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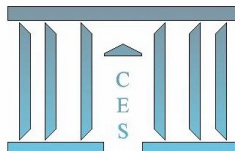
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**Price revelation and existence of financial equilibrium  
with incomplete markets and private beliefs**

Lionel De BOISDEFFRE

**2015.37**



PRICE REVELATION AND EXISTENCE OF FINANCIAL EQUILIBRIUM  
WITH INCOMPLETE MARKETS AND PRIVATE BELIEFS

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***Abstract***

*We consider a pure exchange financial economy, where rational agents, possibly asymmetrically informed, forecast prices privately, with no model of how they are determined. Therefore, agents face both ‘exogenous uncertainty’, on the future state of nature, and ‘endogenous uncertainty’, on the future price. At a sequential equilibrium, all consumers expect the ‘true’ price as a possible outcome and elect optimal strategies at the first period, which clear on all markets, ex post. The paper’s purpose is twofold. First, it defines no-arbitrage prices, which comprise all equilibrium prices, and displays their revealing properties. Second, it shows, under mild conditions, that a sequential equilibrium always exists in this model, whatever agents’ prior beliefs or the financial structure. This outcome suggests that standard existence problems, which followed Hart (1975) and Radner (1979), stem from the rational expectation and perfect foresight assumptions of the classical model.*

**Key words:** sequential equilibrium, temporary equilibrium, perfect foresight, existence, rational expectations, financial markets, asymmetric information, arbitrage.

**JEL Classification:** D52

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# 1 Introduction

The traditional approach to sequential financial equilibrium relies on Radner's (1972-1979) classical, but restrictive, assumptions that agents have the so-called 'rational expectations' of private information signals, and 'perfect foresight' of future prices. Along the former assumption, agents are endowed, quoting Radner, with 'a model' of how equilibrium prices are determined and (possibly) infer private information of other agents from comparing actual prices and price expectations with theoretical values at a price revealing equilibrium. Along the latter assumption, agents anticipate with certainty exactly one price for each commodity (or asset) in each prospective state, which turns out to be the true price if that state prevails. Both assumptions presume much of agents' inference and computational capacities. Both assumptions lead to classical cases of inexistence of equilibrium, as shown by Radner (1979), Hart (1975), Momi (2000), Busch-Govindan (2004), among others.

Under standard regularity conditions and with real assets, the perfect foresight equilibrium is generically locally unique or determinate, as shown by Geanakoplos and Polemarchakis (1986). Thus, with two periods, agents knowing all the primitives of the economy and endowed with sufficient computational capacities, could typically identify prices contingent on each future state, from observing first period prices (provided they be equilibrium prices), and select the corresponding anticipations, as in the Radner classical model.

Making such inferences is yet highly hypothetical and builds on equilibrium being locally unique or determinate. With private beliefs, we argue in Section 4, this outcome no longer holds. Circumventing these inferences, the temporary equilibrium literature, developed by Green (1973) or Grandmont (1977), among others,

drops such anticipation assumptions, but at the cost of losing agents' coordination across periods. In this literature, as time unfolds, agents would typically revise their plans and beliefs, ex post, face bankruptcy, meet unanticipated prices or welfare increasing retrade opportunities. These outcomes are ruled out by the classical sequential equilibrium model, as though there were a tradeoff between making extreme anticipation assumptions or losing coordination across time. As Grandmont (1982) noticed, the temporary and perfect foresight equilibrium literatures followed separate paths, but stood for the two classical streams of general equilibrium theory.

Our belief is that coordination across periods can obtain under much weaker anticipation assumptions than perfect foresight. Pursuing earlier work with Bernard Cornet, we now propose a setting, where rational agents, possibly asymmetrically informed, form their anticipations privately, with no price model, may update their beliefs from observing market prices, and reach equilibrium with correct forecasts.

Dropping both assumptions of rational expectations and perfect foresight, this setting proposes to bridge a link between the two classical concepts of equilibrium. It also improves the existence properties of the standard sequential equilibrium model. Whatever the financial structure or agents' prior beliefs, we show hereafter that a sequential equilibrium exists in our model, as long as consumers take into account the additional uncertainty stemming from their unawareness of other agents' beliefs.

The current model extends one with B. Cornet [4], which dropped rational expectations only. The latter model provided the basic tools, concepts and properties for an arbitrage theory, embedding jointly the symmetric and asymmetric information cases. It turned out to solve the existence problems due to asymmetric information pointed out by Radner (1979). Indeed, we proved in [6] that a financial equilibrium with nominal assets existed in this model, not only generically - as in Radner's

(1979) rational expectations model - but under the very same no-arbitrage condition, with symmetric and asymmetric information, that is, under the generalized no-arbitrage condition introduced in [4]. This result was consistent and extended David Cass' (1984) standard existence theorem to asymmetric information.

Complementing this result in [5], we showed with Bernard Cornet how asymmetrically informed agents, endowed with no expectations a la Radner (1979), could refine their information from observing prices. This former model, however, did not explain how agents reached perfect forecasts of spot prices in realizable states, while they had private anticipations in idiosyncratic states and no Radner price model.

In the current paper, we address this issue and extend our earlier model to all financial structures. As anticipations are now private, agents would no longer be certain which price might prevail tomorrow. Equilibrium prices would typically depend on all agents' private forecasts today. Hence, they would face an additional '*endogenous uncertainty*', referring to the endogenous price variables.

This double uncertainty is encapsured in a two-period pure exchange economy, where agents, possibly asymmetrically informed, face exogenous uncertainty, represented by finitely many states of nature, exchange consumption goods on spot markets, and - nominal or real - assets on financial markets, but also face endogenous uncertainty on prices, in each state they expect. They have private sets of state and price forecasts, distributed along idiosyncratic probability laws, called beliefs.

The current model's equilibrium, or '*correct foresight equilibrium*' (C.F.E.), is reached when all agents, today, anticipate tomorrow's '*true*' price as a possible outcome, and elect optimal strategies, which clear on all markets at both time periods. This equilibrium concept is, indeed, a *sequential* one. It differs from the traditional

*temporary* equilibrium notion, introduced by Hicks (1939) and developed, later, by Grandmont (1977, 1982), Green (1973), Hammond (1983), Balasko (2003), among others. Such typical outcomes of the temporary equilibrium as bankruptcy or a welfare increasing retrade opportunity, ex post, are inconsistent with our concept.

After presenting the model, we propose a notion of no-arbitrage prices, which always exist, encompass equilibrium prices, and may reveal information to agents having no clue of how market prices are determined. We show that any agent can infer enough information from no-arbitrage prices to free markets from arbitrage.

Next, we study the existence issue, and suggest how the correct foresight equilibrium might solve the classical problems, which followed, not only Radner's (1979) rational expectations equilibrium (as we had already shown in [6]), but also Hart (1975), Momi (2001), Busch-Govindan (2004), among others. Namely, we prove that a C.F.E. exists whenever agents' anticipations embed a so-called '*minimum uncertainty set*', corresponding to the incompressible uncertainty which may remain in a private belief economy. Then, equilibrium prices always exist, and reveal to rational agents, whenever required, their own sets of anticipations at equilibrium.

The paper is organized as follows: we present the model, in Section 2, the concept of no-arbitrage prices and the information they reveal, in Section 3, the minimum uncertainty set and the existence Theorem, in Section 4. We prove this theorem, in Section 5, differing to an Appendix the proof of technical Lemmas.

## 2 The basic model

We consider a pure-exchange economy with two periods ( $t \in \{0, 1\}$ ), a commodity market and a financial market, where agents (at  $t = 0$ ) may be asymmetrically

informed and face an endogenous uncertainty on future prices. The sets of agents,  $I := \{1, \dots, m\}$ , commodities,  $\mathcal{L} := \{1, \dots, L\}$ , states of nature,  $S := \{1, \dots, T\}$ , and assets,  $\mathcal{J} := \{1, \dots, J\}$ , are all finite (i.e.,  $(m, L, T, J) \in \mathbb{N}^4$ ).

## 2.1 The model's notations

Throughout, we denote by  $\cdot$  the scalar product and  $\|\cdot\|$  the Euclidean norm on an Euclidean space and by  $\mathcal{B}(K)$  the Borel sigma-algebra of a topological set,  $K$ . We let  $s = 0$  be the non-random state at  $t = 0$  and  $S' := \{0\} \cup S$ . For all set  $\Sigma \subset S'$  and tuple  $(s, l, x, x', y, y') \in \Sigma \times \mathcal{L} \times \mathbb{R}^\Sigma \times \mathbb{R}^\Sigma \times \mathbb{R}^{L\Sigma} \times \mathbb{R}^{L\Sigma}$ , we denote by:

- $x_s \in \mathbb{R}$ ,  $y_s \in \mathbb{R}^L$  the scalar and vector, indexed by  $s \in \Sigma$ , of  $x$ ,  $y$ , respectively;
- $y_s^l$  the  $l^{\text{th}}$  component of  $y_s \in \mathbb{R}^L$ ;
- $x \leq x'$  and  $y \leq y'$  (respectively,  $x \ll x'$  and  $y \ll y'$ ) the relations  $x_s \leq x'_s$  and  $y_s^l \leq y_s'^l$  (resp.,  $x_s < x'_s$  and  $y_s^l < y_s'^l$ ) for each  $(l, s) \in \{1, \dots, L\} \times \Sigma$ ;
- $x < x'$  (resp.,  $y < y'$ ) the joint relations  $x \leq x'$ ,  $x \neq x'$  (resp.,  $y \leq y'$ ,  $y \neq y'$ );
- $\mathbb{R}_+^{L\Sigma} = \{x \in \mathbb{R}^{L\Sigma} : x \geq 0\}$  and  $\mathbb{R}_+^\Sigma := \{x \in \mathbb{R}^\Sigma : x \geq 0\}$ ,  
 $\mathbb{R}_{++}^{L\Sigma} := \{x \in \mathbb{R}^{L\Sigma} : x \gg 0\}$  and  $\mathbb{R}_{++}^\Sigma := \{x \in \mathbb{R}^\Sigma : x \gg 0\}$ ;
- $\mathcal{M}_0 := \{(p_0, q) \in \mathbb{R}_+^L \times \mathbb{R}^J : \|p_0\| + \|q\| = 1\}$ ;
- $\mathcal{M}_s := \{(s, p) \in S \times \mathbb{R}_+^L : \|p\| = 1\}$ , for every  $s \in S$ ;
- $\mathcal{M} := \cup_{s \in S} \mathcal{M}_s$ , a topological subset of the Euclidean space  $\mathbb{R}^{L+1}$ ;
- $B(\omega, \varepsilon) := \{\omega' \in \mathcal{M} : \|\omega' - \omega\| < \varepsilon\}$ , for every pair  $(\omega, \varepsilon) \in \mathcal{M} \times \mathbb{R}_{++}$ ;
- $P(\pi) := \{\omega \in \mathcal{M} : \pi(B(\omega, \varepsilon)) > 0, \forall \varepsilon > 0\}$ , the support of a probability,  $\pi$ , on  $(\mathcal{M}, \mathcal{B}(\mathcal{M}))$ ;
- $\pi(P)$ , for any closed set,  $P \subset \mathcal{M}$ , the set of probabilities on  $(\mathcal{M}, \mathcal{B}(\mathcal{M}))$ , whose support (as defined above) is  $P$ .



## 2.2 The commodity and asset markets

The  $L$  consumption goods,  $l \in \mathcal{L}$ , may be exchanged by consumers, on the spot markets of both periods. In each state,  $s \in S$ , an expectation of a spot price,  $p \in \mathbb{R}_+^L$ , or the spot price,  $p$ , in state  $s$  itself, are denoted by the pair  $\omega_s := (s, p) \in S \times \mathbb{R}_+^L$ . Since we are only concerned about relative prices on each spot market, we will take admissible prices and price forecasts into the set  $\mathcal{M}$ .

Each agent,  $i \in I$ , is granted an endowment,  $e_i := (e_{is}) \in \mathbb{R}_+^{LS'}$ , which secures her the commodity bundle,  $e_{i0} \in \mathbb{R}_+^L$  at  $t = 0$ , and  $e_{is} \in \mathbb{R}_+^L$ , in each state  $s \in S$ , if this state prevails at  $t = 1$ . To harmonize notations, for every triple  $(i, s, \omega) \in I \times S' \times \mathcal{M}_s$ , we will also refer to  $e_{i\omega} := e_{is}$ . Ex post, the generic  $i^{\text{th}}$  agent's welfare is measured by a continuous utility index,  $u_i : \mathbb{R}_+^{2L} \rightarrow \mathbb{R}_+$ , over her consumptions at both dates.

The financial market permits limited transfers across periods and states, via  $J$  assets, or securities,  $j \in \mathcal{J} := \{1, \dots, J\}$ , which are exchanged at  $t = 0$  and pay off at  $t = 1$ . In any contingent state, assets pay off in a fixed amount of account units and/or commodities. For any forecast  $\omega \in \mathcal{M}$ , the cash payoffs,  $v_j(\omega) \in \mathbb{R}$ , of all assets,  $j \in \{1, \dots, J\}$ , conditional on the occurrence of (state and) price  $\omega$ , define a row vector,  $V(\omega) = (v_j(\omega)) \in \mathbb{R}^J$ . For example, given  $\omega := (s, p) \in \mathcal{M}$ , if asset  $j \in \mathcal{J}$  promises to pay exactly the commodity bundle  $v_j^s \in \mathbb{R}_+^L$  in a state  $s \in S$ , then,  $v_j(\omega) = p \cdot v_j^s$ . This specification makes the mapping  $\omega \mapsto V(\omega)$  continuous. The market may be incomplete, in the sense that the span of payoffs,  $\{(V(\omega_s) \cdot z)_{s \in S} : z \in \mathbb{R}^J\}$  may have lower rank (for any price collection  $(\omega_s) \in \prod_{s \in S} \mathcal{M}_s$ ) than the number of states,  $\#S$ .

As we show later, equilibrium with private beliefs is consistent with agents having uncountable sets of anticipations in any state. Insuring the price risk completely, via contingent contracts, if possible, would not be far less demanding than setting

the complete market of contingent goods in the Debreu model. In the end, it is likely that the contingent contracts on goods would be issued and preferred by agents to the contingent contracts on prices only. Why, indeed, would it be possible to insure today the price of a desired quantity of one good on a particular spot market tomorrow, and not be able to exchange the corresponding contingent good directly today? Thus, we would resume the Debreu model, which is inconsistent with incomplete markets. To simplify exposition, we have, therefore, assumed that assets provided no insurance against endogenous uncertainty.

Agents can take unrestrained positions (positive, if purchased; negative, if sold), in each security, which are the components of a portfolio,  $z \in \mathbb{R}^J$ . Given an asset price,  $q \in \mathbb{R}^J$ , a portfolio,  $z \in \mathbb{R}^J$ , is thus a contract, which costs  $q \cdot z$  units of account at  $t = 0$ , and promises to pay  $V(\omega) \cdot z$  units tomorrow, for each expectation  $\omega \in \mathcal{M}$ , if  $\omega$  obtains. Similarly, we normalize first period prices,  $\omega_0 := (p_0, q)$ , to the set  $\mathcal{M}_0$ .

### 2.3 Information and beliefs

Ex ante, the generic agent,  $i \in I$ , is endowed with a private idiosyncratic set of anticipations,  $P_i \subset \mathcal{M}$ , according to which she believes tomorrow's true state and price (i.e., which will prevail at  $t = 1$ ) will fall into  $P_i$ . Consistently with [4], this set,  $P_i \subset S \times \mathbb{R}^L$ , encompasses a private information signal,  $S_i \subset S$ , that the true state will be in  $S_i$  (i.e.,  $P_i \subset S_i \times \mathbb{R}^L$ ). Agents are assumed to receive no wrong signal, that is, no state will prevail tomorrow, out of the pooled information set,  $\underline{S} := \cap_i S_i$ .

We believe that a typical rational agent would not forecast a single price in each state she expects, since prices would now depend on all other agents' private forecasts. Yet, from observing markets, she might update her beliefs. Using sub-Section 2.1's notations, these features are encapsulated in the following definitions.

**Definition 1** A closed subset of  $(S \times \mathbb{R}_{++}^L) \cap \mathcal{M}$  is called an anticipation set. Its elements are called anticipations, expectations or forecasts. We denote by  $\mathcal{A}$  the set of all anticipation sets. A collection  $(P_i) \in \mathcal{A}^m$  is called an anticipation structure if:

(a)  $\bigcap_{i=1}^m P_i \neq \emptyset$ .

We denote by  $\mathcal{AS}$  the set of anticipation structures. A structure,  $(P'_i) \in \mathcal{AS}$ , is said to refine, or to be a refinement of  $(P_i) \in \mathcal{AS}$ , and we denote it by  $(P'_i) \leq (P_i)$ , if:

(b)  $P'_i \subset P_i, \forall i \in I$ .

A refinement,  $(P'_i) \in \mathcal{AS}$ , of  $(P_i) \in \mathcal{AS}$ , is said to be self-attainable if:

(c)  $\bigcap_{i=1}^m P'_i = \bigcap_{i=1}^m P_i$ .

A belief is a probability,  $\pi$ , on  $(\mathcal{M}, \mathcal{B}(\mathcal{M}))$ , whose support is an anticipation set, i.e.,  $P(\pi) \in \mathcal{A}$  (as denoted in sub-Section 2.1). A structure of beliefs is a collection of beliefs,  $(\pi_i)$ , whose supports define an anticipation structure (i.e.,  $(P(\pi_i)) \in \mathcal{AS}$ ).

We denote by  $\mathcal{B}$  and  $\mathcal{SB}$ , respectively, the sets of beliefs and structures of beliefs. A structure,  $(\pi'_i) \in \mathcal{SB}$ , is said to refine  $(\pi_i) \in \mathcal{BS}$ , which we denote  $(\pi'_i) \leq (\pi_i)$ , if  $(P(\pi'_i)) \leq (P(\pi_i))$ . The refinement,  $(\pi'_i)$ , is self-attainable if  $\bigcap_{i=1}^m P(\pi'_i) = \bigcap_{i=1}^m P(\pi_i)$ .

*Remark 1* Along the above Definition, an anticipation set is a closed set of spot prices (at  $t = 1$ ), whose values are never zero. A belief is a probability distribution on  $(\mathcal{M}, \mathcal{B}(\mathcal{M}))$ , which cannot put a positive weight on arbitrarily low prices. Agents' anticipations or beliefs form a structure when they have some forecasts in common. The set of common forecasts is left unchanged at a self-attainable refinement.

## 2.4 Consumers' behavior and the notion of equilibrium

Agents implement their decisions at  $t = 0$ , after having reached their final beliefs,  $(\pi_i) \in \mathcal{SB}$ , from observing market prices,  $\omega_0 := (p_0, q) \in \mathcal{M}_0$ , along a rational behavior described in Section 3, below. Hereafter, the final prices and beliefs at  $t = 0$  are

given, and markets, consistently, are assumed to have eliminated useless deals, that is,  $\{(z_i) \in \mathbb{R}^{Jm} : \sum_{i=1}^m z_i = 0, V(\omega_i) \cdot z_i = 0, \forall (i, \omega_i) \in I \times P(\pi_i)\} = \{0\}$ . The generic  $i^{th}$  agent's consumption set is that of continuous mappings from  $\{0\} \cup P(\pi_i)$  to  $\mathbb{R}_+^L$ :

$$\underline{X(\pi_i) := \mathcal{C}(\{0\} \cup P(\pi_i), \mathbb{R}_+^L)}.$$

Thus, her consumptions,  $x \in X(\pi_i)$ , are mappings, relating  $s = 0$  to a consumption decision,  $x_0 := x_{\omega_0} \in \mathbb{R}_+^L$ , at  $t = 0$ , and, continuously on  $P(\pi_i)$ , every expectation,  $\omega := (s, p) \in P(\pi_i)$ , to a consumption decision,  $x_\omega \in \mathbb{R}_+^L$ , at  $t = 1$ , which is conditional on the joint observation of state  $s$ , and price  $p$ , on the spot market. The generic  $i^{th}$  agent elects a strategy,  $(x, z) \in X(\pi_i) \times \mathbb{R}^J$ , she can always afford with her endowment.

This defines her budget set as follows:

$$\underline{B_i(\omega_0, \pi_i) := \{(x, z) \in X(\pi_i) \times \mathbb{R}^J : p_0 \cdot (x_0 - e_{i0}) \leq -q \cdot z \text{ and } p_s \cdot (x_\omega - e_{i\omega}) \leq V(\omega) \cdot z, \forall \omega := (s, p_s) \in P(\pi_i)\}}.$$

Each agent  $i \in I$  has preferences represented by the V.N.M. utility function:

$$\underline{x \in X(\pi_i) \mapsto U_i(\pi_i, x) := \int_{\omega \in P(\pi_i)} u_i(x_0, x_\omega) d\pi_i(\omega)}.$$

The generic  $i^{th}$  agent elects an optimal strategy in her budget set. The above economy is denoted by  $\mathcal{E}$ . It retains the standard small consumer price-taker hypothesis, along which no single agent's belief, or strategy, may alone have a significant impact on prices. It is said to be standard if, moreover, it meets the following Conditions:

- **Assumption A1:** for each  $i \in I$ ,  $e_i \gg 0$ ;
- **Assumption A2:** for each  $i \in I$ ,  $u_i$  is continuous and strictly concave;
- **Assumption A3:** for any  $(i, l, t) \in I \times \mathcal{L} \times \{0, 1\}$ , the mapping  $(x_0, x_1) \mapsto \partial u_i(x_0, x_1) / \partial x_t^l$  is defined and continuous on  $\{(x_0, x_1) \in \mathbb{R}_+^{2L} : x_t^l > 0\}$ , and  $(\inf_A \partial u_i(x_0, x_1) / \partial x_t^l) > 0$ , for every bounded subset  $A \subset \{(x_0, x_1) \in \mathbb{R}_+^{2L} : x_t^l > 0\}$ .

*Remark 2* Assumption  $A1$  is the standard strong survival's. Assumption  $A2$  could be weakened on strict concavity, but is retained to alleviate a tedious proof in Section 5. Assumption  $A3$  is consistent with agents having positive price forecasts, along Definition 1. It does not require, but allows for the standard Inada Conditions.

The economy's concept of equilibrium is defined as follows:

**Definition 2** *A collection of prices,  $\omega_s \in \mathcal{M}_s$ , defined for each  $s \in \underline{\mathbf{S}}'$ , beliefs,  $\pi_i \in \mathcal{B}$ , and strategies,  $(x_i, z_i) \in B_i(\omega_0, \pi_i)$ , for each  $i \in I$ , is a sequential equilibrium of the economy  $\mathcal{E}$ , or correct foresight equilibrium (CFE), if the following Conditions hold:*

- (a)  $\forall s \in \underline{\mathbf{S}}, \omega_s \in \cap_{i=1}^m P(\pi_i)$ ;
- (b)  $\forall i \in I, (x_i, z_i) \in \arg \max_{(x,z) \in B_i(\omega_0, \pi_i)} U_i(\pi_i, x)$ ;
- (c)  $\forall s \in \underline{\mathbf{S}}', \sum_{i=1}^m (x_{i\omega_s} - e_{i\omega_s}) = 0$ ;
- (d)  $\sum_{i=1}^m z_i = 0$ .

*Under the above conditions, the beliefs,  $\pi_i$ , for each  $i \in I$ , or the prices,  $\omega_s$ , for each  $s \in \underline{\mathbf{S}}'$ , are said to support the equilibrium.*

*Remark 3* In the case where  $\#P(\pi_i) = \#S_i$ , for every  $i \in I$ , the above notion of equilibrium coincides with that of [6], that is, with a perfect foresight equilibrium with (a possible) asymmetric information.

### 3 No-arbitrage prices and the information they reveal

Extending our earlier papers with Bernard Cornet ([4], [5]), we now define and characterize no-arbitrage prices and their revealing properties.

#### 3.1 No-arbitrage prices

Recalling the notations of sub-Section 2.1, we first define no-arbitrage prices.

**Definition 3** Let an anticipation set,  $P \in \mathcal{A}$ , and a price,  $q \in \mathbb{R}^J$ , be given. Price  $q$  is said to be a no-arbitrage price of  $P$ , or  $P$  to be  $q$ -arbitrage-free, if:

(a)  $\nexists z \in \mathbb{R}^J : -q \cdot z \geq 0$  and  $V(\omega) \cdot z \geq 0, \forall \omega \in P$ , with one strict inequality;

We denote by  $Q(P)$  the set of no-arbitrage prices of  $P$ .

Let a structure,  $(P_i) \in \mathcal{AS}$ , and, for each  $i \in I$ , the above price set,  $Q(P_i)$ , be given.

We refer to  $Q_c[(P_i)] := \cap_{i=1}^m Q(P_i)$  as the set of common no-arbitrage prices of  $(P_i)$ . The

structure,  $(P_i)$ , is said to be arbitrage-free (respectively,  $q$ -arbitrage-free) if  $Q_c[(P_i)]$

is non-empty (resp., if  $q \in Q_c[(P_i)]$ ). We say that  $q$  is a no-arbitrage price of  $(P_i)$ ,

and denote it by  $q \in Q[(P_i)]$ , if there exists a refinement,  $(P_i^*)$ , of  $(P_i)$ , such that  $q \in Q_c[(P_i^*)]$ . Moreover, if  $(P_i^*)$  is self-attainable,  $q \in Q_c[(P_i^*)]$  is called self-attainable.

The above definitions and notations extend to any consistent beliefs,  $(\pi_i) \in \Pi_{i=1}^m \pi(P_i)$ ,

as denoted in sub-Section 2.1. We then refer to  $Q(\pi_i) := Q(P_i)$ , for each  $i \in I$ , and to

$Q_c[(\pi_i)] := Q_c[(P_i)]$  and  $Q[(\pi_i)] := Q[(P_i)]$  as, respectively, the sets of no-arbitrage prices

of  $\pi_i$ , and of common no-arbitrage prices, and no-arbitrage prices, of the beliefs  $(\pi_i)$ .

*Remark 4* A symmetric refinement of any structure  $(P_i) \in \mathcal{AS}$ , that is,  $(P'_i) \leq (P_i)$ , such that  $P'_i = P'_1$ , for every  $i \in I$ , is always arbitrage-free along Definition 3. Hence, any structure,  $(P_i) \in \mathcal{AS}$ , admits a self-attainable no-arbitrage price. Indeed, the symmetric refinement,  $(P_i^*) \leq (P_i)$ , such that  $P_1^* = \cap_{i=1}^m P_i$  is arbitrage-free.

Claim 1 states a simple but useful property of arbitrage-free structures.

**Claim 1** An arbitrage-free structure,  $(P_i) \in \mathcal{AS}$ , satisfies the following Assertion:

(i)  $\nexists (z_i) \in \mathbb{R}^{Jm} : \sum_{i=1}^m z_i = 0$  and  $V(\omega) \cdot z \geq 0, \forall \omega \in \cup_{i=1}^m P_i$ , with one strict inequality.

**Proof** Let  $(P_i)$  be an arbitrage-free anticipation structure and  $q \in Q_c[(P_i)] \neq \emptyset$  be given. Assume, by contraposition, that there exists  $(z_i) \in (\mathbb{R}^J)^m$ , such that  $\sum_{i=1}^m z_i = 0$  and  $V(\omega) \cdot z \geq 0$ , for every  $\forall \omega \in \cup_{i=1}^m P_i$ , with one strict inequality, say for  $\omega \in P_1$ . If

$q \cdot z_1 \leq 0$ , then,  $q \notin Q(P_1)$ , which contradicts the fact that  $q \in Q_c[(P_i)]$ . Hence,  $q \cdot z_1 > 0$ , which implies, from the relation  $\sum_{i=1}^m z_i = 0$ , that  $q \cdot z_i < 0$ , for some  $i \in I$ . Then, the above inequalities yield  $q \notin Q(P_i)$ , contradicting the above relation,  $q \in Q_c[(P_i)]$ .  $\square$

### 3.2 Individual anticipations revealed by prices

Claim 2 tackles the notion of information conveyed by prices to individual agents.

**Claim 2** *Let  $(P_i) \in \mathcal{AS}$ , and  $q \in \mathbb{R}^J$ , be given. Then, for each  $i \in I$ , there exists a set,  $\bar{P}_i(q) \in \{\emptyset\} \cup \mathcal{A}$ , said to be revealed by price  $q$  to agent  $i$ , such that:*

(i) *if  $\bar{P}_i(q) \neq \emptyset$ , then,  $\bar{P}_i(q) \subset P_i$  and  $\bar{P}_i(q)$  is  $q$ -arbitrage-free;*

(ii) *every  $q$ -arbitrage-free anticipation set included in  $P_i$  is a subset of  $\bar{P}_i(q)$ .*

**Proof** Let  $i \in I$ ,  $q \in \mathbb{R}^J$  and  $(P_i) \in \mathcal{AS}$  be given. Let  $\mathcal{R}_{(P_i, q)}$  be the set of  $q$ -arbitrage-free anticipation sets included in  $P_i$ . If  $\mathcal{R}_{(P_i, q)} = \emptyset$ , then, the set  $\bar{P}_i(q) = \emptyset$  meets the conditions of Claim 2. If  $\mathcal{R}_{(P_i, q)} \neq \emptyset$ , we let  $P_i^* := \overline{\cup \{P'_i : P'_i \in \mathcal{R}_{(P_i, q)}\}}$  be the closed nonempty set including all elements of  $\mathcal{R}_{(P_i, q)}$ . By construction,  $P_i^*$  is an anticipation set included in  $P_i$  (a closed set), which meets Assertion (ii) of Claim 2.

Assume, by contraposition, that  $P_i^*$  does not meet Assertion (i), that is, there exists  $z \in \mathbb{R}^J$  and  $\bar{\omega} \in P_i^*$ , such that  $-q \cdot z \geq 0$ ,  $V(\omega) \cdot z \geq 0$  for every  $\omega \in P_i^*$ , and  $(V(\bar{\omega}) \cdot z - q \cdot z) > 0$ . From the definition of  $P_i^*$  and the continuity of  $V$ , the relation  $(V(\bar{\omega}) \cdot z - q \cdot z) > 0$  implies that there exists  $P'_i \in \mathcal{R}_{(P_i, q)}$  and  $\omega' \in P'_i$  such that  $(V(\omega') \cdot z - q \cdot z) > 0$ . Since  $P'_i \subset P_i^*$  is  $q$ -arbitrage-free, the above relations,  $-q \cdot z \geq 0$ ,  $V(\omega) \cdot z \geq 0$  for every  $\omega \in P_i^*$ , imply, from Definition 3, that  $-q \cdot z = 0$  and  $V(\omega) \cdot z = 0$  for every  $\omega \in P'_i$ , which contradicts the fact that  $(V(\omega') \cdot z - q \cdot z) > 0$ . This contradiction proves that  $P_i^*$  meets both conditions of Claim 2 and completes the proof.  $\square$

### 3.3 Anticipation structures revealed by prices

The following Claim characterizes the no-arbitrage prices of Definition 3.

**Claim 3** *Let a price,  $q \in \mathbb{R}^J$ , an anticipation structure,  $(P_i) \in \mathcal{AS}$ , and the related set collection,  $(\bar{P}_i(q))$ , of Claim 2, be given. The following statements are equivalent:*

- (i)  $q$  is a no-arbitrage price of  $(P_i)$ ;
- (ii)  $(\bar{P}_i(q))$  is the coarsest  $q$ -arbitrage-free refinement of  $(P_i)$ ;
- (iii)  $(\bar{P}_i(q))$  is a refinement of  $(P_i)$ ;
- (iv)  $(\bar{P}_i(q))$  is an anticipation structure.

*If  $q \in Q[(P_i)]$  is self-attainable, the above refinement,  $(\bar{P}_i(q)) \leq (P_i)$ , is self-attainable.*

**Proof** Assertion (i)  $\Rightarrow$  (ii) Let  $q \in Q[(P_i)]$  be given. From Definition 3, we set as given an arbitrary  $q$ -arbitrage-free refinement,  $(P_i^*)$ , of  $(P_i)$ . Then, for each  $i \in I$ , the set,  $\mathcal{R}_{(P_i, q)}$ , of  $q$ -arbitrage-free anticipation sets included in  $P_i$  is non-empty (for it contains  $P_i^*$ ). From Claim 2, the set,  $\mathcal{R}_{(P_i, q)}$ , admits  $\bar{P}_i(q) \neq \emptyset$  for maximal element, hence,  $P_i^* \subset \bar{P}_i(q) \subset P_i$  and  $q \in Q[\bar{P}_i(q)]$ . The latter relations imply:  $(P_i^*) \leq (\bar{P}_i(q)) \leq (P_i)$  and  $q \in Q_c[(\bar{P}_i(q))]$ . Hence,  $(\bar{P}_i(q))$  is the coarsest  $q$ -arbitrage-free refinement of  $(P_i)$ .

Assertion (ii)  $\Rightarrow$  (iii)  $\Rightarrow$  (iv) The relations are immediate from Definition 1.

Assertion (iv)  $\Rightarrow$  (i) If  $(\bar{P}_i(q)) \in \mathcal{AS}$ , then, from Claim 2,  $(\bar{P}_i(q))$  refines  $(P_i)$  and is  $q$ -arbitrage-free, that is,  $q \in Q_c[(\bar{P}_i(q))] \subset Q[(P_i)]$ .

The end of Claim 3, left to readers, is immediate from Definition 1 and above.  $\square$

**Definition 4** *Let an anticipation structure,  $(P_i) \in \mathcal{AS}$ , and a no-arbitrage price,  $q \in Q[(P_i)]$ , be given. The refinement,  $(\bar{P}_i(q)) \leq (P_i)$ , of Claim 3 is said to be revealed by price  $q$ . A refinement,  $(P'_i)$ , of  $(P_i)$  is said to be price-revealable if it can be revealed by some price, i.e., there exists  $q' \in Q[(P_i)]$  such that  $(P'_i) = (\bar{P}_i(q'))$  along Claim 3. Whenever  $q \in Q_c[(P_i)]$ , we say that  $(P_i)$  is revealed by price  $q$ , hence, price-revealable.*



Extending Cornet-de Boisdeffre (2009), we now examine how agents, endowed with no price model, may update their anticipations from observing market prices.

### 3.4 Sequential refinement through prices

Throughout, we let an anticipation structure,  $(P_i) \in \mathcal{AS}$ , a generic agent,  $i \in I$ , and an asset price,  $q \in \mathbb{R}^J$ , be given. We study how this  $i^{\text{th}}$  agent, endowed with the initial set of anticipations,  $P_i$ , may update her forecasts from observing price  $q$ . In successive steps (denoted by  $n \in \mathbb{N}$ )<sup>2</sup>, she rules out her arbitrage anticipations, namely, those anticipations which would grant her an arbitrage, if correct. She would do so when she believes that price  $q$  reflects an information she misses. The elimination of arbitrage anticipations in one step may result in new arbitrage anticipations in the next step. After finitely many inference steps, however, no arbitrage anticipation remains, that is, the agent's (refined) anticipation set is final.

We thus define, by induction, two sequences,  $\{A_i^n\}_{n \in \mathbb{N}}$  and  $\{P_i^n\}_{n \in \mathbb{N}}$  as follows:

- for  $n = 1$ , we let  $A_i^1 = \emptyset$  and  $P_i^1 := P_i$ ;
- for  $n \in \mathbb{N}$  arbitrary, with  $A_i^n$  and  $P_i^n$  defined at step  $n$ , we let  $A_i^{n+1} := P_i^{n+1} := \emptyset$ , if  $P_i^n = \emptyset$ , and, otherwise,

$$A_i^{n+1} := \{\bar{\omega} \in P_i^n : \exists z \in \mathbb{R}^J, -q \cdot z \geq 0, V(\bar{\omega}) \cdot z > 0 \text{ and } V(\omega) \cdot z \geq 0, \forall \omega \in P_i^n\};$$

$$P_i^{n+1} := P_i^n \setminus A_i^{n+1}, \text{ i.e., the agent rules out anticipations, granting an arbitrage.}$$

**Claim 4** *Let an anticipation structure,  $(P_i) \in \mathcal{AS}$ , an agent,  $i \in I$ , a price,  $q \in \mathbb{R}^J$ , and the information set,  $\bar{\mathbf{P}}_i(q)$ , it reveals along Claim 2, be given. The above set sequences,  $\{A_i^n\}_{n \in \mathbb{N}}$  and  $\{P_i^n\}_{n \in \mathbb{N}}$ , satisfy the following assertions:*

(i)  $\exists N \in \mathbb{N} : \forall n > N, A_i^n = \emptyset$  and  $P_i^n = P_i^N$ ;

(ii)  $P_i^N = \lim_{n \rightarrow \infty} P_i^n = \bar{\mathbf{P}}_i(q)$ .

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<sup>2</sup> We always define the set,  $\mathbb{N}$ , of natural numbers as starting from 1 (and not 0).

**Proof** Let  $(P_i) \in \mathcal{AS}$ ,  $i \in I$ ,  $q \in \mathbb{R}^J$ ,  $\bar{P}_i(q)$ ,  $\{A_i^n\}_{n \in \mathbb{N}}$  and  $\{P_i^n\}_{n \in \mathbb{N}}$  be defined or set as given as in Claim 4, and let  $P_i^* := \bigcap_{n \in \mathbb{N}} P_i^n = \lim_{n \rightarrow \infty} \searrow P_i^n$ .

With a non-restrictive convention that the empty set be included in any other set, we show, first, that the inclusion  $\bar{P}_i(q) \subset P_i^n$  holds for every  $n \in \mathbb{N}$ . It holds, from Claim 2, for  $n = 1$  (since  $\bar{P}_i(q) \subset P_i^1 := P_i$ ). Assume, now, by contraposition, that, for some  $n \in \mathbb{N}$ ,  $\bar{P}_i(q) \subset P_i^n$  and  $\bar{P}_i(q) \not\subset P_i^{n+1}$ . Then, there exist  $\bar{\omega} \in \bar{P}_i(q) \cap A_i^{n+1}$  and  $z \in \mathbb{R}^J$ , such that  $-q \cdot z \geq 0$ ,  $V(\bar{\omega}) \cdot z > 0$  and  $V(\omega) \cdot z \geq 0$ , for every  $\omega \in \bar{P}_i(q) \subset P_i^n$ , which contradicts Claim 2, along which  $\bar{P}_i(q)$  is  $q$ -arbitrage-free, if non-empty. Hence, the relation  $\bar{P}_i(q) \subset P_i^n$ , holds for every  $n \in \mathbb{N}$ .

Assume, first, that  $P_i^* := \bigcap_{n \in \mathbb{N}} P_i^n = \emptyset$ . Since the sequence  $\{P_i^n\}_{n \in \mathbb{N}}$  is non-increasing and made of compact or empty sets (this stems, by induction, from the fact that  $P_i^1$  is compact and  $A_i^{n+1}$  is open in  $P_i^n$  or empty), there exists  $N \in \mathbb{N}$ , such that  $P_i^n = A_i^n = \emptyset$ , for all  $n \geq N$ . Then, from above, Claim 4-(i)-(ii) hold (with  $\bar{P}_i(q) = \emptyset$ ).

Assume, next, that  $P_i^* \neq \emptyset$ . Then,  $P_i^*$ , a non-empty intersection of compact sets, is compact, and, from above,  $\bar{P}_i(q) \subset P_i^*$ .

For every  $n \in \mathbb{N}$ , let  $Z_i^{on} := \{z \in \mathbb{R}^J : V(\omega) \cdot z = 0, \forall \omega \in P_i^n\}$ . Since  $\{P_i^n\}_{n \in \mathbb{N}}$  is non-increasing, the sequence of vector spaces,  $\{Z_i^{on}\}$ , is non-decreasing in  $\mathbb{R}^J$ , hence, stationary. We let  $N \in \mathbb{N}$  be such that  $Z_i^{on} = Z_i^{oN}$ , for every  $n \geq N$ . Assume, by contraposition, that assertion (i) of Claim 4 fails, that is:

$$\forall n \in \mathbb{N}, \exists (\omega_n, z_n) \in P_i^n \times \mathbb{R}^J : -q \cdot z_n \geq 0, V(\omega_n) \cdot z_n > 0 \text{ and } V(\omega) \cdot z_n \geq 0, \forall \omega \in P_i^n.$$

From the definition of the sets  $P_i^n$  and  $P_i^{n+1} \neq \emptyset$ , the above portfolios satisfy, for each  $n \geq N$ , jointly  $z_n \notin Z_i^{on}$  and  $z_n \in Z_i^{o(n+1)}$ , which contradicts the fact that  $Z_i^{o(n+1)} = Z_i^{on}$ . This contradiction proves Claim 4-(i), and we let  $N \in \mathbb{N}$  be such that

$A_i^{N+1} = \emptyset$ . Then, by construction,  $P_i^N = P_i^*$ , and  $P_i^* \subset P_i$  is  $q$ -arbitrage-free (since  $A_i^{N+1} = \emptyset$ ), which yields, from Claim 2,  $P_i^* \subset \bar{P}_i(q)$ , and, from above,  $P_i^* = \bar{P}_i(q)$ .  $\square$

The above inference process is a rational behavior, whereby agents, having no clue of how market prices are determined, update their beliefs from observing them in finitely many inference steps. As long as markets have not reached a no-arbitrage price, traders cannot agree on prices and a sequential equilibrium may not exist. Claims 3 and 4 show that agents have common updated forecasts - a necessary condition for a sequential equilibrium to exist - if, and only if, the observed asset price is a no-arbitrage price. We have seen such prices always exist. We will see below that equilibrium prices are always no-arbitrage prices. Hence, agents may infer their own anticipation sets from observing the current equilibrium price (whenever it exists). We then speak of a price-revealed equilibrium.

We now introduce and discuss the notion of minimum price uncertainty, with private beliefs, and state our Theorem.

## 4 Minimum uncertainty and existence of equilibrium

### 4.1 The existence Theorem

With private idiosyncratic beliefs, a nonempty set of minimum uncertainty exists, any element of which can obtain as an equilibrium price for some beliefs today.

**Definition 5** *Let  $\Omega$  be the set of sequential equilibria (CFE) of the economy,  $\mathcal{E}$ . The minimum uncertainty set,  $\Delta$ , is the subset of prices at  $t = 1$ , which support a CFE, namely:  $\Delta = \{\omega^* = (s^*, p^*) \in \mathcal{M} : s^* \in \underline{\mathbf{S}}, \exists((\omega_s), (\pi_i), [(x_i, z_i)]) \in \Omega, \omega^* = \omega_{s^*}\}$ .*

The following Theorem states existence properties of a standard economy.

**Theorem 1** *Let a standard economy,  $\mathcal{E}$ , its minimum uncertainty set,  $\Delta$ , and an anticipation structure,  $(P_i) \in \mathcal{AS}$ , be given. Then, the following Assertions hold:*

(i)  $\exists \varepsilon > 0 : \forall (s, p) \in \Delta, \forall l \in \mathcal{L}, p^l \geq \varepsilon$ ;

(ii)  $\Delta \neq \emptyset$ .

*Consistently, if  $\Delta \subset P_i$  holds for each  $i \in I$ , then, the following Assertions also hold:*

(iii) *if  $(P_i) \in \mathcal{AS}$  is arbitrage-free, then, any beliefs  $(\pi_i) \in \Pi_{i=1}^m \pi(P_i)$  support a CFE;*

(iv) *if a self-attainable refinement,  $(P_i^*) \leq (P_i)$ , is arbitrage-free (and that refinement exists), any consistent beliefs,  $(\pi_i^*) \in \Pi_{i=1}^m \pi(P_i^*)$ , as denoted in 2.1, support a CFE;*

(v) *if  $(P_i^*) \in \mathcal{AS}$  is a self-attainable price-revealable refinement of  $(P_i)$  (which exists), then, every refinement  $(\pi_i^*) \in \Pi_{i=1}^m \pi(P_i^*)$  supports a price-revealed CFE.*

*Remark 5* Using the notations of Theorem 1, Assertion (iii) implies (iv), by replacing the structure  $(P_i)$  by  $(P_i^*) \leq (P_i)$ . Moreover, we let the reader check, as standard from Assumption  $A\mathcal{B}$ , that whenever a structure  $(\pi_i^*) \in \mathcal{SB}$  and price  $\omega_0 := (p_0, q) \in \mathcal{M}_0$  support a CFE, then,  $q \in Q_c[(\pi_i^*)]$ . Consequently, if a refinement,  $(\pi_i^*) \in \mathcal{SB}$ , is price-revealable and supports a CFE, that CFE is revealed by the equilibrium price. Hence, Assertion (iv) implies (v) and only Assertions (i)-(ii)-(iii) need be proved.

Before discussing the Theorem's Condition,  $\Delta \subset \cap_{i=1}^m P_i$ , we prove Assertion (i).

**Proof of Assertion (i)** Let  $\Omega$  and  $\Delta$  be the sets of Definition 5. Let  $s^* \in \underline{\mathbf{S}}$ ,  $\omega^* := (s^*, p^*) \in \Delta$ , and an equilibrium,  $\mathcal{C} := ((\omega_s), (\pi_i), [(x_i, z_i)]) \in \Omega$ , such that  $\omega^* = \omega_{s^*}$ , be given. The relation  $p^* \gg 0$  is standard from Assumption  $A\mathcal{B}$  and Definition 2-(b).

Let  $e := (\min_{(i,s,l) \in I \times \underline{\mathbf{S}}' \times \mathcal{L}} e_{is}^l) \in \mathbb{R}_{++}$  and  $E := (\max_{(s,l) \in \underline{\mathbf{S}}' \times \mathcal{L}} \sum_{i=1}^m e_{is}^l) \in \mathbb{R}_{++}$  be given. Then, for each  $s \in \underline{\mathbf{S}}'$ , the relations  $(x_{i\omega_s}) \geq 0$  and  $\sum_{i=1}^m (x_{i\omega_s} - e_{is}) = 0$ , which hold from Definition 2-(c), yield  $x_{i\omega_s} \in [0, E]^L$ , for each  $i \in I$ . For each  $l \in \mathcal{L}$ , the above relations

imply that at least one agent, say  $i = 1$ , does not sell the  $l^{\text{th}}$  good in state  $s$ , so that  $x_{1\omega_s}^l \in [e, E]$ . From Assumption  $A\mathcal{B}$ , the mapping,  $(x_0, x_1) \mapsto \partial u_i(x_0, x_1)/\partial x_1^l$ , for each  $i \in I$ , attains a maximum on the set  $X^l := \{(x_0, x_1) \in [0, E]^{2L} : x_1^l \geq e\}$ , and we let:

$\alpha := \inf \partial u_i(x_0, x_1)/\partial x_1^l$ , for  $(i, l, (x_0, x_1)) \in I \times \mathcal{L} \times [0, E]^{2L}$ , and  $\beta := \max \partial u_i(x_0, x_1)/\partial x_1^l$ , for  $(i, l, (x_0, x_1)) \in I \times \mathcal{L} \times X^l$ , be strictly positive numbers. Let  $\gamma = \beta/\alpha$  and  $(l, l') \in \mathcal{L}^2$  be given. Assume, by contraposition, that  $p^{*l}/p^{*l'} > \gamma$  and let  $i \in I$  be an agent, unwilling to sell good  $l \in \mathcal{L}$ , under her consumption decision,  $x_{i\omega^*}$ . We let the reader check, as tedious and standard, that agent  $i$ , starting from  $(x_i, z_i)$ , could find a utility increasing strategy,  $(x_i^*, z_i) \in B_i(\omega_0, \pi_i)$ , modifying her consumptions in state  $s^*$  only, such that  $x_{i\omega^*}^{*l} < x_{i\omega^*}^l$  and  $x_{i\omega^*}^{*l'} > x_{i\omega^*}^{l'}$ . Indeed, with  $p^{*l}/p^{*l'} > \gamma$ , she has an incentive to sell a small amount of commodity  $l$  in exchange for commodity  $l'$ . Hence,  $(x_i, z_i)$  cannot be an equilibrium strategy. This contradiction proves the relation  $p^{*l}/p^{*l'} \leq \gamma$ . Then, we let the reader check from the joint relations  $p^* \gg 0$ ,  $\|p^*\| = 1$  and  $p^{*l}/p^{*l'} \leq \gamma$ , for each pair  $(l, l') \in \mathcal{L}^2$ , which hold from above, that  $p^{*l} \geq \varepsilon = 1/\gamma\sqrt{L}$ , for every  $l \in \mathcal{L}$ .  $\square$

## 4.2 The Theorem's Condition

In Geanakoplos and Polemarchakis (1986), the perfect foresight equilibrium is generically locally unique or determinate. This outcome, it has been argued, would enable rational agents knowing the primitives of the economy to identify prices contingent on each future state, from observing first period equilibrium prices. Such inferences preclude any defect in agents' computations.

It is also well known that perfect foresight equilibrium prices would only obtain, in general, if all such 'sophisticated' agents shared the same beliefs. This setting rules out the possibility that some agents deviated or be uncertain of future prices. Perfect price anticipations need be common and accepted by all consumers (amongst

possibilities), despite contradictory buyer/seller interests. Thus, it is reasonable to believe, although not formally required, that the perfect price forecasts are common knowledge. If not, tomorrow's prices might fail to have been anticipated correctly by anyone, e.g., if some agents' beliefs suddenly and privately changed before trading.

Moreover, the theoretical inferences presented above build on equilibrium prices being locally unique or determinate. This outcome is, by no means, guaranteed under private beliefs. Along Theorem 1, equilibrium is consistent with agents having uncountable sets of anticipations in any state, as well as uncountably many different beliefs, given anticipations. Hence, it seems wise to believe (from Berge's Theorem) that, typically, the nonempty set  $\Delta$  is also uncountable. Then, local uniqueness fails.

As explained below, inferring the set  $\Delta$ , or a bigger set, would not require the primitives of the economy be known, or a price model be used. However, that set should be included in agents' anticipations. Indeed, with price-takers agents seeing other consumers' beliefs as arbitrary, the set,  $\Delta$ , of all possible equilibrium prices, for some structure of beliefs today, may be seen as one of incompressible uncertainty by agents. From Theorem 1, the Condition,  $\Delta \subset \cap_{i=1}^m P_i$ , is sufficient to insure the existence of a CFE. But it might also be a necessary one, especially if beliefs are so unpredictable and erratic to let any price in  $\Delta$  be a possible outcome. We think this situation might arise, in particular, in times of enhanced uncertainty, volatility or erratic change in beliefs, letting no chance to agents to coordinate themselves.

If cautious agents should embed the minimum uncertainty set into their anticipations, the question arises why and how this might happen. As for the refinement mechanism described in Section 3, we suggest this could be achieved, with no price model, from observing markets. Since we are only interested in normalised prices, it is generally possible to observe past prices and reckon their *relative* values on long

time series and in a wide array of economic events, representing virtually all states of nature. For example, the price of many assets are known daily (hence, in the daily state), on several decades. Long statistics also exist for consumption prices.

Along time series, relative prices would vary between observable boundaries. It seems reasonable to assume that they were sequential equilibrium prices along Definition 2. If series are long enough and if we assume that future behaviors may replicate some past ones, then, for instance, the intervals between the lower and upper bounds of the relative prices, observed in the various states, could be thought to embed the set  $\Delta$ . A method of the kind is empirical, based on statistical analysis, not on rational expectations. It does not require a demanding price model or awareness of the economy's primitives. It needs not be implemented by agents individually, but only by a public agent or tradehouse. This method, we think, could have many useful applications in finance. It could also provide estimates for reasonable beliefs.

Thus, from observing markets, we think the set  $\Delta$ , or a bigger set, might be inferred by a public agent. In addition, individual agents may have idiosyncratic uncertainty, given their personal information or feelings. So, neither beliefs, nor their supports, need be symmetric or reduce to  $\Delta$ .

We showed that agents, starting from an anticipation structure,  $(P_i) \in \mathcal{AS}$ , such that  $\Delta \subset \cap_{i=1}^m P_i$ , and observing a self-attainable equilibrium price (which exists), reach a unique equilibrium refinement,  $(P_i^*) \leq (P_i)$ , from making inferences as in Section 3. With no price model, agents cannot infer more. From Theorem 1, once they have reached the refinement  $(P_i^*)$ , all possible equilibrium prices at  $t = 0$ , which are related to beliefs  $(\pi_i^*) \in \Pi_{i=1}^m \pi(P_i^*)$ , along sub-Section 2.1, reveal the same structure,  $(P_i^*)$ . This is another difference with the classical model: possibly different

equilibrium prices at  $t = 0$ , related to all beliefs  $(\pi_i^*) \in \Pi_{i=1}^m \pi(P_i^*)$ , reveal exactly the same anticipation sets,  $(P_i^*) \leq (P_i)$ , which are fixed and may differ across agents.

In all cases, agents' final beliefs,  $(\pi_i^*) \in \Pi_{i=1}^m \pi(P_i^*)$ , remain private. This privacy, agents' fixed expectations sets,  $(P_i^*)$ , and the Theorem's Condition restore existence. There can be no fall in rank problem a la Hart (1975). The generic  $i^{th}$  agent's budget set and strategy are defined *ex ante*, with reference to ex ante conditions, and to a fixed set of anticipations,  $P_i^*$ . So, only her *ex ante span* of payoffs matters, namely,  $\langle V, P_i^* \rangle := \{\omega \in P_i^* \mapsto V(\omega) \cdot z : z \in \mathbb{R}^J\}$ . That span is fixed independently of any equilibrium price,  $p \in \Delta \subset P_i^*$ , whose location in the set  $\Delta$  cannot be predicted at  $t = 0$  and will only be observed at  $t = 1$ . This setting is quite different from Hart's.

## 5 The existence proof

Throughout, we set as given arbitrarily, in a standard economy,  $\mathcal{E}$ , an arbitrage-free anticipation structure,  $(P_i)$ , and related beliefs,  $(\pi_i) \in \Pi_{i=1}^m \pi(P_i)$ , along sub-Section 2.1's notations. These structures are, henceforth, fixed and always referred to. Along Remark 5, we will only prove assertions (ii) and (iii) of Theorem 1.

The proof's principle is to construct a sequence of auxiliary economies, with finite anticipation sets, refining and tending to the initial sets,  $(P_i)$ . Each finite economy admits an equilibrium, which we set as given along Theorem 1 of [6]. Then, the sequence of finite dimensional equilibria yields an equilibrium of the economy  $\mathcal{E}$ .

Each step of the proof uses simple mathematical arguments. Yet, it could not avoid a tedious number of arguments and of subsequent notations, e.g., for specifying the auxiliary economies, whose construction builds on the following Lemma.



**Lemma 1** For each  $i \in I$  and each  $n \in \mathbb{N}$ , there exists a finite partition,  $\mathcal{P}_i^n$ , and a finite subset,  $\Omega_i^n$ , of  $P_i$ , such that the following Assertions hold:

- (i)  $\forall P \in \mathcal{P}_i^n$ ,  $P \in \mathcal{B}(\mathcal{M})$  and  $\pi_i(P) > 0$ , and  $(\Omega_i^1) \in \mathcal{AS}$ ;
- (ii)  $\Omega_i^n \subset \Omega_i^{n+1}$ , and, for every  $P \in \mathcal{P}_i^n$ ,  $P \cap \Omega_i^n$  is a singleton;
- (iii)  $\forall \omega \in P_i$ ,  $\exists \omega' \in \Omega_i^n : \|\omega' - \omega\| \leq L\#S / n$ , and, hence,  $P_i = \overline{\cup_{n \in \mathbb{N}} \Omega_i^n}$ ;
- (iv)  $\exists N \in \mathbb{N}$ ,  $\forall (P_i^*) \leq (P_i)$ ,  $[(\Omega_i^N) \leq (P_i^*)$  and  $\#P_i^* \in \mathbb{N}$ ,  $\forall i \in I] \Rightarrow [Q_c[(P_i^*)] \neq \emptyset]$ .

**Proof** see the Appendix. □

For every integer  $n > N$  along Lemma 1, and any element  $\eta \in ]0, 1]$ , hereafter set as given, we now consider the following auxiliary economies,  $\mathcal{E}_\eta^n$ , and equilibria,  $\mathcal{C}_\eta^n$ .

## 5.2 Auxiliary economies, $\mathcal{E}_\eta^n$

Henceforth, we set as given  $n > N$ , along Lemma 1, and, arbitrarily, a spot price,  $\omega_s^N := (s, p_s^N) \in \mathcal{M}_s$ , for each  $s \in \underline{\mathbf{S}}$ . Then, we define by induction the economy  $\mathcal{E}_\eta^n$ , and an equilibrium price of this economy,  $\omega_s^n \in \mathcal{M}_s$ , for each  $s \in \underline{\mathbf{S}}$ , as follows.

From the previous induction prices,  $(\omega_s^{n-1}) \in \Pi_{s \in \underline{\mathbf{S}}} \mathcal{M}_s$ , the auxiliary economy,  $\mathcal{E}_\eta^n$ , is defined as one of the type described in [6]. Namely, it is a pure exchange economy, with two period ( $t \in \{0, 1\}$ ),  $m$  agents, having incomplete information, and exchanging  $L$  goods and  $J$  nominal assets, under uncertainty (at  $t = 0$ ) about which state of a finite state space,  $S^n$ , will prevail at  $t = 1$ . Formally, referring to [6]:

- The information structure is the collection,  $(S_i^n)$ , of sets  $\underline{S}_i^n := \underline{\mathbf{S}} \cup \tilde{S}_i^n$ , defined, from Lemma 1, by  $\tilde{S}_i^n := \{i\} \times \Omega_i^n$ , for each  $i \in I$ . The set of realizable states is  $\underline{\mathbf{S}} := \cap_{i \in I} S_i = \cap_{i \in I} S_i^n$ . For each agent  $i \in I$ , the set  $\tilde{S}_i^n$  consists of purely formal states, none of which will prevail (but in the  $i^{th}$  agent's mind). The state space of the economy is  $S^n = \cup_{i \in I} S_i^n$ . For notational purposes, we also let  $S'^n := \{0\} \cup S^n$  and  $S_i'^n := \{0\} \cup S_i^n$  (for each  $i \in I$ ) include the first period state,  $s = 0$ .

- The  $S^n \times J$  payoff matrix,  $V^n := (V^n(s^n))$ , is defined, with reference to the payoff mapping,  $\omega \mapsto V(\omega)$ , by the row vectors  $V^n(s^n) := V(\omega) \in \mathbb{R}^J$ , for each  $s^n := (i, \omega) \in S^n \setminus \underline{\mathbf{S}}$ , and  $V^n(s) := V(\omega_s^{n-1}) \in \mathbb{R}^J$ , for each  $s \in \underline{\mathbf{S}}$ . Hence,  $V^n$  is purely nominal.
- In each formal state,  $s^n := (i, (s, p_s)) \in \tilde{S}_i^n$ , the generic agent  $i \in I$  is certain that price  $p_s \in \mathbb{R}_{++}^L$ , and only that price, can prevail on the fictitious  $s^n$ -spot market.
- In each realizable state,  $s \in \underline{\mathbf{S}}$ , the generic agent  $i \in I$  has perfect foresight, i.e., anticipates with certainty the true price, say  $p_s^n \in \mathbb{R}_{++}^L$  (or  $\omega_s^n := (s, p_s^n) \in \mathcal{M}_s$ ).
- The generic  $i^{th}$  agent's endowment,  $e_i^n := (e_{is^n}^n) \in \mathbb{R}_{++}^{LS_i^n}$ , is defined by  $e_{is^n}^n := e_{is}$ , for each  $s^n := s \in \underline{\mathbf{S}}'$ , and  $e_{is^n}^n := e_{is}$ , for each  $s^n := (i, (s, p_s)) \in \tilde{S}_i^n$ .
- For all market prices,  $\omega_0^n := (p_0^n, q^n) \in \mathcal{M}_0$ , at  $t = 0$ , and  $\omega_s^n := (s, p_s^n) \in \mathcal{M}_s$ , for each  $s \in \underline{\mathbf{S}}$ , the generic  $i^{th}$  agent has for budget set and utility function:

$$B_i^n([\omega_s^n]) := \left\{ (x, z) \in \mathbb{R}_+^{LS_i^n} \times \mathbb{R}^J : p_0^n \cdot (x_0 - e_{i0}) \leq -q^n \cdot z \text{ and } p_s^n \cdot (x_s - e_{is}) \leq V^n(s) \cdot z, \forall s \in \underline{\mathbf{S}} \right. \\ \left. \text{and } p_s \cdot (x_{s^n} - e_{is}) \leq V^n(s^n) \cdot z, \forall s^n := (i, (s, p_s)) \in \tilde{S}_i^n \right\};$$

$$u_i^n : x \mapsto \sum_{s^n \in S_i^n} \pi_i^n(s^n) u_i(x_0, x_{s^n}), \text{ where}$$

$$(1 + \#\underline{\mathbf{S}} \eta) \pi_i^n(s^n) := \begin{cases} \pi_i(P) > 0 \text{ where } P \in \mathcal{P}_i^n \text{ satisfies } P \cap \Omega_i^n = \{\omega\}, \text{ if } s^n = (i, \omega) \in \tilde{S}_i^n \\ \eta > 0 \text{ if } s^n \in \underline{\mathbf{S}} \end{cases}$$

Along Theorem 1 of [6], each auxiliary economy admits an equilibrium,  $\mathcal{C}_\eta^n$ , with the properties described in the following Lemma 2.

**Lemma 2** *For each  $n > N$  along Lemma 1, the economy  $\mathcal{E}_\eta^n$  admits an equilibrium,  $\mathcal{C}_\eta^n$ , namely, a collection of prices,  $\omega_0^n := (p_0^n, q^n) \in \mathcal{M}_0$ , at  $t = 0$ , and  $\omega_s^n := (s, p_s^n) \in \mathcal{M}_s$ , in each state  $s \in \underline{\mathbf{S}}$ , and strategies,  $(x_i^n, z_i^n) \in B_i^n([\omega_s^n])$ , for each  $i \in I$ , such that:*

- (i)  $\forall i \in I, (x_i^n, z_i^n) \in \arg \max_{(x, z) \in B_i^n([\omega_s^n])} u_i^n(x);$
- (ii)  $\forall s \in \underline{\mathbf{S}}', \sum_{i=1}^m (x_{is}^n - e_{is}) = 0;$

(iii)  $\sum_{i=1}^m z_i^n = 0$ .

Moreover, the equilibrium,  $\mathcal{C}_\eta^n$ , satisfies the following Assertions:

(iv)  $\forall (n, i, s) \in \mathbb{N} \setminus \{1, \dots, N\} \times I \times \underline{\mathbf{S}}'$ ,  $x_{is}^n \in [0, E]^L$ , where  $E := \max_{(s,l) \in \mathbf{S}' \times \mathcal{L}} \sum_{i=1}^m e_{is}^l$ ;

(v)  $\exists \varepsilon \in ]0, 1[ : p_s^{nl} \geq \varepsilon$ ,  $\forall (n, s, l) \in \mathbb{N} \setminus \{1, \dots, N\} \times \underline{\mathbf{S}} \times \mathcal{L}$ .

**Proof** see the Appendix. □

Along Lemma 2, we set as given an equilibrium of the economy  $\mathcal{E}_\eta^n$ , namely:

$$\mathcal{C}_\eta^n := (\omega_0^n := (p_0^n, q^n), (\omega_s^n), [(x_i^n, z_i^n)]) \in \mathcal{M}_0 \times \prod_{s \in \underline{\mathbf{S}}} \mathcal{M}_s \times \prod_{i=1}^m B_i^n([\omega_s^n]),$$

which is always referred to. The equilibrium prices,  $(\omega_s^n) \in \prod_{s \in \underline{\mathbf{S}}} \mathcal{M}_s$ , permit to pursue the induction and define the economy  $\mathcal{E}_\eta^{n+1}$  in the same way as above, hence, the auxiliary economies and equilibria at all ranks. These meet the following Lemma.

**Lemma 3** *For the above sequence,  $\{\mathcal{C}_\eta^n\}$ , of equilibria, it may be assumed to exist:*

(i)  $\omega_s^\eta = \lim_{n \rightarrow \infty} \omega_s^n \in \mathcal{M}_s$ , for each  $s \in \underline{\mathbf{S}}'$ ;

(ii)  $(x_{is}^\eta) := \lim_{n \rightarrow \infty} (x_{is}^n)_{i \in I} \in \mathbb{R}^{Lm}$ , such that  $\sum_{i \in I} (x_{is}^\eta - e_{is}) = 0$ , for each  $s \in \underline{\mathbf{S}}'$ ;

(iii)  $(z_i^\eta) = \lim_{n \rightarrow \infty} (z_i^n)_{i \in I} \in \mathbb{R}^{Jm}$ , such that  $\sum_{i=1}^m z_i^\eta = 0$ .

Moreover, we define, for each  $i \in I$  and each  $n \in \mathbb{N}$ , the following sets and mappings:

\* the mapping,  $\omega \in P_i \mapsto \arg_i^n(\omega) \in \Omega_i^n$ , such that one  $P \in \mathcal{P}_i^n$  satisfies  $(\omega, \arg_i^n(\omega)) \in P^2$ ;

\* from Assertion (i) and Lemma 2-(v), the belief,  $\pi_i^\eta := \frac{1}{1 + \#\underline{\mathbf{S}}_\eta} (\pi_i + \eta \sum_{s \in \underline{\mathbf{S}}} \delta_s)$ ,

where  $\delta_s$  is (for each  $s \in \underline{\mathbf{S}}$ ) the Dirac's measure of  $\omega_s^\eta$ ;

\*  $P_i^\eta = P_i \cup \{\omega_s^\eta\}_{s \in \underline{\mathbf{S}}}$ , the support of  $\pi_i^\eta \in \mathcal{B}$ ;

\*  $B_i(\omega, z) := \{x \in \mathbb{R}_+^L : p_s \cdot (x - e_{is}) \leq V(\omega) \cdot z\}$ , for every  $\omega := (s, p_s) \in \mathcal{M}$ ,  $z \in \mathbb{R}^J$ .

Then, the following Assertions hold, for each  $i \in I$ :

(iv)  $\{\arg_i^n(\omega)\}_{n \in \mathbb{N}}$  converges to  $\omega$  uniformly on  $P_i$ ;

(v)  $\forall s \in \underline{\mathbf{S}}$ ,  $\{x_{is}^\eta\} = \arg \max u_i(x_{i0}^\eta, x)$ , for  $x \in B_i(\omega_s^\eta, z_i^\eta)$ , along Assertions (i)-(ii)-(iii);

we denote by  $x_{i\omega_s^\eta}^\eta := x_{is}^\eta \in \mathbb{R}_+^L$  a related consumption decision contingent on  $\omega_s^\eta \in \mathcal{M}_s$ ;

(vi) the correspondence  $\omega \in P_i^\eta \mapsto \arg \max u_i(x_{i0}^\eta, x)$ , for  $x \in B_i(\omega, z_i^\eta)$ , is a continuous mapping, denoted by  $\omega \mapsto x_{i\omega}^\eta$ . The mapping,  $x_i^\eta : \omega \in \{0\} \cup P_i^\eta \mapsto x_{i\omega}^\eta$ , defined from Assertions (ii), (v) and above, is a consumption plan, that is,  $x_i^\eta \in X(\pi_i^\eta)$ ;

(vii)  $U_i(\pi_i^\eta, x_i^\eta) = \lim_{n \rightarrow \infty} u_i^n(x_i^n) \in \mathbb{R}_+$ .

**Proof** see the Appendix. □

### 5.3 An equilibrium of the initial economy

We now prove Assertion (ii) of Theorem 1, via the following Claim.

**Claim 5** *The collection of prices,  $(\omega_s^\eta)$ , beliefs,  $(\pi_i^\eta)$ , allocation,  $(x_i^\eta)$ , and portfolios,  $(z_i^\eta)$ , of Lemma 3, defines a C.F.E. of the economy  $\mathcal{E}$ .*

**Proof** Let us define  $\mathcal{C}^\eta := ((\omega_s^\eta), (\pi_i^\eta), [(x_i^\eta, z_i^\eta)])$  as in Claim 5. From Lemma 3-(ii)-(iii)-(v)-(vi),  $\mathcal{C}^\eta$  meets Conditions (c)-(d) of the above Definition 2 of equilibrium. The relation  $\{\omega_s^\eta\}_{s \in \underline{S}} \subset \cap_{i=1}^m P(\pi_i^\eta)$ , that is, Condition (a) of Definition 2, also holds from the definition of the structure  $(\pi_i^\eta) \in \mathcal{SB}$ . To prove that  $\mathcal{C}^\eta$  is a C.F.E., it suffices to show it meets the relation  $[(x_i^\eta, z_i^\eta)] \in \Pi_{i=1}^m B_i(\omega_0^\eta, \pi_i^\eta)$  and Condition (b) of Definition 2.

We show, first, that  $[(x_i^\eta, z_i^\eta)] \in \Pi_{i=1}^m B_i(\omega_0^\eta, \pi_i^\eta)$ , denote  $\omega_0^\eta = (p_0^\eta, q^\eta) \in \mathcal{M}_0$ , and let  $i \in I$  be given. From Lemma 2, the relations  $p_0^\eta \cdot (x_{i0}^\eta - e_{i0}) \leq -q^\eta \cdot z_i^\eta$  hold for each  $n \in \mathbb{N} \setminus \{1, \dots, N\}$ , and yield in the limit (from Lemma 3-(i)-(ii)-(iii) and the continuity of the scalar product):  $p_0^\eta \cdot (x_{i0}^\eta - e_{i0}) \leq -q^\eta \cdot z_i^\eta$ . From Lemma 3-(vi), the relations  $p_s \cdot (x_{i\omega}^\eta - e_{is}) \leq V(\omega) \cdot z_i^\eta$  hold, for every  $\omega = (s, p_s) \in P_i^\eta$ . Then, Lemma 3-(vi) and all above relations yield:  $(x_i^\eta, z_i^\eta) \in B_i(\omega_0^\eta, \pi_i^\eta)$ . We have thus proved that  $\mathcal{C}^\eta$  meets the first of the two desired conditions,  $[(x_i^\eta, z_i^\eta)] \in \Pi_{i=1}^m B_i(\omega_0^\eta, \pi_i^\eta)$ .

Next, we assume, by contraposition, that  $\mathcal{C}^\eta$  fails to meet the second condition, Definition 2-(b). Then, there exist  $i \in I$ ,  $(x, z) \in B_i(\omega_0^\eta, \pi_i^\eta)$  and  $\varepsilon \in \mathbb{R}_{++}$ , such that:

$$(I) \quad \varepsilon + U_i(\pi_i^\eta, x_i^\eta) < U_i(\pi_i^\eta, x).$$

We may assume that there exists  $\delta \in \mathbb{R}_{++}$ , such that:

$$(II) \quad x_\omega^l \geq \delta, \text{ for every } (\omega, l) \in \{0\} \cup P_i^\eta \times \mathcal{L}.$$

If not, for every  $\alpha \in ]0, 1]$ , we let  $(x^\alpha, z^\alpha) := ((1 - \alpha)x + \alpha e_i, (1 - \alpha)z) \in B_i(\omega_0^\eta, \pi_i^\eta)$  meet relations (II), from Assumption A1. Then, from relation (I) and the uniform continuity (on a compact set) of  $(\alpha, \omega) \in [0, 1] \times P_i^\eta \mapsto u_i(x_0^\alpha, x_\omega^\alpha)$ , the strategy  $(x^\alpha, z^\alpha)$  also meets relation (I) for  $\alpha$  small enough. So, we may indeed assume relations (II).

Then, we let the reader check, as tedious but straightforward, from the relations  $(x, z) \in B_i(\omega_0^\eta, \pi_i^\eta)$ ,  $\omega_0^\eta = (p_0^\eta, q^\eta) \in \mathcal{M}_0$ , and  $\pi_i^\eta \in \mathcal{B}$ , from the definitions of  $\mathcal{M}_0$  and  $\mathcal{B}$ , Lemma 2-(v) and Lemma 3-(i), the above relations (I)-(II), Assumptions A1-A2 and continuity arguments, that we may also assume there exists  $\gamma \in \mathbb{R}_{++}$ , such that:

$$(III) \quad p_0^\eta \cdot (x_0 - e_{i0}) \leq -\gamma - q^\eta \cdot z \text{ and } p_s \cdot (x_\omega - e_{is}) \leq -\gamma + V(\omega) \cdot z, \forall \omega := (s, p_s) \in P_i^\eta.$$

From (III), the continuity of the scalar product and of  $\omega \mapsto V(\omega)$ , and from Lemma 3-(i)-(iii)-(vi), there exists  $N_1 \in \mathbb{N} \setminus \{1, \dots, N\}$ , such that, for every  $n \geq N_1$ :

$$(IV) \quad \begin{cases} p_0^n \cdot (x_0 - e_{i0}) \leq -q^n \cdot z \\ p_s^n \cdot (x_{\omega_s^n} - e_{is}) \leq V^n(s) \cdot z, \forall s \in \underline{\mathbf{S}} \\ p_s \cdot (x_\omega - e_{is}) \leq V(\omega) \cdot z, \forall \omega := (s, p_s) \in \Omega_i^n \end{cases} .$$

Along relations (IV) and Lemma 3-(i)-(v)-(iv), for each  $n \geq N_1$ , we define, in the economy  $\mathcal{E}_\eta^n$ , the strategy  $(x^n, z) \in B_i^n([\omega_s^n])$  by:  $x_0^n := x_0$ ,  $x_s^n := x_{\omega_s^n}$ , for every  $s \in \underline{\mathbf{S}}$ , and  $x_{s^n}^n := x_\omega$ , for every  $s^n := (i, \omega) \in \tilde{\mathbf{S}}_i^n$ . We recall that:

- $U_i(\pi_i^\eta, x) := \frac{1}{1 + \#\underline{\mathbf{S}}_\eta} \int_{\omega \in P_i} u_i(x_0, x_\omega) d\pi_i(\omega) + \frac{\eta}{1 + \#\underline{\mathbf{S}}_\eta} \sum_{s \in \underline{\mathbf{S}}} u_i(x_0, x_{\omega_s^\eta});$
- $u_i^n(x^n) := \sum_{s^n \in \tilde{\mathbf{S}}_i^n} u_i(x_0, x_{s^n}^n) \pi_i^n(s^n) + \frac{\eta}{1 + \#\underline{\mathbf{S}}_\eta} \sum_{s \in \underline{\mathbf{S}}} u_i(x_0, x_s^n).$

Then, from above, Lemma 3-(i)-(iv) and the uniform continuity of  $x \in X(\pi_i^\eta)$  and  $u_i$  on compact sets, there exists  $N_2 \geq N_1$  such that:

$$(V) \quad |U_i(\pi_i^\eta, x) - u_i^n(x^n)| < \int_{\omega \in P_i} |u_i(x_0, x_\omega) - u_i(x_0, x_{\arg_i^n(\omega)})| d\pi_i(\omega) < \frac{\varepsilon}{2}, \text{ for every } n \geq N_2.$$

From equilibrium conditions and Lemma 3-(vii), there exists  $N_3 \geq N_2$ , such that:

$$(VI) \quad u_i^n(x^n) \leq u_i^n(x_i^n) < \frac{\varepsilon}{2} + U_i(\pi_i^\eta, x_i^\eta), \text{ for every } n \geq N_3.$$

Let  $n \geq N_3$  be given. The above Conditions (I)-(V)-(VI) yield, jointly:

$$U_i(\pi_i^\eta, x) < \frac{\varepsilon}{2} + u_i^n(x^n) \leq \frac{\varepsilon}{2} + u_i^n(x_i^n) < \varepsilon + U_i(\pi_i^\eta, x_i^\eta) < U_i(\pi_i^\eta, x).$$

This contradiction proves that  $\mathcal{C}^\eta$  is indeed a C.F.E. and Theorem 1-(ii) holds.  $\square$

Claim 6, below, completes the proof of Theorem 1 via the following Lemma.

**Lemma 4** For each  $(i, k) \in I \times \mathbb{N}$ , we let  $\eta_k := \frac{1}{k}$  and denote simply by  $U_i^k$  the mapping  $x \mapsto U_i(\pi_i^{\eta_k}, x)$ , and by  $\mathcal{C}^k = ((\omega_s^k), (\pi_i^k), [(x_i^k, z_i^k)])$  the related C.F.E.,  $\mathcal{C}^k$ , of Claim 5.

For every  $(\omega := (s, p_s), z) \in P_i \times \mathbb{R}^J$ , we let  $B_i(\omega, z) := \{x \in \mathbb{R}_+^L : p_s \cdot (x - e_{is}) \leq V(\omega) \cdot z\}$  be a given set. Then, whenever  $\Delta \subset \cap_{i=1}^m P_i$ , the following Assertions hold for each  $i \in I$ :

(i) for each  $s \in \underline{\mathbf{S}}'$ , it may be assumed to exist prices,  $\omega_s^* = \lim_{k \rightarrow \infty} \omega_s^k \in \mathcal{M}_s$ , such that

$\{\omega_s^*\}_{s \in \underline{\mathbf{S}}} \subset \cap_{i=1}^m P_i$ , and consumptions,  $x_{is}^* = \lim_{k \rightarrow \infty} x_{is}^k$ , such that  $\sum_{i \in I} (x_{is}^* - e_{is}) = 0$ ;

(ii) it may be assumed to exist portfolios,  $z_i^* = \lim_{k \rightarrow \infty} z_i^k$ , such that  $\sum_{i=1}^m z_i^* = 0$ ;

(iii)  $\forall s \in \underline{\mathbf{S}}$ ,  $\{x_{is}^*\} = \arg \max_{x \in B_i(\omega_s^*, z_i^*)} u_i(x_{i0}^*, x)$  along Assertion (i)-(ii); we let  $x_{i\omega_s^*}^* := x_{is}^*$ ;

(iv) the correspondence  $\omega \in P_i \mapsto \arg \max_{x \in B_i(\omega, z_i^*)} u_i(x_{i0}^*, x)$  is a continuous mapping, denoted by  $\omega \mapsto x_{i\omega}^*$ . Its embedding,  $x_i^* : \omega \in \{0\} \cup P_i \mapsto x_{i\omega}^*$ , defined from Assertions

(i)-(ii)-(iii) and above, is a consumption plan, that is,  $x_i^* \in X(\pi_i)$ ;

(v) for all  $x \in X(\pi_i)$ ,  $U_i(\pi_i, x) = \lim_{k \rightarrow \infty} U_i^k(x) \in \mathbb{R}_+$  and  $U_i(\pi_i, x_i^*) = \lim_{k \rightarrow \infty} U_i^k(x_i^k) \in \mathbb{R}_+$ .

**Proof** see the Appendix.  $\square$

**Claim 6** Whenever  $\Delta \subset \cap_{i=1}^m P_i$ , the collection of prices,  $(\omega_s^*) = \lim_{k \rightarrow \infty} (\omega_s^k)$ , beliefs,  $(\pi_i)$ , allocation,  $(x_i^*)$ , and portfolios,  $(z_i^*) = \lim_{k \rightarrow \infty} (z_i^k)$ , of Lemma 4, is a C.F.E.

**Proof** The proof is similar to that of Claim 5. We assume that  $\Delta \subset \cap_{i=1}^m P_i$  and let  $\mathcal{C}^* := ((\omega_s^*), (\pi_i), [(x_i^*, z_i^*)])$  be defined from Lemma 4. Given  $(i, k) \in I \times \mathbb{N}$ , the relations  $\{\omega_s^k\}_{s \in \underline{S}} \subset \Delta \subset \cap_{i=1}^m P_i$  hold from Claim 5, and imply that  $P(\pi_i^k) = P_i$ , hence that,  $B_i(\omega_0^*, \pi_i)$  and  $B_i(\omega_0^k, \pi_i^k)$  may only differ by one budget constraint at  $t = 0$ . From Lemma 4,  $\mathcal{C}^*$  meets Conditions (a)-(c)-(d) of Definition 2. Let us denote  $\omega_0^* = (p_0^*, q^*) \in \mathcal{M}_0$  and  $\omega_0^k = (p_0^k, q^k) \in \mathcal{M}_0$ , for every  $k \in \mathbb{N}$ . Then, for every  $(i, k) \in I \times \mathbb{N}$ , the relations  $p_0^k \cdot (x_{i0}^k - e_{i0}) \leq -q^k \cdot z_i^k$  hold, from Claim 5, and yield, in the limit,  $p_0^* \cdot (x_{i0}^* - e_{i0}) \leq -q^* \cdot z_i^*$ , that is, from Lemma 4-(iv) and above:  $(x_i^*, z_i^*) \in B_i(\omega_0^*, \pi_i)$ . Thus, Claim 6 will be proved if we show that  $\mathcal{C}^*$  meets Definition 2-(b). By contraposition, assume this is not the case, i.e., there exists  $(i, (x, z), \varepsilon) \in I \times B_i(\omega_0^*, \pi_i) \times \mathbb{R}_{++}$ , such that:

$$(I) \quad \varepsilon + U_i(\pi_i, x_i^*) < U_i(\pi_i, x).$$

By the same token as for proving Claim 5, we may assume that the relation:

$$(II) \quad p_0^* \cdot (x_0 - e_{i0}) \leq -\gamma - q^* \cdot z, \text{ holds for some } \gamma \in \mathbb{R}_{++}.$$

From (II), Lemma 4-(i), continuity arguments and the identity of  $B_i(\omega_0^*, \pi_i)$  and  $B_i(\omega_0^k, \pi_i^k)$  on all second period budget constraints, there exists  $K \in \mathbb{N}$ , such that:

$$(III) \quad (x, z) \in B_i(\omega_0^k, \pi_i) = B_i(\omega_0^k, \pi_i^k), \text{ for every } k \geq K.$$

Relations (I)-(III), Lemma 4-(v) and the fact that  $\mathcal{C}^k$  is a C.F.E., yield:

$$(IV) \quad U_i(\pi_i, x) < \frac{\varepsilon}{2} + U_i^k(x) \leq \frac{\varepsilon}{2} + U_i^k(x_i^k) < \varepsilon + U_i(\pi_i, x_i^*) < U_i(\pi_i, x), \text{ for } k \geq K \text{ big enough.}$$

From this contradiction,  $\mathcal{C}^*$  is a CFE; the proof of Theorem 1 is now complete.  $\square$

## Appendix: proof of the Lemmas

**Lemma 1** For each  $i \in I$  and each  $n \in \mathbb{N}$ , there exists a finite partition,  $\mathcal{P}_i^n$ , and a finite subset,  $\Omega_i^n$ , of  $P_i$ , such that the following Assertions hold:

- (i)  $\forall P \in \mathcal{P}_i^n$ ,  $P \in \mathcal{B}(\mathcal{M})$  and  $\pi_i(P) > 0$ , and  $(\Omega_i^1) \in \mathcal{AS}$ ;
- (ii)  $\Omega_i^n \subset \Omega_i^{n+1}$ , and, for every  $P \in \mathcal{P}_i^n$ ,  $P \cap \Omega_i^n$  is a singleton;
- (iii)  $\forall \omega \in P_i$ ,  $\exists \omega' \in \Omega_i^n : \|\omega' - \omega\| \leq L\#S / n$ , and, hence,  $P_i = \overline{\cup_{n \in \mathbb{N}} \Omega_i^n}$ ;
- (iv)  $\exists N \in \mathbb{N}$ ,  $\forall (P_i^*) \leq (P_i)$ ,  $[(\Omega_i^N) \leq (P_i^*)$  and  $\#P_i^* \in \mathbb{N}$ ,  $\forall i \in I] \Rightarrow [Q_c[(P_i^*)] \neq \emptyset]$ .

**Proof** Let  $i \in I$  be given and, for each  $n \in \mathbb{N}$ , let:

$$K_n := \{k_n := (k_n^1, \dots, k_n^L) \in (\mathbb{N} \cap [1, 2^n])^L\};$$

$$P_s^{k_n} := P_i \cap (\{s\} \times \prod_{l \in \{1, \dots, L\}} ]\frac{k_n^l - 1}{2^n}, \frac{k_n^l}{2^n}], \text{ for every } (s, k_n := (k_n^1, \dots, k_n^L)) \in S_i \times K_n.$$

For each  $(s, n, k_n) \in S_i \times \mathbb{N} \times K_n$ , such that  $P_s^{k_n} \neq \emptyset$ , we select a unique  $\omega_s^{k_n} \in P_s^{k_n}$ , and define a set,  $\Omega_i^n := \{\omega_s^{k_n} \in P_s^{k_n} : s \in S_i, k_n \in K_n, P_s^{k_n} \neq \emptyset\}$ , as follows:

- for  $n = 1$ , we select one  $\omega_s^{k_1} \in P_i$ , for each  $s \in S_i$ ; we take  $\omega_s^{k_1} \in \cap_{i=1}^m P_i \neq \emptyset$ , whenever possible, and let  $\Omega_i^1 := \{\omega_s^{k_1} : s \in S_i\}$ ;
- for  $n > 1$  arbitrary, given  $\Omega_i^{n-1} := \{\omega_s^{k_{n-1}} \in P_s^{k_{n-1}} : s \in S_i, k_{n-1} \in K_{n-1}, P_s^{k_{n-1}} \neq \emptyset\}$ , we let, for every  $(s, k_n) \in S_i \times K_n$ , such that  $P_s^{k_n} \neq \emptyset$ ,<sup>3</sup>

$$\omega_s^{k_n} \begin{cases} \text{be equal to } \omega_s^{k_{n-1}}, \text{ if there exists } k_{n-1} \in K_{n-1}, \text{ such that } \omega_s^{k_{n-1}} \in \Omega_i^{n-1} \cap P_s^{k_n} \\ \text{be set fixed in } P_s^{k_n}, \text{ if } \Omega_i^{n-1} \cap P_s^{k_n} = \emptyset \end{cases}$$

This yields, for each  $n \in \mathbb{N}$ , a subset,  $\Omega_i^n := \{\omega_s^{k_n} : s \in S_i, k_n \in K_n, P_s^{k_n} \neq \emptyset\}$ , and a partition,  $\mathcal{P}_i^n := \{P_s^{k_n} : s \in S_i, k_n \in K_n, P_s^{k_n} \neq \emptyset\}$ , of  $P_i$ , satisfying Lemma 1-(i)-(ii)-(iii).

We now prove Lemma 1-(iv), after noticing, from Lemma 1-(i)-(ii), that  $(\Omega_i^N) \in \mathcal{AS}$ .

<sup>3</sup> Up to a shift in the upper boundary of  $P_s^{k_n}$ , if required, we assume costlessly that  $\pi_i(P_s^{k_n}) > 0$  when  $P_s^{k_n} \neq \emptyset$ .



For each  $(i, n) \in I \times \mathbb{N}$ , we define the vector space  $Z_i^n := \{z \in \mathbb{R}^J : V(\omega) \cdot z = 0, \forall \omega \in \Omega_i^n\}$  and its orthogonal,  $Z_i^{n\perp}$ , and, similarly,  $Z_i^* := \{z \in \mathbb{R}^J : V(\omega) \cdot z = 0, \forall \omega \in P_i\}$  and  $Z_i^{*\perp}$ . We show, first, that, for each  $i \in I$ , there exists  $N_i \in \mathbb{N}$ , such that  $Z_i^n = Z_i^*$ , for every  $n \geq N_i$ . Indeed, from Lemma 1-(ii),  $\{Z_i^n\}_{n \in \mathbb{N}}$  is non increasing in  $\mathbb{R}^J$ , hence, stationary, i.e., there exists  $N_i \in \mathbb{N}$ , such that  $Z_i^n = Z_i^{N_i}$ , for all  $n \geq N_i$ . From the definition, the relation  $Z_i^* \subset Z_i^{N_i}$  holds. The converse inclusion, hence,  $Z_i^* = Z_i^{N_i}$ , is immediate from Lemma 1-(iii): for all  $n > N_i$ , take  $z_n \in Z_i^{*\perp} \cap Z_i^n$ , such that  $\|z_n\| = 1$  and derive a contradiction. Then, we define  $N^o = \max_{i \in I} N_i$  and the compact set  $Z := \{(z_i) \in \prod_{i=1}^m Z_i^{*\perp} : \|(z_i)\| = 1, \sum_{i=1}^m z_i \in \sum_{i=1}^m Z_i^*\}$ .

Assume, by contraposition, that Lemma 1-(vi) fails. Then, from above, Definition 3, and from [4] (Definition 2.2, p. 397, and Proposition 3.1, p.401), for every  $n \geq N^o$ , there exist an integer,  $N_n \geq n$ , finite sets,  $P_i^{N_n}$ , defined for each  $i \in I$ , and portfolios,  $(z_i^n) \in Z$ , such that:  $(\Omega_i^{N_n}) \leq (P_i^{N_n}) \leq (P_i)$  and  $V(\omega_i) \cdot z_i^n \geq 0$ , for every  $(i, \omega_i) \in I \times P_i^{N_n}$ , with one strict inequality. The sequence,  $\{(z_i^n)\}_{n \geq N^o}$ , may be assumed to converge in a compact set, say to  $(z_i^*) \in Z$ . From the continuity of the scalar product and Lemma 1-(iii), the above relations on  $\{(z_i^n)\}_{n \geq N^o}$ , imply that  $V(\omega_i) \cdot z_i^* \geq 0$  holds, for every  $(i, \omega_i) \in I \times P_i$ , with one strict inequality, since  $(z_i^*) \in Z$  implies  $\|(z_i^*)\| = 1$ . We let the reader check (on asset prices), this contradicts the fact that  $(P_i)$  is arbitrage-free.  $\square$

**Lemma 2** *For each  $n > N$  along Lemma 1, the economy  $\mathcal{E}_\eta^n$  admits an equilibrium,  $\mathcal{C}_\eta^n$ , namely, a collection of prices,  $\omega_0^n := (p_0^n, q^n) \in \mathcal{M}_0$ , at  $t = 0$ , and  $\omega_s^n := (s, p_s^n) \in \mathcal{M}_s$ , in each state  $s \in \underline{\mathbf{S}}$ , and strategies,  $(x_i^n, z_i^n) \in B_i^n([\omega_s^n])$ , for each  $i \in I$ , such that:*

- (i)  $\forall i \in I, (x_i^n, z_i^n) \in \arg \max_{(x, z) \in B_i^n([\omega_s^n])} u_i^n(x)$ ;
- (ii)  $\forall s \in \underline{\mathbf{S}}', \sum_{i=1}^m (x_{is}^n - e_{is}) = 0$ ;
- (iii)  $\sum_{i=1}^m z_i^n = 0$ .

*Moreover, the equilibrium,  $\mathcal{C}_\eta^n$ , satisfies the following Assertions:*

(iv)  $\forall(n, i, s) \in \mathbb{N} \setminus \{1, \dots, N\} \times I \times \underline{\mathbf{S}}'$ ,  $x_{is}^n \in [0, E]^L$ , where  $E := \max_{(s,l) \in S' \times \mathcal{L}} \sum_{i=1}^m e_{is}^l$ ;

(v)  $\exists \varepsilon \in ]0, 1[ : p_s^{nl} \geq \varepsilon$ ,  $\forall(n, s, l) \in \mathbb{N} \setminus \{1, \dots, N\} \times \underline{\mathbf{S}} \times \mathcal{L}$ .

**Proof** Let  $n > N$  be given along Lemma 1. From Lemma 1-(iv), the payoff and information structure of the economy  $\mathcal{E}_\eta^n$ ,  $[V^n, (S_i^n)]$ , is arbitrage-free, along [4]. Hence, from ([6], Theorem 1 and proof) it admits an equilibrium, or (changing notations) a collection of prices,  $\omega_0^n := (p_0^n, q^n) \in \mathcal{M}_0$ ,  $\omega_s^n := (s, p_s^n) \in \mathcal{M}_s$ , for each  $s \in \underline{\mathbf{S}}$ , and strategies,  $(x_i^n, z_i^n) \in B_i^n([\omega_s^n])$ , for each  $i \in I$ , which satisfy Lemma 2-(i)-(ii)-(iii). The rest of the proof is similar to that of Theorem 1-(i) (simpler) and left to the reader.  $\square$

**Lemma 3** For the above sequence,  $\{\mathcal{C}_\eta^n\}$ , of equilibria, it may be assumed to exist:

(i)  $\omega_s^\eta = \lim_{n \rightarrow \infty} \omega_s^n \in \mathcal{M}_s$ , for each  $s \in \underline{\mathbf{S}}'$ ;

(ii)  $(x_{is}^\eta) := \lim_{n \rightarrow \infty} (x_{is}^n)_{i \in I} \in \mathbb{R}^{Lm}$ , such that  $\sum_{i \in I} (x_{is}^\eta - e_{is}) = 0$ , for each  $s \in \underline{\mathbf{S}}'$ ;

(iii)  $(z_i^\eta) = \lim_{n \rightarrow \infty} (z_i^n)_{i \in I} \in \mathbb{R}^{Jm}$ , such that  $\sum_{i=1}^m z_i^\eta = 0$ .

Moreover, we define, for each  $i \in I$  and each  $n \in \mathbb{N}$ , the following sets and mappings:

\* the mapping,  $\omega \in P_i \mapsto \arg_i^n(\omega) \in \Omega_i^n$ , such that one  $P \in \mathcal{P}_i^n$  satisfies  $(\omega, \arg_i^n(\omega)) \in P^2$ ;

\* from Assertion (i) and Lemma 2-(v), the belief,  $\pi_i^\eta := \frac{1}{1 + \#\underline{\mathbf{S}}_\eta} (\pi_i + \eta \sum_{s \in \underline{\mathbf{S}}} \delta_s)$ ,

where  $\delta_s$  is (for each  $s \in \underline{\mathbf{S}}$ ) the Dirac's measure of  $\omega_s^\eta$ ;

\*  $P_i^\eta = P_i \cup \{\omega_s^\eta\}_{s \in \underline{\mathbf{S}}}$ , the support of  $\pi_i^\eta \in \mathcal{B}$ ;

\*  $B_i(\omega, z) := \{x \in \mathbb{R}_+^L : p_s \cdot (x - e_{is}) \leq V(\omega) \cdot z\}$ , for every  $\omega := (s, p_s) \in \mathcal{M}$ ,  $z \in \mathbb{R}^J$ .

Then, the following Assertions hold, for each  $i \in I$ :

(iv)  $\{\arg_i^n(\omega)\}_{n \in \mathbb{N}}$  converges to  $\omega$  uniformly on  $P_i$ ;

(v)  $\forall s \in \underline{\mathbf{S}}$ ,  $\{x_{is}^\eta\} = \arg \max u_i(x_{i0}^\eta, x)$ , for  $x \in B_i(\omega_s^\eta, z_i^\eta)$ , along Assertions (i)-(ii)-(iii);

we denote by  $x_{i\omega}^\eta := x_{is}^\eta \in \mathbb{R}_+^L$  a related consumption decision contingent on  $\omega_s^\eta \in \mathcal{M}_s$ ;

(vi) the correspondence  $\omega \in P_i^\eta \mapsto \arg \max u_i(x_{i0}^\eta, x)$ , for  $x \in B_i(\omega, z_i^\eta)$ , is a continuous

mapping, denoted by  $\omega \mapsto x_{i\omega}^\eta$ . The mapping,  $x_i^\eta : \omega \in \{0\} \cup P_i^\eta \mapsto x_{i\omega}^\eta$ , defined from

Assertions (ii), (v) and above, is a consumption plan, that is,  $x_i^\eta \in X(\pi_i^\eta)$ ;

(vii)  $U_i(\pi_i^\eta, x_i^\eta) = \lim_{n \rightarrow \infty} u_i^n(x_i^n) \in \mathbb{R}_+$ .

**Proof** Assertions (i)-(ii) result from Lemma 2-(iv) and compactness arguments.  $\square$

Assertion (iii) For each  $i \in I$ , we let  $Z_i^* := \{z \in \mathbb{R}^J : V(\omega) \cdot z = 0, \forall \omega \in P_i\}$  and recall, from Lemma 1's proof, that  $Z_i^n = \{z \in \mathbb{R}^J : V(\omega) \cdot z = 0, \forall \omega \in \Omega_i^n\}$ , for all  $n \geq N$  (assuming  $N > N^0$ ). The sequence  $\{(z_i^n)_{i \in I}\}_{n > N}$  is bounded. Indeed, let  $\delta := \max_{i \in I} \|e_i\|$ . The definition of  $\{\mathcal{C}^n\}_{n > N}$  yields, from budget constraints and clearance conditions:

$$(I) \quad \left[ \sum_{i=1}^m z_i^n = 0 \text{ and } V(\omega_i) \cdot z_i^n \geq -\delta, \forall (i, \omega_i) \in I \times \Omega_i^n \right], \text{ for every } n > N.$$

Assume, by contradiction,  $\{(z_i^n)\}$  is unbounded, i.e., there exists an extracted sequence,  $\{(z_i^{\varphi(n)})\}$ , such that  $n < \|(z_i^{\varphi(n)})\| \leq n+1$ , for every  $n > N$ . From (I), the portfolios  $(\bar{z}_i^n) := \frac{1}{n}(z_i^{\varphi(n)})$  meet the relations  $1 < \|(\bar{z}_i^n)\| \leq 1 + \frac{1}{n}$ , for every  $n > N$ , and:

$$(II) \quad \sum_{i=1}^m \bar{z}_i^n = 0 \text{ and } V(\omega_i) \cdot \bar{z}_i^n \geq -\frac{\delta}{n}, \forall (i, \omega_i) \in I \times \Omega_i^n.$$

From (II), Lemma 1-(ii), the continuity of the scalar product and above, the sequence  $\{(\bar{z}_i^n)\}$  may be assumed to converge, say to  $(z_i^*)$ , such that  $\|(z_i^*)\| = 1$  and:

$$(III) \quad \sum_{i=1}^m z_i^* = 0 \text{ and } V(\omega_i) \cdot z_i^* \geq 0, \forall (i, \omega_i) \in I \times \Omega_i^N.$$

From relations (III), Lemma 1-(iv), ([4], Proposition 3.1) and above, the relation  $(z_i^*) \in \Pi_{i=1}^m Z_i^*$  holds and implies  $(z_i^*) = 0$ , from the elimination of useless deals of Sub-Section 2.4, which contradicts the fact that  $\|(z_i^*)\| = 1$ . Hence, the sequence  $\{(z_i^n)\}$  is bounded and may be assumed to converge, say to  $(z_i^\eta) \in \mathbb{R}^{Jm}$ . Then, the relation  $\sum_{i=1}^m z_i^\eta = 0$  results asymptotically from the clearance conditions of Lemma 2-(iv).  $\square$

Assertions (iv) is immediate from the definition and compactness arguments.  $\square$

Assertion (v) Let  $(i, s) \in I \times \underline{\mathbf{S}}$  be given. For every tuple  $(n, \omega := (s, p_s), \omega', z) \in \mathbb{N} \setminus \{1, \dots, N\} \times \mathcal{M}_s \times \mathcal{M}_s \times \mathbb{R}^J$ , we consider the following (possibly empty) sets:

$B_i(\omega, z) := \{y \in \mathbb{R}_+^L : p_s \cdot (y - e_{is}) \leq V(\omega) \cdot z\}$  and  $B'_i(\omega, \omega', z) := \{y \in \mathbb{R}_+^L : p_s \cdot (y - e_{is}) \leq V(\omega') \cdot z\}$ .

For each  $n > N$ , the fact that  $C_\eta^n$  is an equilibrium of  $\mathcal{E}_\eta^n$  implies, from Lemma 2:

$$(I) \quad (\omega_s^{n-1}, \omega_s^n) \in \mathcal{M}_s^2 \text{ and } x_{is}^n \in \arg \max_{y \in B'_i(\omega_s^n, \omega_s^{n-1}, z_i^n)} u_i(x_{i0}^n, y).$$

As a standard application of Berge's Theorem (see, e.g., [8], p. 19), the correspondence  $(x, \omega, \omega', z) \in \mathbb{R}_+^L \times \mathcal{M}_s \times \mathcal{M}_s \times \mathbb{R}^J \mapsto \arg \max_{y \in B'_i(\omega, \omega', z)} u_i(x, y)$ , which is actually a mapping (from Assumption A2), is continuous at  $(x_{i0}^\eta, \omega_s^\eta, \omega_s^\eta, z_i^\eta)$ , since  $u_i$  and  $B'_i$  are continuous. Moreover, the relation  $(x_{i0}^\eta, x_{is}^\eta, \omega_s^\eta, z_i^\eta) = \lim_{n \rightarrow \infty} (x_{i0}^n, x_{is}^n, \omega_s^n, z_i^n)$  holds from Lemma 2-(i)-(ii)-(iii). Hence, the relations (I) pass to that limit and yield:

$$\{x_{i\omega_s^\eta}^\eta\} := \{x_{is}^\eta\} = \arg \max_{y \in B_i(\omega_s^\eta, z_i^\eta)} u_i(x_{i0}^\eta, y). \quad \square$$

Assertion (vi) Let  $i \in I$  be given. For every  $(\omega, n) \in P_i \times \mathbb{N} \setminus \{1, \dots, N\}$ , the fact that  $C_\eta^n$  is an equilibrium of  $\mathcal{E}_\eta^n$  and Assumption A2 yield:

$$(I) \quad \{x_{i\arg_i^n(\omega)}^n\} = \arg \max u_i(x_{i0}^n, y) \text{ for } y \in B_i(\arg_i^n(\omega), z_i^n).$$

From Lemma 2-(ii)-(iii)-(iv), the relation  $(\omega, x_{i0}^\eta, z_i^\eta) = \lim_{n \rightarrow \infty} (\arg_i^n(\omega), x_{i0}^n, z_i^n)$  holds, whereas, from Assumption A2 and ([8], p. 19), the correspondence  $(x, \omega, z) \in \mathbb{R}_+^L \times P_i \times \mathbb{R}^J \mapsto \arg \max_{y \in B_i(\omega, z)} u_i(x, y)$  is a continuous mapping, since  $u_i$  and  $B_i$  are continuous. Hence, passing to the limit into relations (I) yields a continuous mapping,  $\omega \in P_i \mapsto x_{i\omega}^\eta := \arg \max_{y \in B_i(\omega, z_i^\eta)} u_i(x_{i0}^\eta, y)$ , which, from Lemma 3-(v) and above, is embedded into a continuous mapping,  $x_i^\eta : \omega \in \{0\} \cup P_i^\eta \mapsto x_{i\omega}^\eta$ , i.e.,  $x_i^\eta \in X(\pi_i^\eta)$ .  $\square$

Assertion (vii) Let  $i \in I$  and  $x_i^\eta \in X(\pi_i^\eta)$  be given, along Lemma 3-(vi). Let  $\varphi_i : (x, \omega, z) \in \mathbb{R}_+^L \times P_i \times \mathbb{R}^J \mapsto \arg \max_{y \in B_i(\omega, z)} u_i(x, y)$  be defined on its domain. By the same token as for proving Assertion (vi),  $\varphi_i$  and  $U_i : (x, \omega, z) \in \mathbb{R}_+^L \times P_i \times \mathbb{R}^J \mapsto u_i(x, \varphi_i(x, \omega, z))$  are continuous mappings and, moreover, the relations  $u_i(x_{i0}^\eta, x_{i\omega}^\eta) = U_i(x_{i0}^\eta, \omega, z_i^\eta)$  and

$u_i(x_{i0}^n, x_{i \arg_i^n(\omega)}^n) = U_i(x_{i0}^n, \arg_i^n(\omega), z_i^n)$  hold, for every  $(\omega, n) \in P_i \times \mathbb{N} \setminus \{1, \dots, N\}$ . Then, the uniform continuity of  $u_i$  and  $U_i$  on compact sets, and Lemma 3-(ii)-(iii)-(iv) yield:

$$(I) \quad \forall \varepsilon > 0, \exists N_\varepsilon > N : \forall n > N_\varepsilon, \forall \omega \in P_i,$$

$$| u_i(x_{i0}^\eta, x_{i\omega}^\eta) - u_i(x_{i0}^n, x_{i \arg_i^n(\omega)}^n) | + \sum_{s \in \underline{\mathbf{S}}} | u_i(x_{i0}^\eta, x_{is}^\eta) - u_i(x_{i0}^n, x_{is}^n) | < \varepsilon.$$

Moreover, we recall the following definitions, for every  $n > N$ :

$$(II) \quad U_i(\pi_i^\eta, x_i^\eta) := \frac{1}{1+\#\underline{\mathbf{S}}\eta} \int_{\omega \in P_i} u_i(x_{i0}^\eta, x_{i\omega}^\eta) d\pi_i(\omega) + \eta \frac{1}{1+\#\underline{\mathbf{S}}\eta} \sum_{s \in \underline{\mathbf{S}}} u_i(x_{i0}^\eta, x_{is}^\eta);$$

$$(III) \quad u_i^n(x_i^n) := \frac{1}{1+\#\underline{\mathbf{S}}\eta} \int_{\omega \in P_i} u_i(x_{i0}^\eta, x_{i \arg_i^n(\omega)}^\eta) d\pi_i(\omega) + \eta \frac{1}{1+\#\underline{\mathbf{S}}\eta} \sum_{s \in \underline{\mathbf{S}}} u_i(x_{i0}^\eta, x_{is}^\eta).$$

Then, Lemma 3-(vii) results immediately from relations (I)-(II)-(III) above.  $\square$

**Proof of Lemma 4** It is similar to that of Lemma 3, hence, left to the reader.  $\square$

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