Necessary and Sufficient Conditions for a Solution of the Bellman Equation to be the Value Function: A General Principle

Takashi Kamihigashi, Cuong Le Van

UNIVERSITÉ PARIS 1
PANTHÉON SORBONNE

Documents de Travail du
Centre d'Economie de la Sorbonne

CES Working Papers

Necessary and Sufficient Conditions for a Solution of the Bellman Equation to be the Value Function: A General Principle

Takashi KAMIHIGASHI, Cuong LE VAN

2015.07

Maison des Sciences Économiques, 106-112 boulevard de L'Hôpital, 75647 Paris Cedex 13
http://centredeconomiesorbonne.univ-paris1.fr/
ISSN : 1955-611X

Necessary and Sufficient Conditions for a Solution of the Bellman Equation to be the Value Function: A General Principle*

Takashi Kamihigashi† and Cuong Le Van‡

January 29, 2015

Abstract

In this paper, we give Necessary and Sufficient Conditions for a Solution of the Bellman Equation to be the Value Function. This result is a general principle. It requires no structure beyond the common framework of discrete-time stationary optimization problems with time-additive returns. In particular, the state space X is an arbitrary set.

Keywords: Dynamic programming, Bellman equation, value function, fixed point.

JEL Classification: C61

*

†RIEB, Kobe University, Rokkodai, Nada, Kobe 657-8501 JAPAN. Email: tkamihig@rieb.kobe-u.ac.jp. Tel/Fax: +81-78-803-7015.

‡CNRS, PSE, IPAG, VCREME

1 Introduction

Dynamic programming is one of the most fundamental tools in economic analysis. A typical setup for dynamic programming is a problem of maximizing the infinite sum of discounted returns subject to feasibility constraints. In order to apply a recursive approach, it is commonly assumed that the problem is "stationary": the returns are represented by a time invariant "return function," and the constraints are expressed as a time invariant "feasibility correspondence." The maximum (or supremum) of the infinite sum is called the "value function," and this function can often be computed as a solution to the associated "Bellman equation." When one considers using this approach, two fundamental questions arise:

A. Is the value function a solution to the Bellman equation?

B. Is a solution to the Bellman equation the value function?

Question A is rarely questioned, and it is often taken for granted that the answer is yes. The answer is indeed yes, provided that the Bellman equation is well defined. Stokey and Lucas (1989, Theorem 4.2) show that the value function solves the Bellman equation if the return function is finite, in which case the Bellman equation is always well defined. In a more general nonstationary framework, Kamihigashi (2008) shows that the value function solves the Bellman equation if and only if the Bellman equation is well defined. Thus the answer to Question A is now fully understood.

Concerning Question B, against common wisdom, the straight answer to this question is no. There are obvious solutions to the Bellman equation that are not the value function. For example, the function identically equal to ∞ is a trivial solution to the Bellman equation; so is the function identically equal to $-\infty$. Even if these trivial functions are ruled out, the answer is still no in general unless the space of possible candidate solutions is further restricted. In Subsection 4.1 we present a trivial example that has a continuum of finite solutions to the Bellman equation.

Perhaps the best known result related to Question B is that if the return function is bounded and continuous, and if the feasibility correspondence is continuous and compact-valued, then the Bellman equation has a unique solution in the space of bounded continuous functions, and this solution is the value function (e.g., Stokey and Lucas, 1989, pp. 77–78). In other words, under these conditions, a solution to the Bellman equation in the space of bounded continuous function is the value function. This existence part of this result is shown by using the contraction mapping theorem. Much of the subsequent economic literature on dynamic programming is based on various forms of contractions. Among them are weighted contractions (Boyd, 1990; Alvarez and Stokey, 1998; Durán, 2000), and local contractions (Rinćon-Zapatero and Rodríguez-Palmero, 2003, 2007, 2009; Martins-Da-Rocha and Vailakis, 2010). In these studies, the space of candidate solutions is restricted to those of continuous functions with appropriate topologies.

A more direct approach is used by Le Van and Morhaim (2002), who show that the value function is a unique solution to the Bellman equation in the space of upper semicontinuous functions satisfying transversality-like conditions. Conditions similar to these are used by Stokey and Lucas (1989, Section 4.1) to characterize the value function.

All the aforementioned results regarding Question B assume that the return function and the feasibility correspondence are continuous. Under additional technical assumptions, they also show that iteration of the Bellman equation leads to the value function.

2 A General Principle

The setup is the same as in Kamihigashi (2012). Let X be a set. Let Γ be a nonempty-valued correspondence from X to X. Let D be the graph of Γ:

$$D = \{(x, y) \in X \times X : y \in \Gamma(x)\}. \tag{1}$$

We call a path $\{x_t\}_{t=0}^{\infty}$ *feasible* if Let Π and $\Pi(x_0)$ denote the set of feasible paths and that of feasible paths from x_0, respectively:

$$\Pi = \{\{x_t\}_{t=0}^{\infty} \in X^{\infty} : \forall t \in \mathbb{Z}_+, x_{t+1} \in \Gamma(x_t)\}. \tag{2}$$

$$\Pi(x_0) = \{\{x_t\}_{t=1}^{\infty} \in X^{\infty} : \{x_t\}_{t=0}^{\infty} \in \Pi\}, \qquad x_0 \in X. \tag{3}$$

Let $u : D \to [-\infty, \infty)$. Let $\beta \geq 0$. Given $x_0 \in X$, consider the following optimization problem:

$$\sup_{\{x_t\}_{t=1}^{\infty} \in \Pi(x_0)} \mathop{\mathrm{L}}_{T \uparrow \infty} \sum_{t=0}^{T} \beta^t u(x_t, x_{t+1}), \tag{4}$$

where $\mathrm{L} \in \{\liminf, \limsup\}$. Since $u(x, y) < \infty$ for all $(x, y) \in D$, the objective function is well defined for any feasible path. Define

$$\mathrm{L}^{\star} = \begin{cases} \liminf & \text{if } \mathrm{L} = \limsup, \\ \limsup & \text{if } \mathrm{L} = \liminf. \end{cases} \tag{5}$$

Note that for any sequence $\{a_t\}_{t=0}^{\infty}$ in $[-\infty, \infty]$, we have

$$\mathop{\mathrm{L}}_{t \uparrow \infty} [-a_t] = - \mathop{\mathrm{L}^{\star}}_{t \uparrow \infty} a_t. \tag{6}$$

The value function $v^* : X \to \overline{\mathbb{R}}$ is defined by

$$v^*(x_0) = \sup_{\{x_t\}_{t=1}^{\infty} \in \Pi(x_0)} \mathop{\mathrm{L}}_{T \uparrow \infty} \sum_{t=0}^{T} \beta^t u(x_t, x_{t+1}), \qquad x_0 \in X. \tag{7}$$

We define

$$\Pi^0 = \{\{x_t\}_{t=0}^{\infty} \in \Pi : \mathop{\mathrm{L}}_{T \uparrow \infty} \sum_{t=0}^{T} \beta^t u(x_t, x_{t+1}) > -\infty\}, \tag{8}$$

$$\Pi^0(x_0) = \{\{x_t\}_{t=1}^{\infty} \in \Pi(x_0) : \{x_t\}_{t=0}^{\infty} \in \Pi^0\}, \qquad x_0 \in X, \tag{9}$$

$$X^0 = \{x \in X : \Pi^0(x) \neq \emptyset\}. \tag{10}$$

We follow the convention that $\sup \emptyset = -\infty$. Thus we have

$$\forall x_0 \in X, \qquad v^*(x_0) = \sup_{\{x_t\}_{t=1}^{\infty} \in \Pi^0(x_0)} \mathop{\mathrm{L}}_{T \uparrow \infty} \sum_{t=0}^{T} \beta^t u(x_t, x_{t+1}). \tag{11}$$

Let V be the set of functions from X to $[-\infty, \infty)$. Consider the Bellman equation

$$v(x) = \sup_{y \in \Gamma(x)} \{u(x, y) + \beta v(y)\}, \qquad x \in X, \tag{12}$$

where $v \in V$. We say that a function $v \in V$ *solves, or is a solution of, the Bellman equation* (12) if v satisfies (12). We can also write the Bellman equation simply as $v = Bv$, where B is the operator on V defined by

$$(Bv)(x) = \sup_{y \in \Gamma(x)} \{u(x, y) + \beta v(y)\}, \qquad x \in X. \tag{13}$$

This operator is called the *Bellman operator*. A solution of the Bellman equation is a fixed point of the Bellman operator.

If $v \in V$, then Bv is well defined, but it need not be the case that $Bv \in V$. If $v \notin V$, then Bv need not be well defined. This is because it is possible that $u(x, y) = -\infty$ while $v(y) = \infty$ for some $(x, y) \in D$, in which case Bv is not well defined.[1] For this reason we assume the following for the rest of this paper:

$$v^* \in V. \tag{14}$$

If a solution of the Bellman equation (12) in V coincides with the value function v^*, then (14) must be satisfied. Thus (14) is a minimum requirement to analyze the relation between a solution of the Bellman equation and the value function. Furthermore, if (14) is violated, it is often possible to normalize the problem so that (14) holds; see Subsection 5.4 for an example.

It is well known that the value function solves the Bellman equation in a slightly less general setting (Stokey and Lucas, 1989, Theorem 4.2). This extends to our general framework, and we state it here for later referenc.

Lemma 2.1. *The value function v^* solves the Bellman equation (12).*

Proof. Follows from Kamihigashi (2008). □

We are ready to state the main result of this paper.

Theorem 1. *Let $v \in V$ be a solution of the Bellman equation (12). Then $v = v^*$ if and only if the following two conditions hold:*

(a) For any $\{x_t\} \in \Pi^0$, we have

$$\operatorname*{L^\star}_{T \uparrow \infty} \beta^T v(x_T) \geq 0. \tag{15}$$

(b) For any $x_0 \in X$ and $\epsilon > 0$, there exists $\{x_t\} \in \Pi(x_0)$ such that

$$\forall T \in \mathbb{N}, \qquad v(x_0) \leq \sum_{t=0}^{T-1} \beta^t u(x_t, x_{t+1}) + \beta^T v(x_T) + \epsilon, \tag{16}$$

$$\operatorname*{L^\star}_{T \uparrow \infty} \beta^T v(x_T) \leq \epsilon. \tag{17}$$

[1] See Kamihigashi (2012) for a simple nonstationary problem of this nature.

Proof. See Appendix 6. □

This result is a general principle. It requires no structure beyond the common framework of discrete-time stationary optimization problems with time-additive returns. In particular, the state space X is an arbitrary set.

Since conditions (a) and (b) are necessary and sufficient, they are the weakest possible sufficient conditions for a solution of the Bellman equation (12) to equal v^*. The proof of the sufficiency part of Theorem 1 shows that conditions (a) and (b) have different roles:

Proposition 1. *Let $v \in V$ be a solution of the Bellman equation (12). Then condition (a) of Theorem 1 implies that $v^* \leq v$, and condition (b) implies that $v \leq v^*$.*

Proof. See Lemmas 6.6 and 6.7. □

As we discuss in the next section, the equality version of (15) (with $\mathrm{L}^\star = \lim$) is one of the most commonly used sufficient conditions for a solution of the Bellman equation to equal the value function. By contrast, to our knowledge, condition (b) is new to a large extent. To understand this condition, it is useful to note that any solution of the Bellman equation (12) has the following property.

Lemma 2.2. *Let $v \in V$ be a solution of the Bellman equation (12). Then for any $\epsilon > 0$, there exists $\{x_t\} \in \Pi(x_0)$ satisfying (16).*

Proof. See Stokey and Lucas (1989, p. 73) or Kamihigashi (2012, Lemma 4.1). □

Therefore, condition (b) automatically holds if (17) with $\epsilon = 0$ is assumed to hold for all feasible paths. This assumption is often made indirectly in the literature, as we discuss in the next section. However, there are subtle cases in which the assumption does not hold, and the full strength of condition (b) is needed when one wishes to reject a solution of the Bellman equation as the value function; see Subsections 5.2, 5.3.

One might wonder if X could be replaced by X^0 in condition (b). This is in fact impossible, as we illustrate using a simple example in Section 4.2.

Consider the following assumption, which is often used in the literature.

Assumption. For any $\{x_t\} \in \Pi$, the limit

$$\lim_{T \uparrow \infty} \sum_{t=0}^{T} \beta^t u(x_t, x_{t+1}) \tag{18}$$

exists in $[-\infty, \infty)$.

Under this assumption, the version of Theorem 1 with $\mathrm{L}^\star = \limsup$ and the one with with $\mathrm{L}^\star = \liminf$ are both valid. Thus the value function must satisfy the stronger version of each of conditions (a) and (b):

Corollary 1. *Let Assumption 2 hold. Let $v \in V$ be a solution of the Bellman equation. Then $v = v^*$ if and only if* (a') For any $\{x_t\} \in \Pi^0$, we have

$$\liminf_{T\uparrow\infty} \beta^T v(x_T) \geq 0. \tag{19}$$

(b')For any $x_0 \in X$ and $\epsilon > 0$, there exists $\{x_t\} \in \Pi(x_0)$ satisfying (16) such that

$$\limsup_{T\uparrow\infty} \beta^T v(x_T) \leq \epsilon. \tag{20}$$

Proof. Conditions (a') and (b') are necessary, as mentioned above. They are also sufficient since they imply conditions (a) and (b) of Theorem 1 whether $\mathrm{L} = \liminf$ or $\mathrm{L} = \limsup$. □

With an additional argument, we obtain sufficient conditions weaker than those given by either version of Theorem 1:

Proposition 2. *Let Assumption 2 hold. Let $v \in V$ be a solution of the Bellman equation (12). Then $v = v^*$ if and only if the following two conditions hold:*

(a") For any $\{x_t\} \in \Pi^0$, we have

$$\limsup_{T\uparrow\infty} \beta^T v(x_T) \geq 0. \tag{21}$$

(b") For any $x_0 \in X$ and $\epsilon > 0$, there exists $\{x_t\} \in \Pi(x_0)$ satisfying (16) such that

$$\liminf_{T\uparrow\infty} \beta^T v(x_T) \leq \epsilon. \tag{22}$$

Proof. The "only if" part follows from Corollary 1. To see the 'if" part, assume conditions (a") and (b"). By condition (a") and Proposition 1 with $\mathrm{L} = \liminf$, for any $x_0 \in X$ we have

$$v(x_0) \geq \sup_{\{x_t\}\in\Gamma(x_0)} \liminf_{T\uparrow\infty} \sum_{t=0}^{T} \beta u(x_t, x_{t+1}) \tag{23}$$

$$= \sup_{\{x_t\}\in\Gamma(x_0)} \lim_{T\uparrow\infty} \sum_{t=0}^{T} \beta u(x_t, x_{t+1}) = v^*(x_0), \tag{24}$$

where the first equality uses Assumption 2. It follows that $v^* \leq v$. Likewise we have $v \leq v^*$ by condition (b") and Proposition 1 with $\mathrm{L} = \limsup$. Hence $v = v^*$. □

In what follows, we proceed without Assuming 2, since one of the advantages of the use of the operator L is that one need not worry about the validity of this assumption. Except for Sections 4.5 and 5.1, where we present examples in which the definition of L affects the value function and optimal paths, we simply use L and $\mathrm{L}^\star$ without discussing the possibility that L can be replaced by lim.

3 Sufficient Conditions

In this section we take a solution $v \in V$ of the Bellman equation (12) as given, and present a series of sufficient conditions for v to equal v^* as consequences of Theorem 1. By doing this, we clarify the relationship between Theorem 1 and many of the related results in the literature as well as develop some new results.

We start with the equality version of (15), which is one of the most commonly used sufficient conditions for a solution of the Bellman equation to equal the value function, as mentioned above. The following is a slight generalization of Theorem 4.3 in Stokey and Lucas (1989).

Corollary 2. *We have $v = v^*$ if any $\{x_t\} \in \Pi$ satisfies*

$$\mathop{\mathrm{L}^\star}_{T\uparrow\infty} \beta^T v(x_T) = 0. \tag{25}$$

Proof. Let $x_0 \in X$. Let $\{x_t\} \in \Pi^0(x_0)$. Then (15) follows from (25). Thus condition (a) holds. To see condition (b), let $\epsilon > 0$. By Lemma 2.2 there exists $\{x_t\} \in \Pi(x_0)$ satisfying (16). We also have (17) by (25). Thus condition (b) holds. We now have $v = v^*$ by Theorem 1. □

The above result is particularly useful when v is bounded:

Corollary 3. *Suppose that v is bounded and that $\beta < 1$. Then $v = v^*$.*

Proof. If v is bounded and $\beta < 1$, then (25) holds for any $\{x_t\} \in \Pi$. Thus $v = v^*$ by Corollary 2. □

If u is bounded and $\beta < 1$, then the Bellman equation (12) has a solution that is a bounded function on X To see this, suppose that u is bounded and $\beta < 1$. Then the Bellman operator B maps the Banach space of bounded functions on X (Dunford and Schwartz, 1988, p. 257) to itself, and it can easily be shown that B is a contraction mapping (Stokey and Lucas, 1989, Theorem 3.3). Thus by the contraction mapping theorem (Stokey and Lucas, 1989, Theorem 3.2), B has a unique fixed point in the space of bounded functions on X. This fixed point equals v^* by Corollary 3.

If u is unbounded, however, (25) with $v = v^*$ is easily violated. For example, suppose that there exists $x \in X$ with $x \in \Gamma(x)$ and $u(x, x) = -\infty$. Then the constant path $\{x_t\}$ with $x_t = x$ for all $t \in \mathbb{Z}_+$ violates (25) with $v = v^*$. This can be remedied by restricting the set of feasible paths required to satisfy (25) while introducing an additional condition:

Corollary 4. *Suppose that* (i) *any $\{x_t\} \in \Pi^0$ satisfies (25), and that* (ii) *any $\{x_t\} \in \Pi$ satisfies*

$$\mathop{\mathrm{L}^\star}_{T\uparrow\infty} \beta^T v(x_T) \le 0. \tag{26}$$

Then $v = v^*$.

Proof. Condition (i) above implies condition (a) of Theorem 1. Condition (ii) and Lemma 2.2 imply condition (b) of Theorem 1. Thus $v = v^*$ by Theorem 1. □

Under various additional assumptions, Le Van and Morhaim (2002, Theorem 2) show that conditions (i) and (ii) above are also necessary for v to equal v^*. Their assumptions imply that v^* satisfies (26) for all $\{x_t\} \in \Pi$ (Le Van and Morhaim, 2002, Proposition 4). This implication alone is sufficient to show their result on necessity and sufficiency:

Corollary 5. *Suppose that any $\{x_t\} \in \Pi$ satisfies*

$$\underset{T\uparrow\infty}{\mathrm{L}^\star}\, \beta^T v^*(x_T) \leq 0. \tag{27}$$

Then $v = v^*$ if and only if conditions (i) and (ii) of Corollary 4 hold.

Proof. The "if" part follows from Corollary 4. To see the "only if" part, suppose that $v = v^*$. Then condition (ii) follows from (27). Let $\{x_t\} \in \Pi^0$. By condition (a) of Theorem 1, we have (15). Since we also have (26), we obtain (25). □

Conditions (i) and (ii) of Corollary 4 are implied by the conditions used by Rinćon-Zapatero and Rodríguez-Palmero (2003, DP3'(ii)(b), Theorem 6) and Martins-da-Rocha and Vailakis (2010, DP4(c), DP5) given the existence of a solution of the Bellman equation (12) established by their results. The existence and uniqueness parts of these results are generalized by Kamihigashi (2012, Theorem 2.1), and the conditions used in his result imply condition (a) of Theorem 1 and condition (ii) of Corollary 4:

Corollary 6. *Under condition (a) of Theorem 1 and condition (ii) of Corollary 4, we have $v = v^*$.*

Proof. Assume condition (a) of Theorem 1 and condition (ii) of Corollary 4. Then condition (i) of Corollary 4 holds. Thus $v = v^*$ by Corollary 4. □

As the above proof shows, Corollary 6 is a simple variation of Corollary 4. Using Corollary 6 we obtain the following.

Proposition 3. *We have $v = v^*$ if $\beta < 1$ and the following two conditions hold.*

(i) There exist $\theta > 0$ and $\eta \in \mathbb{R}$ such that

$$\forall (x, y) \in D, \qquad u(x, y) \leq \theta v(x) + \eta. \tag{28}$$

(ii) For any $x_0 \in X$, there exists a sequence $\{\xi_t\}$ in $\mathbb{R}_+$ with $\sum_{t=0}^{\infty} \beta^t \xi_t < \infty$ such that

$$\forall \{x_t\} \in \Pi(x_0), \forall t \in \mathbb{Z}_+, \qquad v(x_t) \leq \xi_t. \tag{29}$$

Proof. See Appendix 7. □

In many cases, condition (i) of Proposition 3 is also a necessary condition. For example, suppose that there exists $\underline{x} \in X$ such that $\underline{x} \in \Gamma(x)$ and $u(x, \underline{x}) > -\infty$ for all $x \in X$. Then $v^*(x) \geq u(x, \underline{x}) + \beta u(\underline{x}, \underline{x})/(1 - \beta)$ for all $x \in X$; hence v^* satisfies condition (i) of Proposition 3. Condition (ii) of Proposition 3 is rather restrictive, but it is satisfied in some important models; see Section 4.4 for an application of Proposition 3.

Since condition (a) of Theorem 1 is a necessary condition, it cannot be weakened any further. On the other hand, condition (ii) of Corollary 4 can still be relaxed. Since the role of this condition is to ensure that $v \leq v^*$, it is not necessary to require all feasible paths to satisfy it. In fact, for each initial state, we only need one feasible path that satisfies it as well as an additional equation:

Corollary 7. *Suppose that* (i) *condition (a) of Theorem 1 holds, and that* (ii) *for any $x_0 \in X$, there exists $\{x_t\} \in \Pi(x_0)$ satisfying (26) such that*

$$\forall t \in \mathbb{Z}_+, \qquad v(x_t) = u(x_t, x_{t+1}) + \beta v(x_{t+1}). \tag{30}$$

Then $v = v^*$.

Proof. Condition (ii) above implies condition (b) of Theorem 1; in fact, (30) implies that the equality version of (16) holds with $\epsilon = 0$, and (26) implies (17) for any $\epsilon > 0$. Thus $v = v^*$ by Theorem 1. □

Corollary 7 is particularly useful when it is easy to find a feasible path satisfying (30) from any initial state. This is the case if Γ is compact-valued and u and v are continuous or upper semicontinuous. In this case, the supremum on the right-hand side of the Bellman equation (12) is achieved at some $y \in \Gamma(x)$ for any $x \in X$, so that from any initial state, it is possible to construct a feasible path satisfying (30). This is assumed in the following result.

Proposition 4. *Suppose that* (i) *any $\{x_t\} \in \Pi^0$ satisfies (25), and that* (ii) *for any $x_0 \in X$, there exists $\{x_t\} \in \Pi(x_0)$ satisfying (30) and*

$$\underset{T\uparrow\infty}{\mathrm{L}^\star}\ \beta^T v(x_T) < \infty. \tag{31}$$

Then $v = v^*$.

Proof. See Appendix 7. □

This result extends the sufficiency part of Theorem 7.4.1 in Le Van and Dana (2003), which is shown in an undiscounted optimal growth model, to our general framework; see Section 5.4 for further discussion on their result.

Finally, the strength of Theorem 1 is that it is applicable even when there is no feasible path satisfying both (26) and (30). In such a case, it is still possible to verify $v = v^*$ provided that there is an approximate solution to (26) and (30); see Subsection 5.3 for an application of the sufficiency part of Theorem 1.

4 Examples with Discounting

In this section we assume that $\beta \in [0, 1)$ though our arguments do not always require this assumption.

4.1 A Trivial Example with Multiple Solutions

Consider the example of Kamihigashi (2012, Section 3.1). Let $X = \mathbb{Z}_+$. Suppose that for all $x \in X$, we have

$$\Gamma(x) = \{x + 1\}, \qquad u(x, x + 1) = 0. \tag{32}$$

For $\alpha \in \mathbb{R}$ and $x \in X$, let $v_\alpha(x) = \alpha\beta^{-x}$. Let $\alpha \in \mathbb{R}$. We have

$$u(x, x+1) + \beta v_\alpha(x+1) = \beta\alpha\beta^{-x-1} = v_\alpha(x). \tag{33}$$

Thus v_α solves the Bellman equation. Let $\{x_t\} \in \Pi$. Then

$$\beta^T v_\alpha(x_T) = \beta^T \alpha\beta^{-x_T} = \beta^T \alpha\beta^{-x_0-T} = \alpha\beta^{-x_0}. \tag{34}$$

Therefore, if $\alpha > 0$, then v_α violates condition (b) of Theorem 1 since $\beta^T v_\alpha(x_T) = \alpha\beta^{-x_0} > 0$ for all $T \in \mathbb{Z}_+$. Similarly, if $\alpha < 0$, then v_α violates condition (a) of Theorem 1. Therefore, if $\alpha \neq 0$, then $v_\alpha \neq v^*$ by Theorem 1. If $\alpha = 0$, then v_α satisfies both conditions (a) and (b). Hence $v^* = v_0$; i.e., $v^*(x) = 0$ for all $x \in X$. This can be seen directly from (32).

4.2 X Cannot Be Replaced with X^0 in Condition (b)

Let $X = \mathbb{Z}_+$. Suppose that for all $x \in X$, we have

$$\Gamma(x) = \{x+1\}, \qquad u(x, x+1) = -\beta^{-x}. \tag{35}$$

For $\alpha \in \mathbb{R}$ and $x \in X$, let

$$v_\alpha(x) = (\alpha + x)\beta^{-x}. \tag{36}$$

Let $\alpha \in \mathbb{R}$. For any $x \in X$, we have

$$u(x, x+1) + \beta v_\alpha(x+1) = -\beta^{-x} + \beta(\alpha + x + 1)\beta^{-x-1} \tag{37}$$

$$= (\alpha + x)\beta^{-x} = v_\alpha(x). \tag{38}$$

Therefore v_α solves the Bellman equation.

Note that for any $\{x_t\} \in \Pi$ and $T \in \mathbb{Z}_+$ we have

$$\sum_{t=0}^{T} \beta^t[-\beta^{-x_t}] = -\sum_{t=0}^{T} \beta^t \beta^{-x_0-t} = -(T+1)\beta^{-x_0} \to -\infty \quad \text{as } T \uparrow \infty. \tag{39}$$

Thus $X^0 = \emptyset$ and $v^*(x_0) = -\infty$ for any $x_0 \in X$. Since $v_\alpha(x) > -\infty$ for all $x \in X$, it follows that $v^* \neq v_\alpha$. This conclusion could not be reached by Theorem 1 if X were replaced by X^0 in condition (b) since $X^0 = \emptyset$ here. However, for any $\{x_t\} \in \Pi$ we have

$$\beta^T v_\alpha(x_T) = \beta^T(\alpha + x_T)\beta^{-x_T} = (\alpha + x_T)\beta^{-x_0} \to \infty \quad \text{as } T \uparrow \infty. \tag{40}$$

Thus v_α in fact violates condition (b), and we have $v_\alpha \neq v^*$ by Theorem 1.

4.3 Optimal Growth with Linear Utility and Production

Let $X = \mathbb{R}_+$ and $\theta > 0$. Suppose that for all $x \in X$, we have

$$\Gamma(x) = [0, \theta x], \qquad \forall y \in \Gamma(x),\ u(x,y) = \theta x - y. \tag{41}$$

Let $v_\theta(x) = \theta x$. Suppose that $\beta\theta \leq 1$. Then

$$u(x,y) + \beta v_\theta(y) = \theta x - y + \beta\theta y \tag{42}$$

$$= \theta x - (1-\beta\theta)y \leq \theta x = v_\theta(x). \tag{43}$$

Thus $v_\theta(x) = \sup_{y\in\Gamma(x)}\{u(x,y) + \beta v_\theta(y)\}$, and the supremum is achieved at $y = 0$. Therefore v_θ solves the Bellman equation. If $\beta\theta < 1$, then any feasible path $\{x_t\}$ satisfies $\beta^T v_\theta(x_T) \leq (\beta\theta)^T x_0 \to 0$ as $T \uparrow \infty$; thus $v_\theta = v^*$ by Corollary 2.

Suppose that $\beta\theta = 1$. In this case, the feasible path given by $x_t = \theta^t x_0$ for all $t \in \mathbb{Z}_t$ with $x_0 > 0$ violates (25); thus Corollary 2 does not apply. On the other hand, since $v_\theta \geq 0$, condition (a) trivially holds. Note from (42) and (43) that any $\{x_t\} \in \Pi$ satisfies (30). Let $\{x_t\} \in \Pi(x_0)$ be given by $x_t = 0$ for all $t \in \mathbb{N}$. Then (26) holds, and condition (ii) of Corollary 7 follows. Therefore $v_\theta = v^*$ by Corollary 7.

4.4 Optimal Growth with Increasing Returns

Let $X = \mathbb{R}_+$ and $\alpha > 0$. Suppose that for all $x \in X$, we have

$$\Gamma(x) = [0, x^\alpha], \qquad \forall y \in \Gamma(x),\ u(x,y) = \ln(x^\alpha - y). \tag{44}$$

We assume that $\alpha\beta < 1$. This problem is considered by Stokey and Lucas (1989, Section 4.4) under the assumption that $\alpha < 1$ (though their argument applies to the case $\alpha > 1$ as well). Our primary concern here is the case $\alpha > 1$, bur since our arguments do not require this assumption, we proceed with the assumption that $\alpha > 0$.

Stokey and Lucas (1989, Theorem 4.14) apply the following general result to this model: Suppose that there exists $\hat{v} \in V$ with the following properties: (I) $B\hat{v} \leq \hat{v}$, (II) $v^* \leq \hat{v}$, (III) the pointwise limit $\tilde{v} \equiv \lim B^n \hat{v}$ is a solution of the Bellman equation (12),[2] and (IV) for any $\{x_t\} \in \Pi$, we have

$$\operatorname*{L^\star}_{T\uparrow\infty} \beta^T \hat{v}(x_T) \leq 0. \tag{45}$$

Then $\tilde{v} = v^*$.

This result can be derived from our results as follows. Conditions (I) and (II) imply $v^* \leq \tilde{v}$ since $v^* = B^n v^* \leq B^n \hat{v}$ for all $n \in \mathbb{N}$ by the monotonicity of B. Since v^* satisfies condition (a) of Theorem 1, it follows that $\tilde{v}$ satisfies condition (a). Since $\tilde{v} \leq \hat{v}$, condition (IV) implies condition (ii) of Corollary 4. Thus $\tilde{v} = v^*$ by Corollary 6.

Following Stokey and Lucas (1989, p. 95), we define $v : X \to [-\infty, \infty)$ by

$$v(x) = \frac{1}{1-\beta}\left[\ln(1-\alpha\beta) + \frac{\alpha\beta\ln(\alpha\beta)}{1-\alpha\beta}\right] + \frac{\alpha \ln x}{1-\alpha\beta}, \qquad x \in X. \tag{46}$$

[2] This limit is well defined since $\{B^n\hat{v}\}$ is a decreasing sequence by condition (I) above.

It can be directly shown that v solves the Bellman equation (12), as noted by Stokey and Lucas (1989, p. 95). They show that $v = v^*$ by verifying conditions (I)–(IV) above with $\tilde{v} = v$. Here we show the same conclusion more easily by applying Proposition 3.

Let $x_0 \in X$. Let $\{x_t\} \in \Pi(x_0)$. Note from (44) that $\ln x_t \leq \alpha \ln x_{t-1} \leq \alpha^t \ln x_0$ for all $t \in \mathbb{N}$. Thus for any $t \in \mathbb{Z}_+$ we have

$$u(x_t, x_{t+1}) = \ln[(x_t)^\alpha - x_{t+1}] \leq \alpha \ln x_t \leq \alpha^t \ln x_0 \leq \alpha^t |\ln x_0|. \tag{47}$$

The first inequality and (46) shows that condition (i) of Proposition 3 holds. For $t \in \mathbb{Z}_+$, let

$$\xi_t = \frac{\alpha^{t+1} |\ln x_0|}{1 - \alpha\beta}. \tag{48}$$

Since $0 < \alpha\beta < 1$, we have $\ln(1 - \alpha\beta) < 0$ and $\ln(\alpha\beta) < 0$. Thus it follows from (46) and (48) that condition (ii) of Proposition 3 holds. Now we have $v = v^*$ by Proposition 3.

4.5 A Discounted Problem with $\liminf \neq \limsup$

In this example we show that the definition of L affects the value function as well as optimal paths. We do not apply our results here.

Let $X = \mathbb{R}$, $\beta > 0$. Suppose that for all $x \in X$, we have $\Gamma(x) = \{-x/\beta\}$, $u(x, y) = x$.

Let $\{x_t\}_t \in \Pi(x_0)$. We have $x_t = (-\beta^{-1})^t x_0$, i.e., $\beta^t x_t = (-1)^t x_0$. Thus

$$\sum_{t=0}^{T} \beta^t u(x_t, x_{t+1}) = \sum_{t=0}^{T} \beta^t x_t = \begin{cases} x_0 & \text{for } T = 0, 2, 4, \ldots, \\ 0 & \text{for } T = 1, 3, 5, \ldots. \end{cases} \tag{49}$$

$$\liminf_{T \uparrow \infty} \sum_{t=0}^{T} \beta^t u(x_t, x_{t+1}) = \min\{x_0, 0\}, \tag{50}$$

$$\limsup_{T \uparrow \infty} \sum_{t=0}^{T} \beta^t u(x_t, x_{t+1}) = \max\{x_0, 0\}. \tag{51}$$

This gives

$$\begin{aligned} \underline{v}^*(x) &= \min\{x, 0\} \\ \overline{v}^*(x) &= \max\{x, 0\} \end{aligned}$$

5 Examples without Discounting

In this section we assume that $\beta = 1$. Undiscounted problems are common in the optimal growth literature, as well as in the literature on environmental issues concerning intergenerational distribution of resources.

5.1 A Trivial Example with $\liminf \neq \limsup$

Let $X = \{-1, 1\}$. Suppose that for all $x \in X$, we have

$$\Gamma(x) = \{-x\}, \qquad u(x, -x) = x. \tag{52}$$

Then for any $x_0 \in X$, there is only one feasible path from x_0, which is given by $x_t = (-1)^t x_0$. Thus for any $T \in \mathbb{Z}_+$ we have (49). The Bellman equation (12) reduces to

$$v(x) = x + v(-x). \tag{53}$$

For $\alpha \in \mathbb{R}$, let $v_\alpha(1) = \alpha$ and $v_\alpha(-1) = \alpha - 1$. Then v_α solves the Bellman equation for any $\alpha \in \mathbb{R}$; indeed, $v_\alpha(1) = 1 + v_\alpha(-1)$ and $v_\alpha(-1) = -1 + v_\alpha(1)$. Define $\underline{v}^*$ and $\overline{v}^*$ as in Section **??**.

Suppose that L = lim inf. Then condition (a) requires that $v_\alpha(1) = \alpha \geq 0$, while condition (b) requires that $v_\alpha(1) = \alpha \leq 0$. Hence $\alpha = 0$ and $\underline{v}^* = v_0$; i.e., $\underline{v}^*(1) = 0$ and $\underline{v}^*(-1) = -1$. This can be directly seen from (49).

Suppose that L = lim sup. Then condition (a) requires that $v_\alpha(-1) = \alpha - 1 \geq 0$, while condition (b) requires that $v_\alpha(-1) = \alpha - 1 \leq 0$. Hence $\alpha = 1$ and $v^* = v_1$; i.e., $v^*(1) = 1$ and $v^*(-1) = 0$. This also directly follows from (49).

5.2 Cake Eating

Consider Gale's (1967, p. 4) "cake eating" problem, which is an undiscounted version of the linear growth model in Section **??**. In particular, let $X = \mathbb{R}_+$ and $\beta = 1$. Suppose that for all $x \in X$, we have

$$\Gamma(x) = [0, x], \qquad \forall y \in \Gamma(x),\ u(x, y) = x - y. \tag{54}$$

For $\alpha \geq 0$, let $v_\alpha(x) = \alpha x$. From the argument of (42) and (43) it follows that v_1 solves the Bellman equation. As in the AK model with $\beta A = 1$, we cannot use Corollary 2, but we obtain $v_1 = v^*$ using Corollary 7.

Let $\alpha > 1$. Note that

$$u(x, y) + v_\alpha(y) = x - y + \alpha y = x + (\alpha - 1)y \leq \alpha x = v_\alpha(x). \tag{55}$$

Thus $v_\alpha(x) = \sup\{u(x, y) + v_\alpha(y)\}$, and the supremum is achieved at $y = x$. Therefore v_α solves the Bellman equation. Since $v^* = v_1$ as mentioned above, we already know that $v_\alpha \neq v^*$, but let us show this using Theorem 1 for illustration purposes. Since $v_\alpha \geq 0$, condition (a) trivially holds. Thus we need to show that condition (b) is violated. To this end, let $\{x_t\} \in \Pi$ with $x_0 > 0$. Let $T \in \mathbb{N}$. Note that

$$\sum_{t=0}^{T-1} u(x_t, x_{t+1}) = \sum_{t=0}^{T-1} (x_t - x_{t+1}) = x_0 - x_T. \tag{56}$$

We have

$$v_\alpha(x_0) - \sum_{t=0}^{T-1} u(x_t, x_{t+1}) - v_\alpha(x_T) = (\alpha - 1)(x_0 - x_T). \tag{57}$$

Let $\epsilon > 0$. Assume (16). Then $x_0 - x_T < \epsilon/(\alpha - 1)$, or $x_T > x_0 - \epsilon/(\alpha - 1)$ for all $T \in \mathbb{N}$. Thus for any $T \in \mathbb{N}$, we have $v_\alpha(x_T) > \alpha x_0 - \alpha\epsilon/(\alpha - 1)$, which is greater than ϵ when ϵ is sufficiently close to zero. This means that it is impossible to satisfy both (16) and (17) when ϵ sufficiently close to zero. Hence by Theorem 1, $v_\alpha \neq v^*$.

5.3 Cake Eating with Strictly Concave Utility

Let $X = \mathbb{R}_+$ and $\beta = 1$ again. Let $g : \mathbb{R}_+ \to \mathbb{R}_+$ be a strictly increasing, strictly concave, continuously differentiable function with $g(0) = 0$ and $\lim_{c\downarrow 0} g'(c) = 1$. Suppose that for all $x \in X$, we have

$$\Gamma(x) = [0, x], \qquad \forall y \in \Gamma(x),\ u(x, y) = g(x - y). \tag{58}$$

Gale (1967, p. 4) shows that there is no optimal path for this problem. Intuitively, by strict concavity of g, an optimal path must be such that consumption $(x_t - x_{t+1})$ is spread evenly over the infinite horizon, but then consumption never takes place, and such a path cannot be optimal.

Even in this pathological case, Theorem 1 can be used to identify the value function. To see this, for $\alpha \in [1, \infty)$ and $x \geq 0$, let $v_\alpha(x) = \alpha x$. For any $x \geq 0$ and $y \in \Gamma(x)$, we have

$$u(x, y) + v(y) = g(x - y) + \alpha y \leq \alpha x. \tag{59}$$

To see the inequality, note that $u(x, y)+v(y)$ is strictly increasing in $y \in [0, x]$ since $-g'(x-y)+\alpha > -g'(0) + \alpha \geq 0$ for all $y \in [0, x)$. It follows that $v_\alpha(x) = \sup_{y\in\Gamma(x)}\{g(x - y) + \alpha y\}$, and the supremum is achieved at $y = x$. Hence v_α solves the Bellman equation.

Let $\{x_t\} \in \Pi$ with $x_0 > 0$. We have

$$\sum_{t=0}^{T-1} u(x_t, x_{t+1}) = \sum_{t=0}^{T-1} g(x_t - x_{t+1}) \leq \sum_{t=0}^{T-1}(x_t - x_{t+1}) = x_0 - x_T, \tag{60}$$

where the inequality holds by concavity of g. It follows that

$$v_\alpha(x_0) - \sum_{t=0}^{T-1} u(x_t, x_{t+1}) - v_\alpha(x_T) \geq (\alpha - 1)(x_0 - x_T). \tag{61}$$

Hence, if $\alpha > 1$, then v_α violates condition (b) for ϵ sufficiently close to zero, as in the "cake eating" problem with linear utility discussed above.

Consider the case $\alpha = 1 = g'(0)$. Note that condition (a) trivially holds. To see that condition (b) also holds, let $x_0 \geq 0$. For $i \in \mathbb{N}$, define $\{x^i_t\}$ by $x^i_t = (1 - t/i)x_0$ for $t = 1, \dots, i$ and $x^i_t = 0$ for $t > i$. Let $T, i \in \mathbb{N}$ and $T_i = \min\{T, i\}$. Then

$$\sum_{t=0}^{T-1} u(x^i_t, x^i_{t+1}) = \sum_{t=0}^{T-1} g(x^i_t - x^i_{t+1}) = T_i g(x_0/i). \tag{62}$$

Since $x_T = (1 - T_i/i)x_0$, it follows that

$$v_\alpha(x_0) - \sum_{t=0}^{T-1} u(x_t^i, x_{t+1}^i) - v_\alpha(x_T^i) \tag{63}$$
$$= x_0 - x_T^i - T_i g(x_0/i) = T_i x_0/i - T_i g(x_0/i) \tag{64}$$
$$= T_i[x_0/i - g(x_0/i)] \le i[x_0/i - g(x_0/i)] \tag{65}$$
$$= x_0[1 - g(x_0/i)/(x_0/i)] \to 0 \quad \text{as } i \uparrow \infty. \tag{66}$$

Let $\epsilon > 0$. Then for i large enough, we have (16) by (63)–(66). Note that $\lim_{T\uparrow\infty} v_1(x_T^i) = 0$ for any $i \in \mathbb{N}$. Therefore condition (b) holds, and $v_1 = v^*$ by Theorem 1.

5.4 Optimal Growth without Discounting

Consider the undiscounted optimal growth model studied by Le Van and Dana (2003). In particular, assume the following: (H1) X is a compact, convex subset of $\mathbb{R}_+^n$ with $n \in \mathbb{N}$ such that X has nonempty interior and $0 \in X$; (H2) Γ is a continuous correspondence, and D is convex; (H3) for any $x, y, x', y' \in X$ such that $y \in \Gamma(x), x' \ge x$, and $y' \le y$, we have $y' \in \Gamma(x')$; (H4) there exist $(x, y) \in D$ with $x \ll y$; (H5) $F(x, y) : (x, y) \in D \mapsto \mathbb{R}$ is continuous, strictly concave, increasing in x, and decreasing in y.

Unless an additional assumption is introduced, the supremum

$$\sup_{\{x_t\}\in\Pi(x_0)} \mathop{\mathrm{L}}_{T\uparrow\infty} \sum_{t=0}^{T} F(x_t, x_{t+1}) \tag{67}$$

need not be finite, and (14) may fail. However, the problem can be normalized to satisfy (14). For this purpose, note from Le Van and Dana (2003, Propositions 7.2.1, 7.2.2) that there exists a unique $\overline{x} \in X$ such that

$$F(\overline{x}, \overline{x}) = \max\{F(x, x) : (x, x) \in D\}. \tag{68}$$

For $(x, y) \in D$, define

$$u(x, y) = F(x, y) - F(\overline{x}, \overline{x}). \tag{69}$$

Le Van and Dana (2003, Theorem 7.2.1) show that for any $x_0 \in X$, we have

$$v^*(x_0) = \sup_{\{x_t\}\in\Pi(x_0)} \mathop{\mathrm{L}}_{T\uparrow\infty} \sum_{t=0}^{T} u(x_t, x_{t+1}) < \infty. \tag{70}$$

Therefore Assumption 14 holds for this normalized problem. A feasible path in Π^0 in this context is known as a good programme.

Le Van and Dana (2003, Theorem 7.4.1) show that $v = v^*$ if and only if v is upper semicontinuous and satisfies (25) for any $\{x_t\} \in \Pi^0$. Since our approach does not make it easier to show that v^* is upper semicontinuous, we focus on the sufficiency part of this result. In particular, we show it as a simple consequence of Proposition 4. To this end, suppose that v is a solution of the Bellman equation such that it is upper semicontinuous and satisfies (25) for any $\{x_t\} \in \Pi^0$. Then condition (i) of Proposition 4 holds. Let $x_0 \in X$. Since v is upper semicontinuous and X is compact, there exists $\{x_t\} \in \Pi(x_0)$ satisfyinig (30); furthermore, $\max_{x\in X} v(x)$ exists and is finite, which implies (31) (with $\beta = 1$). Thus condition (ii) of Proposition 4 holds. Now $v = v^*$ by Proposition 4.

6 Appendix A Proof of Theorem 1

Lemma 6.1. *Let $\{a_t\}_{t=1}^{\infty}$ and $\{b_t\}_{t=1}^{\infty}$ be sequences in $[-\infty,\infty)$. Let $c \in (-\infty,\infty)$. Then*

$$\text{(i) } \forall t \in \mathbb{N},\ a_t + b_t \le c \quad \Longrightarrow \quad \text{(ii) } \mathop{\mathrm{L}}_{t\uparrow\infty} a_t \le c - \mathop{\mathrm{L}^\star}_{t\uparrow\infty} b_t, \tag{71}$$

$$\text{(i) } \forall t \in \mathbb{N},\ a_t + b_t \ge c \quad \Longrightarrow \quad \text{(ii) } \mathop{\mathrm{L}}_{t\uparrow\infty} a_t \ge c - \mathop{\mathrm{L}^\star}_{t\uparrow\infty} b_t. \tag{72}$$

Proof. To see (71), assume (71)(i). Let $t \in \mathbb{N}$. If $b_t > -\infty$, then

$$a_t \le c - b_t. \tag{73}$$

This inequality also holds if $b_t = -\infty$. Applying $\mathrm{L}_{t\uparrow\infty}$ to (73) and recalling (6) we obtain (71)(ii). We have shown (71). To see (72), assume (72)(i). Let $t \in \mathbb{N}$. By (72)(i) we have $b_t > -\infty$. Thus $a_t \ge c - b_t$. Applying $\mathrm{L}_{t\uparrow\infty}$ yields (72)(ii). □

Lemma 6.2. *For any $\{x_t\}_{t=0}^{\infty} \in \Pi$ and $T \in \mathbb{N}$, we have*

$$\mathop{\mathrm{L}}_{\tau\uparrow\infty} \sum_{t=0}^{\tau} \beta^t u(x_t, x_{t+1}) = \sum_{t=0}^{T-1} \beta^t u(x_t, x_{t+1}) + \mathop{\mathrm{L}}_{\tau\uparrow\infty} \sum_{t=T}^{\tau} \beta^t u(x_t, x_{t+1}), \tag{74}$$

$$\mathop{\mathrm{L}}_{\tau\uparrow\infty} \sum_{t=T}^{\tau} \beta^t u(x_t, x_{t+1}) \le \beta^T v^*(x_T). \tag{75}$$

Proof. Let $\{x_t\} \in \Pi$ and $T \in \mathbb{N}$. Note that for any $\tau \ge T$, we have

$$\sum_{t=0}^{\tau} \beta^t u(x_t, x_{t+1}) = \sum_{t=0}^{T-1} \beta^t u(x_t, x_{t+1}) + \sum_{t=T}^{\tau} \beta^t u(x_t, x_{t+1}). \tag{76}$$

Applying $\mathrm{L}_{\tau\uparrow\infty}$ to both sides yields (74). To see (75), note that

$$\mathop{\mathrm{L}}_{\tau\uparrow\infty} \sum_{t=T}^{\tau} \beta^t u(x_t, x_{t+1}) = \beta^T \mathop{\mathrm{L}}_{I\uparrow\infty} \sum_{i=0}^{I} \beta^i u(x_{T+i}, x_{T+i+1}). \tag{77}$$

Since $\{x_{T+i}\}_{i=1}^{\infty} \in \Pi(x_T)$, we obtain (75). □

6.1 Sufficiency

Throughout the proof of the sufficiency part, we let $v \in V$ be a solution of the Bellman equation (12). Since $u : D \to [-\infty, \infty)$ and $v \in V$, we have

$$\forall j \in \mathbb{Z}_+, \qquad \sum_{t=0}^{j} \beta^t u(x_t, x_{t+1}) < \infty, \quad v(x_j) < \infty. \tag{78}$$

Lemma 6.3. *For any $\{x_t\} \in \Pi$, we have*

$$\forall T \in \mathbb{N}, \qquad v(x_0) \geq \sum_{t=0}^{T-1} \beta^t u(x_t, x_{t+1}) + \beta^T v(x_T). \tag{79}$$

Proof. Since v solves the Bellman equation (12), for any $t \in \mathbb{Z}_+$ we have

$$v(x_t) = \sup_{y \in \Gamma(x_t)} \{u(x_t, y) + \beta v(y)\} \geq u(x_t, x_{t+1}) + \beta v(x_{t+1}). \tag{80}$$

By repeated application of this inequality, $v(x_0) \geq u(x_0, x_1) + \beta v(x_1) \geq u(x_0, x_1) + \beta u(x_1, x_2) + \beta^2 v(x_2) \geq \cdots$. Thus (79) follows. $\square$

Lemma 6.4. *Let $\{x_t\} \in \Pi^0$. Suppose that*

$$\exists j \in \mathbb{N}, \qquad v(x_j) > -\infty. \tag{81}$$

Then $-\infty < v(x_0) < \infty$.

Proof. Let $\{x_t\} \in \Pi^0$. Let $j \in \mathbb{N}$ be such that $v(x_j) > -\infty$. Since $\{x_t\} \in \Pi^0$, we have

$$\sum_{t=0}^{j-1} \beta^t u(x_t, x_{t+1}) > -\infty. \tag{82}$$

From (79) with $T = j$, the above inequality, and (81), we obtain $v(x_0) > -\infty$. Since $v \in V$, we also have $v(x_0) < \infty$. $\square$

Lemma 6.5. *Let $\{x_t\} \in \Pi^0$ satisfy (81). Then*

$$\underset{T\uparrow\infty}{\mathrm{L}} \sum_{t=0}^{T-1} \beta^t u(x_t, x_{t+1}) \leq v(x_0) - \underset{T\uparrow\infty}{\mathrm{L}^\star} \beta^T v(x_T). \tag{83}$$

Proof. Let $\{x_t\} \in \Pi^0$ satisfy (81). Then (83) follows from (79), (78), and Lemmas 6.4 and 6.1. $\square$

Lemma 6.6. *If condition (a) of Theorem 1 holds, then $v^* \leq v$.*

Proof. Let $x_0 \in X$. If $x_0 \notin X^0$, then $v^*(x_0) = -\infty \leq v(x_0)$. Suppose that $x_0 \in X^0$. Let $\{x_t\} \in \Pi^0(x_0)$. Note that (15) implies (81). By Lemma 6.5 and (15), we have

$$\underset{T\uparrow\infty}{\mathrm{L}} \sum_{t=0}^{T-1} \beta^t u(x_t, x_{t+1}) \leq v(x_0) - \underset{T\uparrow\infty}{\mathrm{L}^\star} \beta^T v(x_T) \leq v(x_0). \tag{84}$$

Applying $\sup_{\{x_t\}\in\Pi^0(x_0)}$ to the leftmost side and recalling (11), we obtain $v^*(x_0) \leq v(x_0)$ again. Since this is true for any $x_0 \in X$, we have $v^* \leq v$. □

Lemma 6.7. *If condition (b) of Theorem 1 holds, then $v \leq v^*$.*

Proof. Let $\epsilon > 0$. Let $\{x_t\} \in \Pi(x_0)$ satisfy (16) and (17). If $v(x_0) = -\infty$, then we trivially have $v(x_0) \leq v^*(x_0)$. Suppose that $v(x_0) > -\infty$. Then from (16), (78), and Lemma 6.1, we have

$$v(x_0) - \underset{T\uparrow\infty}{\mathrm{L}^\star} \beta^T v(x_T) \leq \underset{T\uparrow\infty}{\mathrm{L}} \sum_{t=0}^{T-1} \beta^t u(x_t, x_{t+1}) + \epsilon \tag{85}$$

$$\leq v^*(x_0) + \epsilon, \tag{86}$$

where the second inequality uses the definition of v^*.

We claim that

$$-\infty < \underset{T\uparrow\infty}{\mathrm{L}^\star} \beta^T v(x_T) \leq \epsilon. \tag{87}$$

The first inequality follows from (85) and (86) since $v^* \in V$ and thus $v^*(x_0) < \infty$. The second inequality in (87) follows from (17).

Now from (85)–(87) we have

$$v(x_0) \leq v^*(x_0) + \epsilon + \underset{T\uparrow\infty}{\mathrm{L}^\star} \beta^T v(x_T) \leq v^*(x_0) + 2\epsilon. \tag{88}$$

Hence $v(x_0) \leq v^*(x_0) + 2\epsilon$. Since ϵ was arbitrary, we have $v(x_0) \leq v^*(x_0)$.

We have shown that $v(x_0) \leq v^*(x_0)$ for all $x_0 \in X$. Hence $v \leq v^*$. □

To complete the proof of the sufficiency part, suppose that v satisfies conditions (a) and (b) of Theorem 1. Then by Lemmas 6.6 and 6.7, we obtain $v^* \leq v$ and $v \leq v^*$, respectively. It follows that $v = v^*$.

6.2 Necessity

Lemma 6.8. *Let $\{x_t\}_{t=0}^\infty \in \Pi^0$. Then*

$$\forall j \in \mathbb{Z}_+, \qquad -\infty < v^*(x_j) < \infty. \tag{89}$$

Proof. Let $j \in \mathbb{Z}_+$. Since $\{x_t\}_{t=0}^\infty \in \Pi^0$, we have

$$-\infty < \operatorname*{L}_{\tau\uparrow\infty} \sum_{t=0}^{\tau} \beta^t u(x_t, x_{t+1}) \tag{90}$$

$$= \sum_{t=0}^{j-1} \beta^t u(x_t, x_{t+1}) + \operatorname*{L}_{\tau\uparrow\infty} \sum_{t=j}^{\tau} \beta^t u(x_t, x_{t+1}), \tag{91}$$

where the equality uses Lemma 6.2. Since $\sum_{t=0}^{j-1} \beta^t u(x_t, x_{t+1}) < \infty$, it follows that

$$-\infty < \operatorname*{L}_{\tau\uparrow\infty} \sum_{t=j}^{\tau} \beta^t u(x_t, x_{t+1}) \tag{92}$$

$$= \beta^j \operatorname*{L}_{I\uparrow\infty} \sum_{i=0}^{I} \beta^i u(x_{j+i}, x_{j+i+1}) \leq \beta^j v^*(x_j), \tag{93}$$

where the last inequality holds since $\{x_{j+i}\}_{i=1}^\infty \in \Pi(x_j)$. It follows that $-\infty < v^*(x_j)$. We have $v^*(x_j) < \infty$ since $v^* \in V$. □

Lemma 6.9. *Let $x_0 \in X^0$. For any $\epsilon > 0$, there exists $\{x_t\} \in \Pi^0(x_0)$ such that*

$$\forall T \in \mathbb{N}, \qquad v^*(x_0) \leq \sum_{t=0}^{T-1} \beta^t u(x_t, x_{t+1}) + \beta^T v^*(x_T) + \epsilon, \tag{94}$$

$$\operatorname*{L^\star}_{T\uparrow\infty} \beta^T v^*(x_T) \leq \epsilon. \tag{95}$$

Proof. Let $\epsilon > 0$. Let $\{x_t\} \in \Pi^0(x_0)$ be such that

$$v^*(x_0) \leq \operatorname*{L}_{\tau\uparrow\infty} \sum_{t=0}^{\tau} \beta^t u(x_t, x_{t+1}) + \epsilon. \tag{96}$$

Such $\{x_t\}$ exists by (11). By Lemma 6.2, for any $T \in \mathbb{N}$ we have

$$\operatorname*{L}_{\tau\uparrow\infty} \sum_{t=0}^{\tau} \beta^t u(x_t, x_{t+1}) \leq \sum_{t=0}^{T-1} \beta^t u(x_t, x_{t+1}) + \beta^T v^*(x_T). \tag{97}$$

Now (94) follows from (96) and (97).

Let $T \in \mathbb{N}$. Note from Lemma 2.1 that v^* solves the Bellman equation (12). Since $x_0 \in X^0$, by Lemma 6.8 we have (89), which implies (81). Thus by Lemma 6.5 we have

$$\operatorname*{L}_{T\uparrow\infty} \sum_{t=0}^{T-1} \beta^t u(x_t, x_{t+1}) \leq v^*(x_0) - \operatorname*{L^\star}_{T\uparrow\infty} \beta^T v^*(x_T). \tag{98}$$

From this and (96) we obtain

$$v^*(x_0) - \epsilon \leq v^*(x_0) - \operatorname*{L^\star}_{T\uparrow\infty} \beta^T v^*(x_T). \tag{99}$$

It follows by (89) that

$$-\epsilon \leq - \operatorname*{L^{\star}}_{T\uparrow\infty} \beta^T v^*(x_T), \tag{100}$$

which is equivalent to (95). $\square$

Lemma 6.10. *Let $x_0 \in X \setminus X^0$. For any $\epsilon > 0$, there exists $\{x_t\} \in \Pi(x_0)$ satisfying (94) and (95).*

Proof. Let $\epsilon > 0$. Since $v^*(x_0) = -\infty$, we have (94) for *any* $\{x_t\} \in \Pi(x_0)$. Thus it suffices to show that there exists $\{x_t\} \in \Pi(x_0)$ satisfying (95). To this end, let $\{x_t\} \in \Pi(x_0)$. If $x_t \notin X^0$ for all $t \in \mathbb{N}$, then $v^*(x_t) = -\infty$ for all $t \in \mathbb{N}$; thus (95) trivially holds. Suppose that $x_\tau \in X^0$ for some $\tau \in \mathbb{N}$. Then by Lemma 6.9, there exists $\{z_j\}_{j=1}^\infty \in \Pi^0(x_\tau)$ such that

$$\operatorname*{L^{\star}}_{J\uparrow\infty} \beta^J v^*(z_J) < \beta^{-\tau}\epsilon. \tag{101}$$

For $t \in \mathbb{Z}_+$, let

$$x'_t = \begin{cases} x_t & \text{for } t \leq \tau, \\ z_{t-\tau} & \text{for } t \geq \tau + 1. \end{cases} \tag{102}$$

Then $\{x'_t\} \in \Pi(x_0)$ and

$$\operatorname*{L^{\star}}_{T\uparrow\infty} \beta^T v^*(x'_T) = \beta^\tau \operatorname*{L^{\star}}_{J\uparrow\infty} \beta^J v^*(z_J) < \epsilon, \tag{103}$$

where the inequality uses (101). The proof is now complete. $\square$

Lemma 6.11. *For any $\{x_t\} \in \Pi^0$, we have*

$$\operatorname*{L^{\star}}_{T\uparrow\infty} \beta^T v^*(x_T) \geq 0. \tag{104}$$

Proof. Let $\{x_t\} \in \Pi^0(x_0)$. Let $\delta \geq 0$ be such that

$$v^*(x_0) = \operatorname*{L}_{\tau\uparrow\infty} \sum_{t=0}^{\tau} \beta^t u(x_t, x_{t+1}) + \delta. \tag{105}$$

Let $T \in \mathbb{N}$. From (105) and Lemma 6.2 we have

$$v^*(x_0) \leq \sum_{t=0}^{T-1} \beta^t u(x_t, x_{t+1}) + \beta^T v^*(x_T) + \delta. \tag{106}$$

By Lemma 6.8, both $v^*(x_0)$ and $v^*(x_T)$ are finite. Thus from (106), (78), and Lemma 6.1, we have

$$v^*(x_0) - \operatorname*{L^{\star}}_{T\uparrow\infty} \beta^T v^*(x_T) \leq \operatorname*{L}_{T\uparrow\infty} \sum_{t=0}^{T-1} \beta^t u(x_t, x_{t+1}) + \delta = v^*(x_0), \tag{107}$$

where the equality uses (105). Recalling that $v^*(x_0)$ is finite, we see from (107) that

$$-\underset{T\uparrow\infty}{\mathrm{L}^\star}\, \beta^T v^*(x_T) \leq 0, \tag{108}$$

which is equivalent to (104). □

To complete the proof of the necessity part, it suffices to notice that v^* satisfies conditions (a) and (b) of Theorem 1 with $v = v^*$. Indeed, condition (a) follows from Lemma 6.11, while condition (b) follows from Lemmas 6.9 and 6.10.

7 Appendix B Proofs of Propositions

7.1 Proof of Proposition 3

Assume conditions (i) and (ii). Fix $x_0 \in X$. Let $\{\xi_t\}$ be given by condition (ii). Let $m = \sum_{t=0}^{\infty} \beta^t \xi_t$. Let $\{x_t\} \in \Pi(x_0)$. Since $\beta^T v(x_T) \leq \beta^T \xi_T$ for all $T \in \mathbb{Z}_+$, we have

$$\limsup_{T\uparrow\infty} \beta^T v(x_T) \leq 0. \tag{109}$$

For $r \in \mathbb{R}$, define

$$H_r^+ = \{t \in \mathbb{Z}_+ : \beta^t v(x_t) \geq r\}, \tag{110}$$

$$H_r^- = \{t \in \mathbb{Z}_+ : \beta^t v(x_t) < r\}. \tag{111}$$

Let $T \in \mathbb{N}$ and $r < 0$. Note from condition (ii) that

$$\sum_{t\in H_r^+ : t<T} \beta^t v(x_t) \leq m. \tag{112}$$

It follows from (28) that

$$\begin{aligned}
\sum_{t=0}^{T} \beta^t [u(x_t, x_{t+1}) - \eta]/\theta &\leq \sum_{t=0}^{T} \beta^t v(x_t) && (113)\\
&= \sum_{t\in H_r^- : t\leq T} \beta^t v(x_t) + \sum_{t\in H_r^+ : t\leq T} \beta^t v(x_t) && (114)\\
&\leq \sum_{t\in H_r^- : t\leq T} \beta^t v(x_t) + m, && (115)
\end{aligned}$$

where the last inequality uses (112). Suppose that $\{x_t\} \in \Pi^0(x_0)$. Then from (113)–(115) we have

$$-\infty < \sum_{t=0}^{T} \beta^t [u(x_t, x_{t+1}) - \eta]/\theta \leq \sum_{t\in H_r^- : t\leq T} \beta^t v(x_t) + m. \tag{116}$$

It follows that

$$\sum_{t \in H_r^- : t \leq T} \beta^t v(x_t) > -\infty. \tag{117}$$

This implies that $\beta^t v(x_t) < r$ only finitely many times (otherwise the above sum would be $-\infty$); thus

$$\liminf_{T \uparrow \infty} \beta^T v(x_T) \geq r. \tag{118}$$

Since $r < 0$ was arbitrary, it follows that

$$\liminf_{T \uparrow \infty} \beta^T v(x_T) \geq 0. \tag{119}$$

We have shown that (119) holds for any $\{x_t\} \in \Pi^0(x_0)$, and that (109) holds for any $\{x_t\} \in \Pi(x_0)$. Hence condition (a) of Theorem 1 and condition (ii) of Corollary 4 hold. Now we have $v = v^*$ by Corollary 6.

7.2 Proof of Proposition 4

Suppose that any $\{x_t\} \in \Pi^0$ satisfies (25). This implies condition (a) of Theorem 1. Thus to conclude that $v = v^*$, it suffices to show condition (ii) of Corollary 7. To this end, let $x_0 \in X$. Let $\{x_t\} \in \Pi(x_0)$ satisfy (30) and (31). To show condition (ii) of Corollary 7, it suffices to verify (26). If $v(x_t) = -\infty$ for all $t \in \mathbb{Z}_+$, then (26) trivially holds. Suppose that $v(x_t) > -\infty$ for some $t \in \mathbb{Z}_+$. We assume that $t = 0$ without loss of generality. By (30) we have

$$v(x_0) = \sum_{t=0}^{T-1} \beta^t u(x_t, x_{t+1}) + \beta^T v(x_T). \tag{120}$$

From (120) and Lemma 6.1 we have

$$\mathop{\mathrm{L}}_{T \uparrow \infty} \sum_{t=0}^{T-1} \beta^t u(x_t, x_{t+1}) = v(x_0) - \mathop{\mathrm{L}^\star}_{T \uparrow \infty} \beta^T v(x_T) > -\infty, \tag{121}$$

where the inequality holds in (121) holds by (31) since $v(x_0) > -\infty$. It follows from (121) that $\{x_t\} \in \Pi^0$, and thus (25) holds, which implies (26).

References

Brock, W.A., 1969, On existence of weakly maximal programmes in a multi-sector economy, Review of Economic Studies 37, 275–280.

Dunford, N., Schwartz, J.T., 1988, Linear Operators, Part I: General Theory, John Wiley & Sons, New York.

Gale, D., 1967, On optimal development in a multi-sector economy, Review of Economic Studies 32, 1–18.

Kamihigashi, T., 2008, On the principle of optimality for nonstationary deterministic dynamic programming, International Journal of Economic Theory 4, 519–525.

Kamihigashi, T., 2012, Existence and uniqueness of a fixed point for the Bellman operator in deterministic dynamic programming, RIEB Discussion Paper DP2012-05.

Le Van, C., Dana, R.-A., 2003, Dynamic Programming in Economics, Kluwer Academic Publishers, Boston.

Le Van, C., Morhaim, L, 2002, Optimal growth models with bounded or unbounded returns: a unifying approach, Journal of Economic Theory 105, 158–187.

Le Van, C., Vailakis, Y., 2011, Monotone concave (convex) operators: applications to stochastic dynamic programming with unbounded returns, memeo, University of Paris 1 and University of Exeter Business School.

Martins-da-Rocha, V.F., Vailakis, Y., 2010, Existence and uniqueness of a fixed point for local contractions, Econometrica 78, 1127–1141.

Rinćon-Zapatero, J.P., Rodríguez-Palmero, C., 2003, Existence and uniqueness of solutions to the Bellman equation in the unbounded case, Econometrica 71, 1519–1555.

Rinćon-Zapatero, J.P., Rodríguez-Palmero, C., 2009, Corrigendum to "Existence and uniqueness of solutions to the Bellman equation in the unbounded case" Econometrica, Vol. 71, No. 5 (September, 2003), 1519–1555, Econometrica 77, 317–318.

Stokey, N., Lucas, R.E., Jr., 1989, Recursive Methods in Economic Dynamics. Harvard University Press, Cambridge, MA.