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Strategic Voting under Committee Approval: A Theory

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Abstract

We propose a theory of strategic voting under “Committee Approval”: a fixed-sized committee of M members is to be elected; each voter votes for as many candidates as she wants, and the M candidates with the most votes are elected. We assume that voter preferences are separable and that there exists a tiny probability that any vote might be misrecorded.

We show that best responses involve voting by pairwise comparisons. Two candidates play a critical role: the *weakest expected winner* and the *strongest expected loser*. Expected winners are approved if and only if they are preferred to the strongest expected loser and expected losers are approved if and only if they are preferred to the weakest expected winner.

At equilibrium, if any, a candidate is elected if and only if he is approved by at least half of the voters. With single-peaked preferences, an equilibrium always exists, in which the first M candidates according to the majority tournament relation are elected.

1 Introduction

In many instances, societies choose, by voting, a group of representatives. Voting rules for these kinds of elections are more complex than rules designed to elect one and only one candidate, and are much less studied in the theoretical literature. Cases of interest include Parliamentary elections and committee selection.

Parliamentary elections often proceed by dividing the electorate in subgroups, usually on a geographical basis, and by electing one or several MEPs in each such district (Blais and Massicotte 2002). The number of delegates is usually fixed for each district, although it may be adjusted in view of the overall results (such is the case in Germany). The set of candidates is usually structured with party lists and, possibly, independent candidates.

Committee selection offers another kind of complexity because, contrary to most Parliamentary elections, some structure is often imposed on the set of elected candidates. For example the chosen committee must reflect some gender or status balance.

The present paper will concentrate on the simplest case where the number of seats to be filled is fixed and has no particular structure, and where the electorate is not divided (and the voters are anonymous). Then a natural rule is that each voter can vote for several candidates and the candidates with the largest numbers of votes are elected. Under “Plain Committee Approval” each voter can vote for as many candidates as she wishes, but can only give one vote to each candidate (no “cumulative voting”). Under “Limited Committee Approval” a voter cannot cast more than a fixed number of votes (usually set to the number of candidates, but this needs not be). The contribution of this paper is twofold. First, since there exists so far no complete and testable theory of strategic voting under Committee Approval at the individual level (best-responses), we propose such a theory, drawing on previous works for the standard case of Approval voting for electing a single candidate (Laslier 2009, Nuñez 2010a). Second, we derive equilibrium predictions.

Related literature The literature on Committee Approval voting has mainly focused on the different ways approval-type ballots can be counted for electing a committee (of fixed size or not). Electing the candidates with the largest approval scores is the simplest but not the only idea one can have (Fishburn 1981). Kilgour (2010) surveys the many proposals which have been made, and Laffond and Lainé (2010) survey the representativeness issue under an assumption of separable preferences. This issue of the representativeness of an electorate by a committee is often tackled in the theoretical literature under the assumption that the committee size is not fixed, which makes the problem similar to a multiple referendum problem. In this vein, see Gehrlein (1985), Bock et al. (1998), Brams et al. (1997, 1998), Brams et al. (2007). We here focus on the case — often met in practice — of a fixed-size committee.

The issue of the voter’s behavior (which ballot to cast?) is not addressed by the previously mentioned studies and one question left pending is to describe “sincere” and “rational” behavior in these elections and to evaluate the level of strategic voting induced by such a voting rule. One exception is Cox (1984) who studies the special case of multi-member districts with two members to be elected and three candidates, when the voter is allowed to cast up to two votes. He shows that depending on the context (anticipations about other voters’ behavior and own preferences), strategic voting in such an election entails either voting for one’s preferred candidate only, or voting for one’s two preferred candidates. In this paper, we will characterize best-responses for any configuration about the number of seats, the number of candidates, and the maximal number of votes a voter is allowed to cast. We will also consider equilibrium predictions.

Section 2 describes the model. We will assume that voter preferences over committees are separable and that there exists a tiny probability that any vote might be mis-recorded. This latter assumption will guarantee that the voter is uncertain about the realized scores of the candidates, even when she knows other voters' strategies (as is standard in strategic voting models, see for example Myerson and Weber 1993). Section 3 provides some preliminary results on the probability of some critical pivot events. Focusing first on the case of "Plain Committee Approval", Section 4 studies best responses and Section 5 studies equilibria. Section 6 is devoted to the case where a limit is set on the number of candidates a voter can approve ("Limited Committee Approval"). Section 7 concludes. Long proofs are relegated in an appendix. A companion paper (Lachat et al. 2014) tests the theory on Swiss data.

2 A model of Committee Approval

In the first sections of the paper, we will focus on the case of "Plain Committee Approval", where each voter can vote for as many candidates as she wishes, with no limit on the maximal number of votes she can cast. The case of "Limited Committee Approval" will be tackled in Section 6.

(Plain) Committee Approval M seats have to be filled. The set of candidates, of size $K > M$, is denoted by \mathfrak{C} . There are N voters, $i = 1, \dots, N$.

Voters vote by casting votes for candidates; they can give at most one vote to a candidate (no "cumulative voting") but can vote for several candidates. The M candidates with the highest numbers of votes are elected. Ties, if any, are randomly broken.

Voter preferences Voters preferences over committees are supposed to be separable across candidates in the following sense: Voter i has a utility function u_i for candidates, and the utility for the committee C is the sum $\sum_{c \in C} u_i(c)$, where C is any subset of size M of the set of candidates \mathfrak{C} . We assume for simplicity that preferences over the set of candidates (as described by the utility function u_i) are strict.

Preferences are common knowledge and there is no uncertainty about the size (N) of the electorate.

Voter strategies For $i = 1, \dots, N$, a strategy for voter i is a vector $s_i = (s_{i,c})_{c \in \mathfrak{C}} \in \{0, 1\}^K$, where for all c , $s_{i,c} = 1$ if voter i casts a vote in favor of candidate c , and $s_{i,c} = 0$ if voter i does not cast a vote for candidate c (we will also use the terminology "casts a vote against candidate c " or simply "votes against c ").

Small mistakes As is standard in strategic voting models (see for example Myerson and Weber 1993), we assume that the voter is uncertain about the

realized scores of the candidates, even when she knows other voters' strategies. Uncertainty is modelled as follows. As described above, preferences are common knowledge and there is no uncertainty about the size (N) of the electorate. But, for any vote which is actually cast for a candidate by a voter, there is a tiny possibility of mistake, a mistake resulting in that vote not being recorded. Conversely, even if a voter has not voted for a candidate, there is a tiny probability that this is wrongly recorded as a vote. We assume that the mistakes are made independently across voters and across candidates.¹

More formally: We suppose that there exists a number $\varepsilon > 0$ such that, for each ballot cast by a voter, and for each candidate c :

- if i votes for c , this vote is recorded with probability $(1 - \varepsilon)$, and with probability ε this vote is not recorded;
- if i does not vote for c , this is correctly recorded with probability $(1 - \varepsilon)$, and with probability ε a vote for candidate c is instead recorded.

For example, with $K = 3$ candidates, assume that a voter has cast the ballot $(1, 1, 0)$. Given our assumptions about the small mistakes made while recording the votes, this ballot is correctly recorded as such with a probability $(1 - \varepsilon)^3$, it is recorded as $(0, 1, 0)$ with probability $(1 - \varepsilon)^2 \cdot \varepsilon$ (one mistake), ..., and recorded as $(0, 0, 1)$ with probability ε^3 (three mistakes).

These assumptions guarantee that, for any profile of ballots cast by the voters, all electoral outcomes (realized scores of candidates) have a positive probability.

Voters' beliefs Preferences and the structure of the game, including the possibilities of mistakes described above, are common knowledge among the voters.

We will assume that the voters in their computation of best responses neglect the possibility of three-way ties; a cognitive assumption which seems realistic for an individual taking part to a large election.

When needed we also assume that the expected scores of any two candidates differ by at least three votes. These assumptions are well suited for large elections (typically, political ones) but would not be reasonable if one wanted to study small electorates.

3 Pivotal events with minimal requirements

In order to determine her best response against the other voters' strategies, the voter will have to estimate the probability of the different events where her vote might be pivotal (that is, change the outcome of the election). Before turning to the study of best responses (Section 4) and equilibria (Section 5), we start by computing the order of magnitude of some critical pivot events. We will first

¹There is no independence across candidates in Nuñez (2010b).

introduce the notion of *minimal requirement*, then use it to estimate the order of magnitude of some critical events involving ties between candidates (Lemma 1 and Lemma 2).

Distribution of realized scores For a profile of ballots $s = (s_i)_{i=1,\dots,N}$ and for a candidate c , denote by $\widehat{s}(c) = \sum_i s_{i,c}$ the number of voters who vote for c , and by $S(c)$ the random variable describing the realized score of candidate c (taking into account the possibility of mistakes) obtained from these ballots.

For any two candidates c and c' , $S(c)$ and $S(c')$ are independent random variables, with expectations $\widehat{s}(c)$ and $\widehat{s}(c')$ respectively. Note that the random variable $S(c)$ can be written as:

$$S(c) = \sum_i [s_{i,c}(1 - \omega_{i,c}) + (1 - s_{i,c})\omega_{i,c}], \quad (1)$$

where the $\omega_{i,c}$, for $i = 1, \dots, N$ and $c \in \mathfrak{C}$, are $N \cdot K$ independent random draws which take value 0 with probability $(1 - \varepsilon)$ and 1 with probability ε . Here, $\omega_{i,c} = 1$ means that a mistake is made when recording voter i 's vote about candidate c .

Write $|\omega| = \sum_{i,c} \omega_{i,c}$ the number of mistakes corresponding to the elementary event ω . The probability of any elementary event ω is:

$$\Pr[\omega] = \varepsilon^{|\omega|} \cdot (1 - \varepsilon)^{NK - |\omega|} = \sum_{k=0}^{k=NK - |\omega|} (-1)^k \binom{NK - |\omega|}{k} \varepsilon^{|\omega| + k}. \quad (2)$$

Notice that this is a polynomial in ε whose first term of lowest degree is $\varepsilon^{|\omega|}$: when the probability of mistake ε goes to zero, the probability of the elementary event ω is asymptotically equivalent to $\varepsilon^{|\omega|}$.

Definition of the “requirement of an event” Any event E can be expressed with the help of the elementary events ω and thus has a probability which is a polynomial in ε . For any event E , let us denote by $A(E)\varepsilon^{m(E)}$ the term of lowest degree of this polynomial, where $m(E)$ is the smallest number of mistakes required to realize E , and $A(E)$ is the number of ways to realize E with $m(E)$ mistakes. The exponent $m(E)$ will be called the **requirement** of event E . Note that, from (2), for any elementary event, $m(\omega) = |\omega|$ and $A(\omega) = 1$.

The requirement of an event is an indicator of how unlikely this event is to happen. Indeed, between two events E and E' of requirements m and m' with $m < m'$, the probability of E' is “vanishingly small” compared to the probability of E , meaning that when ε tends to 0, the ratio $\Pr[E']/\Pr[E]$ tends to 0. This concept of requirement will play an important role when deriving best responses.

Computation of the requirement of critical events The following two lemmas will give some insights about the requirement of some critical events involving ties between candidates.

Lemma 1 *Given a profile of strategies (ballots) $s = (s_i)_{i=1,\dots,N}$, for any two candidates c and c' , the requirement of the event “ $S(c) = S(c')$ ” is $|\widehat{s}(c) - \widehat{s}(c')|$.*

Lemma 1 states that given the ballots cast by the voters, the probability of candidates c and c' obtaining the exact same realized scores is asymptotically equivalent to $\varepsilon^{|\widehat{s}(c) - \widehat{s}(c')|}$, where $|\widehat{s}(c) - \widehat{s}(c')|$ is the absolute value of the difference in expected scores between candidate c and candidate c' . The proof of the lemma is provided in the appendix (section 8.1).

Consider now any candidate c . We will say that realized scores are such that candidate c is *caught in an exact tie for election* if whether c is elected or not has to be determined by a random draw (at least two candidates, including c , tie for the M -th position). The following lemma provides the requirement of such an event, for all candidates.

Lemma 2 *Given a profile of strategies $s = (s_i)_{i=1,\dots,N}$, assume that candidates are labelled in such a way that:*

$$\widehat{s}(c_1) > \widehat{s}(c_2) > \dots > \widehat{s}(c_M) > \widehat{s}(c_{M+1}) > \dots > \widehat{s}(c_K).$$

(i) *If $k \leq M$, the requirement of the event “Candidate c_k is caught in an exact tie for election” is $\widehat{s}(c_k) - \widehat{s}(c_{M+1})$. Besides, any event of minimal requirement where c_k is caught in an exact tie for election involves a tie with candidate c_{M+1} .*

(ii) *If $k \geq M + 1$, the requirement of the event “Candidate c_k is caught in an exact tie for election” is $\widehat{s}(c_M) - \widehat{s}(c_k)$. Besides, any event of minimal requirement where c_k is caught in an exact tie for election involves a tie with candidate c_M .*

Note the crucial role played by two candidates: c_M and c_{M+1} . The former is the candidate whose expected score is the M -th largest — we will call this candidate the *weakest expected winner* and the latter the candidate whose expected score is the $(M + 1)$ -th largest — we will call this candidate the *strongest expected loser*. The proof of the lemma is provided in the appendix (section 8.2).

4 Best responses

4.1 Characterization

We first describe a voter’s, say voter i ’s, best response against a profile of strategies $s_{-i} = (s_j)_{j \neq i}$ by the other $N - 1$ voters. Given this profile s_{-i} , for all c , denote by $\widehat{s}_{-i}(c) = \sum_{j \neq i} s_{j,c}$ the number of voters (other than voter i) who vote for c . Given our model of uncertainty, $\widehat{s}_{-i}(c)$ is the expected score of candidate c , not taking into account the vote of voter i . Proposition 3 describes the voter’s best response in the case where the expected vote difference between any two candidates is at least 3.

Proposition 3 Let \widehat{s}_{-i} denote the vector of expected scores obtained by the candidates from the votes of all the voters except voter i . Let the candidates be labelled in such a way that:

$$\widehat{s}_{-i}(c_1) > \widehat{s}_{-i}(c_2) > \dots > \widehat{s}_{-i}(c_M) > \widehat{s}_{-i}(c_{M+1}) > \dots > \widehat{s}_{-i}(c_K). \quad (3)$$

Assume that the expected vote difference between any two candidates is at least 3, that is, $\widehat{s}_{-i}(c_k) - \widehat{s}_{-i}(c_{k+1}) \geq 3$ for all $k = 1, \dots, K - 1$.

For ε small enough, the best response of voter i is the following:

- For $1 \leq k \leq M$: Voter i votes for c_k if and only if $u_i(c_k) > u_i(c_{M+1})$,
- For $M + 1 \leq k \leq K$: Voter i votes for c_k if and only if $u_i(c_k) > u_i(c_M)$.

With assumption (3) regarding the ranking of candidates, the first M candidates are the expected winners, and the other candidates are the expected losers.²

Proposition 3 states that the voter should vote for an expected winner if and only if she prefers that candidate to the candidate ranked $M + 1$, that is, the *strongest expected loser*. Symmetrically, the voter should vote for an expected loser if and only if she prefers that candidate to the candidate ranked M , that is, the *weakest expected winner*. Best responses are thus quite easy to describe: they entail voting by pairwise comparison with those two critical candidates: the strongest expected loser and the weakest expected winner.

The proof of the proposition is presented in the appendix (section 8.3); but the intuition is quite simple. It mostly derives from Lemma 2. Lemma 2 states that the requirement of the event "Candidate c_k is caught in an exact tie for election" is $\widehat{s}(c_k) - \widehat{s}(c_{M+1})$ if $k \leq M$ and $\widehat{s}(c_M) - \widehat{s}(c_k)$ if $k \geq M + 1$.³

- Therefore, the most likely tie for election occurs between candidates c_M (the weakest expected winner) and c_{M+1} (the strongest expected loser), since the requirement of this event is $\widehat{s}(c_M) - \widehat{s}(c_{M+1})$. If voter i is pivotal, it will most likely be in deciding who between candidate c_M and candidate c_{M+1} will be elected.⁴ Therefore, if she prefers candidate c_M to candidate c_{M+1} ($u_i(c_M) > u_i(c_{M+1})$), she should vote for candidate c_M and not vote for candidate c_{M+1} . Similarly, if $u_i(c_M) < u_i(c_{M+1})$, she should

²The additional assumption that the expected vote difference between any two candidates in \widehat{s}_{-i} is at least 3 guarantees that the expected winners and losers in the election remain the same whatever the ballot chosen by voter i .

³Lemma 2 takes into account the votes of all voters, including voter i . To derive the best response of voter i , the argument should be adjusted to take into account that voter i considers only the votes of other voters as given. These adjustments are made in the proof in the appendix, but the intuition about the orders of magnitude of the different pivot events remains similar.

⁴Lemma 2 deals with exact ties for election. A voter can also be pivotal in case of a near tie (one vote margin) for election between two candidates. Noting that a requirement of a near tie is no larger than the requirement of an exact tie plus one, the arguments carry through when explicitly taking into account the possibility of near ties (which is done in the proof).

vote for candidate c_{M+1} and not vote for candidate c_M . Her decision about candidates c_M and c_{M+1} is thus decided by this pairwise comparison between the two candidates.

- Consider now candidate c_k , $k < m$, an expected winner. If candidate k is caught in a tie, it will most likely be against candidate c_{M+1} (Lemma 2). Therefore the voter should vote for candidate c_k if and only if $u_i(c_k) > u_i(c_{M+1})$: the vote for candidate c_k if decided by a pairwise comparison with the strongest expected loser c_{M+1} .
- Similarly, if candidate c_k , $k > M + 1$, is caught in a tie, it will most likely be against candidate c_M , and therefore the voter should vote for candidate c_k if and only if $u_i(c_k) > u_i(c_M)$: the vote for candidate c_k if decided by a pairwise comparison with the weakest expected winner c_M .

4.2 Properties of best responses: Sincere and non-sincere voting

According to the usual definition in the Approval Voting literature (Brams 1982), a ballot is “sincere” for a voter if, when the voter approves a candidate c , she also approves all the candidates she strictly prefers to c . Proposition 4 characterizes the parameters of the electoral context such that a best response always entails casting a sincere ballot, whatever the voter’s preferences and the other voters’ strategies.

Proposition 4 *Consider the best response function described in Proposition 3, for $\varepsilon \rightarrow 0$.*

- *If $M = 1$ or $M = K - 1$, the best response always entails casting a **sincere ballot**, whatever the voter’s preferences and the other voters’ strategies.*
- *Otherwise, there exists voter’s preferences and other voters’ strategies, such that the best response entails casting a **non-sincere ballot**.*

Laslier (2009) noticed that a best response always entails sincere voting when there is one single candidate to elect ($M = 1$). Cox (1984) noticed that a best response always entails sincere voting when there are two candidates to be elected from a set of three candidates ($M = 2$ and $K = 3$).⁵ Proposition 4 generalizes the result by Cox (1989) by showing that this holds true whenever the number of running candidates exceeds only by one the number of candidates to be elected ($M = K - 1$).

⁵To be precise, the voting rule studied by Cox was slightly different from the one considered here, since voters are only allowed to cast up to two votes (“Limited Committee Approval”). Yet, it is straightforward to check that strategic voting implies never voting for one’s least prefer candidate, therefore, when there are only three candidates, a best response entails casting at most two votes. The two rules are therefore equivalent from a strategic point of view, for three candidates.

But it also shows that these two cases ($M = 1$ and $M = K - 1$) are rather specific in the sense that in any other configuration about the number of seats and the number of candidates, there will be situations (preferences and anticipations about other voters' behavior) such that strategic voting is non-sincere. The intuition for the existence of non-sincere ballots is that the strategic recommendation entails voting by pairwise comparisons, but that expected winners and expected losers are compared to two different candidates (the strongest expected loser and the weakest winner respectively). Note that is all candidates where compared to the same benchmark, sincere voting would result (as is basically the case when $M = 1$: all candidates -but himself- are compared to the expected winner).

The proof of the proposition is in the appendix (section 8.4).

5 Equilibrium

5.1 Characterization

Let us now study the nature of equilibria consistent with the strategic behavior described in Proposition 3. For simplicity, we assume that N is odd .

Denote by $N(c, c')$ the number of voters who prefer⁶ candidate c to candidate c' . For $i = 1, \dots, N$, denote by $N_{-i}(c, c')$ the number of voters, other than voter i , who prefer candidate c to candidate c' . We assume that for all i and for all $(c, c'), (c'', c''')$, with $(c, c') \neq (c'', c''')$, the following condition is satisfied: $|N_{-i}(c, c') - N_{-i}(c'', c''')| \geq 3$. Clearly, this is not totally general. But this simplification is reasonable when the number of voters is large. The following characterization of an equilibrium will be useful in the sequel.

Proposition 5 *A profile of strategies $(s_i)_{i=1, \dots, N}$ is a pure equilibrium if and only if, up to a permutation in the candidates, there exists a partition of the set of candidates into two candidates (c_M et c_{M+1}) and two subsets of candidates, $\{c_1, \dots, c_{M-1}\}$ and $\{c_{M+2}, \dots, c_K\}$ such that:*

1. $N(c_M, c_{M+1}) > N(c_{M+1}, c_M)$,
2. $k < M \implies N(c_k, c_{M+1}) > N(c_M, c_{M+1})$,
3. $k > M + 1 \implies N(c_k, c_M) < N(c_{M+1}, c_M)$,
4. For $i = 1, \dots, N$, s_i is the best response described in Proposition 3 against expected scores (from the $N - 1$ other voters)

$$\begin{aligned} \widehat{s}_i(c_k) &= N_{-i}(c_k, c_{M+1}) \text{ if } k \leq M, \\ \widehat{s}_i(c_k) &= N_{-i}(c_k, c_M) \text{ if } k \geq M + 1. \end{aligned}$$

Then the (expected) winners are the members of the set $\{c_1, \dots, c_M\}$. The expected scores are $\widehat{s}(c_k) = N(c_k, c_{M+1})$ if $k \leq M$ and $\widehat{s}(c_k) = N(c_k, c_M)$ if $k \geq M + 1$.

⁶Remember we assume strict preferences over the set of candidates.

This characterization makes clear a strong link between approval voting for a committee and a notion of “majority rule”, as noted in the following remark, whose transparent proof is provided.

Remark 6 *In a pure equilibrium (if any), a candidate is an expected winner if and only if he is approved by at least half of the voters.*

Proof. Consider an equilibrium, where the (expected) scores of the candidates are:

$$\widehat{s}(c_1) > \widehat{s}(c_2) > \dots > \widehat{s}(c_M) > \widehat{s}(c_{M+1}) > \dots > \widehat{s}(c_K).$$

From the characterization above, the expected scores are $\widehat{s}(c_k) = N(c_k, c_{M+1}) > N(c_M, c_{M+1})$ if $k \leq M$ and $\widehat{s}(c_k) = N(c_k, c_M) < N(c_{M+1}, c_M)$ if $k \geq M + 1$.

From condition (1) in the characterization of a pure equilibrium: $N(c_M, c_{M+1}) > N(c_{M+1}, c_M)$. Since $N(c_M, c_{M+1}) + N(c_{M+1}, c_M) = N$, this implies that $\widehat{s}(c_M) = N(c_M, c_{M+1}) > N/2$ and $\widehat{s}(c_{M+1}) = N(c_{M+1}, c_M) < N/2$.

Thus that $\widehat{s}(c_k) > N/2$ for all $k \leq M$ and $\widehat{s}(c_k) < N/2$ for all $k \geq M + 1$. ■

An alternative interpretation: Trembling hand perfection There is a strong link between a pure equilibrium with the model of small mistakes introduced in section 2 and the concept of *trembling hand perfect equilibrium*. Trembling hand perfect equilibrium is a refinement of Nash equilibrium due to Selten (1975). A trembling hand perfect equilibrium is an equilibrium that takes the possibility of off-the-equilibrium play into account by assuming that the players, through a “slip of the hand” or tremble, may choose unintended strategies, albeit with small probability. One may check that a pure equilibrium in our game with recording mistakes is a trembling hand perfect equilibrium in the game with no recording mistakes (with trembles occurring with probabilities consistent with the model of small mistakes described here).

5.2 Existence and uniqueness

The following two remarks provide the theoretical answers to the questions of existence and uniqueness of equilibrium.

Remark 7 Non existence of equilibrium. *Whenever $M \geq 1$ and $K \geq M + 2$, there may exist no pure equilibrium.*

Proof. Take $M = 1$. It is easy to check that a pure equilibrium exists if and only if there exists a Condorcet winner. Indeed, from the characterization of equilibrium above (Proposition 5), there must exist some candidates c_1, c_2 such that conditions 1 and 3 are satisfied (condition (2) is empty). Condition (1) yields $N(c_1, c_2) > \frac{N}{2}$. Condition (3) yields $k \geq 3 \implies N(c_k, c_1) < N(c_2, c_1)$. Since $N(c_2, c_1) < \frac{N}{2}$, one sees that c_1 is a Condorcet winner. Since a Condorcet winner may not exist, there will be profiles of preferences for which there is no pure equilibrium as soon as there are at least three candidates.

For $M \geq 2$, counter-examples are easily found by considering a preference profile with $M - 1$ candidates who Pareto-dominates all the others, and no Condorcet winner among the remaining candidates, which is possible as soon as there are at least $M + 2$ candidates. ■

Remark 8 *Multiplicity of equilibria.* For $M = 1$, if there is an equilibrium, it is unique. For $M \geq 2$, there may exist several pure equilibria.

Proof. Take $M = 2$ and $K = 4$. Let a, b, c and d denote the candidates. Consider the following matrix g .

g	a	b	c	d
a	0	4	5	1
b	-4	0	2	6
c	-5	-2	0	3
d	-1	-6	-3	0

We know from Debord (1987) that there exists a preference profile for which the majorities $N(x, y)$ are positive affine transformations of $g(x, y)$. Since our characterization of equilibrium only involves comparisons between the numbers $N(x, y)$, we do not need to know exactly the preference profile and we can simply use the matrix g .

One can check that the following three situations are equilibria:

$$\left(\begin{array}{c} a \ (5) \\ b \ (2) \\ c \ (-2) \\ d \ (-6) \end{array} \right), \left(\begin{array}{c} b \ (6) \\ a \ (1) \\ d \ (-1) \\ c \ (-5) \end{array} \right), \left(\begin{array}{c} c \ (3) \\ a \ (1) \\ d \ (-1) \\ b \ (-4) \end{array} \right).$$

In the first case, a and b are expected winners with respectively 5 and 2 (relative) votes, and c and d are rejected with respectively -2 and -6 votes. These numbers are precisely the pairwise scores of a and b compared to c , and of c and d compared to b . This situation is thus an equilibrium. The reader can check that the other situations, in which the elected candidates are again a and b , or are c and a are also equilibria.

The same example can easily be extended to larger values of M by adding Pareto-dominant candidates.

For $M = 1$, it was proven in the proof of Remark 7 that a pure equilibrium exists if and only if there exists a Condorcet winner. Without indifferences or ties in the vote matrix, there cannot be two Condorcet winners. Denote by c^* the unique Condorcet winner. At equilibrium, the expected score of $c \neq c^*$ is $N(c, c^*)$. Denoting by c_2 the candidate such that $c_2 = \arg \max_{c \neq c^*} N(c, c^*)$, the expected score of c^* is $N(c^*, c_2)$. So that uniqueness of pure equilibrium holds. ■

5.3 Majority-transitive and single-peaked preference profiles

If the majority tournament is transitive, a pure equilibrium exists for any committee size. More exactly the following result holds.

Proposition 9 *Suppose that there exists a set of M candidates such that any candidate in this set beats, according to pairwise-majority voting, any candidate not in this set. Then there exist an equilibrium in which these M candidates are elected.*

Proof. Let C be the set of candidates that beat the others, and let $D = \mathfrak{C} \setminus C$. Let $c \in C$ and $d \in D$ be two candidates such that

$$N(c, d) = \min_{x \in C, y \in D} N(x, y)$$

We will check that the expected scores vector defined by $\widehat{s}(x) = N(x, d)$ for all $x \in C$ and $\widehat{s}(y) = N(y, c)$ for all $y \in D$ is an equilibrium. By definition of C , for all $x \in C$, $\widehat{s}(x) \geq \widehat{s}(c) = N(c, d)$. Likewise, for all $y \in D$, $\widehat{s}(y) = N(y, c) = N - N(c, y) \leq N - N(c, d) = N(d, c)$. Moreover, $N(c, d) > N/2 > N(d, c)$ hence \widehat{s} correctly ranks all the candidates. ■

So existence of equilibrium is guaranteed in that case, but there can be many equilibria. When the majority tournament associated with the preference profile is transitive, the proposition applies to the M first candidates according to the majority tournament order, and thus an equilibrium exists for any M . The example we used previously (Remark 8) to demonstrate the possible multiple equilibria is in fact a transitive tournament, as can be easily seen on the matrix g . Notice that the example shows that different equilibria may not only results in different (expected) scores vectors but also in different elected committees.

A nice application is the case of single-peaked preferences. This point is stated in a separate proposition. It can be derived from the previous one, but in the appendix we provide a direct proof, which provides a more detailed result: there can be at most two equilibria, and the elected committee is unique.

Proposition 10 *Assume that the candidates can be ordered (in a one-dimensional space) in such a way that voters have single-peaked preferences over the set of candidates. In that case, there exists one or two pure equilibria, and the expected winners are always the first M candidates according to the majority tournament.*

Proposition 10 highlights again the strong relationship between equilibrium under strategic committee approval and the majority rule. Remark 6 stated that in a pure equilibrium, all expected winners are approved by a majority of voters. Proposition 10 states that when preferences are single-peaked, the first M candidates according to the majority tournament are expected winners at equilibrium. This makes committee approval a normatively appealing rule to elect a committee, in the single-peaked case.

6 Limited Approval

This section briefly tackles the rule called “ V -limited Approval” where a voter can only approve up to V candidates. The case $V = 1$ is thus simple Plurality rule. Plain Committee Approval corresponds to any V larger than K , the number of candidates. The case $V = M$ (the number of votes equals the number of seats) seems natural but does not seem to have any specific theoretical property, as will be seen.

We keep the same model as in Section 2, with a slight change in the definition of the strategies. For $i = 1, \dots, N$, a strategy for voter i is a vector $s_i = (s_{i,c})_{c \in \mathcal{C}} \in \{0, 1\}^K$, such that $\sum_{c \in \mathcal{C}} s_{i,c} \leq V$, where for all c , $s_{i,c} = 1$ if voter i casts a vote in favor of candidate c , and $s_{i,c} = 0$ if voter i does not cast a vote for candidate c . We keep the description of mistakes made when recording the votes for candidates exactly the same as in Section 2 (in particular, we keep the assumption that mistakes are independent across candidates, meaning that we do not rule out the possibility that strictly more than V (positive) votes are recorded).

6.1 Best responses

Proposition 11 *Let \widehat{s}_{-i} denote the vector of expected scores obtained by the candidates from the votes of all the voters except voter i . Let the candidates be labelled in such a way that:*

$$\widehat{s}_{-i}(c_1) > \widehat{s}_{-i}(c_2) > \dots > \widehat{s}_{-i}(c_M) > \widehat{s}_{-i}(c_{M+1}) > \dots > \widehat{s}_{-i}(c_K).$$

Assume that for any pair of candidates (c, c') , $|\widehat{s}_{-i}(c) - \widehat{s}_{-i}(c')| \geq 3$. For ε small enough, the best response of voter i , when he has at most V votes, can be characterized as follows:

1. *The voter identifies the set of expected winners (c_1 to c_M) and that of expected losers (c_{M+1} to c_K).*
2. *If $1 \leq k \leq M$, define candidate c_k 's “main contender” as c_{M+1} and if $M + 1 \leq k \leq K$, define c_k 's “main contender” as c_M .*
3. *The voter ranks the candidates according to (the inverse of) their distance, in terms of expected votes, to their main contender.*
4. *The voter considers all the candidates in turn, according to the priority order defined at the previous step. As long as she does not hit the vote-budget constraint (V votes), she votes for a candidate if and only if her utility for this candidate is larger than her utility for its main contender.*

This Proposition is a generalization of Proposition 3, the only difference being, the appearance, in Step 4, of the vote constraint. The proof of Proposition 11 follows the same reasoning as the proof of Proposition 3 (see Section 8.3, Remark 12).

As noticed above, the only difference between "Plain Approval" and "Limited Approval" is the appearance, in Step 4, of the vote constraint. Given the limited number of votes, the voter has to consider the candidates lexicographically, in the order defined in Step 3. In this order, candidates are ranked according to their distance to their most likely contender (in numbers of expected votes). This is equivalent to ranking them by decreasing probability of them being caught in a tie for election. Indeed, as noticed when describing the intuitive content of Proposition 3, the most likely pivot-event is a tie between the two candidates who are expected to rank M -th and $M+1$ -th (here candidates c_M to c_{M+1}). What is the next most likely pivot-event? Note that all the other pivot events imply some order reversals among candidates, compared to the expected order. Which is the next pair of candidates between which the voter is most likely to be pivotal? Our assumptions imply that it will be either the pair $\{c_M, c_{M+2}\}$ or the pair $\{c_{M-1}, c_{M+1}\}$, depending on whether the difference in expected scores between c_M and c_{M+2} is larger or smaller than the difference in expected scores between c_{M-1} and c_{M+1} . Indeed, they are the two pairs which require the less order reversals compared to the expected outcome. Similarly, other pivot-events can be ranked by decreasing probability of occurrence.

Note that in that case, there is no reason to expect that the strategic recommendation will entail sincere voting. Indeed, there are now two potential causes as to why the strategic recommendation might not be sincere:

(1) The expected winners are compared to the strongest expected loser, whereas the expected losers are compared to the weakest expected winner (note that is all candidates where compared to the same benchmark, sincere voting would result – neglecting the constraint on the number of votes). This fact was exploited to construct counter-examples in the proof of Proposition 4.

(2) The constraint on the number of votes is binding. The voter, if given the opportunity to cast more votes, would vote for candidates higher in her preferences. But she has used all her votes on candidates with higher probability to be caught in a tie for election. One extreme case is $M = V = 1$ (simple plurality to elect one candidate), where the voter should vote for her preferred candidate among the two candidates who are expected to receive the most votes: she should desert her preferred candidate whenever he is not one of the two main candidates.

6.2 Equilibrium

Consider the following example with $M = 2$, $K = 4$ candidates and 65 voters. Denote the candidates by a, b, c, d . The next Table indicates that, for instance, 35 voters prefer a to b , b to c , and c to d .

(45)	(10)	(20)	(10)
a^*	b^*	c^*	c^*
b^*	c^*	d^*	b
c	a	b	a
d	d	a	d

This preference profile is single-peaked with respect to the order $a < b < c < d$. Assume that the voters vote for the starred alternatives. In that case, the resulting expected scores are:

$$\begin{aligned}
\widehat{s}(b) &= 45 + 10 = 55, \\
\widehat{s}(a) &= 45, \\
\widehat{s}(c) &= 10 + 20 + 10 = 40, \\
\widehat{s}(d) &= 20.
\end{aligned}$$

The reader can check that the described ballots (voters vote for the starred alternatives) are in equilibrium if the number of allowed votes is at least two per voter ($V \geq 2$). For example, consider a voter i with the first ranking. If she expects other voters to vote for the starred alternatives, her anticipations about expected scores are as follows:

$$\begin{aligned}
\widehat{s}_{-i}(b) &= 44 + 10 = 54, \\
\widehat{s}_{-i}(a) &= 44, \\
\widehat{s}_{-i}(c) &= 10 + 20 + 10 = 40, \\
\widehat{s}_{-i}(d) &= 20.
\end{aligned}$$

The weakest expected winner is candidate a and the strongest expected loser c . According to Step 3 in Proposition 11, the resulting order of priority for considering the candidates is the following: first, consider the two critical candidates (a and c), second, consider candidate b (whose distance to his main challenger c is $54 - 40 = 14$), third, consider candidate d (whose distance to his main challenger a is $44 - 20 = 24$). If $V \geq 2$, the strategic recommendation is to vote for a and b (preferred to c). One can similarly check that if $V \geq 2$, the described ballots are in equilibrium.

Now suppose that another candidate shows up but that the number of allowed votes is set to two ($V = 2$). If the voters stick to the previous strategies, the new candidate obtains no vote at all, and this is an equilibrium for the constrained $V = 2$ voting rule. This remark holds true whatever the preferences of the voters for the new candidate are. For instance this candidate could be the top choice of all the voters and still not be elected at this equilibrium.

It is not difficult to build such counter examples for any number V so that one can conclude that, at least in theory, the Limited Approval voting rule suffers pathologies similar to that of the Plurality rule (severe coordination problems).

7 Conclusion

We proposed a model of strategic voting under Committee Approval. This model requires that the voters know their own preferences and evaluate the relative likelihoods of the possible electoral outcomes. It rests on a number of cognitive hypothesis: voters are only interested in the result of the election (no expressive motives), they have separable preferences, they are essentially rational, and they neglect three-way ties. All these hypothesis are questionable but they together have the virtue of producing definite predictions.

Equipped with these predictions, one can tackle positive and normative questions: Do people really behave like the model suggests? If yes, is it a good thing? Leaving the positive question to empirical research (see a companion paper Lachat *et al.* 2014 for a first test of the theory), this paper provides some element for a normative discussion.

We noticed that under "Plain Approval" (with no limit on the number of votes), the equilibrium properties of the model were very much in the spirit of an implementation of a generalization of the Condorcet principle to the case of a committee. We find that whatever $M \geq 1$, a candidate is elected if and only if it is supported by more than half of the electorate. Besides, when the majority tournament is transitive, there exists an equilibrium where the first M candidates according to the tournament are elected. This extends the finding by Laslier 2009, which showed that simple Approval Voting, the case $M = 1$, implements the Condorcet principle in that it elects the Condorcet winner whenever it exists.

But it should also be highlighted that an important property of Approval Voting (and of the idea of a Condorcet winner) is lost when we go from $M = 1$ to $M > 1$.

Suppose that the same political party proposes, in a single-member district ($M = 1$), two candidates instead of one, and suppose that the preferences of the voters are such that the voters are chiefly interested in the parties, so that these two fellow candidates are ranked next to each other in every voter's preference. This manipulation⁷ does not alter the fact that this party has a majority or not against an other party. In the very same manner, Approval Voting, by definition, lets the voter vote for several candidates if she wishes to and is thus immune to vote splitting or candidate duplication.

Now suppose that, in a district with $M > 1$ seats, all parties send M candidates, instead of only one. Then the Condorcet-winning party on its own will gather all the seats. In other words, candidate duplication is ineffective for simple Approval Voting but is effective for Committee Approval.

Let us now comment on the differences between "Plain Approval Committee" and "Limited Approval Committee". We have seen that when the majority

⁷This kinds of variation in the preference profile has a long history in Choice Theory; see the "Axiom 2.6" in Milnor (1951), the "Independence of Clones" of Tideman (1987), the "Composition-Consistency" of Laffond *et al.* (1996).

tournament is transitive (in particular, when preferences are single-peaked), the first M candidates according to the majority tournament are expected winners at equilibrium. This makes committee approval a normatively appealing rule to elect a committee in that case. This property is lost whenever there is a limit on the number of votes a voter is allowed to cast. The example in section 6.2 highlights that, at least in theory, the Limited Approval voting rule suffers from pathologies similar to that of the Plurality rule, in particular, from potential severe coordination problems. For instance, there can be situations where a candidate is the top choice of all the voters, and still is not elected at this equilibrium. From a normative point of view, "Plain Approval Committee" seems more attractive than "Limited Approval Committee".

8 Appendix

8.1 Proof of Lemma 1

Take as given the profile of strategies (ballots) of the voters $s = (s_i)_{i=1,\dots,N}$. For any two candidates c and c' ,

$$\Pr[S(c) = S(c')] = \sum_{t=0}^{t=N} \Pr[S(c) = S(c') = t],$$

and, by independence:

$$\Pr[S(c) = S(c')] = \sum_{t=0}^N (\Pr[S(c) = t] \cdot \Pr[S(c') = t]).$$

Without loss of generality, assume that $\widehat{s}(c) \geq \widehat{s}(c')$.

Consider first the case where $t > \widehat{s}(c)$. The first order probability of the event $S(c) = t$ is

$$\binom{N - \widehat{s}(c)}{t - \widehat{s}(c)} \varepsilon^{t - \widehat{s}(c)}. \quad (4)$$

Indeed, as one can easily check, the event $S(c) = t$ requires at least $t - \widehat{s}(c)$ mistakes, and can indeed result from that precise number of mistakes. One can and must pick $t - \widehat{s}(c)$ individuals who voted against c , among the $N - \widehat{s}(c)$ who voted against c , and change their votes to a YES vote in favor of candidate c . Thus the probability (4). A similar argument holds for the probability that c' get t votes, therefore, the first order probability of the event $S(c) = S(c') = t$ is:

$$\binom{N - \widehat{s}(c)}{t - \widehat{s}(c)} \binom{N - \widehat{s}(c')}{t - \widehat{s}(c')} \varepsilon^{2t - \widehat{s}(c) - \widehat{s}(c')}.$$

Similarly, when $t < \widehat{s}(c')$, the first order probability of the event $S(c) = t$ is $\binom{\widehat{s}(c)}{\widehat{s}(c) - t} \varepsilon^{\widehat{s}(c) - t}$ (pick $\widehat{s}(c) - t$ individuals who voted for c among the $\widehat{s}(c)$

who voted for c , and change their votes to a NO vote for candidate c), and therefore the first order probability of the event $S(c) = S(c') = t$ is:

$$\binom{\widehat{s}(c)}{\widehat{s}(c) - t} \binom{\widehat{s}(c')}{\widehat{s}(c') - t} \varepsilon^{\widehat{s}(c) + \widehat{s}(c') - 2t}.$$

Last, when $\widehat{s}(c') \leq t \leq \widehat{s}(c)$, the first order probability of the event $S(c) = S(c') = t$ is :

$$\binom{\widehat{s}(c)}{\widehat{s}(c) - t} \binom{N - \widehat{s}(c')}{t - \widehat{s}(c')} \varepsilon^{\widehat{s}(c) - \widehat{s}(c')}.$$

When $t > \widehat{s}(c)$, $2t - \widehat{s}(c) - \widehat{s}(c') > \widehat{s}(c) - \widehat{s}(c')$ and when $t < \widehat{s}(c')$, $\widehat{s}(c) + \widehat{s}(c') - 2t > \widehat{s}(c) - \widehat{s}(c')$. Therefore, one can see that the event $S(c) = S(c')$ has first order probability

$$\left(\sum_{t=\widehat{s}(c')}^{\widehat{s}(c)} \binom{\widehat{s}(c)}{\widehat{s}(c) - t} \binom{N - \widehat{s}(c')}{t - \widehat{s}(c')} \right) \cdot \varepsilon^{\widehat{s}(c) - \widehat{s}(c')}$$

so that the requirement of the event $S(c) = S(c')$ is here $\widehat{s}(c) - \widehat{s}(c')$. **Q.E.D.**

8.2 Proof of Lemma 2

Given a profile of strategies $(s_i)_{i=1, \dots, N}$, denote by S_M the random variable describing the M -th largest score obtained from the realized votes of all voters. Formally: for any vector of realized scores $(S(c))_{c \in \mathfrak{C}}$, let S_M be the unique number which satisfies the following two conditions:

1. $|\{c \in \mathfrak{C} : S(c) > S_M\}| \leq M - 1$,
2. $|\{c \in \mathfrak{C} : S(c) \geq S_M\}| \geq M$.

Candidates with scores strictly larger than S_M are elected, candidates with scores strictly smaller are not elected, and a candidate with score S_M is elected either for sure (if he is the only candidate with realized score S_M) or with some probability (in case of a tie with other candidates).

The event "Candidate c is caught in an exact tie for election" is the event " $S(c) = S_M$ and there exists at least one other $c' \neq c$ such that $S(c) = S(c') = S_M$ ".

Consider first the case where $k \leq M$. Let us show that the requirement of the event " $S(c_k) = S_M$ and there exists at least another $k' \neq k$ such that $S(c_k) = S(c_{k'}) = S_M$ " is $\widehat{s}(c_k) - \widehat{s}(c_{M+1})$.

Note that $\widehat{s}(c_k) - \widehat{s}(c_{M+1})$ mistakes (from reference scores \widehat{s}) are sufficient to reach this outcome. Indeed, if out of the $\widehat{s}(c_k)$ voters who did vote for c_k , one picks $\widehat{s}(c_k) - \widehat{s}(c_{M+1})$ of them and change their votes (no other mistake

being made), the resulting scores are $S(c) = \widehat{s}(c)$ for all $c \neq c_k$ and $S(c_k) = \widehat{s}(c_{M+1}) = S(c_{M+1})$. Note that this situation involves a two-way tie between candidate c_k and candidate c_{M+1} .

One can check that any other vector of mistakes inducing that candidate c_k is caught in an exact tie for election implies at least as many mistakes, therefore the requirement of the event

$$\begin{aligned} & "S(c_k) = S_M \text{ and there exists at least one other } k' \neq k \text{ such that} \\ & S(c_k) = S(c_{k'}) = S_M" \end{aligned}$$

is exactly $\widehat{s}(c_k) - \widehat{s}(c_{M+1})$. Besides, one may check that the event

$$\begin{aligned} & "S(c_k) = S_M \text{ and there exists at least one other } k' \neq k \text{ and } k' \neq \\ & M + 1 \text{ such that } S(c_k) = S(c_{k'}) = S_M" \end{aligned}$$

(that is, not having candidate c_{M+1} part of the tie for the M th position) involves strictly more mistakes.⁸

Consider now the case $k \geq M + 1$. Let us show that the requirement of the event " $S(c_k) = S_M$ and there exists at least another $k' \neq k$ such that $S(c_k) = S(c_{k'}) = S_M$ " is $\widehat{s}(c_M) - s(\widehat{s}_k)$.

Note that $\widehat{s}(c_M) - \widehat{s}(c_k)$ mistakes (from reference scores \widehat{s}) are sufficient to reach the outcome $S(c_k) = S(c_M) = S_M$. Indeed, if out of the $N - \widehat{s}(c_k)$ voters who did not vote for c_k , one picks $\widehat{s}(c_M) - \widehat{s}(c_k)$ and change their votes (no other mistakes being made), the resulting scores are $S(c) = \widehat{s}(c)$ for all $c \neq c_k$ and $S(c_k) = \widehat{s}(c_M) = S(c_M)$.

One can check that any other vector of mistakes inducing this outcome implies at least as many mistakes, therefore the requirement of the event

$$\begin{aligned} & "S(c_k) = S_M \text{ and there exists at least one other } k' \neq k \text{ such that} \\ & S(c_k) = S(c_{k'}) = S_M" \end{aligned}$$

is exactly $\widehat{s}(c_M) - \widehat{s}(c_k)$. Besides, one may check that the event

$$\begin{aligned} & "S(c_k) = S_M \text{ and there exists at least one other } k' \neq k \text{ and } k' \neq M \\ & \text{such that } S(c_k) = S(c_{k'}) = S_M" \end{aligned}$$

involves strictly more mistakes.⁹

Q.E.D.

⁸Note nevertheless that there are events with requirement $\widehat{s}(c_k) - \widehat{s}(c_{M+1})$ where candidate c_k is caught in a tie for election with candidate c_{M+1} but also with another candidate. Indeed, consider an event where $\widehat{s}(c_k) - \widehat{s}(c_M)$ votes for c_k are not recorded, and where $\widehat{s}(c_M) - \widehat{s}(c_{M+1})$ NO votes for c_{M+1} are wrongly recorded as YES votes for c_{M+1} , no other mistake being made. The requirement of this event is $\widehat{s}(c_k) - \widehat{s}(c_{M+1})$ and it involves a three-way tie for election between c_M , c_{M+1} and c_k . As mentioned in the description of the model (Section 2), we assume that the voter neglects this type of events involving three-way ties.

⁹Here again, note that there exists an event with requirement $\widehat{s}(c_M) - \widehat{s}(c_k)$ involving a three-way tie for election between c_M , c_{M+1} and c_k . We will assume that the voter neglects this type of events involving three-way ties.

8.3 Proof of Proposition 3

Consider a profile of strategies (ballots) from voters other than voter i : $s_{-i} = (s_j)_{j \neq i}$. Let \widehat{s}_{-i} denote the vector of expected scores obtained by the candidates from the votes of all the voters except voter i . Let the candidates be labelled in such a way that:

$$\widehat{s}_{-i}(c_1) > \widehat{s}_{-i}(c_2) > \dots > \widehat{s}_{-i}(c_M) > \widehat{s}_{-i}(c_{M+1}) > \dots > \widehat{s}_{-i}(c_K).$$

Assume that the expected vote difference between any two candidates are at least 3, that is, $\widehat{s}_i(c_k) - \widehat{s}_i(c_{k+1}) \geq 3$ for all $k = 1, \dots, K - 1$.

To start the proof, consider a voter who contemplates any ballot s_i she could cast. Given the strategies s_{-i} of the other voters, the *ex post* utility that voter i derives from ballot s_i depends on the realization of the random variable ω describing the mistakes made when recording the ballots (remember $\omega_{j,c} = 1$ means that a mistake is made when recording voter j 's vote about candidate c , see section 3). Denote this *ex post* utility by $U_i(s_i, s_{-i}, \omega)$. The expected utility derived from strategy s_i is $\sum_{\omega} U_i(s_i, s_{-i}, \omega) \Pr[\omega]$.

Consider two ballots, s_i and s'_i , the voter prefers s_i to s'_i if and only if

$$\Delta = \sum_{\omega} U_i(s_i, s_{-i}, \omega) \Pr[\omega] - \sum_{\omega} U_i(s'_i, s_{-i}, \omega) \Pr[\omega] \geq 0.$$

Obviously all the elementary events ω such that $U_i(s_i, s_{-i}, \omega) = U_i(s'_i, s_{-i}, \omega)$ cancel in this inequality so that the sum can run over elementary events such that $U_i(s_i, s_{-i}, \omega) \neq U_i(s'_i, s_{-i}, \omega)$. This remark, with the fact that the probabilities $\Pr[\omega]$ are polynomials in ε (the requirement of event ω being $|\omega|$), provides the technique for finding best responses to an expected score vector \widehat{s}_{-i} when ε is small. Let m be the requirement of the event $U_i(s_i, s_{-i}, \omega) \neq U_i(s'_i, s_{-i}, \omega)$. Then:

$$\begin{aligned} \Delta &= \sum_{\substack{\omega: U_i(s_i, s_{-i}, \omega) \neq U_i(s'_i, s_{-i}, \omega) \\ |\omega|=m}} [U_i(s_i, s_{-i}, \omega) - U_i(s'_i, s_{-i}, \omega)] \Pr[\omega] \\ &+ \sum_{\substack{\omega: U_i(s_i, s_{-i}, \omega) \neq U_i(s'_i, s_{-i}, \omega) \\ |\omega|>m}} [U_i(s_i, s_{-i}, \omega) - U_i(s'_i, s_{-i}, \omega)] \Pr[\omega] \end{aligned}$$

The first part, where the sum runs over elementary events ω with requirement m , is a polynomial in ε of leading term $G\varepsilon^m$, where

$$G = \sum_{\substack{\omega: U_i(s_i, s_{-i}, \omega) \neq U_i(s'_i, s_{-i}, \omega) \\ |\omega|=m}} [U_i(s_i, s_{-i}, \omega) - U_i(s'_i, s_{-i}, \omega)]$$

does not depend on ε . The leading term of the second part has a strictly higher exponent, hence $G = \lim_{\varepsilon \rightarrow 0} \Delta \varepsilon^{-m}$. It follows that, for ε small enough, the sign of Δ is the sign of G . This implies that, in order to know whether s_i yields

larger expected utility than s'_i , one can restrict attention to those events which realize $U_i(s_i, s_{-i}, \omega) - U_i(s'_i, s_{-i}, \omega)$ with the smallest number of mistakes. Those events will involve ties (or near ties, with a one vote margin) for election of some candidates.

Given s_{-i} , what are the ballots s_i and s'_i and the events ω which realize $U_i(s_i, s_{-i}, \omega) - U_i(s'_i, s_{-i}, \omega)$?

A necessary condition is that the ballots s_i and s'_i differ on a candidate which is caught in a tie (or a near tie) for election. Under our assumption that the voters in their computation of best responses neglect the possibility of three-way ties, we will focus on ties and near ties involve two candidates. Two candidates are said to be caught in an exact tie for election if realized scores given the votes of all the voters other than i , are such that the candidates both receive the M th highest scores; they are said to be caught in a near tie for election if realized scores given the votes of all the voters other than i , are such that one of the candidate get the M -th highest score and the other one less vote. In both types of events, by voting for one of these candidate but not the other can change the outcome of the election. Note that the difference between requirement of a tie and requirement of a near tie, for any given two candidate, is at most two.

Now, what are the events and ballots which realize $U_i(s_i, s_{-i}, \omega) - U_i(s'_i, s_{-i}, \omega)$ with the smallest number of mistakes?

Lemma 2 provides the answer. A straightforward adaptation of Lemma 2 states that, given the strategies s_{-i} of all voters but i , the requirement of the event "Candidate c_k is caught in an exact tie for election (not taking into account the vote of voter i)" is $\widehat{s}_{-i}(c_k) - \widehat{s}(c_{M+1})$ if $k \leq M$ and $\widehat{s}_{-i}(c_M) - \widehat{s}(c_k)$ if $k \geq M + 1$. Therefore, the most likely exact tie for election occurs between candidates c_M (the weakest expected winner) and c_{M+1} (the strongest expected loser), since the requirement of this event is $\widehat{s}_{-i}(c_M) - \widehat{s}_{-i}(c_{M+1})$. Given our assumption that the expected vote difference between any two candidates are at least 3, the most likely *near* tie (that is, with a one vote margin) for election occurs between candidates c_M and c_{M+1} . Therefore, if voter i is pivotal, it will most likely be in deciding who between candidate c_M and candidate c_{M+1} will be elected. Therefore, if she prefers candidate c_M to candidate c_{M+1} ($u_i(c_M) > u_i(c_{M+1})$), she should vote for candidate c_M and not vote for candidate c_{M+1} . Similarly, if $u_i(c_M) < u_i(c_{M+1})$, she should vote for candidate c_{M+1} and not vote for candidate c_M . Her decision about candidates c_M and c_{M+1} is thus decided by this pairwise comparison between the two candidates.

What is the next most likely pivot-type event, involving at least one candidate other than candidate c_M and candidate c_{M+1} ?

Again, lemma 2 provides the answer. It will be either a tie (or near tie) for election between c_{M-1} and c_{M+1} , or a tie (or near tie) for election between c_M and c_{M+2} , depending on whether $\widehat{s}_{-i}(c_{M-1}) - \widehat{s}_{-i}(c_{M+1})$ is smaller or larger than $\widehat{s}_{-i}(c_M) - \widehat{s}_{-i}(c_{M+2})$. More generally, the results in lemma 2 allow us to rank the different two-way ties for election involving candidates other than

candidates c_M and c_{M+1} . Most specifically, if $1 \leq k \leq M$, define candidate c_k 's "main contender" as c_{M+1} and if $M+1 \leq k \leq K$, define c_k 's "main contender" as c_M . Then, rank the candidates according to (the inverse of) their distance, in terms of expected votes, to their main contender. As seen above, candidates c_M and c_{M+1} share the first rank in this ordering.

Consider now the candidate with the second position (either c_{M-1} or c_{M+2}), call this candidate $c(2)$. The next most likely pivot-type event involves a tie (or a near tie) between $c(2)$ and its main contender. Therefore, the voter should vote for $c(2)$ if and only if she prefers $c(2)$ to its main contender. Remember that the vote for or against $c(2)$'s main contender (c_M or c_{M+1}) has already previously been decided by the pairwise comparison candidates c_M and c_{M+1} . Indeed the event " $c(2)$'s main contender is caught in a tie for election with candidate $c(2)$ " is much less likely than a tie for election between c_M and c_{M+1} .

What is the next most likely pivot-type event, involving at least one candidate other than candidates $c(2)$, c_M and c_{M+1} ? Denoting by $c(k)$, for $2 \leq k \leq K-1$ the candidate with the k 's position in the ordering defined in the previous paragraph, one may check that the next most likely pivot-type event, involving at least one candidate other than candidates $c(2)$, c_M and c_{M+1} is a tie (or a near tie) between $c(3)$ and its main contender. Therefore, the voter should vote for $c(3)$ if and only if she prefers $c(2)$ to its main contender.

The same reasoning can be generalized by considering all the candidates in turn. Thus the strategic recommendation described in Proposition 3.

Remark 12 *In Section 6, we tackle the rule called "V-limited Approval", whereby a voter can only approve up to V candidates. Note that the proof above also characterizes the best response in that case. Indeed, in that case, the voter considers all the candidates in turn, according to the priority order defined in the proof. Note that the assumption that for any pair of candidates (c, c') , $|\widehat{s}_{-i}(c) - \widehat{s}_{-i}(c')| \geq 3$ in Proposition 10 guarantees that there is no ambiguity when defining this priority order. As long as she does not hit the vote-budget constraint (V votes), she votes for a candidate if and only if her utility for this candidate is larger than her utility for its main contender.*

Q.E.D.

8.4 Proof of Proposition 4

Consider a profile of strategies (ballots) from voters other than voter $i : (s_j)_{j \neq i}$. Let \widehat{s}_{-i} denote the vector of expected scores obtained by the candidates from the votes of all the voters except voter i . Let the candidates be labelled in such a way that:

$$\widehat{s}_{-i}(c_1) > \widehat{s}_{-i}(c_2) > \dots > \widehat{s}_{-i}(c_M) > \widehat{s}_{-i}(c_{M+1}) > \dots > \widehat{s}_{-i}(c_K).$$

- For $M = 1$ (one person to be elected), the best response described in Proposition 3 prescribes (i) to identify the critical candidates (c_1 and c_2), (ii) for $k \geq 2$, to approve c_k if and only if $u_i(c_k) > u_i(c_1)$, (iii) to approve c_1 if and only if $u_i(c_1) > u_i(c_2)$. This recommendation prescribes voting for all candidates strictly preferred to c_1 if $u_i(c_1) < u_i(c_2)$, and voting for all candidates weakly preferred to c_1 if $u_i(c_1) > u_i(c_2)$. This always produces a sincere ballot, whatever the voter's preferences over the candidates. This property for $M = 1$ was already noticed in Laslier (2009).
- For $M = K - 1$, this rule always produces a sincere ballot. Indeed, if $u_i(c_M) > u_i(c_{M+1}) = u_i(c_K)$: for any candidate c , she should vote for c if and only if she strictly prefers c to c_K . This always produces a sincere ballot. If $u_i(c_M) < u_i(c_{M+1})$: the voter should vote for a candidate c if and only if she weakly prefers c to c_K . This always produces a sincere ballot.
- Whenever $M \geq 2$ and $K \geq M + 2$, there exist preferences for voter i such that strategic voting entails casting a non-sincere ballot. Suppose that voter i has preferences over the candidates such that:

$$u_i(c_M) > u_i(c_K) > u_i(c_1) > u_i(c_{M+1}),$$

which is possible whenever $M \geq 2$ and $K \geq M + 2$. The voter should approve the expected winners (c_1, c_2, \dots, c_M) if and only if she prefers them to the strongest expected loser c_{M+1} : given her preferences, this implies in particular voting for c_1 . She should approve the expected losers ($c_{M+1}, c_{M+2}, \dots, c_K$) if and only if she prefers them to the weakest expected winner c , this implies in particular not voting for c_K . One concludes that such a voter should approve c_1 but not c_K , although she prefers c_K to c_1 . This results in a non-sincere ballot.

Q.E.D.

8.5 Proof of Proposition 10

In the single-peaked case, the majority tournament is transitive and the M top candidates according to the majority tournament form a set that we denote by X . The set X forms a segment (in the set of ordered candidates); the Condorcet winner is in this set, then the alternative which beats all the others but the Condorcet winner is located next to the Condorcet winner, either left or right, etc. Let x_L and x_R be the left-most and right-most positions in X , then

$$X = [x_L, x_R]$$

Let Y_L be the set of candidates (strictly) at the left of X and let Y_R be the set of candidates (strictly) at the right of X , so that $Y_L \cup X \cup Y_R$ is the whole set of candidates (Y_L or Y_R might be empty). Let y_L be the right-most position

in Y_L and y_R be the left most position in Y_R , and consider the following two outcomes:

Outcome *A*: The set of expected winners is X , the weakest expected winner is x_L , the strongest expected loser is y_R .

Outcome *B*: The set of expected winners is X , the weakest expected winner is x_R , the strongest expected loser is y_L .

First part of the proof. Let us show that one of these two outcomes is supported by a pure equilibrium.

Consider first outcome *A*. From the characterization in Proposition 5 (section 5.1), outcome *A* is supported by a pure equilibrium if and only if:

1. $N(x_L, y_R) > N/2$,
2. $c \in X \setminus \{x_L\} \implies N(c, y_R) > N(x_L, y_R)$,
3. $c \in Y_L \cup Y_R \setminus \{y_R\} \implies N(c, x_L) < N(y_R, x_L)$.

Since x_L is ranked higher than y_R according to the majority tournament relation, condition (1) is satisfied.

Since preferences are single-peaked, the majority scores of the candidates in X against y_R are decreasing from right (x_R) to left (x_L), therefore condition (2) is satisfied.

Since preferences are single-peaked, the majority scores of the candidates in $Y_R \setminus \{y_R\}$ against x_L are decreasing from left (y_R) to right, therefore $c \in Y_R \setminus \{y_R\} \implies N(c, x_L) < N(y_R, x_L)$.

Since preferences are single-peaked, the majority scores of the candidates in Y_L against x_L are decreasing from right (y_L) to left, therefore property ($c \in Y_L \implies N(c, x_L) < N(y_R, x_L)$) is true if and only if $N(y_L, x_L) < N(y_R, x_L)$. In words, the latter condition states that y_L is weaker than y_R against x_L .

One therefore concludes that outcome *A* is an equilibrium outcome if and only if:

$$N(y_L, x_L) < N(y_R, x_L). \quad (5)$$

Similarly, one might show that outcome *B* is a pure equilibrium outcome if and only if:

$$N(y_R, x_R) < N(y_L, x_R). \quad (6)$$

We will now check that one of these inequalities must be true.

Indeed, assume that inequality (5) is not true, that is, $N(y_R, x_L) \leq N(y_L, x_L)$. Since preferences are single-peaked with $x_L < x_R < y_R$, $N(x_L, y_R) < N(x_R, y_R)$, which is equivalent to

$$N(y_R, x_R) < N(y_R, x_L). \quad (7)$$

Since preferences are single-peaked with $y_L < x_L < x_R$, $N(x_L, y_L) > N(x_R, y_L)$, which is equivalent to

$$N(y_L, x_L) < N(y_L, x_R). \quad (8)$$

Combining $N(y_R, x_L) \leq N(y_L, x_L)$ with inequalities (7) and (8), one gets that $N(y_R, x_R) < N(y_R, x_L) \leq N(y_L, x_L) < N(y_L, x_R)$ and inequality (6) is true.

Similarly, assume that inequality (6) is not true, that is, $N(y_L, x_R) \leq N(y_R, x_R)$. Combining with inequalities (7) and (8), this implies that $N(y_L, x_L) < N(y_L, x_R) \leq N(y_R, x_R) < N(y_R, x_L)$ and inequality (5) is true.

This concludes the proof that A or B is an equilibrium.

Second part of the proof. We will now see that there can be no other equilibrium.

Denote by w_L and w_R the two elected candidates at the left-most and right-most positions. By single-peakedness, any candidate y such that $w_L < y < w_R$ must beat, under majority rule w_L or w_R . But w_L and w_R , being winners, are both compared to the strongest loser, and beat that candidate; hence, no such candidate y can be the strongest loser. It follows that c_{M+1} , the strongest loser, is outside the segment $[w_L, w_R]$.

By symmetry, we can suppose that the strongest loser is at the left of $[w_L, w_R]$: $c_{M+1} < w_L$. Then, the candidates in X have scores that rank them from right to left, like we have seen in the first part of the proof. In this case the weakest winner is $c_M = x_L$.

By single-peakedness, c_{M+1} being beaten by w_R implies that c_{M+1} is also beaten by any alternative x such that $c_{M+1} < x \leq w_R$, that is at least M alternatives. But, by transitivity of the tournament, c_{M+1} is beaten by precisely the M alternatives of the set X . Therefore the set of elected candidates must be exactly X . We are thus in the situation A of the first part of the proof. **Q.E.D.**

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