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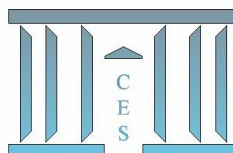
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**A note on the characterization of optimal allocations in
OLG economies with multiple goods**

Jean-Marc BONNISSEAU, Lalaina RAKOTONINDRAINY

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A note on the characterization of optimal allocations in OLG economies with multiple goods

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Abstract

We consider a pure exchange overlapping generations economy with finitely many commodities and consumers per period having possibly non-complete non transitive preferences. We provide a geometric and direct proof of the Balasko-Shell characterization of Pareto optimal allocation (See, [2]). *JEL classification:* C62, D50, D62.

Keywords: Overlapping generations model, preference set, normal cone, equilibrium, Pareto optimality.

1 Introduction

As already well known, OLG equilibria may lack in being Pareto optimal. This market failure was established by Samuelson [11], who attributes this phenomenon as a lack of double coincidence of wants, and suggests to solve this by using the role of money. Geanakoplos [8] clarifies this phenomenon as a result of two facts: generations overlap, and infinite horizon. He points some links between the Samuelson model and the Arrow-Debreu model, and established how a durable good such as money, or an infinitely lived asset like a land, could restore the market failure.

Balasko and Shell [2] provide a criterion based on the asymptotic behavior of the inverse of the norm of the prices to characterize Pareto optimal allocation without durable good or infinitely lived asset. Burke [3] revisits this criterion by focusing in particular on the right definition of the Gaussian curvature of the indifference curve.

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Our purpose in this paper is to provide a simpler, direct proof of the Balasko-Shell Criterion, which also allows to consider several consumers for each generation and non-complete, non-transitive preferences. Note that the structure of the proof is based strongly on Balasko-Shell's one.

Actually, in [2] and [3], the authors provide a proof with a first step considering the special case of a single commodity per period. Then, the generalization to several commodities is only sketched. We simplify the proof by tackling directly the multi commodity case and by using a geometric approach.

To avoid some smoothness assumption, we replace the assumption on the curvature of the indifference surface by geometric properties of preferred sets. In particular, we use the notion of prox-regularity, introduced in variational analysis by Rockafellar and Poliquin in [9], which is extensively studied by Thibault and Colombo in [4]. This allows us to translate geometrically that the curvature is bounded above. As for the lower bound on the curvature, our assumption is related to the strict convexity of preferred sets.

Details about the model and assumptions are described in Section 2. Some preliminary results are provided in Section 3. The end of this Section explains how the multi-consumer case can be simplified by considering aggregate feasible Pareto improving transfer. In Section 4, we provide the proof and we show which assumptions are used for the if part and for the only if part of the criterion.

2 The model

We consider an OLG economy \mathcal{E} with infinitely many dates $t = 1, 2, \dots$. At each date there is a finite and identical set of commodities \mathcal{L} , and we denote by L its cardinal.

At each date $t \in \mathbb{N}$, a finite set of individuals \mathcal{I}_t is born, living for two periods, young at date t and old at date $t+1$. We start with the first generation 0 which lives for only one period and consists of the old agents at date 1. $\mathcal{I} = \cup_{t=0}^{\infty} \mathcal{I}_t$ denotes the set of all individuals and $\mathcal{I}_{-0} = \cup_{t=1}^{\infty} \mathcal{I}_t$.

At each date t , we denote by $x^i = (x_t^i, x_{t+1}^i)$ the consumption by an individual i in \mathcal{I}_t , which is an element of the consumption set $X = \mathbb{R}_+^L \times \mathbb{R}_+^L$. The consumption set of consumers of generation 0 is \mathbb{R}_+^L .

Consumers preferences are represented by a (strict) preference relation $P^i : X \rightarrow X$: for all $i \in \mathcal{I}_t$, $t \geq 1$:

$\xi^i \in P^i(x^i)$ means that ξ^i is strictly preferred to x^i .

$\xi^i \in \bar{P}^i(x^i)$, the closure of $P^i(x^i)$ means that ξ^i is preferred or indifferent to x^i .

For all $t \geq 0$, $i \in \mathcal{I}_t$, we denote by $N_{\bar{P}^i(x^i)}(x^i)$ the normal cone¹ of $\bar{P}^i(x^i)$ at x^i .

We now posit the main assumption, which is maintained throughout the paper.

Assumption A.

- a) For all individual i in \mathcal{I} , $P^i(x^i)$ is open in X , convex, $x^i \in \bar{P}^i(x^i)$ and $x^i \notin P^i(x^i)$. For all $i \in \mathcal{I}_{-0}$ $P^i(x^i) + (\mathbb{R}_+^L \times \mathbb{R}_+^L) \subset P^i(x^i)$ and for $i \in \mathcal{I}_0$, $P^i(x^i) + \mathbb{R}_+^L \subset P^i(x^i)$.
- b) Each consumer is endowed with some strictly positive endowments of the goods during his lifetime: for all $i \in \mathcal{I}_{-0}$, $e^i \in \mathbb{R}_{++}^L \times \mathbb{R}_{++}^L$ and for all $i \in \mathcal{I}_0$, $e^i \in \mathbb{R}_{++}^L$.
- c) For all $i \in \mathcal{I}$, for all x_i in the interior of X_i , $-N_{\bar{P}^i(x^i)}(x^i)$ is a half line $\{t\gamma^i(x^i) \mid t \geq 0\}$, defined by $\gamma^i(x^i)$ which is a continuous mapping on the interior of X^i satisfying $\|\gamma^i(x^i)\| = 1$. For all x_i in the interior of X^i , for all $i \in \mathcal{I}_{-0}$, $\gamma^i(x^i) \in \mathbb{R}_{++}^L \times \mathbb{R}_{++}^L$ and for all $i \in \mathcal{I}_0$, $\gamma^i(x^i) \in \mathbb{R}_{++}^L$.

Assumption A is a classical assumption in a standard finite economy. If the preferences are represented by a utility function, it means that it is continuous, quasi-concave, strictly increasing and smooth on the interior of X^i .

At each date t , there is a spot market for the L commodities. The spot price vector p is an element of $\prod_{t=1}^{\infty} \mathbb{R}_{++}^L$ and $p_{t\ell}$ is the spot price of commodity ℓ at date t . We consider the set of normalized prices $\Delta := \{p \in \prod_{t=1}^{\infty} \mathbb{R}_{++}^L \mid \|p_1\| = 1\}$. We denote by $\Pi_t = (p_t, p_{t+1})$, for $t \geq 1$, and $\Pi_0 = p_1$.

Budget Constraints

The budget constraint at a given price p , for each agent $i \in \mathcal{I}_{-0}$ is given by:

$$p_t \cdot x_t^i + p_{t+1} \cdot x_{t+1}^i \leq p_t \cdot e_t^i + p_{t+1} \cdot e_{t+1}^i,$$

and for each agent $i \in \mathcal{I}_0$,

$$p_1 \cdot x_1^i \leq p_1 \cdot e_1^i.$$

3 Preliminary results

Let us recall some basic definitions of equilibrium and optimal allocations in a standard pure exchange OLG economy with multiple commodities.

¹ $N_{\bar{P}^i(x^i)}(x^i) = \{q \in \mathbb{R}^L \times \mathbb{R}^L \mid q \cdot (z - x^i) \leq 0, \forall z \in \bar{P}^i(x^i)\}$

Definition 1 An equilibrium of the economy \mathcal{E} is a list $(p^*, (x^{i*}))$ in $\Delta \times \prod_{t=0}^{\infty} \prod_{i \in \mathcal{I}_t} X^i$ such that:

a) for all $i \in \mathcal{I}$, (x^{i*}) is a maximal element for P^i in the budget set associated to the equilibrium price p^* , that is, for all $i \in \mathcal{I}_{-0}$, $p_t^* \cdot x_t^{i*} + p_{t+1}^* \cdot x_{t+1}^{i*} \leq p_t^* \cdot e_t^i + p_{t+1}^* \cdot e_{t+1}^i$, and for all $x^i \in P^i(x^{i*})$, $p_t^* \cdot x_t^i + p_{t+1}^* \cdot x_{t+1}^i > p_t^* \cdot e_t^i + p_{t+1}^* \cdot e_{t+1}^i$ and for all $i \in \mathcal{I}_0$, $p_1^* \cdot x_1^{i*} \leq p_1^* \cdot e_1^i$, and for all $x^i \in P^i(x^{i*})$, $p_1^* \cdot x_1^i > p_1^* \cdot e_1^i$.

b) the allocation (x^{i*}) is feasible:

$$\sum_{i \in \mathcal{I}_{t-1} \cup \mathcal{I}_t} x_t^{i*} = \sum_{i \in \mathcal{I}_{t-1} \cup \mathcal{I}_t} e_t^i, \text{ for } t \geq 1$$

Proposition 1 Under Assumption C, the OLG economy \mathcal{E} has an equilibrium.

Proof. The existence of an equilibrium can be proved following the same procedure as in [1], using the existence result in finite dimension of Gale and Mas-Colell [6], [7], see also Florenzano [5]. ²

Definition 2 The feasible allocation x in $\prod_{i \in \mathcal{I}} X^i$ is Pareto optimal (PO) (resp. weakly Pareto optimal (WPO)) if there is no (y^i) in $\prod_{i \in \mathcal{I}} X^i$ such that:

$$\sum_{i \in \mathcal{I}_{t-1} \cup \mathcal{I}_t} y_t^i = \sum_{i \in \mathcal{I}_{t-1} \cup \mathcal{I}_t} e_t^i, \text{ for } t \geq 1$$

and for all $i \in \mathcal{I}$, $y^i \in \bar{P}^i(x^i)$, with $y^i \in P^i(x^i)$ for at least one individual i (resp. and $y^i = x^i$ for all $i \in \mathcal{I}_t$, except for a finite number of t).

We remark that a PO allocation is WPO.

Definition 3 Let $x = (x^i)$ be an allocation in $\prod_{i \in \mathcal{I}} X^i$. The price $p \in \Delta$ is said to support x if for each $t \in \mathbb{N}$, for all $i \in \mathcal{I}_t$ and for all $\xi^i \in P^i(x^i)$, $\Pi_t \cdot \xi^i > \Pi_t \cdot x^i$.

Remark 1 Every competitive allocation $x^* = (x^{i*})$ associated with the equilibrium price $p^* \in \Delta$ is supported by p^* .

If $x = (x^i)$, an interior allocation, is supported by the price p , then for all $i \in \mathcal{I}$, there exists $\lambda^i > 0$ such that $\gamma^i(x^i) = \lambda^i \Pi_t$.

Lemma 1 If $x = (x^i)$, an interior allocation, is WPO, then, for all $t \geq 0$, $\gamma^i(x^i) = \gamma^j(x^j)$ for all $i, j \in \mathcal{I}_t$.

Proof. Let $t \geq 0$. Since (x^i) is WPO, $\sum_{i \in \mathcal{I}_t} x^i \notin \sum_{i \in \mathcal{I}_t} P^i(x^i)$. Indeed, if it would not hold, then there exists $(\xi^i)_{i \in \mathcal{I}_t} \in \prod_{i \in \mathcal{I}_t} P^i(x^i)$, such that $\sum_{i \in \mathcal{I}_t} x^i = \sum_{i \in \mathcal{I}_t} \xi^i$. Then, one easily checks that the allocation (y^i) defined by $y^i = x^i$ for

²Main steps of the proof are provided in Appendix

$i \notin \mathcal{I}_t$ and $y^i = \xi^i$ for $i \in \mathcal{I}_t$ is feasible and Pareto dominates (x^i) , which is in contradiction with the weak Pareto optimality of (x^i) .

By Assumption A, $(P^i(x^i))_{i \in \mathcal{I}_t}$ are convex and nonempty, and for all i , $x^i \notin P^i(x^i)$ and $x^i \in \bar{P}^i(x^i)$. So, for each $i \in \mathcal{I}_t$, there exists a sequence $(\xi^{i\nu})$ of $P^i(x^i)$, which converges to x^i . The set $\sum_{i \in \mathcal{I}_t} P^i(x^i)$ being convex, so by using the standard separation theorem for convex sets in finite-dimensional space for $\sum_{i \in \mathcal{I}_t} x^i$ and $\sum_{i \in \mathcal{I}_t} P^i(x^i)$, there exists $q \neq 0$ such that $q \cdot \sum_{i \in \mathcal{I}_t} x^i \leq q \cdot \sum_{i \in \mathcal{I}_t} \xi^i$ for all $(\xi^i) \in \prod_{i \in \mathcal{I}_t} P^i(x^i)$. Consider an individual i_0 in \mathcal{I}_t . Then for all $\xi^{i_0} \in P^{i_0}(x^{i_0})$, $q \cdot x^{i_0} + q \cdot \sum_{i \in \mathcal{I}_t, i \neq i_0} x^i \leq q \cdot \xi^{i_0} + q \cdot \sum_{i \in \mathcal{I}_t, i \neq i_0} \xi^{i\nu}$. By taking the limit, we obtain: $q \cdot x^{i_0} \leq q \cdot \xi^{i_0}$, which means that q belongs to the cone $-N_{\bar{P}^{i_0}(x^{i_0})}(x^{i_0})$, thus $q = \|q\| \gamma^{i_0}(x^{i_0})$. By repeating the same reasoning for all individuals of \mathcal{I}_t , we establish that $\gamma^i(x^i) = \frac{1}{\|q\|} q$ for all $i \in \mathcal{I}_t$. \square

Lemma 2 *The interior allocation $x = (x^i)$ is WPO if and only if there exists a price sequence $p \in \Delta$ which supports $x = (x^i)$.*

Proof.

Let x be supported by a price sequence p . Assume that x is not WPO, then there exists a feasible allocation (y^i) and some t' such that $x^i = y^i$ for all $i \in \mathcal{I}_t$, $t \geq t'$, where $y^i \in \bar{P}^i(x^i)$ for all $i \in \mathcal{I}$ and $y^i \in P^i(x^i)$ for at least an individual. Therefore, for all t and for all $i \in \mathcal{I}_t$, $p_t \cdot y_t^i + p_{t+1} \cdot y_{t+1}^i \geq p_t \cdot x_t^i + p_{t+1} \cdot x_{t+1}^i$ with at least one strict inequality for $i_0 \in \mathcal{I}_{t_0}$ where $t_0 < t'$. Thus, we have:

$$\begin{aligned} p_1 \cdot \sum_{i \in \mathcal{I}_0 \cup \mathcal{I}_1} y_1^i + p_2 \cdot \sum_{i \in \mathcal{I}_1 \cup \mathcal{I}_2} y_2^i + \dots + p_{t'-1} \cdot \sum_{i \in \mathcal{I}_{t'-1} \cup \mathcal{I}_{t'-2}} y_{t'-1}^i + p_{t'} \cdot \sum_{i \in \mathcal{I}_{t'-1}} y_{t'}^i > \\ p_1 \cdot \sum_{i \in \mathcal{I}_0 \cup \mathcal{I}_1} x_1^i + p_2 \cdot \sum_{i \in \mathcal{I}_1 \cup \mathcal{I}_2} x_2^i + \dots + p_{t'-1} \cdot \sum_{i \in \mathcal{I}_{t'-1} \cup \mathcal{I}_{t'-2}} x_{t'-1}^i + p_{t'} \cdot \sum_{i \in \mathcal{I}_{t'-1}} x_{t'}^i \end{aligned}$$

which is in contradiction with the feasibility of (y^i) implying $\sum_{i \in \mathcal{I}_{t-1} \cup \mathcal{I}_t} x_t^i = \sum_{i \in \mathcal{I}_{t-1} \cup \mathcal{I}_t} y_t^i$, and $y_t^i = x_t^i$ for all $i \in \mathcal{I}_t$, $t \geq t'$.

Conversely, let x be a WPO allocation. We first truncate the economy at a finite horizon t by considering the t first generations. Denote $\mathcal{J}_{t-1} = \prod_{s=0}^{t-1} \mathcal{I}_s$. We shall prove the result by induction on the truncation at t .

First, consider the truncated economy \mathcal{E}_1 at date $t = 1$, which consists of the generation 0, \mathcal{I}_0 . From Lemma 1, $p_1 = \gamma^i(x^i)$ for any $i \in \mathcal{I}_0$ supports $(x^i)_{i \in \mathcal{I}_0}$.

Now, suppose that (p_1, \dots, p_t) is supporting $(x^i)_{i \in \mathcal{J}_{t-1}}$, and let us prove that there is a unique $p_{t+1} \gg 0$ such that (p_1, \dots, p_{t+1}) supports $(x^i)_{i \in \mathcal{J}_t}$. From Lemma 1, for any $i_0 \in \mathcal{I}_t$, $\gamma^{i_0}(x^{i_0})$ supports x^i for all $i \in \mathcal{I}_t$. So, it suffices to prove that $\gamma^{i_0}(x^{i_0})$ is collinear to p_t and then to choose $p_{t+1} = \frac{\|p_t\|}{\|\gamma^{i_0}(x^{i_0})\|} \gamma^{i_0}(x^{i_0})$.

We consider a reduced economy with L commodities, the individuals in $\mathcal{I}_{t-1} \cup \mathcal{I}_t$ and the preferences defined by $Q^i(\xi^i) = P^i(x_{t-1}^i, \xi^i)$ for $i \in \mathcal{I}_{t-1}$ and

$Q^i(\xi^i) = P^i(\xi^i, x_{t+1}^i)$ for $i \in \mathcal{I}_t$. We remark that the allocation $(x_t^i)_{i \in \mathcal{I}_{t-1} \cup \mathcal{I}_t}$ is Pareto optimal in this finite economy. We also remark that $-N_{Q^i(x_t^i)}(x_t^i) = \{\lambda \gamma_t^i(x_{t-1}^i, x_t^i) \mid \lambda \geq 0\}$ for $i \in \mathcal{I}_{t-1}$ and $-N_{Q^i(x_t^i)}(x_t^i) = \{\lambda \gamma_t^i(x_t^i, x_{t+1}^i) \mid \lambda \geq 0\}$ for $i \in \mathcal{I}_t$. So, using the same argument as in the proof of Lemma 1, we prove that the vectors $(\gamma_t^i(x_{t-1}^i, x_t^i))_{i \in \mathcal{I}_{t-1}}$ and $(\gamma_t^i(x_t^i, x_{t+1}^i))_{i \in \mathcal{I}_t}$ are colinear. Since (p_{t-1}, p_t) supports x^i for any $i \in \mathcal{I}_{t-1}$, p_t is colinear to $\gamma_t^i(x_{t-1}^i, x_t^i)$, so it is also colinear to $\gamma_t^i(x_t^i, x_{t+1}^i)$ for all $i \in \mathcal{I}_t$. \square

Definition of aggregate Pareto improving transfers

The proof of the main result studies the behavior of Pareto improving transfers that we now introduce. In the following, we distinguish transfers and aggregate transfers. For an allocation (x^i) , for each generation t , we define an aggregate preferred set as follows:

$$\bar{P}_t((x^i)) := \sum_{i \in \mathcal{I}_t} \bar{P}^i(x^i)$$

From Assumption A, $\bar{P}_t((x^i))$ is a closed convex subset of $\mathbb{R}_+^L \times \mathbb{R}_+^L$ (or \mathbb{R}_+^L for the generation 0) and satisfies $\bar{P}_t((x^i)) + (\mathbb{R}_+^L \times \mathbb{R}_+^L) \subset \bar{P}_t((x^i))$ (or $\bar{P}_t((x^i)) + \mathbb{R}_+^L \subset \bar{P}_t((x^i))$ for the generation 0). The closedness of $\bar{P}_t((x^i))$ is ensured by the fact that $\bar{P}^i(x^i)$ are closed subsets of $\mathbb{R}_+^L \times \mathbb{R}_+^L$ for all $i \in \mathcal{I}_t$.

Let $\bar{x}^t := \sum_{i \in \mathcal{I}_t} x^i$, then one easily checks that ³:

$$N_{\bar{P}_t((x^i))}(\bar{x}^t) = \bigcap_{i \in \mathcal{I}_t} N_{\bar{P}^i(x^i)}(x^i)$$

Definition 4 (a) For a given allocation $x = (x^i)$, the sequence of commodity transfers $h = (h^i) \in \prod_{i \in \mathcal{I}_0} \mathbb{R}^L \times \prod_{t=1}^{\infty} \prod_{i \in \mathcal{I}_t} (\mathbb{R}^L \times \mathbb{R}^L)$ is feasible if $(x^i + h^i)$ is feasible, which means that $(x^i + h^i)$ belongs to $\prod_{t=0}^{\infty} \prod_{i \in \mathcal{I}_t} X^i$ and $\sum_{i \in \mathcal{I}_{t-1} \cup \mathcal{I}_t} h_t^i = 0$.

(b) The sequence of commodity transfers $h = (h^i)$ is Pareto improving upon $x = (x^i)$ if h is feasible and $x + h$ Pareto dominates x , that is for all $t \geq 1$ and all $i \in \mathcal{I}_t$, $x^i + h^i \in \bar{P}^i(x^i)$, with $x^i + h^i \in P^i(x^i)$ for at least one agent i .

(c) An aggregate transfer $\bar{h} \in \mathbb{R}^L \times \prod_{t=1}^{\infty} \mathbb{R}^L \times \mathbb{R}^L$ is feasible if $\bar{h}_t^{t-1} = -\bar{h}_t^t$ for all $t \geq 1$ and Pareto improving upon the allocation $x = (x^i)$ if:

i) for all t , $\sum_{i \in \mathcal{I}_t} x^i + \bar{h}^t \in \bar{P}_t((x^i))$

ii) there exists \underline{t} such that $\sum_{i \in \mathcal{I}_{\underline{t}}} x^i + \bar{h}^{\underline{t}} \in \text{int}(\bar{P}_{\underline{t}}((x^i)))$

By the very definition of Pareto optimality, the allocation $x = (x^i)$ is Pareto optimal if and only if there exists no feasible Pareto improving transfer upon

³see Appendix

x . But, we also remark that the allocation $x = (x^i)$ is Pareto optimal if and only if there exists no feasible aggregate Pareto improving transfer upon x . So, in the next section, we will be able to work only on aggregate feasible transfers and not on feasible transfers.

If there exists \bar{h} , an aggregate Pareto improving transfer, then let $t \geq 0$, by definition of $\bar{P}_t((x^i))$, there exists $(\xi^i)_{i \in \mathcal{I}_t}$ in $\prod_{i \in \mathcal{I}_t} \bar{P}^i(x^i)$ such that $\sum_{i \in \mathcal{I}_t} x^i + \bar{h}^t = \sum_{i \in \mathcal{I}_t} \xi^i$; By letting $h^i = \xi^i - x^i$, we easily check that $x^i + h^i \in \bar{P}^i(x^i)$ for all $i \in \mathcal{I}_t$.

Furthermore, for $t = \underline{t}$, $\sum_{i \in \mathcal{I}_{\underline{t}}} x^i + \bar{h}^{\underline{t}} - \alpha(\mathbf{1}_L \times \mathbf{1}_L) \in \bar{P}^{\underline{t}}((x^i))$, for $\alpha > 0$ small enough. Then there exists $(\xi^i)_{i \in \mathcal{I}_{\underline{t}}}$ in $\prod_{i \in \mathcal{I}_{\underline{t}}} \bar{P}^i(x^i)$ such that $\sum_{i \in \mathcal{I}_{\underline{t}}} x^i + \bar{h}^{\underline{t}} - \alpha(\mathbf{1}_L \times \mathbf{1}_L) = \sum_{i \in \mathcal{I}_{\underline{t}}} \xi^i$. Let us consider the individual $i_0 \in \mathcal{I}_{\underline{t}}$, we can then write $\bar{h}^{\underline{t}} = \sum_{i \neq i_0} (\xi^i - x^i) + (\xi^{i_0} - x^{i_0}) + \alpha(\mathbf{1}_L \times \mathbf{1}_L)$. Take $h^i = \xi^i - x^i$ and $h^{i_0} = \xi^{i_0} - x^{i_0} + \alpha(\mathbf{1}_L \times \mathbf{1}_L)$. We note that $x^{i_0} + h^{i_0} = \xi^{i_0} + \alpha(\mathbf{1}_L \times \mathbf{1}_L) \in P^{i_0}((x^{i_0}))$. So, h is a Pareto improving transfer and x is not Pareto optimal.

Conversely, if x is not Pareto optimal, there exists h a Pareto improving transfer and we easily check that \bar{h} defined by $\bar{h}^t = \sum_{i \in \mathcal{I}_t} h^i$ is an aggregate Pareto improving transfer.

4 Characterization of Pareto-optimal allocations

We now state the main result of the paper. It provides a condition on the supporting price of a weak Pareto optimal allocation, which is necessary and sufficient for the Pareto optimality of the given allocation.

Proposition 2 *Let $x = (x^i) \in \prod_{i \in \mathcal{I}} X^i$ be a WPO allocation supported by the price sequence $p = (p_1, p_2, \dots, p_t, \dots)$. We suppose that:*

Assumption C: there exists $\underline{r} > 0$ such that for all $\xi^i \in \bar{B}(x^i + \underline{r}\gamma^i(x^i), \underline{r}) \setminus \{x^i\}$, we have $\xi^i \in P^i(x^i)$, for all $i \in \mathcal{I}$;

Assumption C': there exists $\bar{r} > 0$ such that for all $\xi^i \leq (\bar{e}_t, \bar{e}_{t+1})$ (or $\xi^i \leq \bar{e}_1$ for generation 0) with $\xi^i \in P^i(x^i)$, we have $\xi^i \in \bar{B}(x^i + \bar{r}\gamma^i(x^i), \bar{r})$ for all $i \in \mathcal{I}$, where \bar{e}_t and \bar{e}_{t+1} are respectively the total initial endowments at dates t and $t + 1$;

Assumption G: there exists a closed convex cone C in $\mathbb{R}_{++}^{2L} \cup \{0\}$ such that $\gamma^i(x^i) \in C$, for all $i \in \mathcal{I}_{-0}$;

Assumption B: there exist $\bar{e} \in \mathbb{R}_{++}^L$ and $\underline{x} \in \mathbb{R}_{++}^L$ such that for all $t \in \mathbb{N}^$, $\bar{e}_t \leq \bar{e}$, $(\underline{x}, \underline{x}) \leq x^i$ for all $i \in \mathcal{I}_{-0}$ and $\underline{x} \leq x^i$ for all $i \in \mathcal{I}_0$.*

Then, x is Pareto optimal if and only if:

$$\sum_{t \in \mathbb{N}^*} \frac{1}{\|p_t\|} = +\infty.$$

Remark 2 *i)* Assumptions C means that the preferences are smooth and uniformly strictly convex. While Assumptions C and C' in [2] are stated in terms of curvature, we have chosen a more geometric approach. Assumptions C and C' suggest that the boundary of the preferred sets lie below small closed balls of radius \underline{r} and above bigger closed balls of radius \bar{r} .

ii) Note that Assumption B implies that the number of individuals in uniformly bounded above at each generation. Indeed, if I_t is the number of individual of the generation t , then $I_t(\underline{x}, \underline{x}) \leq \sum_{i \in \mathcal{I}_t} x^i \leq (\bar{e}_t, \bar{e}_{t+1}) \leq (\bar{e}, \bar{e})$. We denote by \bar{I} , an upper bound of the number of individual at each generation.

iii) Assumptions C and C' still hold when we aggregate the finitely many consumers at each period by considering the set $\bar{P}^t((x^i))$. Actually, we have $\sum_{i \in \mathcal{I}_t} \bar{B}(x^i + \underline{r}\gamma^i(x^i), \underline{r}) \subset \bar{P}^t((x^i))$ and all $\xi \in \bar{P}^t((x^i))$ such that $\xi \leq (\bar{e}_t, \bar{e}_{t+1})$ (or $\xi \leq \bar{e}_1$ for $t = 0$) belongs to $\sum_{i \in \mathcal{I}_t} \bar{B}(x^i + \bar{r}\gamma^i(x^i), \bar{r})$. Thus, at the aggregate level, we have $\bar{B}(\bar{x}^t + \underline{\rho}\gamma^t, \underline{\rho}) \subset \bar{P}^t((x^i))$ and all $\xi \in \bar{P}^t((x^i))$ such that $\xi \leq (\bar{e}_t, \bar{e}_{t+1})$ (or $\xi \leq \bar{e}_1$ for $t = 0$) belongs to $\bar{B}(\bar{x}^t + \bar{\rho}\gamma^t, \bar{\rho})$, where $\bar{x}^t = \sum_{i \in \mathcal{I}_t} x^i$, $\underline{\rho} = \bar{I}\underline{r}$, $\bar{\rho} = \bar{I}\bar{r}$, and $\gamma^i(x^i) = \gamma^t$ for all $i \in \mathcal{I}_t$.

iv) By considering the compact set Δ_C , the intersection of C and the unit sphere of \mathbb{R}_+^{2L} , we find a constant P such that for all $q = (q_1, q_2) \in C \setminus \{0\}$:

$$0 < P \leq \frac{q_s^\ell}{\|q\|}, \text{ for } s = 1, 2, \ell = 1, 2, \dots, L$$

which in turn implies that:

$$0 < P \leq \frac{q_s^\ell}{\|q_s\|}, \text{ for } s = 1, 2, \ell = 1, 2, \dots, L$$

Furthermore, it is easy to prove that:

$$\frac{q_s^\ell}{\|q_s\|} \leq 1, \text{ for } s = 1, 2, \ell = 1, 2, \dots, L$$

Thus Assumption G is equivalent to Property G of Balasko and Shell in [2], since for all t , there exists $\lambda^i > 0$ such that $\Pi_t = \lambda^i \gamma^i(x^i) \in C$, for all $i \in \mathcal{I}_t$.

Remark 3 For each individual i , consider the closed set F^i defined as the complementery of $P^i(x^i)$, $F^i := \mathbb{C}P^i(x^i)$. Then Assumption C is similar to the prox-regularity of F . Indeed, let $r > 0$ and $\beta > 0$. The set F is called (r, β) -prox-regular at $\bar{x} \in F$, if for any $y \in F \cap B(\bar{x}, \beta)$ and any $v \in N^P(F, y) \cap B$,

$y \in \text{Proj}_F(y + rv)$, B is the unit ball in \mathbb{R}^{2L} and $N^P(F, y)$ is the proximal normal cone to F at y , that is:

$$N^P(F, y) = \{v \in \mathbb{R}^{2L} \mid \exists \rho > 0, y \in \text{Proj}_F(y + \rho v)\}$$

In our framework, under Assumption A,

$$N^P(F, y) = \{\mu \gamma(y) \mid \mu \geq 0\}$$

The notion of prox-regularity was introduced by Poliquin and Rockafellar, as a new important regularity in variational analysis, see [9], and extensively studied by L. Thibault in [4].

Proof of Proposition 2 . Since the allocation $x = (x^i)$ is Pareto optimal if and only if there exists no feasible aggregate Pareto improving transfer upon x , the proof will be established by constructing and characterizing a sequence of aggregate Pareto improving transfers.

First, it is important to note that since $x = (x^i)$ is WPO and supported by p , then we have seen from the previous section that $\frac{\Pi_t}{\|\Pi_t\|} = \gamma^i(x^i)$ for all $i \in \mathcal{I}_t$.

Let \bar{h} be a feasible aggregate Pareto improving transfer, that is, for all t , $\bar{h}^t = (\bar{h}_t^i, \bar{h}_{t+1}^i)$ is an element of $\mathbb{R}^L \times \mathbb{R}^L$, and $\bar{h}_{t+1}^i = -\bar{h}_t^{i+1}$.

Set $\eta^t := \Pi_t \cdot \bar{h}^t$, the net present value of the aggregate transfer \bar{h} at each date t .

Remark 4 If \bar{h} is a feasible aggregate Pareto improving transfer upon (x^i) , then by Definition 4, $\sum_{i \in \mathcal{I}_t} x^i + \bar{h}^t \in \bar{P}^t((x^i))$, so thanks to Assumption C', $\Pi_t \cdot \bar{h}^t > 0$, thus $\eta^t > 0$.

Let us define the sequence α by:

$$\alpha^t := \frac{\|\bar{h}^t\|^2 \|\Pi_t\|}{\Pi_t \cdot \bar{h}^t} = \frac{\|\bar{h}^t\|^2 \|\Pi_t\|}{\eta^t}$$

Remark 5 i) Note that $\|\bar{h}^t\|^2 = \frac{\alpha^t \eta^t}{\|\Pi_t\|} \leq \frac{\alpha^t \|\Pi_t\| \|\bar{h}^t\|}{\|\Pi_t\|}$, thus $\|\bar{h}^t\| \leq \alpha^t$. Thus, if α is bounded then \bar{h} is also bounded.

ii) $\frac{\alpha^t}{2}$ actually represents the radius of the sphere $S(\bar{x}^t + \frac{\alpha^t}{2} \gamma^t, \frac{\alpha^t}{2})$ which is tangent to $\bar{P}^t((x^i))$ at \bar{x}^t , and contains $\bar{x}^t + \bar{h}^t$. Indeed, we easily check that:

$$\|\bar{x}^t + \bar{h}^t - (\bar{x}^t + \frac{\alpha^t}{2} \frac{\Pi_t}{\|\Pi_t\|})\| = \frac{\alpha^t}{2}$$

We prepare the proof by three lemmas and then we prove the necessary and the sufficient condition in two additional lemmas.

Lemma 3 *If the feasible aggregate transfer \bar{h} is Pareto improving upon an allocation $x = (x^i)$, and if Assumption C' holds, then the sequence α is bounded from above by $2\bar{\rho}$.*

Proof. Since \bar{h} is a feasible aggregate Pareto improving transfer, $\bar{y}^t = \bar{x}^t + \bar{h}^t$ is feasible so $\bar{y}^t \leq (\bar{e}_t, \bar{e}_{t+1})$, and $\bar{y}^t \in \bar{P}^t((x^i))$ lies in $\bar{B}(\bar{x}^t + \bar{\rho} \frac{\Pi_t}{\|\Pi_t\|}, \bar{\rho})$ from Assumption C' and Remark 2, so:

$$\begin{aligned} \|\bar{x}^t + \bar{h}^t - (\bar{x}^t + \bar{\rho} \frac{\Pi_t}{\|\Pi_t\|})\| &\leq \bar{\rho} \\ \|\bar{h}^t \Pi_t - \bar{\rho} \frac{\Pi_t}{\|\Pi_t\|}\|^2 &\leq \bar{\rho}^2 \end{aligned}$$

This implies that $\|\bar{h}^t\|^2 - 2\bar{\rho} \frac{\bar{h}^t \cdot \Pi_t}{\|\Pi_t\|} + \bar{\rho}^2 \leq \bar{\rho}^2$, thus $\|\bar{h}^t\|^2 \leq 2\bar{\rho} \frac{\bar{h}^t \cdot \Pi_t}{\|\Pi_t\|}$, thus $\alpha^t := \frac{\|\bar{h}^t\|^2 \|\Pi_t\|}{\Pi_t \cdot \bar{h}^t} \leq 2\bar{\rho}$. \square

Lemma 4 *Note that if \bar{h} is a feasible Pareto improving aggregate transfer, then:*

$$\alpha^t \geq \frac{\|\Pi_t\|}{\eta^t} \|\bar{h}^t\|^2 = \frac{\|\Pi_t\|}{\eta^t} \left[\frac{1}{\|p_t\|^2} (\eta^0 + \dots + \eta^{t-1})^2 + \frac{1}{\|p_{t+1}\|^2} (\eta^0 + \dots + \eta^t)^2 \right]$$

Proof. Indeed, the construction of η and the feasibility of the transfer h allow us to write:

$$\begin{aligned} \eta^0 &= p_1 \cdot \bar{h}^0 = p_1 \cdot \bar{h}_1^0 = -p_1 \cdot \bar{h}_1^1 \\ \eta^1 &= \Pi_1 \cdot \bar{h}^1 = p_1 \cdot \bar{h}_1^1 + p_2 \cdot \bar{h}_2^1 = -p_1 \cdot \bar{h}_1^0 + p_2 \cdot \bar{h}_2^1 \\ &\dots \\ \eta^t &= \Pi_t \cdot \bar{h}^t = p_t \cdot \bar{h}_t^t + p_{t+1} \cdot \bar{h}_{t+1}^t = -p_t \cdot \bar{h}_t^{t-1} + p_{t+1} \cdot \bar{h}_{t+1}^t \end{aligned}$$

By summing up, we obtain that: $\eta^0 + \eta^1 + \dots + \eta^t = p_{t+1} \cdot \bar{h}_{t+1}^t$, with $\bar{h}_t^t = -\bar{h}_t^{t-1}$

By definition,

$$\alpha^t = \frac{\|\Pi_t\|}{\eta^t} (\|\bar{h}_t^t\|^2 + \|\bar{h}_{t+1}^t\|^2)$$

By Schwarz inequality, $p_t \cdot \bar{h}_t^t \leq \|p_t\|^2 \|\bar{h}_t^t\|^2$, that is: $\left(\frac{p_t \cdot \bar{h}_t^t}{\|p_t\|} \right)^2 \leq \|\bar{h}_t^t\|^2$, we then obtain that:

$$\alpha^t \geq \frac{\|\Pi_t\|}{\eta^t} \left[\frac{(p_t \cdot \bar{h}_t^t)^2}{\|p_t\|^2} + \frac{(p_{t+1} \cdot \bar{h}_{t+1}^t)^2}{\|p_{t+1}\|^2} \right]$$

Hence,

$$\alpha^t \geq \frac{\|\Pi_t\|}{\eta^t} \left[\frac{1}{\|p_t\|^2} (\eta^0 + \dots + \eta^{t-1})^2 + \frac{1}{\|p_{t+1}\|^2} (\eta^0 + \dots + \eta^t)^2 \right]$$

\square

Lemma 5 Let η be a positive sequence in \mathbb{R} . Let us define an aggregate transfer \bar{h} in $\prod_{t=0}^{\infty}(\mathbb{R}^L \times \mathbb{R}^L)$ and the associated α respectively by:

$$\bar{h}_{t+1}^t = (\eta^0 + \eta^1 + \dots + \eta^t) \frac{p_{t+1}}{\|p_{t+1}\|^2} \text{ and } \bar{h}_t^t = -\bar{h}_t^{t-1}$$

$$\alpha^t = \frac{\|\Pi_t\|}{\eta^t} \|\bar{h}^t\|^2 = \frac{\|\Pi_t\|}{\eta^t} \left[\frac{1}{\|p_t\|^2} (\eta^0 + \dots + \eta^{t-1})^2 + \frac{1}{\|p_{t+1}\|^2} (\eta^0 + \dots + \eta^t)^2 \right]$$

Under Assumptions B and C, if α is bounded then there exists $\mu > 0$ such that for all t , $\mu\alpha^t < \underline{\rho}$ and $\mu\bar{h}$ is a feasible Pareto improving aggregate transfer.

Remark 6 In Lemma 5, \bar{h} is computed in such a way that it is feasible, that $\Pi_t \cdot \bar{h}^t = \eta^t$ and the associated α^t is the smallest feasible one.

Proof. Let μ be taken small enough so that $\mu\alpha^t < 2\underline{\rho}$, for all t , which is feasible since $\alpha = (\alpha^t)$ is bounded.

From the formula above and Remark 5 ii), $\bar{x}^t + \mu\bar{h}^t$ belongs to the sphere $S(\bar{x}^t + \frac{\mu\alpha^t}{2}\gamma^t, \frac{\mu\alpha^t}{2}) \subset \bar{B}(\bar{x}^t + \underline{\rho}\gamma^t, \underline{\rho})$, since $\frac{\mu\alpha^t}{2} \leq \underline{\rho}$. Since $\bar{h}^t \neq 0$, $\bar{x}^t + \mu\bar{h}^t \in \bar{B}(\bar{x}^t + \underline{\rho}\gamma^t, \underline{\rho}) \setminus \{\bar{x}^t\}$. From Assumption C, $\bar{x}^t + \mu\bar{h}^t \in \bar{P}^t((x^i))$, that is $\mu\bar{h}$ is a feasible aggregate Pareto improving transfer upon the allocation (x^i) .

□

Lemma 6 Given the positive price sequence p , if x is not PO, then under Assumption C', $\sum_{t \in \mathbb{N}^*} \frac{1}{\|p_t\|} < +\infty$.

Proof. Since x is not PO, there exists a Pareto improving aggregate transfer \bar{h} . From Lemma 3, the associated α is bounded from above by $2\bar{\rho}$, thus, from Lemma 4:

$$2\bar{\rho} \geq \alpha^t \geq \frac{\|\Pi_t\|}{\eta^t \|p_{t+1}\|^2} [\eta^0 + \dots + \eta^t]^2 \geq \frac{1}{\eta^t \|p_{t+1}\|} [\eta^0 + \dots + \eta^t]^2$$

since $\frac{\|\Pi_t\|}{\|p_{t+1}\|} = \sqrt{1 + \frac{\|p_t\|^2}{\|p_{t+1}\|^2}} \geq 1$. Thus:

$$\frac{1}{\|p_{t+1}\|} \leq \frac{2\bar{\rho}\eta^t}{[\eta^0 + \dots + \eta^t]^2}$$

But, we notice that:

$$\frac{\eta^t}{[\eta^0 + \dots + \eta^t]^2} \leq \frac{1}{\eta^0 + \dots + \eta^{t-1}} - \frac{1}{\eta^0 + \dots + \eta^t}$$

This implies that:

$$\sum_{t=1}^{\infty} \frac{\eta^t}{[\eta^0 + \dots + \eta^t]^2} \leq \frac{1}{\eta^0}$$

Hence

$$\sum_{t=1}^{\infty} \frac{1}{\|p_t\|} = \frac{2\bar{\rho}}{\eta^0} < +\infty$$

□

The last step of the proof of Proposition 2 is given by the following lemma.

Lemma 7 *Under Assumptions B, C and G, if the positive price sequence p satisfies $\sum_t \frac{1}{\|p_t\|} < +\infty$, then x is not PO.*

Proof. Consider $\eta^t = \frac{1}{\|p_t\|}$ for all t , and denote by $\bar{\eta}^t := \sum_{s=1}^t \eta^s$. Note that $(\bar{\eta}^t)$ is bounded. Let the corresponding aggregate transfer \bar{h} and the associated sequence α defined by the formula of Lemma 5. From this Lemma, it remains to show that α is bounded.

From Lemma 5, we have: $\bar{h}^t = (\bar{\eta}^{t-1} \frac{p_t}{\|p_t\|^2}, \bar{\eta}^t \frac{p_{t+1}}{\|p_{t+1}\|^2})$. Consequently, since $\bar{\eta}^{t-1} \leq \bar{\eta}^t$, $\|\bar{h}^t\|^2 = (\bar{\eta}^{t-1})^2 \frac{1}{\|p_t\|^2} + (\bar{\eta}^t)^2 \frac{1}{\|p_{t+1}\|^2} \leq (\bar{\eta}^t)^2 \left(\frac{1}{\|p_t\|^2} + \frac{1}{\|p_{t+1}\|^2} \right)$. Since $\eta^t = \frac{1}{\|p_t\|}$, the associated α defined in Lemma 5 satisfies:

$$\begin{aligned} \alpha^t &= \frac{\|\Pi_t\|}{\eta^t} \|\bar{h}^t\|^2 \leq \|\Pi_t\| \|p_t\| (\bar{\eta}^t)^2 \left(\frac{1}{\|p_t\|^2} + \frac{1}{\|p_{t+1}\|^2} \right) \\ &\leq \sqrt{\|p_t\|^2 + \|p_{t+1}\|^2} \frac{(\bar{\eta}^t)^2}{\|p_t\|} \left(1 + \frac{\|p_t\|^2}{\|p_{t+1}\|^2} \right) \\ &\leq (\bar{\eta}^t)^2 \sqrt{1 + \frac{\|p_{t+1}\|^2}{\|p_t\|^2}} \left(1 + \frac{\|p_t\|^2}{\|p_{t+1}\|^2} \right) \end{aligned}$$

Assumption G implies that $\frac{\|p_t\|}{\|p_{t+1}\|}$ is bounded. Since $\bar{\eta}^t$ is bounded, α is so.

□

Appendix

Proof of the existence of an equilibrium

We will proceed as in exchange economies (see Balasko et al. [2]) to establish the existence of equilibrium in \mathcal{E} .

Step 1: Truncate the economy at a finite horizon τ : we thus consider individuals born up to period $\tau - 1$ and group them all together into $\mathcal{I}_0^{\tau-1}$. The truncated economy with a finite horizon $\mathcal{E}_\tau = (X^{\tau i}, P^{\tau i}, e^{\tau i})_{i \in \mathcal{I}_0^{\tau-1}}$ where, for each $t = 1, \dots, \tau - 1$, for each $i \in \mathcal{I}_t$,

$$X^{\tau i} = \{x \in \prod_{t=1}^{\tau} \mathbb{R}_+^L \mid x_{t'} = 0, \forall t' \neq t, t+1\},$$

$$P^{\tau i}(x) = P^i(x^i),$$

$$e^{\tau i} = (e_{t'}^{\tau i})_{t'=1}^{\tau} \text{ such that } e_t^{\tau i} = e_t^i, e_{t+1}^{\tau i} = e_{t+1}^i \text{ and } e_{t'}^{\tau i} = 0 \text{ if } t' \neq t, t+1$$

satisfies the sufficient conditions for the existence of pseudo-equilibrium in finite

dimension of Gale and Mas-Colell [6], [7], where the production set is $Y = \{0\}$. Indeed, the $X^{\tau i}$'s are closed (as a sum of closed subsets of \mathbb{R}_+^{2L}), convex, non-empty and bounded below, the $P^{\tau i}(x)$'s are open in X and convex, and the income functions $\alpha^{\tau i}$'s defined by $\alpha^{\tau i}(p) = p \cdot e^{\tau i}$ satisfy $\alpha^{\tau i}(p) > \inf p \cdot X^{\tau i} = 0$. The difference between a pseudo-equilibrium and an equilibrium is that we do not require the market clearing condition at the last period τ and we artificially increase the initial endowments by adding those of the consumers of the generation τ . We also recall that if $\tau' > \tau$, then the restriction of a pseudo-equilibrium of $\mathcal{E}_{\tau'}$ to the $\tau - 1$ first generations is a pseudo-equilibrium of \mathcal{E}_{τ} .

Step 2: We prove that prices and allocations remain in a compact space of a suitable linear space and we finally show that an equilibrium of the OLG economy is a limit point of pseudo-equilibria for the esquence of truncated economies.

Relation between the aggregate normal cone and the normal cones to the individual preferred sets

$$N_{\bar{P}_t((x^i))}(\bar{x}^t) = \bigcap_{i \in \mathcal{I}_t} N_{\bar{P}^i(x^i)}(x^i)$$

Proof. Indeed, let $q \in \cap_{i \in \mathcal{I}_t} N_{\bar{P}^i(x^i)}(x^i)$, then $q \cdot (z^i - x^i) \leq 0$ for all $z^i \in \bar{P}^i(x^i)$, $i \in \mathcal{I}_t$. Thus by summing up, $q \cdot \sum_{i \in \mathcal{I}_t} (z^i - x^i) = q \cdot (\sum_{i \in \mathcal{I}_t} z^i - \sum_{i \in \mathcal{I}_t} x^i) \leq 0$, where $\sum_{i \in \mathcal{I}_t} z^i \in \bar{P}_t((x^i))$. Consequently, $\cap_{i \in \mathcal{I}_t} N_{\bar{P}^i(x^i)}(x^i) \subset N_{\bar{P}_t((x^i))}(\bar{x}^t)$.

Conversely, let $q \in N_{\bar{P}_t((x^i))}(\bar{x}^t)$, then for all $z \in \bar{P}_t((x^i))$, $q \cdot (z - \bar{x}^t) \leq 0$. Since $P_t((x^i)) = \sum_{i \in \mathcal{I}_t} \bar{P}^i(x^i)$, for all $i \in \mathcal{I}_t$, there exists $z^i \in \bar{P}^i(x^i)$ such that $z = \sum_{i \in \mathcal{I}_t} z^i$. By linearity of the scalar product, $\sum_{i \in \mathcal{I}_t} q \cdot (z^i - x^i) \leq 0$. By taking $z^i = x^i$ for all $i \neq 1, i \in \mathcal{I}_t = \{1, 2, \dots, I_t\}$ and $z^1 \in \bar{P}^1(x^1)$, then $q \cdot (z^1 - x^1) \leq 0$ which means that $q \in N_{\bar{P}^1(x^1)}(x^1)$. By repeating the same reasoning for $i = 2, \dots, I_t$, we obtain that $q \cdot (z^i - x^i) \leq 0$ for all $i \in \mathcal{I}_t$, thus $N_{\bar{P}_t((x^i))}(\bar{x}^t) \subset \cap_{i \in \mathcal{I}_t} N_{\bar{P}^i(x^i)}(x^i)$. \square

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