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Tae-Hwan Kim, Christophe Muller. A Particular Form of Non-Constant Effect in Two-Stage Quantile Regression. 2015. halshs-01157552v2

**HAL Id: halshs-01157552**

**<https://shs.hal.science/halshs-01157552v2>**

Preprint submitted on 6 Jan 2016 (v2), last revised 10 Apr 2017 (v3)

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WP 2015 - Nr 22

# A Particular Form of Non-Constant Effect in Two-Stage Quantile Regression

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December 2015

**Abstract:** For linear quantile regression under endogeneity, it is often believed that the fitted-value approach (i.e., replacing endogenous regressors with their exogenous fitted-values) implies constant effect (that is: the coefficients of the covariates do not depend on the considered quantile, except for the intercept), as discussed in Lee (2007). We show the possibility of a particular form of non-constant effect models with the fitted-value approach under a weakened IV condition. However, only the constant-effect coefficients of the model can be consistently estimated.

**JEL Codes:** C13, C21, C31.

**Key Words:** Two-Stage Estimation, Quantile Regression, Fitted-Value Approach, Non Constant Effect.

This work was carried out thanks to the support of the A\*MIDEX project (No. ANR-11-IDEX-0001-02) funded by the "Investissements d'Avenir" French Government program, managed by the French National Research Agency (ANR). We are grateful to P. Lavergne for a useful discussion and to participants in a seminar at the Toulouse School of Economics and at the 2015 ERCIM conference in London for their comments. Remaining errors are ours.

# 1 Introduction

## 1.1 The issue

This paper considers the estimation of a structural linear equation using quantile regression in the presence of endogeneity problems. Since the seminal work by Koenker and Bassett (1978), the literature on quantile regressions has expanded a great deal. There are two trends in this literature when dealing with endogeneity. The first one, denoted the ‘structural approach,’ corresponds to models directly specified in terms of the conditional quantile of the structural equation of interest. In that case, semi-parametric restrictions needed for identification are directly imposed on the structural errors terms, or on the structural model if there are no separable errors. The second trend, denoted the ‘fitted-value approach,’ is based on the conditional quantile of the reduced-form equation. Accordingly, the restrictions are imposed on the reduced-form errors when they can be separated. In this latter approach, the analysts substitute the endogenous regressors with the fitted-values obtained from ancillary equations based on some exogenous variables. In this paper, we exhibit a particular case of non-constant effect (i.e., quantile-dependent coefficients) for linear quantile regression with the fitted-value approach, a situation generally believed to be ruled out (that is: it is believed that all coefficients, except the intercept must be the same for all quantiles), as discussed in Lee (2007). The latter criticism has not been addressed in the recent literature, other than by falling back to the structural approach, or by assuming constant effect for the true quantile process. We deal with this gap.

The literature on the structural approach for linear quantile regressions is abun-

dant,<sup>1</sup> while it meets computation difficulties for correcting endogeneity issues.<sup>2</sup> On the other hand, the fitted-value approach corresponds to a simple two-step quantile regression procedure that is analogous to the 2SLS method. Such a procedure has been employed by empirical researchers.<sup>3</sup> Partial theoretical results had been obtained by Amemiya (1982) and Powell (1983), who analyse the two-stage least-absolute-deviations estimators in simple settings, and by redefining the dependent variable. Chen and Portnoy (1996) and Kim and Muller (2004, 2015) investigate such two-stage quantile regressions with diverse first-step estimators (LS, LAD, and trimmed least squares estimators) and in general settings.

However, all these authors deal with constant effect specifications, according to Lee (20007)'s analyses. In contrast, our focus in this paper is the occurrence of non-constant effect with the fitted-value approach, although heterogeneity will still not be allowed for some coefficients (of endogenous or exogenous variables, depending on hypotheses).

For this, we first show that any separable model can be made to satisfy a quantile restriction for any quantile  $\theta$ , provided an inconsistency term, which we characterize, is included. Second, we show how the inconsistency term can be mobilised to define some weakened IV conditions, allowing for endogenous regressors in a (pseudo) reduced form. Under these new IV conditions, we show that non-constant effect can arise with endogeneity in linear quantile regression with restrictions corresponding to the fitted-value approach. However, in that case, only the constant effect coefficients can be estimated consistently. Finally, we show that the constant effect (respectively

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<sup>1</sup>See for example: Abadie et al. (2002), Chen et al. (2003), Chesher (2003), Hong and Tamer (2003), Chernozhukov and Hansen (2005, 2006, 2008), Imbens and Newey (2006), Ma and Koenker (2006), Chernozhukov, Imbens and Newey (2007), Lee (2007), Jun (2009), etc.

<sup>2</sup>As documented in Kim and Muller (2013).

<sup>3</sup>Arias et al. (2001), Garcia et al. (2001), Chortareas et al. (2012) and Chepatrakul et al. (2012).

non-constant effect) coefficients of the pseudo reduced form can be simply transmitted to constant effect (respectively non-constant effect) coefficients in the structural equation.

## 1.2 Practical examples

We now illustrate our reflections with a few economic examples in which constant effect occurs at least for some variables in the regressions. A first illustration is the assessment of a policy rule that, first, is constant for an identifiable population, and second, affects an outcome variable that is shifted by a quantity proportional to the policy rule. For example, one may wish to assess the impact of a cash transfer program on family earnings. Let  $y_t$  be the total earnings of family  $t$  and  $Y_t$  be the policy treatment, which is here a dummy variable describing whether or not the family is covered by the program. Assume that the treated families can be identified through some observable characteristics, for example, because they live in a specific place or have given socio-demographic characteristics. For example, in France, family allowances are implemented through cash transfers that are exclusively based on the number of children by age class in the family (ADECRI, 2008). The information on family composition is generally observed in household surveys, which could be used to run a regression of family earnings. In that case, the transfer amount is generally not or is ill observed in the surveys, but the treatment dummy variable ( $Y_t$ ) can be observed, while it may be endogenous. Indeed, unobserved characteristics may be present that affect both household income and household composition. For example, families with low work motivation may have more children and less income. This may be the case if having children is used by some families to access social aid and compensate for insufficient incomes. Assume that the cash amount transferred by the

program is unknown to the researcher. This amount is the constant parameter  $\gamma$  to estimate.

Another illustration is the assessment of the impact on household earnings of taxes that are defined in terms of some identifiable categories of household. Assume that these unobserved tax amounts are fixed within each household category. Then, when running a quantile regression with the household net income as a dependent variable, the taxes can be described by a vector of fixed coefficients  $\gamma$  that measures the role of the dummy variables specifying the household categories. Another example is that of racketing extraction tariffs imposed by mafia organisations, which have the same property of being fixed for some given categories of businesses, while their actual amounts are generally unobservable by researchers.

A final illustration is the assessment of the contribution of some unobserved expenses for a discrete good, or a discrete service, to total family expenditure. For example, the acquisition or the ownership of a given durable good with fixed characteristics might be observable, but not its fixed price. Estimating the corresponding coefficient  $\gamma$  would allow some inference about the unobserved price.

In all of these illustrations, we have encountered constant effect for a treatment variable of interest. In these examples of models, some variables  $x_{1t}$  that determine family earnings or expenditure - as location, activity types, education - can be included. For the current discussion, assume that these variables are exogenous<sup>4</sup>, for example because they have been pre-determined. In the considered examples their coefficients  $\beta$  should generally correspond to non-constant effect. Indeed, economic theory does not provide any reason for imposing constant effect for these variables. Such non-constancy is likely to generate a specification bias in quantile regressions that would wrongly assume constant coefficients for these variables. However, if the

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<sup>4</sup>We shall be able to relax this assumption later on.

interest of the researcher is exclusively in the coefficients  $\gamma$  of the structural endogenous regressors, the fact that the coefficients  $\beta$  of the structural exogenous regressors have non-constant effect that cannot be estimated consistently with the fitted-value approach is not an issue as long as constant effect coefficients can be estimated consistently, which we show in this paper.

Finally, some observed variability in the treatment can easily be incorporated in these cases by interacting the treatment variable with some observed characteristics, or by considering subpopulations defined in terms of these characteristics. This is an easy way to relax too stringent specifications of constant effect.

The paper is organized as follows. Section 2 presents the model and the assumptions. In Section 3, we exhibit and analyse a case of non-constant effect for the fitted-value approach. Finally, we conclude the paper in Section 4.

## 2 The Model

Assume that our interest lies in the parameter vector  $\alpha_\theta \equiv (\beta'_\theta, \gamma'_\theta)'$  in the following linear equation for  $T$  observations and an arbitrary quantile index  $\theta \in (0, 1)$  that will denote quantile restrictions introduced later on.

$$\begin{aligned} y_t &= x'_{1t}\beta_\theta + Y'_t\gamma_\theta + u_{t\theta} \\ &= z'_t\alpha_\theta + u_{t\theta}, \end{aligned} \tag{1}$$

where  $[y_t, Y'_t]$  is a  $(G + 1)$ -row vector of endogenous variables,  $x'_{1t}$  is a  $K_1$ -row vector of exogenous variables,  $z_t = [x'_{1t}, Y'_t]'$  and  $u_{t\theta}$  is an error term.

Since we wish to study non-constant effect models, we emphasize that the coefficient vector and the errors may vary with the considered quantile index  $\theta$ . However, to give sense to this notation, we shall introduce below some quantile restrictions at



quantile  $\theta$ , in Assumptions 1 and 4, and in Propositions 2-5. We denote by  $x'_{2t}$  the row vector of the  $K_2$  exogenous variables excluded from (1), and we assume  $K_2 \geq G$ . We further assume that  $Y_t$  can be linearly predicted from the exogenous variables through the following equation, which we assume to be correctly specified.

$$Y'_t = x'_t \Pi + V'_t, \quad (2)$$

where  $x'_t = [x'_{1t}, x'_{2t}]$  is an unbounded  $K$ -rows vector with  $K = K_1 + K_2$ ,  $\Pi$  is a  $K \times G$  matrix of unknown parameters, and  $V'_t$  is a  $G$ -row vector of unknown error terms. Again, some stochastic assumptions on the errors  $V_t$  must be made so as to complete (2) for a correctly specified model. For example, to fix ideas, one may assume that the conditional expectation of  $V_t$  is zero, as for OLS estimation, thus ensuring the consistency of the fitted-value for  $Y_t$ . To avoid absurdities, we assume that the columns in  $x_t$  are linearly independent. Using (1) and (2),  $y_t$  can also be expressed as:

$$y_t = x'_t \pi_\theta + v_{t\theta}, \quad (3)$$

where

$$\begin{aligned} \pi_\theta &= H(\Pi) \alpha_\theta \text{ with } H(\Pi) = \left[ \begin{pmatrix} I_{K_1} \\ 0 \end{pmatrix}, \Pi \right] \\ \text{and } v_{t\theta} &= u_{t\theta} + V'_t \gamma_\theta. \end{aligned} \quad (4)$$

Again here, the vector of coefficients  $\pi_{0\theta}$  may vary with the quantile  $\theta$ . The fitted-value approach corresponds to quantile restrictions on the reduced form. As noted in Blundell and Powell (2006): “The reduced form for  $y_t$  may be of interest if the values of IVs are control variables for the policy maker.” However, the reduced form can also be viewed as an intermediary stage for calculations. Amemiya (1974) pointed out that, while substitution of fitted values in nonlinear structural functions generally yields

inconsistent estimates of the structural parameters, consistent estimation methods that substitute fitted values into the structural function can rely on linearity of the regression, where the model is based on the reduced form error  $v_t = u_t + V_t' \gamma$ , with similar stochastic properties to the structural error  $u_t$ . We first consider the following quantile restriction on the reduced-form errors for a given quantile  $\theta_0$ . Later on, we shall also discuss the restriction when quantile restrictions are imposed over the whole quantile process altogether.

**Assumption 1:**  $E(\psi_{\theta_0}(v_{t\theta_0})|x_t) = 0$ , for an arbitrary given  $\theta_0 \in (0, 1)$ , where  $\psi_{\theta}(z) \equiv \theta - 1_{[z \leq 0]}$ , and  $1_{[\cdot]}$  is the indicator function.

Assumption 1 imposes that zero is the given  $\theta_0^{th}$ -quantile of the conditional distribution of  $v_{t\theta_0}$ . This assumption allows the identification of the coefficients  $\alpha_{\theta}$  of the quantile regression model centered in quantile  $\theta_0$ . It is associated with the fitted-value approach in which, first, the conditional quantile restriction is placed on the reduced-form error  $v_{t\theta_0}$ , and second, the information set used for the conditional restriction exclusively consists of the exogenous variables  $x_t$ . It has been used in Amemiya (1982), Powell (1983), Chen and Portnoy (1996), and Kim and Muller (2004, 2015). In particular, Kim and Muller (2015) provide an analysis of asymptotic properties based on this assumption and with broad stochastic conditions. One issue is how quantile restrictions for different quantiles come together so as to define a unique quantile process for a single model; that is, so that all these restrictions are compatible. The analysis we pursue will clarify this point, which is at the core of understanding the possible non-constancy of effects.

Although directly imposing conditional quantile restrictions on the structural error term is more popular, the fitted value approach is an alternative way of exploring the

conditional distribution of the dependent variable  $y_t$  through the reduced-form error. In that case, the structural effects are recovered in the second stage of the estimation. The link of structural and reduced-form parameters is described by (4). Identification of the two-stage procedure is allowed by the following assumption.

**Assumption 2:**  $H(\Pi_0)$  is of full column rank.

Assumption 2 is the usual necessary condition for IV regressions, for example, obtained through usual exclusion restrictions. This is an identification condition for simultaneous linear equations models. In the fitted-value approach, a first-stage estimator of  $\Pi$  in (2),  $\hat{\Pi}$ , allows the construction of the fitted value  $\hat{Y}_t' = x_t' \hat{\Pi}$ , which is substituted for  $Y_t'$  in the final estimation stage. In this paper, we are not only interested in an estimator of the structural parameter  $\alpha_\theta$ , but also in an estimator of the reduced-form parameter  $\pi_\theta$ , denoted  $\hat{\pi}_\theta$ , for each given  $\theta \in (0, 1)$

Let us now define two-stage quantile regressions based on the fitted-value approach in our setting, which allows us to deal with endogeneity problems. For any quantile  $\theta \in (0, 1)$ , we define  $\rho_\theta(z) = z\psi_\theta(z)$ . If the orthogonality condition,  $E(z_t\psi_\theta(u_{t\theta})) = 0$ , is satisfied, then the quantile regression estimator (QR) for quantile  $\theta$  is consistent. However, when  $u_{t\theta}$  and  $Y_t$  (a sub-vector of  $z_t$ ) are statistically linked under weak endogeneity of  $Y_t$ , these orthogonality conditions might not be satisfied. In that case, the QR of  $\alpha_\theta$  is generally not consistent, and usual quantile regressions cannot be used.

We define, for any quantile  $\theta$ , the two-stage quantile regression estimator  $\hat{\alpha}_\theta$  of  $\alpha_\theta$  as a solution to the following program:

$$\min_{\alpha} S_T(\alpha, \hat{\Pi}, \theta) = \sum_{t=1}^T \rho_\theta(y_t - x_t' H(\hat{\Pi})\alpha). \quad (5)$$

The estimator is straightforward to calculate using typical quantile regression algorithms. In contrast, structural approaches involve computational complications, such as grid search for simulation techniques (Chernozhukov and Hansen, 2006) or preliminary nonparametric estimation plagued by the ‘multidimensionality curse’ (Abadie et al., 2002, Chen et al., 2003, Lee, 2007).

However, the fitted-value approach has also a shortcoming: It is believed that the fitted-value approach used with quantile regressions exclusively implies the constant effect (Lee, 2007). We show in the next section that, provided a relatively weak independence assumption, some kind of non-constant effect can be exhibited in the structural model with the fitted-value approach.

### 3 Non-constant Effect in the Fitted-Value Approach

#### 3.1 Regularities and quantile restrictions

Let us start again with Equations (1) and (2), with possible non-constant effect in structural and reduced-form equations, *but without a priori imposing Assumption 1*. In order to deal with unique quantile values so as to simplify the discussion, we make the following continuity and monotonicity assumption for a starting value  $\theta_0$  of the quantile index.

**Assumption 3:** *For a given quantile index  $\theta_0$ , the cdf of  $v_{t\theta_0}$  conditional on  $x_t$ , denoted  $F_{v_{t\theta_0}|x_t}$ , the cdf of  $v_{t\theta_0}$  conditional on  $x_{1t}$ , denoted  $F_{v_{t\theta_0}|x_{1t}}$ , and the marginal cdf of  $x_{2t}$ , denoted  $F_{x_{2t}}$ , are continuous and strictly increasing.*

First, note that, under Assumption 3, an inverse cdf term can always be isolated in the reduced-form equation, for  $\theta_0$ , by denoting:  $y_t = x_t' \pi_{\theta_0} + v_{t\theta_0} = F_{v_{t\theta_0}|x_t}^{-1}(\theta) +$

$x_t' \pi_{1\theta_0} + v_{t\theta}^*$  and  $v_{t\theta}^* \equiv v_{t\theta_0} - F_{v_{t\theta_0}|x_t}^{-1}(\theta)$ . Then, by construction,  $v_{t\theta}^*$  is characterized by the conditional quantile restriction:  $E(\psi_\theta(v_{t\theta}^*)|x_t) = \theta - P[v_{t\theta}^* \leq 0|x_t] = \theta - P[v_{t\theta_0} \leq F_{v_{t\theta_0}|x_t}^{-1}(\theta)|x_t] = \theta - \theta = 0$ . As a consequence, one can always obtain a quantile regression restriction at  $\theta$ , even distinct from  $\theta_0$ , for the reduced-form, provided one accepts the introduction of a possible nuisance inconsistency term  $F_{v_{t\theta_0}|x_t}^{-1}(\theta)$  that can affect all the coefficients of the model. We here investigate the conditions in which this inconsistency takes simplified and useful forms. Note that  $v_{t\theta}^*$  depends both on  $\theta$  and on  $\theta_0$ . In the next subsection, we weaken the quantile restriction at quantile index  $\theta_0$ , so as to allow enough flexibility for the non constant effect.

### 3.2 Generating non-constant effect

Let us now return to our maintained model, but *instead of Assumption 1*, we now consider the following weaker restriction.

**Assumption 4:** *For a given quantile  $\theta_0$  and under Assumption 3:*

$$F_{v_{t\theta_0}|x_t}^{-1}(\theta) = F_{v_{t\theta_0}|x_{1t}}^{-1}(\theta), \quad (6)$$

for all  $\theta \in (0, 1)$ .

**Proposition 1:** *Under Assumption 3, Assumption 4 is equivalent to:  $v_{t\theta_0}$  is independent on  $x_{2t}$ , conditionally on  $x_{1t}$ .*

**Proof:** Let us first consider Assumption 4 without the variables  $x_{1t}$ , that is,  $F_{v_{t\theta_0}|x_{2t}}^{-1}(\theta) = F_{v_{t\theta_0}}^{-1}(\theta)$ , for all  $\theta$ , which we wish to show to be equivalent to:  $v_{t\theta_0}$  is independent of  $x_{2t}$ . By definition, under Assumption 3, the latter statement means that  $f_{v_{t\theta_0}, x_{2t}}(v, w) = f_{v_{t\theta_0}}(v) f_{x_{2t}}(w)$ , for all  $v$  and  $w$ , and where  $f_{v_{t\theta_0}, x_{2t}}$  is the joint pdf

of  $v_{t\theta_0}$  and  $x_{2t}$ ,  $f_{v_{t\theta_0}}$  is the marginal pdf of  $v_{t\theta_0}$ , and  $f_{x_{2t}}$  is the marginal pdf of  $x_{2t}$ . Since, under Assumption 3, there is no level of  $x_{2t}$  with vanishing marginal pdf, we can rewrite the equality as  $f_{v_{t\theta_0}}(v) = f_{v_{t\theta_0}, x_{2t}}(v, w)/f_{x_{2t}}(w)$ , for all  $v$  and  $w$ . Then, by integration with respect to  $v$  on both sides of the equality, we obtain  $\theta = F_{v_{t\theta_0}}(v) = F_{v_{t\theta_0}|x_{2t}}(v | x_{2t} = w)$ , for all  $w$ . By inversion, thanks to Assumption 3, this is equivalent to  $F_{v_{t\theta_0}|x_{2t}}^{-1}(\theta) = F_{v_{t\theta_0}}^{-1}(\theta) = v$ , for all  $v$ , i.e., for all  $\theta = F_{v_{t\theta_0}}(v)$ . Introducing variables  $x_{1t}$  and conditioning on them is straightforward by considering the different cdfs for each level of  $x_{1t}$ . QED.

Assumption 4 is akin to conditions in the control function literature, with here the control function known to depend on  $x_{1t}$  only. We do not require complete exogeneity, as we will discuss later. Assumption 4 is also related to the notion of ‘conditional exogeneity’ in White and Chalak (2010). Note that the restriction in Assumption 4 implies that the OLS in the reduced form may be inconsistent in the allowed case where the  $x_{1t}$  are endogenous. One may also have  $E(\psi_\theta(v_{t\theta})|x_{1t}) \neq 0$  under this hypothesis. This means that  $x_{1t}$  may be endogenous in the sense of the quantile regressions of quantile  $\theta$  for equation (3). In such situation, equation (3) no longer characterizes a genuine ‘reduced form’ based only on exogenous regressors, and we instead denote it the ‘*pseudo reduced-form equation*.’ In the remainder of this section, we show how a non-constant effect can be obtained for conditional quantiles of the pseudo reduced-form, consistently estimated asymptotically, and then conveyed to the conditional quantiles of the structural form.

An example of such a setting is a structural wage equation for a labour market study in which the dependent variable ( $y_t$ ) is the logarithm of individual wage rate for a sample of workers, while the two independent variables in this equation are the industrial sector dummy ( $x_{1t}$ ) and the workers’s education level ( $Y_t$ ), and the

regressors in the pseudo reduced form are again the industrial sector dummy and the worker's birth quarter ( $x_{2t}$ ). The birth quarter is used as an instrument for her education level that is assumed to be the sole endogenous independent variable in the structural model (e.g., in Angrist and Krueger, 1991). In such an empirical problem, one may expect the log wage rate to be positively correlated with the capital of the firm, omitted from the model, while incorporated in the error  $v_t$ , and which should be typically correlated with the industrial sector dummy. Then,  $x_{1t}$  and  $v_t$  should be correlated, while  $x_{2t}$  and  $v_t$  should be independent according to the usual reasons justifying the use of quarter of birth as an instrument in wage equations.<sup>5</sup>

Alternatively, Assumption 4 can be imposed only for a limited range of quantiles  $\theta$  of interest, still for a given  $\theta_0$  indexing the cdfs. Indeed, as Koenker states, instruments can be valid for some quantiles only.<sup>6</sup> In addition, Kim and Muller (2013) show for UK consumption data that exogeneity may, even under constant effect, be satisfied only for a limited range of quantiles. Since  $v_{t\theta} = u_{t\theta} + V'_{t\theta}\gamma_\theta$ , Assumption 4 for all  $\theta$  can also be obtained, for example, by assuming that  $u_{t\theta}$  and  $V_t$  are independent of  $x_{2t}$ , conditional on  $x_{1t}$ . Thus, this condition can also be seen as the consequence of a natural while strong instrumental variable characterization of  $x_{2t}$  for the structural model.

We now show that Assumption 4 implies that there are constant effects in the quantile regressions of the pseudo reduced-form equation of the coefficients of the variables in  $x_{2t}$ , but not necessarily of the coefficients of the variables in  $x_{1t}$ . Indeed,

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<sup>5</sup>Conditioning on the industrial sector might make the hypothesis of independence of the birth quarter and the error more plausible if the sector was a common determinant of the latter two variables, although it is unclear why this should be the case.

<sup>6</sup>(p. 285, Koenker, 2005): "When instruments are only effective over a limited quantile range, then mean effects are unidentified unless one makes stringent assumptions about the homogeneity of effects across quantiles."

consider *another* quantile index  $\theta$  different from the previously chosen  $\theta_0$  and let us impose Assumption 1 to it, that is: the typical quantile restriction for a quantile  $\theta$ . In that way, we investigate the features of a quantile regression model centered on a quantile index  $\theta$ . Then, we examine how this conditional quantile restriction can coexist with Assumption 4 assumed with the chosen  $\theta_0$ .

The conditional quantile restriction at  $\theta$  is:  $P[v_{t\theta} \leq 0 \mid x_t] = \theta$ , i.e.,  $P[y_t \leq x_t' \pi_\theta \mid x_t] = \theta$ , which implies that  $P[x_{1t}' \pi_{K_1\theta_0} + x_{2t}' \pi_{K_2\theta_0} + v_{t\theta_0} \leq x_{1t}' \pi_{K_1\theta} + x_{2t}' \pi_{K_2\theta} \mid x_t] = \theta$ ,

where  $\pi_{K_1\theta_0}, \pi_{K_2\theta_0}, \pi_{K_1\theta}, \pi_{K_2\theta}$  are the respective components of  $\pi_{\theta_0}$  and  $\pi_\theta$  according to the partition of  $x_t$  into  $x_{1t}$  and  $x_{2t}$ . By regrouping, we obtain

$$P[v_{t\theta_0} \leq x_{1t}'(\pi_{K_1\theta} - \pi_{K_1\theta_0}) + x_{2t}'(\pi_{K_2\theta} - \pi_{K_2\theta_0}) \mid x_t] = \theta,$$

which in turn implies that  $F_{v_t|x_t}[x_{1t}'(\pi_{K_1\theta} - \pi_{K_1\theta_0}) + x_{2t}'(\pi_{K_2\theta} - \pi_{K_2\theta_0})] = \theta$ . Finally, under Assumption 3, we have

$$x_{1t}'(\pi_{K_1\theta} - \pi_{K_1\theta_0}) + x_{2t}'(\pi_{K_2\theta} - \pi_{K_2\theta_0}) = F_{v_{t\theta_0}|x_t}^{-1}(\theta) = F_{v_{t\theta_0}|x_{1t}}^{-1}(\theta), \quad (7)$$

where the latter equality is obtained using Assumption 4.

**Proposition 2:** *Under Assumption 3 and 4, for a quantile regression centered at  $\theta$  for (3):*

(a) *There is constant effect for the variables in  $x_{2t}$ .*

(b) *Non-constant effect is possible for the variables in  $x_{1t}$ .*

(c)  *$F_{v_{t\theta_0}|x_{1t}}^{-1}(\theta)$  must be linear in  $x_{1t}$ .*



Proof: (a) Since  $F_{v_{t\theta_0}|x_{1t}}^{-1}(\theta)$  does not depend on  $x_{2t}$ , equation (7) implies  $\pi_{K_2\theta} = \pi_{K_2\theta_0}$ . As a consequence, we drop the dependence of  $\pi_{K_2\theta}$  on  $\theta$ , which becomes  $\pi_{K_2}$ . (b) Since there is no restriction on the effect of the variables in  $x_{1t}$ , their coefficients may vary with  $\theta$  in that case. To be consistent with (7), we must also assume (c), which is in fact a consequence of the hypotheses. QED.

Result (c) is implied in particular by the often used model of the ‘linear location-scale hypothesis’ in the quantile regression literature on non-constant effect (e.g., Koenker, 2005, pp 104-105). In cases where Result (c) would be judged too unrealistic, this assumption can be easily relaxed by incorporating polynomial terms in  $x_{1t}$ , or splines, in the model, as is usual for approximating nonlinear functions. Alternatively, one can specify a pseudo reduced form (3) as being partially linear in  $x_{2t}$ , and possibly nonlinear in  $x_{1t}$ , with an unknown nonlinear functional form. The above reasoning delivering constant effects for  $x_{2t}$  and unrestricted (nonlinear) effect for  $x_{1t}$  remains valid. In that case, Result (c) need not be necessary. However, this would lead us toward nonparametric estimation methods, which is not what we discuss in this paper. Finally, instead of imposing Assumption 4, one may assume that all covariates are endogenous and test for which coefficients the hypothesis of constant effect is not rejected in a quantile regression estimation. This would guide the specification of Assumption 4 in the considered data.

In the next subsection, we analyse the asymptotic properties of a quantile regression estimator in this context, with a special attention to its consistency. Indeed, beyond exhibiting the characteristics of the model, we want to assess which of them can be consistently estimated by using quantile regressions.

### 3.3 Asymptotics for the Quantile Process

Generally, authors dealing with quantile regressions assume Assumption 1 for the whole quantile process. In that case, there is no room for relaxing the restriction by imposing Assumption 4 instead at one or several quantiles  $\theta_0$ , and a constant effect is obtained for all coefficients, apart from the intercept if it exists. This is an obvious consequence of Proposition 2, applied to a case where there are no variables in  $x_{1t}$ .

However, it may be possible to impose Assumption 4 for all quantiles, instead of imposing Assumption 1. This would preserve the possibility of non-constant effect for some coefficients. To discuss the consistency, and other asymptotic properties, of the quantile estimation of some coefficients under this assumption, we need to consider the following asymptotic representation.

**Proposition 3:** *Suppose that the following assumptions hold:*

- (i):  $T^{1/2}(\hat{\Pi} - \Pi) = O_p(1)$ .
- (ii): *The sequence  $\{(x'_t, u_{t\theta}^*, v_{t\theta}^*)\}$  is  $m$ -dependent.*
- (iii) *Let  $f_{t\theta}^*(\cdot|x)$  be the conditional probability density function (PDF) of  $v_{t\theta}^*$ . The conditional PDF  $f_{t\theta}^*(\cdot|x)$  is assumed to be Lipschitz continuous for all  $x$ , strictly positive and bounded by a constant  $f_0$  (i.e.,  $f_{t\theta}^*(\cdot|x) < f_0$ , for all  $x$ ).*
- (iv) *The matrices  $Q = \lim_{T \rightarrow \infty} E \left[ \frac{1}{T} \sum_{t=1}^T x_t x'_t \right]$  and  $Q_0^* = \lim_{T \rightarrow \infty} E \left[ \frac{1}{T} \sum_{t=1}^T f_{t\theta}^*(0|x_t) x_t x'_t \right]$  are finite and positive definite.*
- (v) *There exists a positive number  $C > 0$  such that  $E(\|x_t\|^3) < C < \infty$  for any  $t$ .*

*Then, the asymptotic representation for our estimator is:*

$$T^{1/2}(\hat{\alpha}_\theta^* - \alpha_\theta^*) = RT^{-1/2} \sum_{t=1}^T x_t \psi_\theta(v_{t\theta}^*) - RQ_0^* T^{1/2}(\hat{\Pi} - \Pi)\gamma + o_p(1),$$

where  $R = Q_{zz}^{*-1}H(\Pi)'$  and  $Q_{zz}^* = H(\Pi)'Q_0^*H(\Pi)$ .

**Proof:** Direct application of the result in Kim and Muller (2015) to the translated errors  $v_{t\theta}^*$ . Indeed, in that case the quantile restriction at  $\theta$  is satisfied by construction of  $v_{t\theta}^*$ . QED.

From the formula of the asymptotic representation, it is possible to unwrap the translated errors  $v_{t\theta}^*$  into their components to see how the true values of the coefficients may vary with  $\theta$ . The next proposition discusses the consistency of quantile regression estimators under Assumptions 3 and 4.

**Proposition 4:** *Under Assumptions 3 and 4, the inconsistencies in the  $\theta^{th}$  conditional quantile regression estimator of the pseudo reduced-form linear model can be confined to the first  $K_1$  variables, for any quantile index  $\theta$  possibly distinct from  $\theta_0$ .*

A hint about the reason for the results in Proposition 4 is that under  $v_t$  independent on  $x_{2t}$  conditionally on  $x_{1t}$ , only the coefficient of  $x_{2t}$  can be identified through solving these  $K_2$  orthogonality conditions, even conditionally on  $x_{1t}$ .

As mentioned before, we have  $v_{t\theta}^* \equiv v_{t\theta_0} - F_{v_{t\theta_0}|x_t}^{-1}(\theta)$ , so that by definition of the quantile regression at quantile  $\theta$ , we have  $E(\psi_\theta(v_{t\theta}^*)|x_t) = 0$ . Variable  $v_{t\theta}^*$  is therefore a good choice for  $v_{t\theta}$  and an associated quantile restriction at  $\theta$ . Another perspective on this can be seen by starting from  $P(v_{t\theta} < 0|x_t) = \theta$ . We have successively  $\theta = P(y_t - x_t'\pi_\theta < 0|x_t) = P(x_t'(\pi_{\theta_0} - \pi_\theta) + v_{t\theta_0} < 0|x_t) = F_{v_{t\theta_0}|x_t}(x_t'(\pi_\theta - \pi_{\theta_0}))$ , which implies under Assumption 4:  $x_t'\pi_\theta = x_t'\pi_{\theta_0} + F_{v_{t\theta_0}|x_t}^{-1}(\theta)$ .

We can avail of the typical asymptotic representation of quantile regression while using the error  $v_{t\theta}^*$  since  $E(\psi_\theta(v_{t\theta}^*)|x_t) = 0$ . The limit vector  $\pi_\theta^*$ , to which tends asymptotically the quantile regression estimator is then obtain by ‘unwrapping’ the inconsistency term  $-F_{v_{t\theta_0}|x_t}^{-1}(\theta)$  and regrouping it with  $\pi_\theta$ .

It is useful to distinguish the definition of the true quantile restriction  $E(\psi_\theta(v_{t\theta}^*)|x_t) = 0$  and the quantile restriction used for the estimation algorithm of the linear quantile

regression,  $E(\psi_\theta(y_t - x'_t \pi_\theta) | x_t) = 0$ . The first restriction defines the ‘true coefficients’ of the conditional quantile at  $\theta$ . In addition, we have assumed that: (1) the conditional quantile is linear, and, (2) that the term  $-F_{v_{t\theta_0}|x_t}^{-1}(\theta)$  is also linear. This describes our theoretical perspective. These ‘theoretical FOCs’ under linearity assumptions and Assumption 4 are therefore:

$E(\psi_\theta(y_t - x'_t \pi_{\theta_0} - F_{v_{t\theta_0}|x_{1t}}^{-1}(\theta)) | x_t) = E(\psi_\theta(y_t - x'_t \pi_{\theta_0} - x'_{1t} \pi_{1F\theta}) | x_t) = 0$ , where  $\pi_{1F\theta}$  is the coefficient vector corresponding to the linearity of  $F_{v_{t\theta_0}|x_{1t}}^{-1}(\theta)$ , and which implies

$$E(x_t \psi_\theta(y_t - x'_{2t} \pi_{2\theta_0} - x'_{1t} (\pi_{1\theta_0} + \pi_{1F\theta}))) = 0. \quad (8)$$

In contrast, the second restriction defines what is estimated, i.e. the solution to the linear quantile regression FOCs:

$$\sum_{t=1}^T x_t \psi_\theta(y_t - x'_t \pi_\theta^*) = 0_k, \quad (9)$$

of which solution is the estimator  $\hat{\pi}_\theta$ .

*Proof of Proposition 4:* In these conditions the only difficulty is to apply Assumption 4 to the asymptotic expansion of the quantile regression estimator, instead of applying the complete quantile regression restrictions. Indeed, under Assumption 4, we have:

$$E(x_{1t} \psi_\theta(y_t - x'_{2t} \pi_{2\theta_0} - x'_{1t} (\pi_{1\theta_0} + \pi_{1F\theta}))) \neq 0 \quad (10)$$

$$\text{and } E(x_{2t} \psi_\theta(y_t - x'_{2t} \pi_{2\theta_0} - x'_{1t} (\pi_{1\theta_0} + \pi_{1F\theta}))) = 0. \quad (11)$$

As for usual situations of quantile regressions, the restrictions of orthogonality with respect to  $x_{2t}$ , (11), are satisfied by the quantile regression estimator under

Assumption 4. In contrast, the restrictions of orthogonality with respect to  $x_{1t}$  are no longer satisfied, as (10) shows. Therefore, there is no consistency of the usual quantile regression estimator under Assumption 4. The fact that the orthogonality restrictions with respect to  $x_{1t}$  cannot be used for identification is what generates the inconsistency in the quantile regression estimator of the coefficients of  $x_{1t}$ . However, the restrictions (11) are sufficient to identify the coefficients of  $x_{2t}$ . Indeed, by aggregating the term in  $x_{1t}$  with the error term in the equation, when have exactly the same identifying conditions than for a typical quantile regression where the only covariates are the  $x_{2t}$ ; that is:  $K_2$  restrictions for  $K_2$  parameters to estimate. Thus, one can obtain consistent estimators  $\hat{\pi}_{2\theta}$  of the coefficients of  $x_{2t}$  in the quantile regression of interest.

In these conditions, as seen before, a non-constant effect is possible for the conditional quantiles of the pseudo reduced form for  $\pi_{K_1\theta}^*$ , even though Assumption 4 is made only for the value  $\theta_0$ . There is no constant effect for  $\pi_{K_2}$  that has been shown not to vary with  $\theta$ . For example, a special case of non-constant effect quantile regression can be parameterized such as:  $\pi_{K_1\theta}^* = \pi_{K_1}^* + \theta$ . However, even though the exhibited non-constant effect in the pseudo reduced form allows for generalization of the model as compared to constant effect cases, the non-constant effect cannot be estimated consistently with usual quantile regression estimators. Though, the other parameters of the model can be consistently estimated. Asymptotic normality and asymptotic variance-covariance matrix are identical to the case of typical quantile regression estimation for these parameters. Then, constant effect coefficients may often be more accurately estimated by using central quantiles, such as LAD estimators, than with extreme quantiles.

One may also want to consider fully specified quantile regressions only at some quantiles. In that case, one may require Assumption 1 for these quantiles only, keeping

Assumption 4 for other quantiles. Jun (2008) studies variations across quantiles of identification through instrument variables. This suggests an interest in applying different semi-parametric IV restrictions at different quantiles. This would allow many distinct models with non-constant effect.

### 3.4 Transmitting the non-constant effect to the structural form

We now assess the consequences for the structural model of the partial occurrence of constant and non-constant effects in the pseudo reduced form. Assumption 4 allows us to rewrite the pseudo reduced form model as follows.

$$y_t = x_t' \pi_\theta^* + v_{t\theta}^*, \quad (12)$$

where  $\pi_\theta^*$  includes an inconsistency term. Eq. (12) can be considered as the proper reduced-form equation for the  $\theta^{th}$  quantile regression since it satisfies  $E(\psi_\theta(v_{t\theta}^*)|x_t) = 0$ . This restriction indicates that both  $x_{1t}$  and  $x_{2t}$  can be here considered as exogenous in the quantile regression sense for the quantile index  $\theta$ , provided the inconsistency term is included in the definition of the parameter. Then, a quantile regression estimator  $\hat{\pi}_\theta^*$  in this equation would converge toward  $\pi_\theta^*$ , under usual conditions. In that case, the corresponding estimator for the structural model is denoted  $\hat{\alpha}_\theta^*$  and converges to a value  $\alpha_\theta^* = (\beta_\theta^{*'}, \gamma_\theta^{*'})'$  that may also incorporate an inconsistency term.

To investigate this, we now decompose the link of the reduced-form and the structural-form (possibly biased) parameters, by splitting system (4) into two blocks of equations, partitioning  $\Pi = \begin{bmatrix} \Pi_1 \\ \Pi_2 \end{bmatrix} = [\Pi_1', \Pi_2']'$  according to the partition of  $\pi_\theta$ .

We obtain:

$$\pi_{K_1\theta}^* = \beta_\theta^* + \Pi_1\gamma_\theta^*, \quad (13)$$

$$\pi_{K_2\theta}^* = \Pi_2\gamma_\theta^*, \quad (14)$$

where  $\pi_{K_2\theta}^* = \pi_{K_2}$ , which does not include an inconsistency term and does not depend on  $\theta$ , as seen above. If the system is exactly-identified (i.e.,  $K_2 = G$ ), then  $\gamma_\theta^*$  can be directly expressed in terms of  $\Pi_2$  and  $\pi_{K_2}$ , which implies that  $\gamma_\theta^*$  does not depend on  $\theta$ . For the other over-identifying case (i.e.,  $K_2 > G$ ), there are more equations than is necessary to identify  $\gamma_\theta^*$ . Similarly to 2SLS or GMM, the additional information does not perturb the identification of  $\gamma_\theta^*$ . One can even enhance the finite sample performance of the resulting estimators by choosing an appropriate weighting matrix to combine these equations. In either the exactly-identifying case or the over-identifying case,  $\gamma_\theta^*$  is a function of  $\Pi_2$  and  $\pi_{K_2}$  only. Hence, the property of constant effect in  $\pi_{K_2\theta}^*$ , without inconsistency term, is conveyed to  $\gamma_\theta^* \equiv \gamma$ . On the other hand, since  $\gamma_\theta^*$  is fully determined in (14), (13) shows that  $\beta_\theta^*$  incorporates the non-constant effect from  $\pi_{K_1\theta}^*$  and exactly the same inconsistency term as in  $\pi_{K_1\theta}^*$ . We summarize all these results in the next proposition.

**Proposition 4:** *Under our hypotheses:*

- (a) *The coefficients of the endogenous regressors in the structural model do not vary with the quantile  $\theta$ .*
- (b) *The coefficients of the exogenous regressors in the structural model can vary with the quantile  $\theta$ . They exhibit exactly the same inconsistency term as their respective coefficients in the pseudo reduced-form model.*

When a first-stage estimation is performed based on Eq. (2), the independent variables  $x_t$  consists of vectors  $x_{1t}$  and  $x_{2t}$ . If a non-zero asymptotic inconsistency

term is present only in the coefficients of  $x_{1t}$  in the pseudo reduced-form estimator  $\hat{\pi}_\theta$ , which is the case on which we focus, then the non-zero asymptotic inconsistency in the second-stage estimator  $\hat{\alpha}_\theta = [\hat{\beta}'_\theta, \hat{\gamma}'_\theta]'$  is exclusively confined to the coefficients of  $x_{1t}$ ; that is, only  $\hat{\beta}_\theta$  is inconsistent with exactly the same asymptotic bias as that of  $\hat{\pi}_\theta$ . Therefore, the parameter  $(\gamma_\theta)$  for the endogenous variable  $Y_t$  in the structural equation in (1) can be consistently estimated. In this setting, because  $\pi_{K_2}$  is characterized by constant effect, and because  $\gamma_\theta$  is not connected to  $\pi_{K_1\theta}$ , we have also constant effect for  $\gamma_\theta$  that can be estimated consistently. In contrast, a non-constant effect may occur for  $\beta_\theta$ . However,  $\beta_\theta$  cannot be estimated without an inconsistency, which is exactly equal to the inconsistency in  $\pi_{K_1\theta}$ .

Allowing for the weakened IV condition in Assumption 4 has enabled us to introduce non-constant effect on the vector  $\beta_\theta$ , even though this parameter is estimated inconsistently. This alone is a generalization of the *stricto sensu* constant effect structural quantile regression, which may be useful if the researcher's interest is concentrated on vector  $\gamma$  that can be estimated consistently. Indeed, this approach avoids misspecification of the quantile regression, if the true DGP involves non-constant effect for  $\beta_\theta$  and constant effect for  $\gamma$ .

## 4 Conclusion

In this paper, we have shown how some particular cases of non-constant effect can be obtained with two-stage quantile regressions based on the fitted-value approach under endogeneity. However, we have also established that only the coefficients of constant-effect variables can be consistently estimated, even though non-constant effect is present for the other variables. Finally, we have discussed a few practical examples where our approach would be useful.



Our results are based on relatively little demanding instrumental variable conditions, for example that the (pseudo) reduced-form errors (or the structural errors) are independent of *SOME excluded variables*, conditionally on the other independent variables. Such weakening of the usual IV conditions is potentially useful since convincing instruments are typically difficult to find. Then, any reduction in the set of necessary instruments is valuable. This is also important because some instruments might be only valid over a limited range of quantiles.

Our approach is also interesting because a quantile regression model with constant effect for one coefficient only (or a few coefficients only) is much more general than a general constant effect for all coefficients of a quantile regression model. Thus, our approach corresponds to a specification that is intermediate between the constant effect quantile model and the fully non-constant effect quantile model. Besides, only allowing for inconsistent non-constant effect on the intercept plus on a single other coefficient of the included exogenous variables might already be sufficient in some cases to achieve consistent estimation of the coefficient of the endogenous variables of interest, under correct specification. As a matter of fact, there are a variety of more or less demanding independence conditions that can be used to allow for partially non-constant structural effects without giving up too many consistent estimates of the coefficients of the structural regressors. All that is needed is that constant effect can be assumed for *some* endogenous variables.<sup>7</sup>

Alternatively, one could assume instead that the (pseudo) reduced-form errors (or the structural errors) are independent of some of the included structural regressors, conditional on the excluded independent variables. In that case, one would obtain

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<sup>7</sup>Allowing for an inconsistency only on the intercept is not useful to generate non-constant effect, as the original intercept is already used to center the additive separable error term in the reduced-form quantile regression.

constant effect for the structural coefficients of these selected exogenous regressors and possible non-constant effect for the structural endogenous regressors, although the latter ones would not be estimable without bias. This approach could be useful if the constant effect of interest is that of exogenous regressors. Finally, our analysis can be applied to nonlinear quantile regression models, provided the contributions of variables in  $x_{1t}$  and in  $x_{2t}$  can be additively separated in the model equations.

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