan brings them together to make a square of 3 *chi* sides, diagram **10**.

In what follows, Jia Gongyan deals with the remaining piece, which is a rectangle 3 *chi* long and 1 *chi* wide:

22. There still remains three *chi*.

仍有三尺在。

23. One cuts it in the middle into two portions,

中破之爲兩段，

24. Each [being a rectangle] five *cun* wide and three *chi* long.

各廣五寸，長三尺。

25. One complements them to two sides of the previous square whose sides are three *chi*.

裨於前三尺方兩畔。

26. On each of these sides there are five *cun*.

畔有五寸。

27. Adding [them] on the two sides to the previous three *chi* makes [the sides of square] three *chi* and half [a *chi*].

兩畔并前三尺，爲三尺半。

Thus, in **22** to **24**, Jia Gongyan divides the remaining piece into two parts; each part is a rectangle 3 *chi* long and 5 *cun* wide, the rectangles **d** and **e** in diagram **6**. In **25** to **27**, he brings them to the two sides of the previous square, diagram **10**. He doesn't give the reason for this operation. Based on this analysis of the structure and his commentary, the reason possibly lies in that Jia Gongyan wanted to follow Zheng Xuan's commentary, which gives a quantity that is “three *chi* and nearly a half. Therefore, Jia Gongyan looks for the half *chi*, which is equal to 5 *cun*. In order to account for Zheng Xuan's value, he needs two rectangles whose lengths are 3 *chi* and widths are half a *chi*, that is 5 *cun*. Cutting the remaining piece into two parts makes two rectangles half a *chi* wide. Adding these two rectangles to the two sides of the previous square makes the geometrical figure, diagram **10**.

28. In the tip of the angle, there still lacks a square of five *cun* sides.

角頭仍少方五寸。

29. [The quantity]³² doesn't fit [the problem raised] and does not complete three *chi* and half [a *chi*].³³

不合不整三尺半。

30. *Ji* means nearly. So [Zheng Xuan] says nearly half [a *chi*].³⁴

幾，近也，言近半。

³² This quantity refers to what Jia Gongyan obtained above, which is a little smaller than 3 *chi* and half.

³³ This “three *chi* and half” comes from Zheng Xuan.

³⁴ The referee suggests *Ji* has a meaning “approximately”. That's correct. However, *Ji Ban* is usually used for something is approximately but a little smaller than half. For example, “Li Bai's footmark covers almost half of the world.” 其足跡幾半天下 *Litaibai quanji*, 1977, 1633.

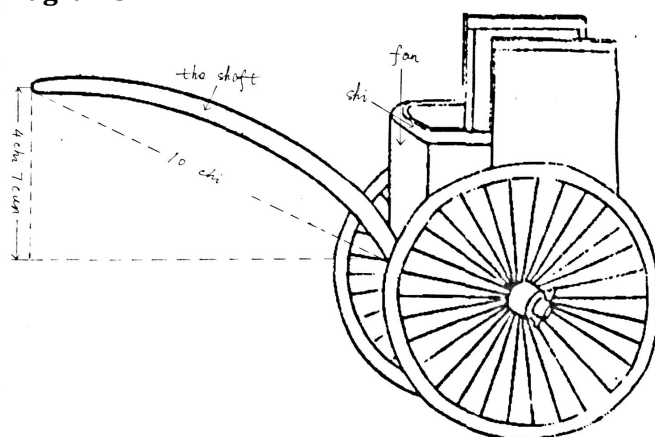
Jia Gongyan states that the resulting figure lacks a small square in the tip of the angle, therefore, the sides of the square cannot be three and a half *chi*. This means a rectangle whose length is 12 *chi* and width is 1 *chi* cannot be transformed into a square whose sides are 3 and a half *chi*, but can be transformed into a square whose sides are 3 *chi* and nearly half a *chi*. Thus Jia Gongyan finished his commentary on Zheng Xuan's quantity.

2.2 Procedure of “ten *chi* in front of a board”

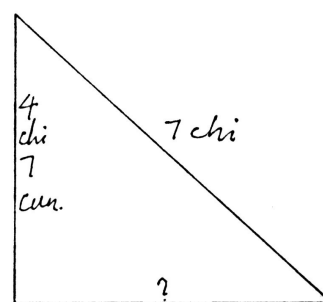
The second piece of the commentary in which Jia Gongyan discusses square root extraction is also related to a carriage. Specifically, it deals with the length of the shaft extending from the *fan*, which is a board used in front of a carriage, diagram 11. The *Rites of Zhou Dynasty* says the length of the shaft in front of the *fan* should be 10 *chi*; Zheng Xuan says in some editions, one uses “7 *chi*” instead of “10 *chi*”. The computation is used here to answer the question as to which of the two versions is correct.

In the *Rites of Zhou Dynasty*, three kinds of horse are mentioned and differentiated by their height: the national horse [*guo ma* 國馬] 8 *chi*, the campestral horse [*tian ma* 田馬] 7 *chi*, and the tardy horse [*nu ma* 弩馬] 6 *chi*. This means the length of the horse here should be more than 8 *chi*.³⁵ For a national horse, the depth of shaft is 4 *chi* 7 *cun*.³⁶ Zheng Xuan finds the long side would be shorter than 6 *chi* by computation if the length in front of the *fan* was 7 *chi*. The data added by Zheng Xuan is that of the length of a horse. Thus, the quantity should be “ten *chi*” and the “seven *chi*” figure is wrong. Similar to the previous commentator's analysis, Zheng Xuan only shapes the problem in terms of a right angle triangle, diagram 12, and gives a final statement³⁷ and, except for this, he doesn't add any details, see the following sentences 31 and 32.

Diagrams 11-12



11: [The length] in front of the *fan*, [a board used at the front of a carriage] is ten *chi*.
The sketch of a carriage is borrowed from *Kaogongji tu* 考



12: Suppose one takes seven [*chi*] as the hypotenuse, four *chi* seven *cun* as the short side, and one looks for the corresponding long side. Hence, the

³⁵ It is reasonable to assume ancient people know the length of a horse is usually more than its height.

³⁶ The depth of shaft of national horse is four *chi* and seven *cun*. 國馬之轡深四尺七寸。 *Shisanjing zhushu*, 1979, 913.

³⁷ The difference lies in that Zheng Xuan doesn't give the exact quantity of the long side.

工記圖 (*Record Inspecting Public Works*). Dai Zhen 戴震. long side is shorter than the
Block-printed edition, 1746. The dotted lines and words are (expected) length of the horse.
added by the author of this article.

The text of the Classic says:

31. [The length] in front of the *fan*, [a board used at the front of a carriage], being ten *chi*, for [the length of the] horse whip, [called *ce*], one halves this.

軌前十尺，而策半之。

32. [Zheng Xuan's] Commentary: this means the length of the shaft attached to the front of the *fan*. *Ce* means the horse whip of the person who drives the horses. Another edition has "seven" instead of "ten". Suppose if one takes seven [*chi*] as the hypotenuse, four *chi* seven *cun* as the short side, and one looks for the corresponding long side, then the long side will be shorter [than the length of a horse], thus "seven" is wrong.

注：謂軌以前之長也。策，御者之策也。十，或作七。令七爲弦，四尺七寸爲鉤，以求其股。股則短矣。七，非也。

Like the previous commentary, Jia Gongyan comments on both the text of the Classic and Zheng Xuan's commentary.

33. [Jia Gongyan's] Explanation: when [the *Rites of Zhou Dynasty*] says "*fan*", this refers to the shape of the carriage.

釋曰：云「軌」，謂車式。

34. When [the *Rites of Zhou Dynasty*] says "[The length] in front of the *shi* [the railing at the top of the *fan*] being ten *chi*", this means the middle line in the curved part of the shaft.

式前十尺，謂轅曲中。

35. When [the *Rites of Zhou Dynasty*] says "for [the length of] the horse whip one halves this", halving it, the horse whip is thus five *chi*.

「軌策半之」，半之策則五尺矣。

36. The reason why [the *Rites of Zhou Dynasty*] speaks of the whip being used to drive the horses, is that it aims to take the combined measure of the lengths of the whip and the shaft to be estimated with respect to each other.

言策者，策以禦馬，欲取策與轅長短相准合度之意也。

In **33** to **36** Jia Gongyan comments on classic and Zheng Xuan's commentary.

37. When [Zheng Xuan] says, "Another edition has "seven" instead of "ten". Suppose if one takes seven [*chi*] as the hypotenuse, four *chi* seven *cun* as the short side, and one looks for the corresponding long side, then the long side will be shorter",

云「十，或作七。令七爲弦，四尺七寸爲鉤，以求其股。股則短矣」者。

38. [This means:] seven times seven being forty-nine makes four *zhang* nine *chi*.

七七四十九，四丈九尺。

39. Four times four being sixteen makes one *zhang* six *chi*.

四四十六，丈六尺。

40. Seven times seven being forty-nine further yields four *chi* nine *cun*.

七七四十九，又得四尺九寸。

41. Adding them together, [one *zhang* six *chi* and four *chi* nine *cun*] [makes] two *zhang* nine *cun*.

并之，二丈九寸。

From **37** to **41**, Jia Gongyan squares the hypotenuse and the short side respectively. In **38**, he calculates $7\text{chi} \times 7\text{chi} = 4\text{zhang}9\text{chi}$, which is the square of hypotenuse. This is the same principle as in **12**. In **39**, **40**, and **41**, he calculates the square of the short side. Since the short side is $4\text{chi}7\text{cun}$, Jia Gongyan should have squared this quantity, that is $(4\text{chi} + 7\text{cun})^2 = (4\text{chi})^2 + 2 \times 4\text{chi} \times 7\text{cun} + (7\text{cun})^2$. Jia Gongyan makes two mistakes here. One mistake is that he only calculated $(4\text{chi})^2 + (7\text{cun})^2$. This means the $2 \times 4\text{chi} \times 7\text{cun}$ is missing. Another mistake Jia Gongyan made is that he computes $4\text{chi} \times 4\text{chi} = 1\text{zhang}6\text{chi}$, and $7\text{cun} \times 7\text{cun} = 4\text{chi}9\text{cun}$ by the same principle, and then he adds them together directly. In fact, according to Jia Gongyan's own idea in the previous commentary, $1\text{zhang}6\text{chi}$ refers to a rectangle whose length is $1\text{zhang}6\text{chi}$ and width is 1chi , diagram **13**, and $4\text{chi}9\text{cun}$ refers to a rectangle whose length is $4\text{chi}9\text{cun}$ and width is 1cun , diagram **14**. The lengths of these two rectangles cannot be added together directly because their widths are not equal. When he actually does this, as $1\text{chi} = 10\text{cun}$, he multiplies the area of the very thin ($4\text{chi}9\text{cun}$ long and 1cun wide) rectangle by ten.³⁸ In other words, what he does is transform this thin rectangle into a new rectangle, which is $4\text{chi}9\text{cun}$ long and 1chi wide, in order to be added to the rectangle, which is $1\text{zhang}6\text{chi}$ long and 1chi wide.

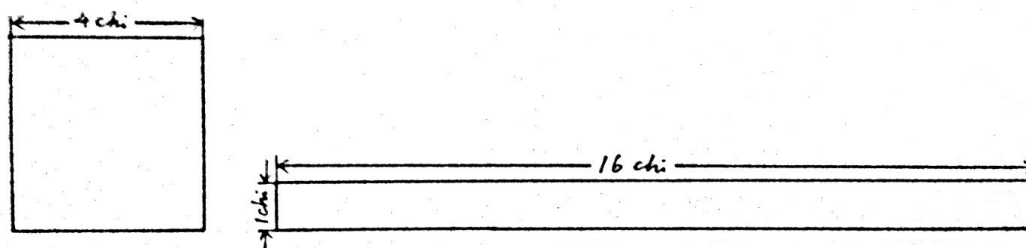


Diagram 13

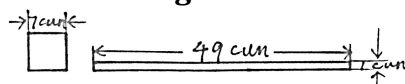


Diagram 14

However, due to the mistake Jia Gongyan made, missing $2 \times 4\text{chi} \times 7\text{cun}$, leads to a smaller result, and while the another mistake he made, multiplying $4\text{chi}9\text{cun}$ by 10 leads to a bigger result. As a consequence, the answer doesn't deviate too much from the correct answer. The correct answer for $4\text{chi}7\text{cun} \times 4\text{chi}7\text{cun}$ is $2\text{zhang}2\text{chi}9\text{fen}$ ³⁹, and Jia Gongyan obtains $2\text{zhang}9\text{cun}$ in **41**. The deviation is only $1\text{chi}1\text{cun}9\text{fen}$. This is probably why he does not notice this deviation. Jia Gongyan then goes on with the computation as follows:

³⁸ Another possibility is that he makes the large rectangle ten times smaller, so he can add these two rectangles together. From his own commentary later, we know he actually makes the small rectangle ten times larger.

³⁹ This statement of the result is based on the ancient meaning. Since $4.7 \times 4.7 = 22.09$, and $4\text{chi}7\text{cun} \times 4\text{chi}7\text{cun} = 4.7\text{chi} \times 4.7\text{chi} = 22.09\text{chi}$, $22.09\text{chi} = 2\text{zhang}2\text{chi}9\text{fen}$, which refers to a rectangle with $2\text{zhang}2\text{chi}9\text{fen}$ long and 1chi side.

42. Computation procedure: one removes the [square of the] short side from the [square of the] hypotenuse.

算法：以勾除弦。

43. If one removes two *zhang* nine *cun* from four *zhang* nine *chi*, leaving two *zhang* eight *chi* one *cun*.

以二丈九寸除四丈九尺，仍有二丈八尺一寸在。

In **42** and **43** Jia Gongyan subtracts the square of the short side from the square of the hypotenuse, yielding 2 *zhang* 8 *chi* 1 *cun*, which is the square of the long side. Sentences **42** and **43** have the same reasoning as **15** and **16**.

44. Then one looks for the corresponding long side.

然後以求其股。

45. [The rectangle is one *chi* wide and] two *zhang* eight *chi* one *cun* [long]; make a square of it.

以二丈八尺一寸，方之。

Sentences **44** and **45** are also the same as **17** and **18**, in which Jia Gongyan reveals his idea on how he perceives the extraction of the square root.

46. One makes a square whose sides are five *chi*.

爲五尺之方。

47. Five times five being twenty five, one uses two *zhang* five *chi* to make a square with sides of five *chi*.

五五二十五，用二丈五尺，爲方五尺也。

Table 1: comparison between two texts on the first step in square root extraction

19. One takes nine <i>chi</i> from one <i>zhang</i> two <i>chi</i> ;	46. One makes a square whose sides are five <i>chi</i> .
20. Each three <i>chi</i> , one cuts once.	47. Five times five being twenty five, one uses two <i>zhang</i> five <i>chi</i> to make a square with sides of five <i>chi</i> .
21. Complementing them with each other yields a square with sides of three <i>chi</i> .	

As illustrated in Table 1, the procedures in **46** and **47** are different from the previous case. In **19** to **21** Jia Gongyan firstly takes 9 *chi* and then makes a square with sides of 3 *chi*. In **46** and **47** Jia Gongyan first makes a square with sides of 5 *chi*, and then uses 2 *zhang* 5 *chi*. There is a minor difference between the two.

48. There remains [a length of/rectangle which is one *chi* wide and] three *chi* one *cun* [long].

餘有三尺一寸。

49. One computes all of these, [*chi* and *cun*], using a square whose sides are one *cun*, yielding [a rectangle which is one *cun* wide and] three hundred ten *cun* [long]; one brings it to the previous square [whose sides are five *chi*] to make a [larger] square of it.

皆以方一寸乘之，得三百一十寸，方之。

50. Taking three hundred *cun* yields [a rectangle which is] six *cun* wide and five *chi* long.

三百寸，得廣六寸，長五尺。

51. One divides it in the middle and joins the [resulting pieces] to the previous square whose sides are five *chi*.

中分之。裨前五尺之方。

52. Three *cun* is added to each side.

一廂得三寸。

In **48** to **55** Jia Gongyan deals with the remaining part of the rectangle, which corresponds to that of **22** to **27**. But the way in which the detailed procedure is carried out is slightly different. As a square whose sides are five *chi* was obtained in the previous procedure, Jia Gongyan now transforms the remaining part into rectangles 5 *chi* long. In **49** Jia Gongyan first transforms the remaining rectangle, which is 3 *chi* 1 *cun* long and 1 *chi* wide into a new rectangle, which is 310 *cun* long and 1 *cun* wide. The reason he does so is to make it easier to find the part of the result, which is measured in *cun*. Since 5 *chi* is equal to 50 *cun*, the first number of the result of the division $310\text{cun} \div 50\text{cun}$ is 6. In **50**, therefore, Jia Gongyan takes 300 *cun*, which refers to a rectangle whose length is 5 *chi* and width is 6 *cun*. Sentences **51** and **52** are the same operation as in **23** to **27**.

53. The tip of the angle is a square whose sides are three *cun*, 3 times 3 is 9, which further takes nine squares whose sides are one *cun*.

角頭方三寸，三三而九，又用一寸之方九。

54. There remains one square whose sides are one *cun* [which has not been taken] yet.

餘有一寸之方一在。

55. In total, this yields a square with sides of three *chi* five *cun*, leaving a square with one *cun* sides.

總得方五尺三寸，餘方一寸。

In **53** Jia Gongyan further takes nine *cun* from three hundred and ten *cun* to fill the tip of the corner. This leaves one *cun* in **54**. Therefore, in **55** “this yields a square with sides of three *chi* five *cun*, leaving a square with one *cun* sides”. Jia Gongyan doesn’t continue with the computation to obtain the part of the result corresponding to a smaller unit as the result is possibly enough for his commentary.

56. Seen from this angle, there would be only five *chi* three *cun* in the front of the *fan*, which would be too short for a horse.

以此言之，則軌前惟有五尺三寸，不容馬。

57. So [Zheng Xuan] says, “Then the long side will be shorter. “Seven” thus is wrong”.

故云「股則短矣，七非也」。

In **56** and **57**, since the horse is 6 *chi* long, the long side of 5 *chi* 3 *cun* would be too short for a horse. Thus, Jia Gongyan finishes his commentary.

2.3 Summary of the previous analysis

In the commentary on the *Rites of Zhou Dynasty* and *Ceremonies and Rites*, Jia Gongyan uses much mathematical knowledge. The method of square root extraction he uses above is one among many other methods he employed in mathematics, and it has the same features as the others.⁴⁰ Moreover, other commentators on the Confucian Canons use similar mathematical methods. For instance, in the seventh century, the Tang government had a huge project writing commentaries on the *Five Confucian Classics*⁴¹, and many scholars took part.⁴² In the *Records of Rites* edited by Kong Yinda and his colleagues, methods for the calculations of the thickness of bands and quantity of grains eaten in death rites are similar to Jia Gongyan's methods in *Ceremonies and Rites*.⁴³

The above fact indicates that Jia Gongyan did have a geometrical method for square root extraction,⁴⁴ which looks for the sides of a square equal to the area of a given rectangle with known length and width. The geometrical operation can be divided into many steps. Based on the previous analysis, the method has three main points:

1. The procedure is based on the geometrical interpretation of a quantity, that can represent a length, an area or a volume depending on the context, diagrams **6** to **8**. The principle of the relation between a quantity and its geometrical meaning is important because the numbers Jia Gongyan computes are used with measuring units. This principle makes it possible to give a geometrical meaning to square root extraction, diagrams **9** and **10**.
2. The procedure is a geometrical operation based on a correlated computation table and a table of measuring units. When Jia Gongyan executed his procedure, he needed to do calculations using tables. For example, when he calculated $4\ chi \times 4\ chi = 1\ zhang\ 6\ chi$ in sentence **12**, he needed “four times four is sixteen”, “*chi* times *chi* is *chi*”, and “one *zhang* is ten *chi*”, which came from multiplication tables and tables of measuring units.
3. The procedure is general, and the executions and the results can be different depending on the different radicands. This statement is on the modern mathematical analysis of Jia Gongyan's method.⁴⁵ Moreover, there are two possibilities for approximation for the results: approximations by excess and by

⁴⁰ I have done some research on this issue. For example, this article analyses Jia Gongyan's computation in a comparison between volume and capacity of a vessel: A Preliminary Research on the Mathematical Knowledge in Confucian Classics——Take a Excerpt of Jia Gongyan's commentary on the *Rites of Zhou Dynasty* as an example, *Studies in the Natural Sciences*, forthcoming.

⁴¹ The Five canons in early the Tang period refer to *Odes* [Shi 詩], *Documents* [Shu 書], *Records of Rites* [Li Ji 禮記], *Changes* [Yi 易], and *Spring and Autumn* [Chun Qiu 春秋]. For further information, see: Michael Nyland, 2010.

⁴² Simply speaking, this project had two phases: 633-642 and 651-653, respectively under the leadership of Kong Yingda [孔穎達 576-648] and Zhangsun Wuji [張孫無忌 ?-695]. Jia Gongyan assisted in both phases of the project.

⁴³ I have analyzed the two cases in another paper, see: Another Culture of Computation from Seventh Century China, forthcoming.

⁴⁴ The referee asked a question: why is the more straightforward explanation false? Jia Gongyan was not schooled in mathematics, and was unable to extract a square root by any method at all. The above fact answers this question.

⁴⁵ We can use modern mathematical notations to analyse Jia Gongyan's method, and it's not difficult to give the generalization of it. However, since we only have two cases of square root extraction in documents, it's better not to go too far with this.

defect. That is, the result is an approximation by excess in the first case,⁴⁶ while it is by defect in the second case.⁴⁷ The phrase “nearly half” shows Jia Gongyan understood the difference.

3. Analysis of Zhen Luan’s and Li Chunfeng’s procedures

While Jia Gongyan commented on the *Rites of Zhou Dynasty*, Li Chunfeng and his colleagues⁴⁸ did similar work on the *Ten Mathematical Books*.⁴⁹ One of Li’s colleagues, Wang Zhenru, commented on the Confucian Canons with Jia Gongyan,⁵⁰ and later commented on mathematical books with Li Chunfeng.⁵¹ Furthermore, in one of the *Ten Mathematical Books*, namely the *Mathematical Procedure of the Five Canons* [*Wujing Suanshu* 五經算術], the same problem as discussed in the first case of Jia Gongyan’s procedure was raised. The author of the book, Zhen Luan [甄鸞, 6th century], gave a procedure for square root extraction, which is carried out using counting rods. Li Chunfeng commented on the Classic and Zhen Luan’s method. He confirms Zhen Luan’s method and further rearranges the structure of the problem. Thus, it is necessary to present this text, and compare the mathematical knowledge in the Confucian Canons and the *Ten Mathematical Books*.

The text of the Classic says:

58. One trisects the length of the rib, taking one of them [the pieces obtained] as *zun*.

參分弓長，以其一爲之尊。⁵²

59. [Zheng Xuan’s] Commentary says: *Zun* means height. For a rib of six *chi*, the higher horizontal part near the centre of the canopy is two *chi*, hence one extremity of the claw is lower by two *chi* than the centre of the canopy. Two *chi* being taken as the short side, four *chi* being taken as the hypotenuse, one looks for the corresponding long side. The [square of the] long side is twelve [*chi*]. One divides it by extracting the square root, which makes the sides, [of the square of its long side] three *chi* and nearly half [a *chi*].

注云：尊，高也。六尺之弓，上近部平者二尺，爪末下於部二尺。二尺爲句，四尺爲弦，求其股。股十二。開方除之，面三尺幾半。

⁴⁶ Jia Gongyan obtains three *chi* and a half *chi* as the result of the square root of 12 *chi*. The result is more than the actual square root, which is approximately 3.46 *chi*.

⁴⁷ Jia Gongyan obtains 5 *chi* and 3 *cun* as the result of extracting the square root of 2 *zhang* 8 *chi* 1 *cun*. The result is less than the actual square root, which is just over 5 *chi* and 3 *cun*.

⁴⁸ Li Chunfeng’s colleagues were Li Shu 梁述 and Wang Zhenru 王真儒. At that time, Li Chunfeng was the Grand Astrologer of the Astrological Service [Taishiling 太史令]; Li Shu was an Erudite of Mathematics [Suanxue Boshi 算學博士]; Wang Zhenru was an Assistant at the National University [Taixue Zhujiao 太學助教]. Liu Xu, *Jiu Tangshu*, 1975, 2719. Ouyang Xiu, Song Qi, *Xin Tangshu*, 1975, 5798.

⁴⁹ Li Chunfeng and his colleagues carried out their work in 656. Liu Xu, *Jiu Tangshu*, 1975, 2719.

⁵⁰ Ouyang Xiu, Song Qi, *Xin Tangshu* 1975, 1428.

⁵¹ When Wang Zhenru commented on the Confucian classics, he was an Assistant at the School of the Four Gates [Simen Zhujiao 四門助教]. According to *the Old Books of Tang Dynasty*, this role is a little lower than the Assistant of the National School. Therefore, I assume Wang’s commentary on the mathematical books is later than on the Confucian classics.

⁵² *Wujing suanshu* 五經算術 (*Mathematical Procedures of the Five Canons*). In: [算經十書 (*Ten Classical Books of Mathematics*). Qian Baocong 錢寶琮 (ed.), 1963. Zhonghua Book Company, Beijing, 437-486]. In this paper, all texts from the *Mathematical Procedures of the Five Canons* come from this edition.

Sentence **58** is the same as **1**. Sentence **59** is almost the same as **2** to **6**. A slight difference is that in **6** it says, “one divides it” whereas here it is “one divides it by extracting the square root”. Since the earliest edition of the *Mathematical Procedure of the Five Canons* is an 18th edition, that is the edition in the *Complete Works in the Four Treasuries* [*Siku Quanshu* 四庫全書], the difference may be caused by collation.

3.1 Zhen Luan’s method

60. Zhen Luan’s Note: [According to] the *Gou Gu* method⁵³, the horizontal [line] is taken as *gou* [the short side]; the vertical [line] is taken as *gu* [the long side]; and, the oblique [line] is taken as *xian* [the hypotenuse]. If the short side is three and [the corresponding] long side is four, then the [corresponding] hypotenuse is five. These [numbers] are *lǚ*⁵⁴ [conform to] nature. If now in the canopy of this carriage, the short side is two; the hypotenuse is four; then the [integer of] the long side is three. These [numbers] are equally *lǚ* [corresponding] to nature. The procedure for looking for [the answer] is: One multiplies the short side and the long side respectively by themselves, adding them [the results] together, then dividing this [the result] by extracting the square root, thus makes the hypotenuse. The long side multiplied by itself being subtracted from the hypotenuse being multiplied by itself, one thus divides this [the result] by extracting the square root, which makes the short side. The short side being multiplied by itself being subtracted from the hypotenuse multiplied by itself, one then divides this [the result] by extracting the square root, which makes the long side.

甄鸞按：句股之法，橫者爲句，直者爲股，邪者爲弦。若句三，則股四而弦五，此自然之率也。今此車蓋，句二、弦四則股三，此亦自然之率矣。求之法，句股各自乘，并，而開方除之，即弦也。股自乘，以減弦自乘，其餘開方除之，即句也。句自乘，以減弦自乘，其餘開方除之，即股也。

61. Suppose: one multiplies the three of the short side by itself, which yields nine; one multiplies the four of the long side by itself, which yields sixteen; adding them together yields twenty-five. One divides this by extracting the square root, which makes five that is the hypotenuse. One multiplies the four of the long side by itself, which makes sixteen; one multiplies the five of the hypotenuse by itself, which makes twenty-five; subtracting sixteen from twenty-five, leaves nine. One divides this by extracting the square root, which yields three that is the short side. One multiplies the three of the short side by itself, which yields nine; one multiplies the five of the hypotenuse by itself, which yields twenty-five; subtracting nine from twenty-five, leaves sixteen. One divides this by extracting the square root, which makes four, that is the long side.

假令句三自乘得九，股四自乘得十六，併之，得二十五。開方除之，得五，弦也。股四自乘，得十六，弦五自乘得二十五，以十六減之，餘九。開方除之，得三，句也。句三自乘得九，弦五自乘得二十五，以九減之，餘十六，開方除之，得四，股也。

⁵³ *Gou* refers to the short side in a right angle triangle. *Gu* refers to the long side in a right angle triangle. The *Gou Gu* method was given in the *Nine Chapters on Mathematical Procedures*. It is for finding the unknown side of a right angle triangle from two known sides.

⁵⁴ *Lǚ* 率 was a complex concept in Chinese mathematics. See: Karine Chemla, Guo Shuchun, 2004, 956.

In 60 Zheng Luan introduces the *Gou Gu* method. By saying, “If now in the canopy of this carriage, the short side is two, the hypotenuse is four, and then the [integer of the] long side is three”, he connects the canopy problem to the *Gu Gu* method. In 61 he gives further explanation in three examples.

61. Now the height of the centre of the carriage canopy is two *chi* and the rib is four *chi*. Two *chi* [by which the extremity of the claw is] lower than the centre [of the canopy of the carriage], being taken as the short side and, four *chi* for the rib being taken as the hypotenuse, one looks for the corresponding long side. The method of looking for the long side is: multiplying two *chi* of the short side by itself yields four; multiplying four *chi* of the hypotenuse by itself yields sixteen; subtracting four from sixteen, leaves twelve.

今車蓋崇二尺，弓四尺。以崇下二尺爲句，弓四尺爲弦，爲之求股。求股之法，句二尺自乘得四，弦四尺自乘得十六，以四減十六，餘十二。

In this paragraph, Zhen Luan first obtains the squares of the long side and the hypotenuse. His computations “ $2chi \times 2chi = 4$, $4chi \times 4chi = 16$ ” are different from Jia Gongyan’s, which can be expressed as “ $2chi \times 2chi = 4chi$, $4chi \times 4chi = 1zhang6chi$ ”. One obvious difference is that the results of Zhen Luan’s computations have no measuring units. The reason maybe that using counting rods means the numbers are usually without measuring units and are nearly abstract numbers.⁵⁵

62. Dividing it by extracting the square root⁵⁶ yields three, which means the long side has three *chi*, with remainder three. Doubling the divisor of the square, three, called *fang fa*, yields six. One further adds the nethermost divisor, one, called *xia fa*, to it, making seven. That is three and three parts in a *chi* that is divided into seven *chi*⁵⁷ of the long side. This is why [Zheng Xuan] says “nearly half [a *chi*]”.

開方除之，得三，即股三尺也。餘三。倍方法三，得六。又以下法一從之，得七。即股三、七分尺之三，故曰幾半也。

The terms Zhen Luan used in this paragraph reveal he relies on counting rods to execute the procedure. On the basis of previous research, the procedure is shown in diagram 15.

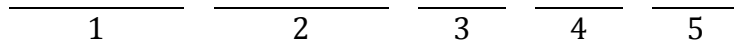
Diagram 15

The result	
The radicand	
The divisor of the square	
The nethermost divisor	

⁵⁵ This is my argument. When one uses counting rods, it there are two possibilities: either it means the use abstract numbers since rods represent numbers, or it refers to quantities with a measuring unit of rods. For the first possibility, we have evidence from Chinese mathematical writings. Calendric calculations bear witness to the second possibility.

⁵⁶ Reference to Karine Chemla’s translation. And I further borrow “divisor of square”, “nethermost divisor” from her.

⁵⁷ This refers to the value $3/7$ *chi*.



Step 1: Place 12 [the radicand] and 1 [the nethermost] on the computational surface.

Step 2: Estimate the first digit of the result is 3, then place 3 on the uppermost line and 3 on the line of the divisor of the square.

Step 3: Compute $12 - 3 \times 3 = 3$, which makes the remainder.

Step 4: Double the divisor of the square, which makes 6.

Step 5: Add the nethermost divisor to the divisor of the square, which gives $1 + 6 = 7$. Diagram 5 is the result of the whole operation, $3 + 3/7$.⁵⁸

This casts light on different ideas for Zheng Xuan's statement. Jia Gongyan wants to prove the result is smaller than three and half a *chi* by using a geometrical operation. Zhen Luan wants to get the value by an arithmetical operation, and then to prove this value is nearly three and half a *chi*.⁵⁹ That is to say, $3 + 3/7$ is actually less than both the square root of 12 and $3 + 1/2$, and this cannot prove that the square root of 12 is less than $3 + 1/2$. Zhen Luan is only able to make his statement if he believes $3 + 3/7$ is the exact result of the square root of 12.

3.2 Li Chunfeng's Commentary

Next, Li Chunfeng's Commentary on this problem is considered:

63. Li Chunfeng et al's comment: this problem should be raised as: the length of the rib of the canopy of the carriage is six *chi*. The higher two *chi* part near [the centre of the canopy], joining the centre of the canopy and being horizontal, is taken as the height. Four *chi* of the line from the articulation of the rib, called *yu qu*, in an oblique downwards direction is taken as the hypotenuse. Two *chi*, by which the extremity of the claw is lower [than the centre of the canopy], is taken as the short side. One looks for the corresponding long side. One asks: how long is the long side? Answer: three *chi* three [parts] of a *chi* that is divided into seven parts. The procedure is: the short side multiplied by itself is subtracted from the hypotenuse

⁵⁸ In Liu Hui's commentary on "Kai Fang Shu" in the *Nine Chapters on Mathematical Procedures*, he gives two ways of approximation for square root extraction, respectively saying "Command one doesn't add the borrowed counting rod [to the divisor of the result] and names a fraction, then [the result will be] always a little smaller [than the correct result]; if one adds the borrowed counting rod [to the divisor of the result], [the result will be] always a little bigger [than the correct result]". "令不加借算而命分，則常微少；其加借算而命分，則又微多。" The Song edition of the *Nine Chapters*. That is to say there are two methods for approximation. One is to not add the borrowed counting rod [i.e. the nethermost divisor in the previous operation] and get the approximation by excess, e.g. $3 + 3/6$ *chi* in this case of the 4th step. Another is to add the borrowed counting rod and get the approximation by defect, e.g. $3 + 3/7$ *chi* in this case of the 5th step. Apparently, Zhen Luan adopts the first method of approximation. The reason why Zhen Luan doesn't adopt the second method lies in two aspects: 1, Zhen Luan's aim is to prove Zheng Xuan's commentary, in which Zheng Xuan says "three *chi* and nearly half [a *chi*]". If Zhen Luan got $3 + 3/6$ *chi*, this would mean "a half", rather than "nearly half [a *chi*]"; 2, "nearly half [a *chi*]" means smaller than a half, i.e. one should obtain a value smaller than $3 + 1/2$ *chi*. Therefore, Zhen Luan has to adopt the method of "adding the borrowed counting rods".

⁵⁹ In modern mathematics, Zhen Luan is wrong, because he obtains an approximation, which is $3 + 3/7$ *chi*. His aim is to prove $\sqrt{12} < 3 + 1/2$. However, the value he obtained $3 + 3/7$ *chi* is less than $\sqrt{12}$ *chi*. Based on $3 + 3/7 < 3 + 1/2$, and $3 + 3/7 < \sqrt{12}$, Zhen Luan would fail to prove $\sqrt{12} < 3 + 1/2$, which is Zheng Xuan's statement "three *chi* and nearly half [a *chi*]". Therefore, only if he believes $3 + 3/7 = \sqrt{12}$, that is he obtains the exact result, does he prove Zheng Xuan's statement. Because of this, it casts light on how he understands the procedure. He does not think it's an approximation value. That is to say he believes by these procedures [diagram 15], an exact value is obtained. And he further finds it's just "three *chi* and nearly half [a *chi*]".

multiplied by itself. Dividing the remainder by extracting the square root thus yields the long side.

臣淳風等謹按：其問宜云：車蓋之弓長六尺，近上二尺連部而平爲高。四尺邪下宇曲爲弦，爪末下于部二尺爲句，欲求其股。問股幾何？曰：三尺七分尺之三。術曰：句自乘以減弦自乘，其餘開方除之，即得股也。

In this paragraph, Li Chunfeng obtains the same result as Zhen Luan. This implies that Li Chunfeng's procedure is the same as Zhen Luan's, and he further confirms Zheng Xuan's procedure. What is noteworthy is that Li Chunfeng reorganizes the structure, and makes it look like a mathematical problem. That is to say, Li Chunfeng reshapes the Confucian Canon texts into Mathematical Canon texts.⁶⁰

4. Geometrical bases of square root extraction

A primary comparison makes it clear that Jia Gongyan's and Li Chunfeng's procedures are different. On the one hand, Jia's method is geometrical, in which geometrical figures⁶¹ are to be cut for computation. The operation on figures uses multiplication and measuring-unit tables, on the basis of the geometrical interpretation of a quantity. Jia doesn't use counting rods. On the other hand, Li's method is arithmetical, and relies on the use of counting rods. That is to say, different mathematical instruments are used in both their procedures.

Besides computations, there are geometrical bases. In order to address the original question, the bases need to be discussed further. Karine Chemla has argued that Liu Hui's computation based on the formula for the calculation of the area of a circle has two facets: one is the meaning of the magnitude and another is to interpret the meaning of the value.⁶² For the operation of square root extraction, the radicand similarly has two facets: a rectangle for computation, and a square for geometrical interpretation.⁶³

Li Chunfeng especially distinguishes two characters: *Ji* 積 and *Mi* 冪. *Ji* refers to the magnitude that is a rectangle, while *Mi* refers to the meaning of value that is a square. Therefore, Li Chunfeng's point is similar to Liu Hui's. However, for Jia Gongyan, the two facets are the same. To make a square from a rectangle is not only for computation, it also has a geometrical meaning. In Tables 2 and 3, the meanings

⁶⁰ My statement on this is that by reshaping canon texts into a mathematical form, Li Chunfeng makes mathematical writings become canons. Further discussion will be shown in my other paper, "Another Culture of Computation from Seventh Century China".

⁶¹ Karine Chemla has argued that diagrams are not ideal materials for early times [3rd century]. I agree with this point. See: Karine Chemla, 2010b.

⁶² Using this example, she proves that the two facets are different. For $A = \frac{C}{2} \cdot \frac{D}{2}$ where A is the area of a circle; C, the circumference; and, D the diameter, the result will be exact. However the magnitude will always be an approximation unless the procedures can continue indefinitely. Her analysis for the case of squares and square root extraction are similar. See: Karine Chemla, 2005.

⁶³ This can also be witnessed in Liu Hui's case. In the *Nine Chapters*, in his comments on texts, the commentator Liu Hui says, "The short side being multiplied by itself makes the vermilion square, the long side multiplied by itself makes the blue square" [句自乘爲朱方，股自乘爲青方。Karine Chemla, Guo Shuchun, 2004, 706.] Liu Hui gives the geometrical meaning to the square [operation] that is to obtain a square [area]. However, for computation, Liu Hui needs another facet for the operation in order to make the quantity meaningful and to obtain a rectangle.

of magnitude and value for squares and square roots for each commentator are compared.

Table 2: the meanings of magnitude for geometrical interpretation













Operations	Square	Square root extraction
Jia Gongyan	 → 	 → 
Liu Hui and Li Chunfeng	____ → 	 → ____

Table 3: the meanings of value for computations

Operations	Square	Square root extraction
Jia Gongyan	 → 	 → 
Liu Hui and Li Chunfeng	____ → 	 → ____

5. Conclusion and Discussion

At the beginning of this paper, several questions were raised: what did ancient practitioners view as the operation of square root extraction? What is the relationship between the operation and its various procedures? What is the relationship between the arithmetical procedures and the geometrical procedures? Which terminology was involved in the procedures? Such questions are all, in part, related to the mathematical philosophy of the time and are not easy to answer. However, based on previous analyses and comparisons between Jia Gongyan's and Li Chunfeng's procedures, there have already been opportunities to address this issue.

It is shown that in seventh century China, two scholars, Jia Gongyan and Li Chunfeng, had different ideas and procedures for the same problem given in the Confucian Canons. Their differences lie in many aspects and their aims for commentating are different. Jia Gongyan wants to prove Zheng Xuan's Commentary, as does Li Chunfeng. Moreover he wants to shape the problem into a mathematical form in order to show the special role of mathematics in commentating on the Confucian Canons. Jia Gongyan seldomly uses counting rods, which were the typical instruments for doing mathematics at that time.

Key terms used for square root extraction are different in the mathematical texts and the Confucian texts.⁶⁴ This means the way the procedures are written are different. Zheng Xuan uses the term "Chu 除" to refer to the operation of square root extraction. Jia Gongyan comments on "Yue 約" which has a very close meaning to "Chu 除". Zhen Luan and Li Chunfeng both use "Kai Fang Chu 開方除", the same term as in the *Nine Chapters*. For the procedures, "Kai Fang Chu 開方除" is used by Zhen Luan and Li Chunfeng. On the other hand, Jia Gongyan interprets the aim of the operation as being the making of a square, namely the procedure "Fang Zhi 方

⁶⁴ Karine Chemla gave me suggestions, to differentiate between the operation and its different procedures.

之”。They actually establish two different links between the operation and the procedures, “Fang Zhi” and “Kai Fang Chu” respectively.⁶⁵

The uses of geometrical figures are different in the two kinds of texts. Li Chunfeng and other scholars of mathematics use geometry as an instrument to illustrate and interpret the arithmetical procedure carried out with counting rods. This is the geometrical basis for “Kai Fang”. For Jia Gongyan, geometrical figures are not only used to interpret the procedure, but are also necessary for the execution of the procedure. Jia Gongyan and Li Chunfeng have different geometrical interpretations for the operation of square root extraction and, the instruments used are different, Tables 2 and 3.

Furthermore, three layers of mathematical practices appear in the texts under discussion. The basic layer is the geometrical interpretation of the operation of square root extraction, Tables 2 and 3. Secondly, a procedure to execute the operation is needed. For Jia Gongyan, the procedure is geometrical, using tables. For Li Chunfeng, the procedure is arithmetical, relying on counting rods. Finally, the procedure is written in texts with technical terms. The key terms are different in the two kinds of texts.

It is seen that **geometry**, **counting rods**, and **written texts** are three elements used in different layers of mathematical practice and they have different roles. They can exist individually, or two or all three can co-exist, depending on different cultures of computation. **Geometry**, and **written texts** are used by Jia Gongyan and scholars of Confucianism. **Geometry**, **counting rods**, and **written texts** usually exist simultaneously, and have a close relationship for Li Chunfeng and scholars of mathematics.

To sum up, it is possible to say that in seventh century China there was a divergence on Zheng Xuan’s commentary about square root extraction. There was no uniform understanding of this operation.⁶⁶ Jia Gongyan’s and Li Chunfeng’s methods represent different cultures of computation. These two cultures, in which the elements of mathematical practice interact, structured mathematics in different ways. The overlap of the cultures lays in the general terminology, tables and systems of measurement, and reveals the common mathematical feature shared by scholars at that time.

Returning to the issue of square root extraction, the different ways of carrying out square root extraction used by two colleagues in the Tang dynasty have been seen. These methods are based on geometry, counting rods and written texts, which imply that the relationships between the operation and procedures, and between geometry and arithmetic, depended on the different ways these elements were structured by the cultures. These cultures of computation, in turn, were probably

⁶⁵ In the Confucian Canons, when scholars use “Kai Fang”, this term has a different meaning.

⁶⁶ It is true Jia Gongyan and Li Chunfeng shared some common ground on square root extraction. Both aimed to prove Zheng xuan’s commentary correct. Also, some general terms for operations are the same, for example, Cheng 乘, which means “to multiply”. They both use the Nine- Nine table and the same system of measuring units and, they both accept the key relation between a quantity and its geometrical interpretation, however, the different ways they use them show the diversity of ancient Chinese computational cultures.

the result of different interpretations of the interplay between these elements of mathematical practice. Moreover, the interplay among cultures of computation is very complex and needs to be studied more in the future.

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