Price dynamics on a risk averse market with asymmetric information

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Abstract

A market with asymmetric information can be viewed as a repeated exchange game between an informed sector and an uninformed sector. The case where all agents in the market are risk neutral was analyzed in De Meyer [2010]. The main result of that paper was that the price process in this risk neutral environment should be a particular kind of Brownian martingale called CMMV. This type of dynamics is due to the strategic use of their private information by the informed agents. In this paper, we generalize this analysis to the case of a risk averse market. Our main result is that the price process is still a CMMV under a martingale equivalent measure.

JEL Classification: G14, C72, C73, D44
Keywords: Asymmetric information, Price dynamics, Martingales of maximal variation, Repeated games, Martingale equivalent measure, Risk aversion

1 Introduction

Information asymmetries are omnipresent in financial markets. We don't mean here insider trading which is illegal and hopefully quite marginal but, de facto, there is an asymmetry between agents: institutionals have typically a better access to information than private investors. They have access to more information, quicker, and they are better skilled to analyze it. In such a situation everybody is aware that informational asymmetries exist and knows who are the informed agents.

Actions by the informed agents on the market are therefore analyzed by the uninformed agents in order to deduce the informative content behind these actions. As suggested previous papers (De Meyer [2010] and De Meyer and Saley [2003]), this phenomenon could partially explain the kind of price dynamics observed on the market.

In De Meyer and Saley [2003], the game between two market makers with asymmetric information was analyzed. This paper proved in a very particular risk neutral case that

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the price process converges to a continuous martingale involving a Brownian motion in its description. This result gives thus an endogenous justification for the appearance of the Brownian term in the price dynamics: it is seen as an aggregation of the random noises introduced strategically day after day by the informed agents on their moves to hide their private information.

This idea was generalized in De Meyer [2010] to a broader setting and the main result of that paper was that the price dynamics must be a so called "Continuous martingale with maximal variation" (CMMV), see Definition 1 below. These martingales also involve a Brownian motion. The market is analyzed there as a repeated exchange game between an informed sector and an uninformed one. Both sectors exchange a risky asset R with a numeraire in counterpart. In this game, the informed sector plays as a unique risk neutral player who wants to maximize the expected value of his final portfolio. To modelize the market in the broadest generality, an exchange mechanism T is introduced. T maps couples of actions (i, j) to a resulting transfer $T_{i,j} = (A_{i,j}, B_{i,j})$ of risky asset and money between the sectors. To make a realistic modelization, a set **(H)** of five hypothesis is made on T:

$$
\textbf{(H)} \begin{cases} 1\text{: Invariance with respect with the numeraire scale} \\ 2\text{: Invariance with respect to the risk-less part of the risky asset} \\ 3\text{: Existence of the value.} \\ 4\text{: Positive value of information} \\ 5\text{: Continuity of } V_1. \end{cases}
$$

De Meyer [2010] noticed that a profit for the informed player results in a loss for the uninformed player. The exchanges between the players are typically zero sum (no creation of goods). It is then natural to consider in a first analysis the case of two risk neutral players. In this framework the price at period q is naturally identified with the conditionnal expectation of the final value of the risky asset given the public information at that period. The main result is that under **(H)**, the price process at equilibrium p^n, (considered as a process in continuous time $t \to p^n_{\lfloor nt \rfloor}$) converges to a CMMV as n goes to infinity. The same martingale appears no matter the choice of the mechanism T satisfying **(H)**. The class of CMMV is therefore very robust as it doesn't depend on the peculiarity of the trading mechanism.

In the third chapter of his thesis Gensbittel [2010] studied the case of risk neutral players on a market with multiple risky assets and asymmetric information. In the particular case where all assets are derivatives that depends monotonically on a given underlying asset, the price dynamics of all assets are again CMMV.

The question we adress in the present paper is also about the robustness of the CMMV class: will this class still appear without the hypothesis that the uninformed sector is risk neutral? Since it typically represents big institutional investors, it is natural to model the informed sector as risk neutral agent. The problem is more about player 2 which is an aggregation of small uninformed agents. As individual, they typically display risk aversion. We will model this aggregated sector by a single representative agent called player 2, and it seems then natural to assume him to be risk averse.

This risk aversion is modeled with the introduction of a non linear utility function in player 2's payoff. Due to this utility function, we are not in front of a zero sum game anymore as it was the case in De Meyer [2010]. This makes the analysis more involved, the notions of value and optimal strategies are here to be replaced by the notion of Nash equilibrium.

In this model, since player 2 is risk averse, it makes no sense to define the price as the expectation of the final value of the risky asset. What would be the price in this risk averse setting with a general (abstract) trading mechanism? We chose to bypass this question by considering a particular exchange mechanism that naturally involves prices.

The mechanism considered in this paper is very simple: the uninformed sector is represented here by a risk averse market maker called player 2. At each period $q \in \{1 \dots, n\}$, he chooses a price p_q for one share of the risky asset, and player 1 will have to decide whether he wants to sell or to buy. Both players try to maximize their utility for the liquidation value of their final portfolio.

We first prove the existence of Nash equilibrium for a game with fixed length n. We then analyze the price dynamics at equilibrium. Let P_n denote the law of the price process at equilibrium. P_n is referred to as the historical probability measure. We then prove that this price dynamics is compatible with the financial theory of no-arbitrage (Harrison and Pliska [1981]): There exists a probability measure Q_n, equivalent to the historical one P_n $\left(\frac{\partial P_q}{\partial Q_n} = y_n > 0\right)$, such that the price process p_t is a martingale under Q_n.

We then analyze the asymptotics of the price dynamics as the lenght n increases to infinity. Our main result is that under Q_n, the price process p^n converges in finite dimensional distribution to a CMMV Z_t with probability distribution Q. We further prove that the historical distributions P_n converge to a limit distribution P which is equivalent to Q.

This paper seems therefore to indicate that under the martingale equivalent probability measure, the actualized price processes on the market should be CMMV. This class of martingales seems therefore to be a natural class of stochastic processes that could be used to develop pricing and hedging models.

We now give a precise definition of CMMV:

Definition 1. *A continuous martingale of maximal variation (CMMV) is a stochastic process* Π*, which is a martingale satisfying:*

$$\Pi_t = f(B_t, t)$$

where B *is a standard Brownian motion and* $f : \mathbb{R} \times [0,1] \to \mathbb{R}$ *is a deterministic function which is increasing in its first variable.*

Remark 2. *Due to Itô's formula, this definition implies in particular that f must satisfy the time reversed heat equation:*

$$\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2} = 0$$

Remark 3. *The terminology "CMMV" was introduced in De Meyer [2010] due to the following result. The n-variation of a martingale $(X_t)_{t\in[0,1]}$ is:*

$$V_X^n = \sum_{q=0}^{n-1} \|X_{\frac{q+1}{n}} - X_{\frac{q}{n}}\|_{L^1}$$

Consider the problem $\mathscr{M}_n$ of maximizing the n-variation V_X^n on the class of martingales X with final distribution μ ($X_1 \sim \mu$). It is proved in De Meyer [2010] that the martingales that solve $\mathscr{M}_n$, (i.e. martingales of maximal variation) converge in finite distributions, as n goes to infinity, to a process that satisfies the above Definition 1.

Remark 4. *Note that if B is a Brownian motion on a filtration $(\mathscr{F}_t)$ and g is an increasing function $\mathbb{R} \to \mathbb{R}$, then $X_t := E[g(B_1)|\mathscr{F}_t]$ is a CMMV. Indeed, due to the Markov property of the Brownian motion, we have $X_t = E[g(B_1)|\mathscr{F}_t] = E[g(B_1)|B_t]$. We get therefore $X_t = f(B_t, t)$, where $f(x,t) = E_z[g(x + z\sqrt{1-t})]$, with $z \sim \mathscr{N}(0,1)$. Note that f is the convolution of g with a normal density kernel. This convolution preserves the class of increasing functions and f is thus indeed increasing in x.*

2 Description of the model

In the game we are considering, player 1 receives initially some private information about the risky asset R. This information will be publicly disclosed at a future date (for example at the next shareholder meeting). At that date, the value L of the risky asset will depend on the information initially received. Since the liquidation value is the only part of the information that is relevant for R, we can model the whole situation as follows: (1) Nature draws initially L, once for all, with a lottery μ. (2) Player 1 observes L, not player 2. (3) All this process, including μ, is common knowledge.

We define $G_n(\mu)$, $n \geq 1$, as the n-times repeated game where two players are exchanging at each round a risky asset R for a numeraire. At period $q \in \{1,\ldots,n\}$, player 1 decides to buy ($u_q = 1$) or to sell ($u_q = -1$) one unit of the risky asset. $u_q \in \{+1,-1\}$ is thus the action of player 1. Simultaneously, player 2 selects the price $p_q \in \mathbb{R}$ of the transaction at stage q.

Remark 5. *Choices are thus considered to be simultaneous: in our model, player 1 does not observe player 2's action before deciding whether to sell or buy. This can be surprising at first glance. Indeed, one usually assumes that the trader will buy or sell after observing the market maker's prices. In fact, we argue that this sequential model where player 1 reacts to the price posted by player 2 is in fact equivalent to our model. Indeed, we prove in section 4 that, due to Jensen's inequality, the equilibrium strategy of player 2 in the simultenaous game considered here, is a pure strategy. Player 2's move p_q is thus completely forecastable for player 1 at period q. Player 1 would get no benefit from observing p_q before selecting u_q.*

We denote h_q the history of plays until round q, i.e. $h_q = (u_1, p_1, \dots, u_q, p_q)$ and $\mathbb{H}_q$ the set of all possible histories until round q. At the end of stage q, u_q and p_q are publicly revealed. Then both player know and remember all the past actions taken by both of them. Since the game is with perfect recall we can apply Kuhn's theorem and assume, without loss of generality, that players use behavioral strategies.

A behavioral strategy for player 1 in this game is $\sigma = (\sigma_1, \dots, \sigma_n)$ with $\sigma_q : (h_{q-1}, L) \to \sigma_q(h_{q-1}, L) \in \Delta(\{-1, +1\})$. A behavioral strategy for player 2 is $\tau = (\tau_1, \dots, \tau_n)$ with $\tau_q : h_{q-1} \to \tau_q(h_{q-1}) \in \Delta(\mathbb{R})$. A triple (μ, σ, τ) induces a unique probability distribution on (L, h_n). When X is a random variable, we denote $E_{\mu,\sigma,\tau}[X]$ its expectation with respect to this probability.

In this paper, player 1 is risk neutral. His payoff in $G_n(\mu)$ is then the expected value of his final portfolio:

$$g_1(\sigma, \tau) = E_{\mu,\sigma,\tau}\left[\frac{1}{\sqrt{n}} \sum_{q=1}^{n} u_q(L - p_q)\right] \tag{1}$$

Note that we introduced the normalization factor $\frac{1}{\sqrt{n}}$. The interest of this normalization appears in De Meyer [2010]. Indeed, without this normalization factor, this particular game corresponds to an exchange mechanism T defined by:

$$i \in \{-1, +1\}, j \in \mathbb{R} \text{ and } T_{i,j} = (i, -i \times j)$$

It can be easily shown that this mechanism satisfies the five hypothesis mentionned above, and we know from De Meyer [2010] that the value V_n of the game is such that $\frac{V_n}{\sqrt{n}}$ converges to a finite quantity. This result points out that payoffs should be normalized by a factor $\frac{1}{\sqrt{n}}$. This normalization has no effect on player 1 since he is risk neutral, but it is important for player 2.

The particularity of the current paper is that we consider a risk averse player 2. The payoff he aims to minimize (we keep the formalism of the zero sum games where player 2 is a minimizer) is thus:

$$g_2(\sigma, \tau) = E_{\mu,\sigma,\tau}\left[H\left(\frac{1}{\sqrt{n}} \sum_{q=1}^{n} u_q(L - p_q)\right)\right] \tag{2}$$

where H is a risk aversion function, (convex and increasing).

Throughout this paper, we will make the following assumptions on μ and H:

A1: μ is a probability measure on $[0, 1]$ absolutely continuous with respect to the Lebesgue measure. Its density function f_μ is strictly positive and C^1.

A2: H is a strictly positive, strictly convex and C^2 function and H' is Lipschitz-continuous and satisfies: $\exists K, \epsilon > 0$ such that for all $x \in \mathbb{R} : \epsilon < H'(x) < K$.

Observe that in **A1** we assume that L takes only values in the $[0,1]$ interval. We could obviously change this assumption to any compact interval by just a renormalization.

3 Results and structure of the paper

In the first part of the paper (sections 4, 5, 6), we analyse the game $G_n(\mu)$ for a fixed number n of stages. We first prove in section 4 that some equilibria of $G_n(\mu)$ can be found among the equilibria of the simpler game $\overline{G}_n(\mu)$ where the informed player 1 does not observe the actions of player 2. We then focus on the reduced game $\overline{G}_n(\mu)$.

This game can be completly reformulated: a strategy of player 1 can be identified with a probability Π_n on the pair (ω, L) where $\omega = (u_1, \ldots, u_n)$. Due to Jensen's inequality we prove in Proposition 10 that player 2 can restrict himself to pure strategies.

We argue in section 5 that such a pure strategy can be identified with a map X_n from $\Omega_n := \{-1,1\}^n$ to $\mathbb{R}$ that satisfies $E_{\lambda_n}[X_n] = 0$ where λ_n is the uniform probability on Ω_n. The link between X_n and the corresponding pure strategy is made precise in subsection 5.1.

We show in subsection 5.2 that, in order to be an equilibrium in $\overline{G}_n(\mu)$, (Π_n, X_n) must satisfy the conditions **(C1)** to **(C4)**, with $\overline{\Pi}_n$ denoting the marginal of Π_n on (L, S_n), with $S_n := \frac{1}{\sqrt{n}} \sum_{k=1}^n u_k$, and Ψ_n being a convex function such that $X_n(\omega) = \Psi_n(S_n(\omega))$.

(C1): $E_{\overline{\lambda}_n}[\Psi_n(S_n)] = 0$, where $\overline{\lambda}_n$ is the law of S_n under λ_n.

(C2): The marginal distribution $\overline{\Pi}_{n|L}$ of L under $\overline{\Pi}_n$ is μ.

(C3): The marginal distribution $\overline{\Pi}_{n|S_n}$ of S_n under $\overline{\Pi}_n$, denoted ν_n, is such that the density $\frac{\partial \overline{\lambda}_n}{\partial \nu_n}$ is proportional to $E_{\overline{\Pi}_n}[H'(LS_n - \Psi_n(S_n)) \mid S_n]$.

(C4) $\overline{\Pi}_n(L \in \partial\Psi_n(S_n)) = 1$ where $\partial\Psi_n$ denotes the subgradient of the convex function Ψ_n.

Conversely, one can always associate an equilibrium to a pair $(\overline{\Pi}_n, \Psi_n)$ satisfying the four conditions.

Using the link between X_n and the corresponding pure strategy of player 2 (which is the price process at equilibrium), we prove in the subsection 5.3 that if conditions **(C1)** to **(C4)** are satisfied, there exists a unique probability Q_n, equivalent to the historical probability P_n, such the price process $(p_q^n)_{q=1,\ldots,n}$ is a martingale under Q_n.

We next turn to the problem of existence of these reduced equilibria. In fact, the existence could be proved by classical methods, using Nash-Glicksberg's theorem. We give here an alternative proof introducing in section 6 an operator $T_\lambda : \Delta(\mathbb{R})$ to $\Delta(\mathbb{R})$. This operator will be useful both to prove existence, and to study the asymptotic properties of these equilibria.

The idea behind this operator T_λ is as follow: as explained in section 6, the problem of finding a pair $(\overline{\Pi}_n, \Psi_n)$ satisfying **(C1)** to **(C4)** can be further reformulated. First observe that the marginal $\nu_n = \overline{\Pi}_{n|S}$ completely determines $\overline{\Pi}_n$ and Ψ_n as long as **(C1)**, **(C2)** and **(C4)** are satisfied.

Indeed, due to Fenchel lemma, condition **(C4)** can heuristically be interpreted by saying that S_n is an increasing function[1] g of L: $S_n = g(L)$. When L is μ-distributed, $g(L)$ must have distribution ν_n. There is essentially a unique function g which satisfies that condition[2] and we find therefore heuristically that $L = \Psi_n'(S_n) = g^{-1}(S_n)$. This fix $\overline{\Pi}_n$ which is then the joint law of $(L, g(L))$ when L is μ-distributed. This also fix Ψ_n up to a constant which can be determined in a unique way to satisfy **(C1)**.

So, from the above discussion, given a measure λ on $\mathbb{R}$, for all measure ν on $\mathbb{R}$ there exists a unique pair $(\overline{\Pi}_\nu, \Psi_{\nu,\lambda})$ satisfying **(C1)** with $\overline{\lambda}_n$ replaced by λ, **(C2)**, **(C4)** and such that the marginal distribution of S_n under $\overline{\Pi}_\nu$ is ν.

So we are now seeking a ν for which condition **(C3)** is further satisfied. There exists a unique probability ρ such that $\frac{\partial\rho}{\partial\nu}$ is proportional to $E_{\overline{\Pi}_\nu}[H'(LS - \Psi_{\nu,\lambda}(S)) \mid S]$. Call T_λ the map $\nu \to T_\lambda(\nu) := \rho$. With these notations, finding an equilibrium in $\overline{G}_n(\nu)$ is equivalent to find a measure ν satisfying the equation:

$$T_{\overline{\lambda}_n}(\nu) = \overline{\lambda}_n \tag{3}$$

The existence of equilibrium in the game $\overline{G}_n(\mu)$ is finally proved in section 6 by showing that the operator $T_{\overline{\lambda}_n}$ is onto the space of measures. We first prove the continuity of the operator $T_{\overline{\lambda}_n}$ in term of Wasserstein distance W_2. The onto property of $T_{\overline{\lambda}_n}$ results then from an application of the KKM theorem. Indeed, it follows from its definition that $T_{\overline{\lambda}_n}(\nu)$ is absolutely continuous with respect to ν. So, if one consider the restriction of $T_{\overline{\lambda}_n}(\nu)$ to the set of measures having the same finite support K as $\overline{\lambda}_n$, $T_{\overline{\lambda}_n}(\nu)$ is essentially a continuous map from the $|K|$-dimensional simplex to itself that preserves the faces.

Section 7 of the paper is devoted to the asymptotics of ν_n. Remember that $\overline{\lambda}_n$ is the law of $\frac{1}{\sqrt{n}}\sum_{q=1}^n u_q$ when u_q are independent and centered. Due to the central limit theorem, $\overline{\lambda}_n$ converges (in Wasserstein distance W_2) to the normal law $\overline{\lambda}_\infty$.

On the other hand, by a compactness argument we can prove that any sequence $(\nu_n)_{n\in\mathbb{N}}$ of solutions ν_n of equation (3) has an accumulation point ν satisfying $T_{\overline{\lambda}_\infty}(\nu) = \overline{\lambda}_\infty$. The convergence of $(\nu_n)_{n\in\mathbb{N}}$ is then obtained by proving in subsection 7.4 that there is a unique solution to this equation.

We first prove that if ν is solution to $T_{\overline{\lambda}_\infty}(\nu) = \overline{\lambda}_\infty$ then $\Psi_{\nu,\overline{\lambda}_\infty}$ is a C^2 function. Since $\Psi'_{\nu,\overline{\lambda}_\infty}(S_n)$ is μ-distributed when S_n is ν-distributed, we get that $\Psi'_{\nu,\overline{\lambda}_\infty}(S_n) = F_\mu^{-1}(F_\nu(S_n))$ where F_μ and F_ν are the cumulative distribution functions of μ and ν. Differentiating this equation with respect to S_n gives, with f_ν and f_μ the density functions of ν and μ:

$$\Psi''_{\nu,\overline{\lambda}_\infty}(S_n) = \frac{f_\nu(S_n)}{f_\mu(F_\mu^{-1}(F_\nu(S_n)))} = \frac{f_\nu(S_n)}{f_\mu(\Psi'_{\nu,\overline{\lambda}_\infty}(S_n))}$$

[1] We remain very heuristical in our explanation at this point because $\partial\Psi_n^\sharp$ is actually a correspondence and not a single valued function.

[2] Would ν have no atom, we would have $g(\ell) = F_\nu^{-1}(F_\mu(\ell))$, where F_μ and F_ν are the cumulative distribution functions of μ and ν.

The equation $T_{\overline{\lambda}_\infty}(\nu) = \overline{\lambda}_\infty$ indicates that $\frac{f_{\overline{\lambda}_\infty}}{f_\nu} = \frac{\partial \overline{\lambda}_\infty}{\partial \nu} = cH'(LS_n - \Psi_{\nu,\overline{\lambda}_\infty}(S_n))$.

This is exactly the equation appearing in the differential problem $(\mathscr{D})$ of Proposition 40. The proof of the uniqueness of the solution of that system is technically too involved to be described in this introduction. It is done in section 7 (see Theorem 41).

Remark 6. *The difficulty of this uniqueness result comes on one hand from the fact that the equation is strongly non linear due to both f_μ and H. On the other hand, it is usual in game theory to prove the convergence of the values of discrete time repeated games to a limit using the concept of viscosity solution and uniqueness result to these differential problems. Nonetheless the corresponding equation are usually of second order. Our equation is apparently also of the second order, but it involves an additional constant c. Such an equation can be considered as a integral version of a third order equation without c. But the theory of viscosity solution does not apply to third order equations and this explains the technical difficulty of our proof.*

With the convergence of ν_n proved in section 7, we can prove in section 8 that the historical laws P_n converge. We also get the convergence of the law Q_n of this process under the martingale equivalent measure. More specifically, the discrete time price process $(p_1^n, \ldots, p_n^n)$ can be represented by the continuous time price process $t \to p^n_{\lceil nt \rceil}$.

We first show that the processes $p^n_{\lfloor nt \rfloor}$ under the law Q_n can be represented (Skorokhod embedding) on the natural filtration $\mathscr{F}$ of a Brownian motion on a probability space $(\tilde{\Omega}, \mathscr{F}, \tilde{P})$. More precisely, there exists a sequence of variables $\tilde{p}^n_q$ and an increasing sequence of stopping times τ^n_q such that $\tilde{p}^n_q$ is $\mathscr{F}_{\tau^q_n}$-mesurable, and has the same distribution as p^n_q. We show in Theorem 49 that $\tilde{p}^n_{\lfloor nt \rfloor}$ converges in finite dimensional distribution to a limit process Z_t defined on the space $(\tilde{\Omega}, \mathscr{F}, \tilde{P})$ and results to be a CMMV with law Q.

We also show the convergence in finite dimensional distribution of $p^n_{\lfloor nt \rfloor}$ under the historical probability P_n. More precisely we show in Lemma 52 that the density $y_n = \frac{\partial P_n}{\partial Q_n}$ converges in L^1 to a density y, with y bounded. We can then show in Theorem 53 that the historical law $P_n = y_n Q_n$ of the process $p^n_{\lfloor nt \rfloor}$ converges to the law $P = y.Q$.

4 Reduced equilibrium

Definition 7. *The reduced game $\overline{G}_n(\mu)$ is the game where player 1 does not observe player 2's actions and player 2 is not allowed to randomize his moves (he only uses pure strategies).*

In this paper pure strategies of player 2 will be denoted $\overline{p}$. Such a strategy $\overline{p}$ is thus a sequence of maps $(\overline{p}_1, \ldots, \overline{p}_n)$ where $\overline{p}_q$ is a map $\mathbb{H}_{q-1} \to \mathbb{R}$. $\overline{p}_q(h_{q-1})$ denotes then the deterministic action taken by player 2 after history h_{q-1}. Remark however that since player 2 does not randomize before stage q, the action he will take at stage q is just a deterministic function of previous moves of player 1. Therefore, in this paper, a pure strategy of player 2 will be considered as a sequence $(\overline{p}_1, \ldots, \overline{p}_n)$ where $\overline{p}_q$ is a function $\{-1, +1\}^{q-1} \to \mathbb{R}$.

In this section we show that any equilibrium in $\overline{G}_n(\mu)$ is an equilibrium in $G_n(\mu)$. These equilibria will be referred to as reduced equilibrium. Let σ be a reduced strategy of player 1 and let τ be any strategy of player 2. $E_{\mu,\sigma,\tau}[p_q \mid u_1, \ldots, u_{q-1}]$ is just some real valued function $f_q(u_1, \ldots, u_{q-1})$ (defined only on histories $(u_1, \ldots, u_n)$ with positive probability under (μ, σ)). The sequence $(f_1, \ldots, f_n)$ of those functions determines a pure strategy for player 2. Let us call $p_{\sigma,\tau}$ this trategy. The point is that $p_{\sigma,\tau}$ has a version that does not depend on the reduced strategy σ. Indeed, strategy σ being reduced, we can consider that the whole sequence of actions $u_1, \ldots, u_n$ is selected by player 1 after observing L and before player 2 starts to play. Then τ_1 will fix the law of the action p_1 of player 2. The law of p_2 conditionally on u_1 will be given by $\tau_2(u_1, p_1)$ when p_1 is τ_1-distributed and independent of u_1. And so forth: $\tau_1, \ldots, \tau_q$ fix the law of $p_1, \ldots, p_q$ given $u_1, \ldots, u_{q-1}$. Therefore σ does not appear in the computation of the law of $p_1, \ldots, p_q$ given $u_1, \ldots, u_{q-1}$. As announced, $E_{\mu,\sigma,\tau}[p_q \mid u_1, \ldots, u_{q-1}]$ does not depend on σ. We write therefore p_τ instead of $p_{\sigma,\tau}$, and we get thus for all reduced σ:

$$p_{\tau,q} = E_{\mu,\sigma,\tau}[p_q \mid u_1, \ldots, u_{q-1}]$$

Moreover, $p_1, \ldots, p_q$ do not contain more information on $L, u_1, \ldots, u_n$ than the information already contained in $u_1, \ldots, u_{q-1}$. In other words, the law of $L, u_q, \ldots, u_n$ is independent of $p_1, \ldots, p_q$ given $u_1, \ldots, u_{q-1}$.

We now compare the payoffs induced by a strategy τ with those induced by the corresponding strategy p_τ.

Lemma 8. *For any τ a strategy of player 2, p_τ is such that for all reduced strategy σ of player :*

$$\begin{cases} g_2(\sigma, \tau) \geq g_2(\sigma, p_\tau) \\ g_1(\sigma, \tau) = g_1(\sigma, p_\tau) \end{cases}$$

Proof. To simplify notations, the expectation $E_{\mu,\sigma,\tau}$ is denoted E.

$$g_2(\sigma, \tau) = E[H(\sum_{q=1}^{n} u_q(L - p_q))] = E[E[H(\sum_{q=1}^{n} u_q(L - p_q)) \mid u_1, \ldots, u_n, L]]$$

We now apply Jensen's inequality with the convex function H, and take into account the fact that u_q and L are $(u_1, \ldots, u_n, L)$-measurable:

$$g_2(\sigma, \tau) \geq E[H(E[\sum_{q=1}^{n} u_q(L - p_q) \mid u_1, \ldots, u_n, L])] = E[H(\sum_{q=1}^{n} u_q(L - E[p_q \mid u_1, \ldots, u_n, L]))]$$

As noticed above, p_q is independent of $(L, u_q, \ldots, u_n)$ conditionally to $(u_1, \ldots, u_{q-1})$. Therefore $E[p_q \mid u_1, \ldots, u_n, L] = E[p_q \mid u_1, \ldots, u_{q-1}]$, and we get:

$$g_2(\sigma, \tau) \geq E[H(\sum_{q=1}^{n} u_q(L - E[p_q \mid u_1, \ldots, u_{q-1}]))] = g_2(\sigma, p_\tau)$$

Similarly, we have:

$$
\begin{aligned}
g_1(\sigma, \tau) =& E_{\mu,\sigma,\tau}[u_q(L - p_q)] \\
=& E_{\mu,\sigma,\tau}[\sum_{q=1}^{n} u_q(L - E_\tau[p_q \mid u_1, \ldots, u_{q-1}])] \\
=& E_{\mu,\sigma,\tau}[\sum_{q=1}^{n} u_q(L - p_{\tau,q})] \\
=& g_1(\sigma, p_\tau)
\end{aligned}
$$

□

The next lemma highlights another property satisfied by the strategy p_τ (as a pure reduced strategy).

Lemma 9. *Let $\overline{p}$ be a pure strategy of player 2 and σ any strategy of player 1 (even non reduced). Then, there exists a reduced strategy of player 1 denoted $\tilde{\sigma}_{(\sigma,\overline{p})}$ which gives him the same payoff as σ against $\overline{p}$, i.e. :*

$$g_1(\tilde{\sigma}_{(\sigma,\overline{p})}, \overline{p}) = g_1(\sigma, \overline{p})$$

Proof. The strategy σ is not reduced, so σ_q depends on $(u_1, \ldots, u_{q-1}, \overline{p}_1, \ldots, \overline{p}_{q-1})$. But player 2 is completely deterministic since he uses strategy $\overline{p}$. Therefore he plays action $p_q = \overline{p}_q(u_1, \ldots, u_{q-1})$, and the whole history $p_1, \ldots, p_{q-1}$ is just a deterministic function of $u_1, \ldots, u_{q-2}$. In the arguments of σ_q, we can replace $p_1, \ldots, p_{q-1}$ by this function and we get in this way:

$$\tilde{\sigma}_{(\sigma,p),q}(u_1, \ldots, u_{q-1}) := \sigma(u_1, \ldots, u_{q-1}, \overline{p}_1, \ldots, \overline{p}_{q-1}(u_1, \ldots, u_{q-2}))$$

which is a reduced strategy and clearly: $g_1(\tilde{\sigma}_{(\sigma,\overline{p})}, \overline{p}) = g_1(\sigma, \overline{p})$ □

Proposition 10. *If $(\sigma^\star, \overline{p}^\star)$ is an equilibrium in $\overline{G}_n(\mu)$, then $(\sigma^\star, \overline{p}^\star)$ is an equilibrium in $G_n(\mu)$.*

Proof. For all player 2's strategy τ in $G_n(\mu)$, we have:

$$g_2(\sigma^\star, \overline{p}^\star) \leq g_2(\sigma^\star, p_\tau) \leq g_2(\sigma^\star, \tau)$$

where p_τ is defined above. Indeed the first equality just indicates that the pure strategy p_τ is not a profitable deviation from the equilibrium strategy $\overline{p}^\star$ in $\overline{G}_n(\mu)$. The second inequality comes from Lemma 8.

Let σ be any strategy of player 1. With the notation of Lemma 9 we get:

$$g_1(\sigma, \overline{p}^\star) = g_1(\tilde{\sigma}_{(\sigma,\overline{p}^\star)}, \overline{p}^\star) \leq g_1(\sigma^\star, \overline{p}^\star)$$

where the inequality follows from the fact that $\tilde{\sigma}$ is a reduced strategy and can thus not be a profitable deviation from $\sigma^\star$ for player 1. □

Based on the previous proposition, equilibria in $\overline{G}_n(\mu)$ will be referred to as the reduced equilibria in $G_n(\mu)$. We will only focus on this paper on the reduced equilibria of $G_n(\mu)$.

5 Characterisation of equilibrium

In subsection 5.1 we give an other representation of the strategy spaces in $\overline{G}_n(\mu)$. Next, in subsections 5.2 we provide necessary and sufficient conditions on a pair of strategies to be an equilibrium.

5.1 Alternative representation of the strategy spaces

When playing a reduced strategy player 1 does not observe player 2's actions and we can therefore assume that he selects his actions after getting the information L and before the first move of player 2. Thus, joint with μ, a reduced strategy σ induces a joint law Π_n on (L, ω) where $\omega = (u_1, \dots, u_n)$ belongs to $\Omega_n := \{-1, +1\}^n$. The marginal $\Pi_{n|L}$ of Π_n on L is clearly μ. We can further recover the strategy σ from Π_n computing the conditional probabilities given L. Therefore the player 1's strategy space may be seen as the set of Π_n in $\Delta(\mathbb{R} \times \Omega_n)$ such that $\Pi_{n|L} = \mu$.

Let us now consider the set of pure strategies $\mathscr{P}$ of player 2. If $\overline{p} \in \mathscr{P}$, then $\overline{p}_q$ is a function $\Omega_n \to \mathbb{R}$ which is measurable with respect to $(u_1, \dots, u_{q-1})$. Note that the strategy $\overline{p}$ only appears in the payoff functions (see equations 1 and 2) thought the quantity $X_{n,\overline{p}}(\omega) := \frac{1}{\sqrt{n}} \sum_{q=1}^n u_q \overline{p}_q(\omega)$. We can therefore identify the strategy space of player 2 with the set $\mathbb{X}_n := \{X_{n,\overline{p}} | \overline{p} \in \mathscr{P}\} \subset \mathbb{R}^\Omega$.

Next lemma characterizes this set. Let λ_n be the uniform probability on Ω_n. Under λ_n, $(u_q)_{q=1,\dots,n}$ are mutually independent, and have zero expectation.

Lemma 11. $\mathbb{X}_n = \{X \in L^1(\lambda_n) \mid E_{\lambda_n}[X] = 0\}$

Proof. Let $X \in \mathbb{X}_n$. Then $X = X_{n,\overline{p}}$ for some $\overline{p} \in \mathscr{P}$. Since Ω_n is a finite set, X as a map from Ω_n to $\mathbb{R}$ belongs to $L^1(\lambda_n)$. Moreover, using that $\overline{p}_q$ is $(u_1, \dots, u_{q-1})$ measurable:

$$E_{\lambda_n}\left[\frac{1}{\sqrt{n}} \sum_{q=1}^n u_q p_q\right] = E\left[\frac{1}{\sqrt{n}} \sum_{q=1}^n E_{\lambda_n}[u_q p_q | u_1, \dots, u_{q-1}]\right] = E\left[\frac{1}{\sqrt{n}} \sum_{q=1}^n p_q E_{\lambda_n}[u_q]\right] = 0$$

We thus have proved that $\mathbb{X}_n \subseteq \{X \in L^1(\lambda_n) \mid E_{\lambda_n}[X] = 0\}$.
Suppose now that $X \in L^1(\lambda_n)$ is such that $E_{\lambda_n}[X] = 0$. For $k \in \{1, \dots, n-1\}$, we denote $X^k(u_1, \dots, u_k) := E_{\lambda_n}[X \mid u_1, \dots, u_k]$. Let $\mathbf{1}_{\{u_k=1\}}$ denotes the random variable that takes the value 1 if $u_k = 1$ and 0 otherwise. An easy computation shows that $\mathbf{1}_{\{u_k=1\}} = \frac{u_q+1}{2}$. One gets therefore

$$X^k(u_1, \dots, u_k) = \mathbf{1}_{\{u_k=1\}} X^k(u_1, \dots, u_k - 1, 1) + \mathbf{1}_{\{u_k=-1\}} X^k(u_1, \dots, u_k - 1, -1)$$

$$= \frac{u_k + 1}{2} X^k(u_1, \dots, u_{k-1}, 1) + \frac{1 - u_k}{2} X^k(u_1, \dots, u_{k-1}, -1)$$

$$= \frac{u_k \overline{p}_k(\omega)}{\sqrt{n}} + \frac{X^k(u_1, \dots, u_{k-1}, 1) + X^k(u_1, \dots, u_{k-1}, -1)}{2},$$

where:

$$\overline{p}_k(\omega) = \frac{X^k(u_1,\dots,u_{k-1},1) - X^k(u_1,\dots,u_{k-1},-1)}{2/\sqrt{n}} \tag{4}$$

Now observe that $X^{k-1}(u_1,\dots,u_{k-1}) = E_{\lambda_n}[X^k|u_1,\dots,u_{k-1}] = \frac{X^k(u_1,\dots,u_{k-1},1)+X^k(u_1,\dots,u_{k-1},-1)}{2}$. Therefore $X^k(u_1,\dots,u_k) - X^{k-1}(u_1,\dots,u_{k-1}) = \frac{u_k\overline{p}_k(\omega)}{\sqrt{n}}$.
Summing up those equalities for $k=1$ to n, we get:

$$X^n(u_1,\dots,u_n) = \frac{\sum_{k=1}^n u_k\overline{p}_k(\omega)}{\sqrt{n}} + X_0$$

But $X^n(u_1,\dots,u_n) = X$ and $X_0 = E_\lambda(X) = 0$. We get thus:

$$X = \frac{\sum_{k=1}^n u_k\overline{p}_k(\omega)}{\sqrt{n}} = X_{n,\overline{p}},$$

for the strategy $\overline{p}$ defined in 4. □

Let us make more precise the relation between X and the strategy p such that $X = X_{n,\overline{p}}$.

Proposition 12. *Let $X \in \mathbb{X}_n$. There exists a unique pure reduced strategy $\overline{p}$ such that $X = X_{n,\overline{p}}$. Moreover we have the explicit formula:*

$$\overline{p}_q(u_1,\dots,u_{q-1}) = \sqrt{n}E_{\lambda_n}[u_qX \mid u_1,\dots,u_{q-1}] \tag{5}$$

Proof. Let $\overline{p}_j$ be $(u_1,\dots,u_{j-1})$-measurable. Then observe that if $j<q$:

$$E_{\lambda_n}[\overline{p}_j u_q u_j|u_1,\dots,u_{q-1}] = \overline{p}_j u_j E_{\lambda_n}[u_q|u_1,\dots,u_{q-1}] = 0$$

On the other hand, if $j>q$,

$$\begin{aligned} E_{\lambda_n}[\overline{p}_j u_q u_j|u_1,\dots,u_{q-1}] =& E_{\lambda_n}[E_{\lambda_n}[\overline{p}_j u_q u_j|u_1,\dots,u_{j-1}]|u_1,\dots,u_{q-1}] \\ =& E_{\lambda_n}[\overline{p}_j u_q E_{\lambda_n}[u_j|u_1,\dots,u_{j-1}]|u_1,\dots,u_{q-1}] \\ =& 0 \end{aligned}$$

We get thus $E_{\lambda_n}[\overline{p}_j u_q u_j|u_1,\dots,u_{q-1}] = \overline{p}_q$ if $j=q$ and 0 otherwise. Let now X be in $\mathbb{X}_n$. According to the previous lemma, $X = X_{n,\overline{p}}$ for some $\overline{p}$. We can therefore write $E_{\lambda_n}[u_qX \mid u_1,\dots,u_{q-1}] = E_{\lambda_n}[u_q\frac{\sum_{i=1}^n \overline{p}_i u_i}{\sqrt{n}} \mid u_1,\dots,u_{q-1}] = \frac{1}{\sqrt{n}}\sum_{i=1}^n E_{\lambda_n}[\overline{p}_i u_q u_i \mid u_1,\dots,u_{q-1}] = \frac{\overline{p}_q}{\sqrt{n}}$ as announced.

□

We can now reformulate the completely reduced game $\overline{G}_n(\mu)$ as follow: player 1 select $\Pi_n \in \Delta(\Omega_n \times \mathbb{R})$ such that $\Pi_{n|L} = \mu$. Simultaneously player 2 chooses $X \in \mathbb{X}_n$.
The payoff functions are now given by the formula:

$$\begin{cases} g_1(\Pi_n, X) = E_{\Pi_n}[LS_n - X] \\ g_2(\Pi_n, X) = E_{\Pi_n}[H(LS_n - X)] \end{cases}$$

where $S_n(\omega) = \frac{1}{\sqrt{n}}\sum_{k=1}^n u_q$.

5.2 Characterization of equilibrium strategies in $\overline{G}_n(\mu)$

In this section we provide necessary conditions for a pair $(\Pi_n^\star, X^\star)$ to be an equilibrium in $\overline{G}_n(\mu)$.

Our first result is Proposition 13 that claims that any history ω has a positive probability at equilibrium. We next argue in Proposition 14 that if a strategy X of player 2 is such that there exists a best response of player 1 which charges all history ω, then X has a very particular form: $X(\omega)$ is a convex function of $S_n(\omega)$.

The next result is that if player 2 plays a strategy $X = \Psi_n(S_n)$ for a convex function Ψ then Π_n is a best response to X if and only if $\Pi_n(L \in \partial\Psi_n(S_n)) = 1$.

Finally we express the fact that the strategy $\Psi(S)$ is a best reply to the probability Π in Proposition 16.

We say that a strategy $\Pi_n^\star$ of player 1 is completely mixed if for all $\omega \in \Omega^n : \ \Pi_n^\star(\omega) > 0$.

Proposition 13. *If player 2 has a best reply to a strategy $\Pi_n^\star$ of player 1 in $\overline{G}_n(\mu)$ then $\Pi_n^\star$ is completely mixed.*

Proof. $\Pi_n^\star$ is a probability on (L, ω) where $\omega = (u_1, \dots, u_n)$. It induces therefore a marginal distribution on $\omega_q = (u_1, \dots, u_q)$. Denote Γ_q the set of ω_q such that $\Pi_n^\star(\omega_q) > 0$. We want to prove that $\Gamma_n = \Omega_n$. Assume on the contrary that $\Gamma_n \neq \Omega_n$. We can then define $q^\star$ as the smallest q such that $\Gamma_q \neq \Omega_q = \{-1, +1\}^q$. There is then a history $(u_1, \dots, u_{q^\star-1}) \in \Gamma_{q^\star-1}$ such that one of the histories $(u_1, \dots, u_{q^\star-1}, 1)$ or $(u_1, \dots, u_{q^\star-1}, -1)$ does not belong to $\Gamma_q^\star$. Whence, this history $(u_1, \dots, u_{q^\star-1})$ has a positive probability under $\Pi_n^\star$ and is followed by a deterministic move of player 1 at stage $q^\star$. But after observing this history, player 2 could increase his benefit by posting a higher or lower price according to the forecoming deterministic move of player 1. This contradicts the hypothesis that there is a best reply against $\Pi_n^\star$. Therefore, assuming $\Gamma_n \neq \Omega_n$ leads to a contradiction. □

Proposition 14. *If player 1 has a completely mixed best reply $\Pi_n^\star$ to a strategy $X^\star$ of player 2 in $\overline{G}_n(\mu)$, then $X^\star = \Psi_n(S_n(\omega))$ where Ψ_n is a convex function such that $E_{\lambda_n}[\Psi_n(S_n)] = 0$*

Proof. Suppose that player 2 is playing $X^\star$ and player 1 wants to maximize his payoff $E_{\Pi_n}[LS - X^\star]$. After observing $L = \ell$, he will select an history $\omega \in V_\ell$ where V_ℓ is the set of ω that solve the maximization problem $A(\ell)$:

$$A(\ell) = \max_{\omega' \in \Omega_n} \ell S_n(\omega') - X^\star(\omega'). \tag{6}$$

Therefore

$$\Pi_n^\star(\omega \in V_L) = 1. \tag{7}$$

Since all history ω has a positive probability under $\Pi_n^\star$ we conclude that the set of values ℓ such that $\omega \in V_\ell$ can not be empty. Otherwise ω would never be selected by player 1 and would have zero probability under $\Pi_n^\star$.

Now remark that it follows from the definition of A that for all ℓ and for all ω:

$$A(\ell) \geq \ell S_n(\omega) - X^\star(\omega) \tag{8}$$

Therefore, for all ω, for all ℓ:

$$X^\star(\omega) \geq \ell S_n(\omega) - A(\ell)$$

and thus for all ω:

$$X^\star(\omega) \geq \sup_{\ell \in \mathbb{R}} \quad \ell S_n(\omega) - A(\ell).$$

As observed above for all ω, the set of ℓ such that $\omega \in V_\ell$ is not empty.
For those ℓ, equation (8) is an equality, and thus:

$$X^\star(\omega) = \sup_{\ell \in \mathbb{R}} \quad \ell S_n(\omega) - A(\ell).$$

We get therefore $X^\star(\omega) = \Psi_n(S_n(\omega))$ with $\Psi_n : s \in \mathbb{R} \to \sup_{\ell \in \mathbb{R}} \quad \ell s - A(\ell)$. Observe that as supremum of affine functions of s, the map $s \to \Psi_n(s)$ is convex.
Finally, since $X \in \mathbb{X}_n$ we get with Lemma 11 that $E_{\lambda_n}[\Psi_n(S_n(\omega))] = 0$. □

Note that the function $A(\ell)$ was also defined as a maximum of a finite number of affine functions of ℓ and is therefore also a piece-wise linear concave function of ℓ. The Fenchel transform $A^\sharp$ of A is defined as:

$$A^\sharp(s) = \sup_\ell \ell s - A(\ell)$$

The function Ψ_n introduced in previous proposition is just the Fenchel transform $A^\sharp$ of A. As it is well known from Fenchel lemma (see Rockafellar [1970]), $A(\ell) = \Psi_n^\sharp(\ell)$ and we have the equivalence between the following claims:

- **(1)** ℓ is optimal in $A^\sharp(s) = \sup_\ell \ell s - A(\ell)$
- **(2)** s is optimal in $\Psi_n^\sharp(\ell) = \sup_s \ell s - \Psi_n(s)$
- **(3)** $\ell \in \partial\Psi(s)$ where $\partial\Psi_n(s)$ is defined as the subgradient of the convex function Ψ_n at s: $\partial\Psi_n(s) = \{\ell | \forall z \Psi_n(z) \geq \Psi_n(s) + \ell(z-s)\}$.
- **(4)** $s \in \partial A(\ell)$
- **(5)** $A(\ell) + \Psi_n(s) = s\ell$

Now coming back to the definition of V_ℓ as the set of ω that are optimal in 6 with $X^\star = \Psi_n(S_n)$, we claim that:

$$\omega \in V_\ell \Leftrightarrow \ell \in \partial\Psi_n(S_n(\omega)) \tag{9}$$

Indeed $\omega \in V_\ell$ if and only if $A(\ell) = S_n(\omega)\ell - X^\star(\omega)$. On the other hand we also have for all r, $A(r) \geq S_n(\omega)r - X^\star(\omega)$. Combining these two relations, we have that $A(r) \geq S(\omega)(r-\ell) + A(\ell)$ and thus $S(\omega) \in \partial A(\ell)$ or equivalently $\ell \in \partial\Psi_n(S_n(\omega))$.
Conversely assume that $\ell \in \partial\Psi_n(S_n(\omega))$. With condition **(5)** in Fenchel lemma above, this means that $A(\ell) = S_n(\omega)\ell - \Psi_n(S_n(\omega)) = S_n(\omega)\ell - X^\star(\omega)$. According to the definition

of A_ℓ in (6) as $\max_{\omega' \in \Omega_n} \ell S_n(\omega') - X^\star(\omega')$, the last equality means that ω is an optimal ω' and thus belongs to V_ℓ.

We are now ready to prove the following proposition:

Proposition 15. *Let X be a strategy of the form $X(\omega) = \Psi_n(S_n(\omega))$ where Ψ_n is a convex function. Then a strategy $\Pi_n^\star$ of player 1 is a best response to X if and only if:*

$$\Pi_n^\star(L \in \partial\Psi_n(S_n)) = 1$$

Proof. We proved above in equation (7) that $\Pi_n^\star$ is a best response to X if and only if $\Pi_n^\star(\omega \in V_L) = 1$. With our claim (9), this is equivalent to $\Pi_n^\star(L \in \partial\Psi_n(S_n)) = 1$. □

The next proposition expresses the first order conditions of player 2 optimization problem. $\Pi_{n|\omega}^\star$ just denotes the marginal distribution of ω under $\Pi_n^\star$.

Proposition 16. *A strategy $X^\star$ is a best reply to a strategy $\Pi_n^\star$ of player 1, if and only if λ_n has a density with respect to $\Pi_{n|\omega}^\star$ given by the formula:*

$$\frac{d\lambda_n}{d\Pi_{n|\omega}^\star} = \alpha_n E_{\Pi_n^\star}[H'(LS_n - X_n^\star) \mid \omega]$$

for a constant α_n.

Proof. Suppose that $X^\star$ is a best reply to a strategy $\Pi_n^\star$ of player 1. Then $X^\star$ is a solution to the minimization problem of player 2:

$$\min_{X \in \mathbb{X}_n} E_{\Pi_n^\star}[H(LS_n - X)].$$

Note that the map $X \to E_{\Pi_n^\star}[H(LS_n - X)]$ is convex in X and we are in front of a convex minimization problem. In such a problem the first order conditions are both necessary and sufficient. We get these first order conditions considering for fixed $\delta \in \mathbb{X}_n$ the map $G : \epsilon \in \mathbb{R} \to G(\epsilon) := E_{\Pi_n^\star}(H(LS_n - X^\star + \epsilon\delta))$. This map must reach a minimum at $\epsilon = 0$.

Observe now that H is C^1 and so is G. We get then $G'(0) = E_{\Pi_n^\star}[H'(LS_n - X^\star)\delta]$, and therefore, for all $\delta \in \mathbb{X}_n$:

$$E_{\Pi_n^\star}[H'(LS_n - X^\star)\delta] = 0$$

Since δ is just a function of ω, this equality can also be written as:

$$0 = E_{\Pi_n^\star}[E[H'(LS_n - X_n^\star)\delta|\omega]] = E_{\Pi_n^\star}[\delta Y_n]$$

where $Y_n(\omega) := E_{\Pi_n^\star}[H'(LS^n - X^{n,\star}) \mid \omega]$. $Y_n(\omega) > 0$ because $H' > \epsilon > 0$ according to **A2**.

Since λ_n puts a positive weight on every history, $\Pi_{n|\omega}^\star$ is absolutely continuous with respect to λ_n and has a density $y_n = \frac{d\Pi_{n|\omega}^\star}{d\lambda_n}$.

We can rephrase previous conditions as: for all $\delta \in \mathbb{X}_n$,

$$E_{\lambda_n}[y_n Y_n \delta] = 0$$

This relation can interpreted as an orthogonality relation in $L^2(\lambda_n)$ with the scalar product $< A, B >:= E_{\lambda_n}[AB]$. The space $\mathbb{X}_n$ must then be orthogonal to $y_n Y_n$. But Lemma 11 shows that $\mathbb{X}_n = \{1\}^{\perp}$. Therefore $y_n Y_n$ is co-linear with 1: it is equal to a positive constant that we denote $\frac{1}{\alpha_n}$.

Since $y_n = \frac{d\Pi^{\star}_{n|\omega}}{d\lambda_n} > 0$, λ_n is absolutely continuous with respect to $\Pi^{\star}_{n|\omega}$ and we get $\frac{d\lambda_n}{d\Pi^{\star}_{n|\omega}} = \frac{1}{y_n} = \alpha_n Y_n$. □

Propositions 13, 14, 15 and 16 clearly lead to a system of necessary and sufficient conditions for equilibrium which is summurized by the following corollary:

Corollary 17. *A pair of strategy $(\Pi^{\star}_n, X^{\star}_n)$ is an equilibrium in $\overline{G}_n(\mu)$ if and only if $X^{\star}_n = \Psi_n(S_n)$ where Ψ_n is a convex function that jointly satisfy with $\Pi^{\star}_n$ the following conditions **(C1)**,**(C2)**,**(C3)**,**(C4)**.*

$$\begin{cases} \textbf{(C1)} & \Psi_n \text{ is such that } E_{\lambda_n}[\Psi_n(S_n(\omega)] = 0 \\ \textbf{(C2)} & \Pi^{\star}_{n|L} = \mu \\ \textbf{(C3)} & \frac{\partial \lambda_n}{\partial \Pi^{\star}_{n|\omega}} = \alpha_n E_{\Pi^{\star}_n}[H'(LS_n - X^{\star}_n)|\omega] \text{ where } \alpha_n \text{ is a constant} \\ \textbf{(C4)} & \Pi^{\star}_n(L \in \partial \Psi_n(S_n)) = 1 \end{cases}$$

5.3 The price process and the martingale equivalent measure

Before proving the existence of equilibrium in section 6, let us emphasize that the above characterization of equilibrium implies that under an appropriate equivalent measure the price process is a martingale.

Consider an equilibrium $(\Pi^{\star}_n, X^{\star})$. We already know that $X^{\star} = \Psi_n(S_n(\omega))$ for a convex function Ψ_n. Since $X^{\star} \in \mathbb{X}_n$ there exists $\overline{p}$ such that $X = X_{n,\overline{p}}$. The price process posted by player 2 will then be $\overline{p}_1, \overline{p}_2(u_1), \ldots, \overline{p}_n(u_1, \ldots, u_{n-1})$. When $(u_1, \ldots, u_n)$ are randomly selected by player 1 with lottery $\Pi^{\star}_n$, the law of this process $\overline{p}$ is called the historical law. We now prove that if $(u_1, \ldots, u_n)$ are selected under λ_n, the process is a martingale.

Theorem 18. *The price process $(\overline{p}^n_q)_{q=1,\ldots,n}$ is a martingale under the probability λ_n.*

Proof. With equation (5) we have:

$$\begin{aligned} \overline{p}^n_q(u_1, \ldots, u_{q-1}) &= \sqrt{n} E_{\lambda_n}[u_q X^{\star} \mid u_1, \ldots, u_{q-1}] \\ &= \sqrt{n} E_{\lambda_n}[u_q \Psi_n(S_n) \mid u_1, \ldots, u_{q-1}] \\ &= \sqrt{n} E_{\lambda_n}[u_n \Psi_n(S_n) \mid u_1, \ldots, u_{q-1}] \end{aligned} \tag{10}$$

The last equality follows from the fact that, conditionally to $u_1, \ldots, u_{q-1}$, the vector (u_q, S_n) and (u_n, S_n) have the same law under λ_n. The price process $\overline{p}^n$ is written as a conditional expectation of a terminal variable with respect to an increasing sequence of σ-algebras. It is then a martingale under the probability λ_n. □

We further aim to prove that λ_n is the unique probability on Ω_n that makes the price process a martingale. To do so, we need the following lemma.

Lemma 19.

$$\overline{p}_q^n(u_1, \ldots, u_{q-2}, 1) > \overline{p}_q^n(u_1, \ldots, u_{q-2}, -1)$$

Proof. Since Ψ_n is convex, his derivative exists except on a countable number of points, and the following definition is not ambiguous:

$$\chi(x) = \frac{\sqrt{n}}{2} \int_{\frac{-1}{\sqrt{n}}}^{\frac{1}{\sqrt{n}}} \Psi_n'(x+v)dv \tag{11}$$

We also set $S_q := \frac{1}{\sqrt{n}} \sum_{k=1}^{q} u_q$. According to formula (10), we get:

$$\begin{aligned}
\overline{p}_q^n(u_1, \ldots, u_{q-1}) =& E_{\lambda_n}\left[\frac{\sqrt{n}}{2}\left(\Psi_n(S_{n-1} + \frac{1}{\sqrt{n}}) - \Psi_n(S_{n-1} - \frac{1}{\sqrt{n}})\right) \mid u_1, \ldots, u_{q-1}\right] \\
=& E_{\lambda_n}[\chi(S_{n-1})|u_1, \ldots, u_{q-1}] \\
=& E_{\lambda_n}[\chi(S_{n-1})|S_{q-1}] \\
=& E_{\lambda_n}[\chi(S_{q-1} + V)|S_{q-1}]
\end{aligned}$$

where $V = \frac{u_q + \ldots, u_{n-1}}{\sqrt{n}}$. Since V is independent of S_{q-1}, we get therefore $\overline{p}_q^n(u_1, \ldots, u_{q-1}) = r(S_{q-1})$ where $r(x) := E_{\lambda_n}[\chi(x+V)]$. To prove the lemma, we just have to show that χ is strictly increasing on the support $U = \{\frac{-n+2k}{\sqrt{n}} | k \in 0, \ldots, n\}$ of $\overline{\lambda}_n$. Indeed, r will therefore be strictly increasing, and the lemma will follow immediately.

Denote ν_n the marginal law of S_n under $\Pi_n^\star$. The condition **(C3)** indicates that ν_n has the same support as $\overline{\lambda}_n$. Condition **(C4)** implies that $\Pi_n^\star(L \in [\Psi_n'(S_n^-), \Psi_n'(S_n^+)]) = 1$ where $\Psi_n'(s^-)$ and $\Psi_n'(s^+)$ stands respectively for the left and the right limits at s of the derivative Ψ_n'. Indeed for all s, $\partial\Psi_n(s) = [\Psi_n'(s^-), \Psi_n'(s^+)]$.

Note that any point of $x \in U$ has a strictly positive probability under ν_n. Therefore $\Psi_n'(x^-) < \Psi_n'(x^+)$. Otherwise, on the event $\{S_n = x\}$, which has strictly positive probability under $\Pi_n^\star$, we would have $L = \Psi_n'(x)$. This is impossible since μ, the marginal law of L under $\Pi_n^\star$, has no atom.

We next argue that if x, y are two successive values in U, then $\Psi_n'(x^+) = \Psi_n'(y^-)$. Indeed, suppose on the contrary that $\Psi_n'(x^+) < \Psi_n'(y^-)$. There would then exist ℓ_1, ℓ_2 such that $\Psi_n'(x^+) < \ell_1 < \ell_2 < \Psi_n'(y^-)$. Define $\tilde{x} := \inf\{s | \Psi_n'(s^+) > \ell_1\}$, and $\tilde{y} := \sup\{s | \Psi_n'(s^-) < \ell_2\}$. Observe that $\Psi_n'(\tilde{x}) \geq \ell_1$, as it follows from the definition of $\tilde{x}$. Therefore $\Psi_n'(\tilde{x}) > \Psi_n'(x^+)$ which implies $x < \tilde{x}$. A similar argument leads to $y > \tilde{y}$. Let A denote the event $\{\Psi_n'(S_n^-) \leq L \leq \Psi_n'(S_n^+)\}$. We know that $\Pi_n^\star(A) = 1$. Consider then the event $B := \{\ell_1 < L < \ell_2\}$. On $A \cap B$ we then clearly have $\Psi_n'(S_n^-) < \ell_2$ and $\ell_1 < \Psi_n'(S_n^+)$. It results then from the definition of $\tilde{x}$ and $\tilde{y}$ that $A \cap B$ is included in the event $\{\tilde{x} \leq S_n \leq \tilde{y}\}$. Therefore: $\Pi_n^\star(\ell_1 < L < \ell_2) \leq \Pi_n^\star(\tilde{x} \leq S_n \leq \tilde{y}) \leq \Pi_n^\star(x < S_n < y)$. But remember that x and y are two successive points in the support of ν_n, therefore $\Pi_n^\star(x < S_n < y) = 0 = \Pi_n^\star(\ell_1 < L < \ell_2) = \mu(]\ell_1, \ell_2[)$. According to our hypothesis **A1** on μ, this implies that $\ell_1 = \ell_2$ which is in contradiction with our definition of ℓ_1, ℓ_2. It must therefore be the case, as announced, that $\Psi_n'(x^+) = \Psi_n'(y^-)$.

If $x \in U \backslash \{-n, n\}$ then formula (11) joint with the fact that Ψ_n' is constant on $]x, y[$ implies that $\chi(x) = \frac{\Psi_n'(x^+)+\Psi_n'(x^-)}{2}$. For $x \in \{n, -n\}$, one gets: $\chi(-n) \le \frac{\Psi_n'((-n)^+)+\Psi_n'((-n)^-)}{2}$ and $\chi(n) \ge \frac{\Psi_n'(n^+)+\Psi_n'(n^-)}{2}$. Therefore if $x < y$ are two successive points in U, $\chi(x) \le \frac{\Psi_n'(x^+)+\Psi_n'(x^-)}{2} < \Psi_n'(x^+) = \Psi_n'(y^-) < \frac{\Psi_n'(y^+)+\Psi_n'(y^-)}{2} \le \chi(y)$. χ is therefore strictly increasing. □

Theorem 20. *λ_n is the unique probability on Ω_n that makes the price process $(\overline{p}_q)_{q=1,\dots,n}$ a martingale.*

Proof. Indeed, let $\tilde{\lambda}_n$ be a probability on Ω_n under which the price process is a martingale.

We find with the similar computation as that made to get equation (5) that:

$$\overline{p}_q(u_1, \dots, u_{q-1}) = \frac{u_q + 1}{2}\overline{p}_q(u_1, \dots, u_{q-2}, 1) + \frac{u_q - 1}{2}\overline{p}_q(u_1, \dots, u_{q-2}, -1)$$

Since $\overline{p}$ is a martingale under λ_n, we find $\frac{\overline{p}_q(u_1,\dots,1)+\overline{p}_q(u_1,\dots,-1)}{2} = \overline{p}_{q-1}(u_1, \dots, u_{q-2})$

And thus

$$\overline{p}_q(u_1, \dots, u_{q-1}) = \overline{p}_{q-1}(u_1, \dots, u_{q-2}) + c_q(u_1, \dots, u_{q-2})u_{q-1}$$

where $c_q = \frac{\overline{p}_q(u_1,\dots,1)-\overline{p}_q(u_1,\dots,-1)}{2} > 0$.

So if $\overline{p}$ is a martingale under $\tilde{\lambda}_n$ we must have for all q: $E_{\tilde{\lambda}_n}[u_{q-1}|u_1, \dots, u_{q-2}] = 0$. Therefore $\tilde{\lambda}_n = \lambda_n$.

□

From Lemma 19 we remark that observing the price process, one can recover the whole history $u_1, \dots, u_n$. The natural filtration of the process $(\overline{p}_q)_{q=1,\dots,n}$ coincides with the natural filtration of $(u_q)_{q=1,\dots,n}$. There is therefore a one to one correspondence between the laws of the price process $(\overline{p})_{q=1,\dots,n}$ and the law of the process $(u_q)_{q=1,\dots,n}$. The historical law P_n corresponds to the law $\Pi_n^\star$, and the martingale equivalent measure Q_n to λ_n. Our last result claims that Q_n is the unique martingale equivalent measure.

6 Existence of equilibrium

In this section we aim to prove the existence of equilibrium in $G_n(\mu)$. According to section 4 we can focus on the game $\overline{G}_n(\mu)$. According to the last corollary we just have to prove the existence of a pair (Π_n, Ψ_n) such that conditions **(C1)** to **(C4)** are satisfied with $X = \Psi_n(S_n)$.

In the next subsection we prove that we can focus our analysis on the marginal $\overline{\Pi}_n \in \Delta(\mathbb{R}^2)$ of Π_n on (L, S).

6.1 The marginal $\overline{\Pi}_n$

Π_n is a probability on $\Omega_n \times \mathbb{R}$ and it induces a marginal law $\overline{\Pi}_n \in \Delta(\mathbb{R}^2)$ for the pair (S, L). **(C1)**, **(C2)** and **C4)** are in fact conditions on $(\overline{\Pi}_n, \Psi_n)$. **(C3)** is the unique condition that involves the conditional law of L given ω. As proved with the first claim of Lemma 21, it turns out that **(C3)** implies the following necessary condition on $\overline{\Pi}_n$ and Ψ_n:

(C3'): There exists a constant α_n such that $\dfrac{\partial \overline{\lambda}_n}{\partial \overline{\Pi}_{n|S}} = \alpha_n E_{\overline{\Pi}_n}[H'(LS_n - \Psi_n(S_n))|S_n]$

An important part of this paper is devoted to the analysis of the pair $(\overline{\Pi}_n, \Psi_n)$ satisfying **(C1)**, **(C2)**, **(C3')** and **(C4)**. In particular, the asymptotics of these pairs is analyzed in the following sections. In section 8, we will infer the convergence Π_n from the convergence of $\overline{\Pi}_n$.

It is useful to note that various equilibria (Π_n, Ψ_n) could have the same marginal $\overline{\Pi}_n$. On the other hand, we will prove in Corollary 37 the existence of pairs $(\overline{\Pi}_n, \Psi_n)$ that satisfy **(C1)**, **(C2)**, **(C3')** and **(C4)**. To prove the existence of reduced equilibrium in $G_n(\mu)$ we therefore need the second claim of the next lemma:

Lemma 21.
*1/ Any reduced equilibrium (Π_n, Ψ_n) in $G_n(\mu)$ is such that $(\overline{\Pi}_n, \Psi_n)$ satisfies **(C1)**, **(C2)**, **(C3')** and **(C4)**, where $\overline{\Pi}_n = \Pi_{n|(L,S_n)}$.*
*2/ Conversely, to any $(\overline{\Pi}_n, \Psi_n)$ satisfying **(C1)**, **(C2)**, **(C3')** and **(C4)**, there corresponds at least one equilibrium (Π_n, Ψ_n) such that $\Pi_{n|(L,S_n)} = \overline{\Pi}_n$.*

Proof. We start with the first claim. We just have to prove that **(C3)** implies **(C3')**. Let Φ be a continuous and bounded function. According to **(C3)** we have:

$$\begin{aligned} E_{\lambda_n}[\Phi(S_n(\omega))] =& E_{\Pi_{n|\omega}}[\Phi(S_n(\omega))\frac{d\lambda_n}{d\Pi_{n|\omega}}] \\ =& E_{\Pi_n}[\Phi(S_n(\omega))\alpha_n E_{\Pi_n}[H'(LS_n - \Psi_n(S_n))|\omega]] \\ =& E_{\Pi_n}[\Phi(S_n(\omega))\alpha_n H'(LS_n - \Psi_n(S_n))] \\ =& E_{\Pi_n}[\Phi(S_n(\omega))\alpha_n E_{\Pi_n}[H'(LS_n - \Psi_n(S_n))|S]] \end{aligned}$$

Therefore $E_{\overline{\lambda}_n}[\Phi(S_n)] = E_{\lambda_n}[\Phi(S_n(\omega))] = E_{\overline{\Pi}_{n|S}}[\Phi(S_n(\omega))\alpha_n E_{\overline{\Pi}_n}[H'(LS_n - \Psi_n(S_n)|S_n]]$ which is exactly our condition **(C3')**.

We now prove the second claim. Let $(\overline{\Pi}_n, \Psi_n)$ satisfy **(C1)**, **(C2)**, **(C3')** and **(C4)**. Consider then the probability Π_n induced by the following lottery: select first L and S_n according to $\overline{\Pi}_n$. If $S_n = s$, select an history ω with the uniform probability on the set $K_s = \{\omega | S_n(\omega) = s\}$.

The marginal of Π_n on (L, S_n) coincides with $\overline{\Pi}_n$ and (Π_n, Ψ_n) satisfies therefore **(C1)**, **(C2)** and **(C4)**.

Observe then that under Π_n, L is then independent of ω given S_n and therefore the conditional law of (L, S_n) given ω coincides with the conditional law of (L, S_n) given S_n.

So: $E_{\Pi_n}[H'(LS_n - \Psi_n(S_n))|\omega] = E_{\overline{\Pi}_n}[H'(LS_n - \Psi_n(S_n))|S_n]$, and **(C3)** then follows from **(C3')**. $\square$

6.2 Reformulation of (C1), (C2) and (C4)

In this subsection we show that a pair $(\overline{\Pi}_n, \Psi_n)$ satisfying **(C1)**, **(C2)** and **(C4)** is completely determined by the marginal law $\nu := \overline{\Pi}_{n|S_n}$ of S_n.

It will be convenient to introduce the following notation: $\Delta(\mathbb{R}^2, \mu, \nu)$ is the set of probability distributions on $(L, S_n) \in \mathbb{R}^2$ with respective marginal laws μ and ν.

Definition 22. *For $\nu \in \Delta(\mathbb{R})$, we define $\phi_\nu(\ell) := F_\nu^{-1}(F_\mu(\ell))$ and $\gamma_\nu(s) := F_\mu^{-1}(F_\nu(s))$ where F_μ and F_ν are the cumulative distribution functions of μ and ν, and F_μ^{-1} and F_ν^{-1} are their right inverses i.e. $F_\nu^{-1}(y) = \inf\{x \mid F_\nu(x) > y\}$.*

We further define:

$$\Gamma_\nu(s) := \int_0^s \gamma_\nu(t)dt \tag{12}$$

$$\Phi_\nu(\ell) := \int_0^\ell \phi_\nu(t)dt \tag{13}$$

We denote Π_ν the law of the pair $(L, \phi_\nu(L))$ when L is μ-distributed.

Finally we set:

$$\Psi_{\nu,\lambda_n} := \Gamma_\nu(S_n) - E_{\lambda_n}(\Gamma_\nu) \tag{14}$$

Lemma 23. *Let $(\overline{\Pi}, \Psi)$ be a pair such that $\overline{\Pi} \in \Delta(\mathbb{R}^2, \mu, \nu)$ and Ψ is a convex function. Then $(\overline{\Pi}, \Psi)$ satisfies **C1**, **(C2)** and **(C4)** if and only if $(\overline{\Pi}, \Psi) = (\Pi_\nu, \Psi_{\nu,\lambda_n})$*

Proof. First observe that $\overline{\Pi}_\nu \in \Delta(\mathbb{R}^2, \mu, \nu)$. Indeed, according to the definition of $\overline{\Pi}_\nu$ the marginal law of L is μ. On the other hand since μ has no atom $U := F_\mu(L)$ is uniformly distributed and as well known $F_\nu^{-1}(U)$ is ν-distributed. Therefore the marginal law of $\overline{\Pi}_\nu$ on S_n is just ν.

$\Psi_{\nu,\lambda}$ is a convex function since γ_ν is increasing and thus Γ_ν is convex. It further satisfies **(C1)** since, due to equation (22), $E_{\lambda_n}[\Psi_{\nu,\lambda}] = 0$.

$\overline{\Pi}_\nu$ satisfies **(C2)** since it belongs to $\Delta(\mathbb{R}^2, \mu, \nu)$.

γ_ν is right continuous and therefore it follows from the definition of Γ_ν that $\partial\Psi_{\nu,\lambda}(s) = [\gamma_\nu(s^-), \gamma_\nu(s)]$ where $\gamma_\nu(s^-)$ is the left limit of γ_ν at s. Under $\overline{\Pi}_\nu$, $S_n = \phi_\nu(L)$. Therefore, condition **(C4)** is equivalent to:

$$\overline{\Pi}_\nu\left[\gamma_\nu((\phi_\nu(L))^-) \le L \le \gamma_\nu(\phi_\nu(L))\right] = 1 \tag{15}$$

We first prove that for all x:

$$F_\nu((F_\nu^{-1}(x))^-) \le x \le F_\nu(F_\nu^{-1}(x)) \tag{16}$$

Let $A := \{s|F_\nu(s) > x\}$ and $\alpha := F_\nu^{-1}(x)$. It results from the definition of F_ν^{-1} that α is the infimum of A. Furthermore, since F_ν is increasing, $]\alpha, \infty[\subset A \subset [\alpha, \infty[$. Since

F_ν is right continuous, we get $F_\nu(\alpha) = \inf_{s\in A} F_\nu(s)$. But if $s \in A$, $F_\nu(s) > x$, and therefore $\inf_{s\in A} F_\nu(s) \geq x$ and the right hand inequality in (16) is proved.

On the other hand, $F_\nu(\alpha^-) = \lim_{u\to\alpha, u<\alpha} F_\nu(u)$. But if $u < \alpha$, $u \in A^c$ and thus $F_\nu(u) \leq x$. Therefore $F_\nu(\alpha^-) \leq x$ which is the second inequality.

Replace x by $F_\mu(L)$ in (16) to obtain: $F_\nu((\phi_\nu(L))^-) \leq F_\mu(L) \leq F_\nu(\phi_\nu(L))$. Since F_μ is increasing and one to one, we get therefore $F_\mu^{-1}(F_\nu((\phi_\nu(L))^-)) \leq L \leq F_\mu^{-1}(F_\nu(\phi_\nu(L)))$ which is exactly (15) according to the definition of γ_ν, and $(\overline{\Pi}_\nu, \Psi_{\nu,\lambda})$ satisfies thus **(C4)**.

Let now $\overline{\Pi}_n$ belong to $\Delta(\mathbb{R}^2, \mu, \nu)$ and Ψ_n be a convex function such that $(\overline{\Pi}_n, \Psi_n)$ satisfies **(C1)**, **(C2)** and **(C4)**.

The derivative $\rho(s)$ of $\Psi_n(s)$ exists for every s except possibly on a countable set.

The function ρ can always be taken right continuous and we have then for all s,

$\partial\Psi_n(s) = [\rho(s^-), \rho(s)]$. Since $\ell \in \partial\Psi_n(s) \Leftrightarrow s \in \partial\Psi_n^\sharp(\ell)$ according to Fenchel lemma, we get $\partial\Psi_n^\sharp(\ell) = [\rho^{-1}(\ell^-), \rho^{-1}(\ell)]$ where $\rho^{-1}(\ell) := \inf\{s|\rho(s) > \ell\}$.

Condition **(C4)** implies therefore $\overline{\Pi}_n(\rho^{-1}(L^-) \leq S_n \leq \rho^{-1}(L)) = 1$. Observing that ρ^{-1} is an increasing function, there are at most countably many points in $A := \{\ell|\rho^{-1}(\ell^-) \neq \rho^{-1}(\ell)\}$. Since μ is non atomic, $\mu(A) = 0$ and thus $\overline{\Pi}_n[S_n = \rho^{-1}(L)] = 1$. It follows that, under $\overline{\Pi}_n$, (L, S_n) has the same law as $(L, \rho^{-1}(L))$. Since $\overline{\Pi}_n \in \Delta(\mathbb{R}^2, \mu, \nu)$, we conclude that $\rho^{-1}(L)$ is ν-distributed when L is μ-distributed. As observed in the beginning of this proof $\phi_\nu(L) \sim \nu$ when $L \sim \mu$. It turns out that ϕ_ν is the unique right continuous increasing function having that property[3], and we may therefore conclude that $\rho^{-1} = \phi_\nu$.

It follows on one hand that $\overline{\Pi}_n = \overline{\Pi}_\nu$. On the other hand, $\rho = \phi_\nu^{-1} = \gamma_\nu$. Therefore, $\partial\Psi_n(s) = \partial\Gamma_\nu(s)$ for all s. As a consequence $\Psi_n = \Gamma_\nu + c$ where c is a constant. Since Ψ_n satisfies C1, we conclude that $c = -E_{\overline{\lambda}_n}[\Gamma_\nu]$ and thus $\Psi_n = \Psi_{\nu,\overline{\lambda}_n}$ as announced. $\square$

As explained in the introduction of this section, we are seeking for pairs $(\overline{\Pi}_n, \Psi_n)$ satisfying **(C1)**, **(C2)**, **(C3')** and **(C4)**. According to lemma 23, this is equivalent to find ν such that $(\overline{\Pi}_\nu, \Psi_\nu)$ satisfies **(C3')**.

(C3') is a condition on the density of $\overline{\lambda}_n$ with respect to the marginal of $\Pi_{\nu|S_n} = \nu$. Namely it express that this density $\frac{\partial\overline{\lambda}_n}{\partial\nu}$ is proportional to $Y_{\nu,\lambda}$ defined as:

$$Y_{\nu,\lambda} := E_{\overline{\Pi}_\nu}[H'(LS_n - \Psi_{\nu,\lambda}(S_n)) \mid S_n] \tag{17}$$

Since H' is strictly positive, so is $Y_{\nu,\lambda}$. There exists a unique constant $\alpha_{\nu,\lambda}$ such that $\rho := \alpha_{\nu,\lambda} Y_{\nu,\lambda}\nu$ is a probability measure, namely $\alpha_{\nu,\lambda} = \frac{1}{E_\nu[Y_{\nu,\lambda}]}$.

Definition 24. *For $\lambda \in \Delta(\mathbb{R})$, T_λ is defined as the map from $\nu \in \Delta(\mathbb{R})$ to $T_\lambda(\nu) := \alpha_{\nu,\lambda}.Y_{\nu,\lambda}.\nu \in \Delta(\mathbb{R})$.*

[3]Let indeed f_1, f_2 be two right continuous increasing functions such that $f_i(L) \sim \nu$ when $L \sim \mu$. Then for all $a \in \mathbb{R}$, $A_i := \{\ell|f_i(\ell) \geq a\}$ is a closed set. Since f is increasing, A_i must be an half line and we must have therefore $A_i = [\alpha_i, \infty[$. Since $f_i(L) \sim \nu$ and F_μ is continuous, we get:

$$\nu([a, \infty[) = \mu(f_i(L) \geq a) = \mu(L \geq \alpha_i) = 1 - F_\mu(\alpha_i)$$

Therefore $F_\mu(\alpha_1) = F_\mu(\alpha_2)$ and thus $\alpha_1 = \alpha_2$, since F_μ is strictly increasing according to the hypothesis **A1** on μ. As a result, $A_1 = A_2$, or in other words: for all ℓ and for all a, $f_1(\ell) \geq a$ if and only if $f_2(\ell) \geq a$. We conclude therefore that $f_1 = f_2$.

With this definition, we get:

Lemma 25. *For all ν, the pair $(\overline{\Pi}_\nu, \Psi_\nu)$ satisfies **(C3')** if and only if $T_{\overline{\lambda}_n}(\nu) = \overline{\lambda}_n$.*

Proof. Obvious from the definition of T_λ and condition **(C3')**. □

The aim of this section is to prove the existence of such a ν. We first prove in the following subsection that T_λ is a continuous operator for the Wasserstein distance W_2. This result will also be useful for our asymptotic analysis.

6.3 Continuity of T_λ

T_λ is thus a map from $\Delta(\mathbb{R})$ to $\Delta(\mathbb{R})$ and we now analyse its continuity with respect to the Wasserstein metric of order 2. We remind the definition of this concept:

Definition 26. *For $p \in [1, +\infty[$ we define P_p the Wasserstein space of order p as:*

$$P_p(\mathbb{R}) := \{\nu \in \Delta(\mathbb{R}),\ \textit{such that} \int_{\mathbb{R}} \mid x \mid^p \nu(dx) < \infty\}$$

For $\nu_1, \nu_2 \in P_p(\mathbb{R})$ we define:

Definition 27.

$$W_p(\nu_1, \nu_2) = \left[\underset{\pi \in \Delta(\mathbb{R}^2, \nu_1, \nu_2)}{inf} \int_{\mathbb{R}} \mid x - y \mid^p d\pi(x, y) \right]^{\frac{1}{p}}$$

W_p is finite on P_p. Moreover $(P_p(\mathbb{R}), W_p)$ is a metric space. This metric is usefull to deal with weak convergences (see Proposition 30).

Definition 28. *The weak convergence on $P_p(\mathbb{R})$ is defined by: $\nu_k \to \nu$ (weakly in $P_p(\mathbb{R})$) if for any continuous functions Φ such that there exists $C \in \mathbb{R}$ such that $\mid \Phi(x) \mid \leq C(1 + x)^2$, we have $E_{\nu_k}[\phi] \to E_\nu[\phi]$ as $k \to \infty$.*

Definition 29. *The weak convergence on $\Delta(\mathbb{R})$ is defined by: $\nu_k \to \nu$ (weakly in $\Delta(\mathbb{R})$) if and only if for any bounded continuous function $\phi : \mathbb{R} \to \mathbb{R}$, we have $E_{\nu_k}[\phi] \to E_\nu[\phi]$ as $k \to \infty$.*

The following proposition is well known (see for instance theorem 6.9 in Villani [2008], or Mallows [1972] for a proof). It makes the link between weak convergence in $\Delta(\mathbb{R})$ and W_2 convergence.

Proposition 30. *The three following statements are equivalent:*
1/ $W_2(\nu_n, \nu) \to 0$
2/ $\nu_n \to \nu$ (weakly in $P_2(\mathbb{R})$)
3/ $\nu_n \to \nu$ (weakly in $\Delta(\mathbb{R})$) and $E_{\nu_n}(s^2) \to E_\nu(s^2)$.

The next representation formula for W_2 is well known and proved in Dall'Aglio [1956][4].

[4] Actually we just used the fact that minimizing $\int_{\mathbb{R}} \mid x - y \mid^2 d\pi(x, y)$ is equivalent to maximizing $E_\pi(xy)$, then to maximizing $cov(XY)$ with $X \sim \nu_1$ and $Y \sim \nu_2$. This maximum is reached when X and Y can be written as increasing functions of the same uniform random variable, here $F_\mu(L)$ with $L \sim \mu$.

Lemma 31.

$$W_2(\nu_1, \nu_2) = \sqrt{\int_0^1 \left(F_{\nu_1}^{-1}(x) - F_{\nu_2}^{-1}(x)\right)^2 dx}$$

We will prove the continuity of the operator step by step. The following lemma are useful in the proof of the continuity.

Lemma 32. *The mappings $\nu \to \Phi_\nu$ and $\nu \to \Gamma_\nu$ are continuous from $(P_2(\mathbb{R}), W_2)$ to the set of convex functions on respectively $]0,1[$ and $\mathbb{R}$ with the topology of uniform convergence.*

Proof. Let ν_1, ν_2 be two measures in $P_2(\mathbb{R})$ and $x \in]0,1[$.

$$\begin{aligned}
\|\Phi_{\nu_1}(x) - \Phi_{\nu_2}(x)\|_\infty &\leq \left| \int_0^x F_{\nu_1}^{-1}(F_\mu(\ell)) - F_{\nu_2}^{-1}(F_\mu(\ell)) d\ell \right| \\
&\leq \int_0^x \left| F_{\nu_1}^{-1}(F_\mu(\ell)) - F_{\nu_2}^{-1}(F_\mu(\ell)) \right| d\ell \\
&\leq \int_0^x \left| F_{\nu_1}^{-1}(F_\mu(\ell)) - F_{\nu_2}^{-1}(F_\mu(\ell)) \right| \frac{f_\mu(\ell)}{f_\mu(\ell)} d\ell \\
&\leq E_\mu[\frac{\left| F_{\nu_1}^{-1}(F_\mu(L)) - F_{\nu_2}^{-1}(F_\mu(L)) \right|}{f_\mu(L)}] \\
&\leq \sqrt{E_\mu[(F_{\nu_1}^{-1}(F_\mu(L)) - F_{\nu_2}^{-1}(F_\mu(L)))^2]} \sqrt{E_\mu[\frac{1}{f_\mu(L)^2}]}
\end{aligned}$$

the last inequality follows from Cauchy Scwharz theorem. Next observe with Lemma 31 and the fact that $F_\mu(L)$ is uniformly distributed on $[0,1]$ when L is μ-distributed (μ has a density with respect to the Lebesgue measure) that the first factor in the right hand side is equal to the Wasserstein distance between ν_1 and ν_2. Since f_μ is bounded from below by $\epsilon > 0$ (assumption **A1**) we get:

$$\|\Phi_{\nu_1}(x) - \Phi_{\nu_2}(x)\|_\infty \leq W_2(\nu_1, \nu_2) \sqrt{E_\mu[\frac{1}{f_\mu(L)^2}]} \leq W_2(\nu_1, \nu_2) \sqrt{\frac{1}{\epsilon^2}}$$

Then we proved that the mapping $\nu \to \Phi_\nu$ is $\sqrt{\frac{1}{\epsilon^2}}$-Lipschitz continuous for the uniform norm.

We now prove that $\nu \to \Gamma_\nu$ is also Lipschitz continuous.

Observe that $\Gamma_\nu(s) = \Phi_\nu^\sharp(s) - \Phi_\nu^\sharp(0)$. Indeed from the definition of Γ_ν and Φ_ν we get that the $\partial \Phi_\nu(\ell) = [\phi_\nu(\ell^-), \phi_\nu(\ell)]$ and thus by Fenchel lemma :

$$\partial \Phi_\nu^\sharp(s) = [\phi_\nu^{-1}(s^-), \phi_\nu^{-1}(s)] = [\gamma_\nu(s^-), \gamma_\nu(s)] = \partial \Gamma_\nu(s)$$

The two functions $\Phi_\nu^\sharp$ and Γ_ν just differ by a constant, and since $\Gamma_\nu(0) = 0$ we find $\Gamma_\nu(x) = \Phi_\nu^\sharp(x) - \Phi_\nu^\sharp(0)$. As well known Fenchel transform in an isometry for the uniform norm[5]. We conclude that the mapping $\nu \to \Gamma_\nu$ is also Lipschitz continuous for the uniform norm. $\square$

Lemma 33. *If $W_2(\nu_k, \nu) \to 0$ then $W_2(\Pi_{\nu_k}, \Pi_\nu) \to 0$.*

Proof. Let L be a random variable with law μ. Let $X_1 = (L, \phi_{\nu_k}(L))$ and $X_2 = (L, \phi_\nu(L))$. Then $X_1 \sim \Pi_{\nu_k}$ and $X_2 \sim \Pi_\nu$.

$$W_2(\Pi_{\nu_k}, \Pi_\nu)^2 \leq \|X_1 - X_2\|_{L^2}^2 = \|L - L\|_{L^2}^2 + \|\phi_{\nu_k}(L) - \phi_\nu(L)\|_{L^2}^2 = W_2(\nu_k, \nu)^2$$

where the last equality follows from equation (31). $\square$

Lemma 34. *If $W_2(\lambda_k, \lambda) \to 0$ and $W_2(\nu_k, \nu) \to 0$ then:*

1/ $\|\Psi_{\nu_k,\lambda_k} - \Psi_{\nu,\lambda}\|_\infty \to 0$

2/ For all continuous function Θ such that $\frac{\Theta(x)}{1+x^2}$ is bounded, we have:

$$E_{\Pi_{\nu_n}}[\Theta(S_n)H'(S_nL - \Psi_{\nu_k,\lambda_k}(S_n))] \to E_{\Pi_\nu}[\Theta(S_n)H'(S_nL - \Psi_{\nu,\lambda}(S_n))]$$

Proof. 1/

$$\begin{aligned}
\|\Psi_{\nu_k,\lambda_k} - \Psi_{\nu,\lambda}\|_\infty =&\|(\Gamma_{\nu_k} - E_{\lambda_k}[\Gamma_{\nu_k}]) - (\Gamma_\nu - E_\lambda[\Gamma_\nu])\|_\infty \\
\leq&\|\Gamma_{\nu_k} - \Gamma_\nu\|_\infty + \|E_{\lambda_k}[\Gamma_{\nu_k}] - E_{\lambda_k}[\Gamma_\nu]\|_\infty + |E_{\lambda_k}[\Gamma_\nu] - E_\lambda[\Gamma_\nu]| \\
\leq&2\|\Gamma_{\nu_k} - \Gamma_\nu\|_\infty + |E_{\lambda_k}[\Gamma_\nu] - E_\lambda[\Gamma_\nu]|
\end{aligned}$$

The first term of the right hand side goes to zero by Lemma 32. Next observe that $\partial\Gamma_\nu = [\gamma(s^-), \gamma(s)] \in [0,1]$ and $\Gamma_\nu(0) = 0$ therefore, $|\Gamma_\nu(x)| \leq |x| \leq C(1+x^2)$ for a constant C. Since Γ is further continuous as claimed in Proposition 30, the last term goes also to zero.

2/

$$|E_{\Pi_{\nu_k}}[\Theta(S_n)H'(S_nL - \Psi_{\nu_k,\lambda_k}(S_n))] - E_{\Pi_\nu}[\Theta(S_n)H'(S_nL - \Psi_{\nu,\lambda}(S_n))]| \leq I_k + J_k$$

where $I_k := |E_{\Pi_{\nu_k}}[\Theta(S_n)H'(S_nL - \Psi_{\nu_k,\lambda_k}(S_n))] - E_{\Pi_{\nu_k}}[\Theta(S)H'(S_nL - \Psi_{\nu,\lambda}(S_n))]|$ and $J_k := |E_{\Pi_{\nu_k}}[\Theta(S_n)H'(S_nL - \Psi_{\nu,\lambda}(S_n))] - E_{\Pi_\nu}[\Theta(S_n)H'(S_nL - \Psi_{\nu,\lambda}(S_n))]|$.

According to **A1**, H' is Lipschitz continuous. Let $\hat{K}$ denote the Lipschitz constant, then:

$$I_k \leq E_{\Pi_{\nu_k}}[|\Theta(S_n)|\hat{K}\|\Psi_{\nu_k,\lambda_k} - \Psi_{\nu,\lambda}\|_\infty] = \hat{K}\|\Psi_{\nu_k,\lambda_k} - \Psi_{\nu,\lambda}\|_\infty E_{\nu_k}[|\Theta(S_n)|]$$

Since $|\Theta(S_n)| \leq C(1+S^2)$ and $W_2(\nu_k, \nu) \to 0$ we get with Proposition 30 that $E_{\nu_k}[|\Theta(S_n)|] \to E_\nu[|\Theta(S_n)|] < \infty$. On the other hand $\|\Psi_{\nu_k,\lambda_k} - \Psi_{\nu,\lambda}\|_\infty \to 0$ according to the first claim of this lemma. I_k converges therefore to 0.

The map $(L, S_n) \to \Theta(S_n)H'(S_nL - \Psi_{\nu,\lambda}(S_n))$ is continuous and is also bounded by $C(1 + \|(L, S_n)\|^2)$ since H' is itself bounded. Since Π_{ν_k} converges to Π_ν in W_2, it follows from Proposition 30 that J_k goes to zero. $\square$

[5] Let indeed f and g be two lower semi continuous convex functions $\mathbb{R}^n \to \mathbb{R}$, then $\| f^\sharp - g^\sharp \|_\infty = \| f - g \|_\infty$. Indeed, for all $x \in \mathbb{R}$: $f^\sharp(x) = \sup_t x.t - f(t) \leq \sup_t x.t - g(t) + \|f - g\|_\infty = g^\sharp(x) + \|f - g\|_\infty$. Interchanging f and g we get therefore for all x: $|f^\sharp(x) - g^\sharp(x)| \leq \|f - g\|_\infty$. Since the right hand side doesn't depend on x, we get: $\|f^\sharp - g^\sharp\|_\infty \leq \|f - g\|_\infty$. The reverse inequality follows from Fenchel lemma: $f^{\sharp\sharp} = f$ and $g^{\sharp\sharp} = g$. Therefore: $\|f - g\|_\infty = \|f^{\sharp\sharp} - g^{\sharp\sharp}\|_\infty \leq \|f^\sharp - g^\sharp\|_\infty$ as announced.

Corollary 35. *If* $\begin{cases} W_2(\nu_k,\nu) \to 0 \\ W_2(\lambda_k,\lambda) \to 0 \end{cases}$ *then* $W_2(T_{\lambda_k}(\nu_k), T_\lambda(\nu)) \to 0$

Proof. We have to prove that if Φ continuous and satisfies $\mid \Phi(s) \mid \leq C(1+s^2)$, then:

$$E_{T_{\lambda_k}(\nu_k)}(\Phi) \to E_{T_\lambda(\nu)}(\Phi)$$

According to the definition of $T_{\lambda_k}(\nu_k)$ this amounts to show that:

$$E_{\nu_k}(\Phi(S_n).\alpha_{\nu_k,\lambda_k}.Y_{\nu_k,\lambda_k}(S_n)) \to E_\nu(\Phi(S_n).\alpha_{\nu,\lambda}.Y_{\nu,\lambda}(S_n)) \tag{18}$$

We start by proving that:

$$E_{\nu_k}[\Phi(S_n).Y_{\nu_k,\lambda_k}(S_n)] \to E_\nu[\Phi(S_n).Y_{\nu,\lambda}(S_n)] \tag{19}$$

But due to the definition of Y_{ν_k,λ_k} we get:

$$\begin{aligned} E_{\nu_k}[\Phi(S_n).Y_{\nu_k,\lambda_k}(S_n)] =& E_{\nu_k}[\Phi(S_n).E_{\Pi_{\nu_k}}[H'(LS_n - \Psi_{\nu_k,\lambda_k}(S_n))|S_n]] \\ =& E_{\Pi_{\nu_k}}[\Phi(S_n).E_{\Pi_{\nu_k}}[H'(LS_n - \Psi_{\nu_k,\lambda_k}(S_n))|S_n]] \\ =& E_{\Pi_{\nu_k}}[\Phi(S_n).H'(LS_n - \Psi_{\nu_k,\lambda_k}(S_n))] \end{aligned}$$

and we have a similar formula for $E_\nu[\Phi(S_n).Y_{\nu,\lambda}(S_n)]$. We thus have to prove that

$$E_{\Pi_{\nu_k}}[\Phi(S_n).H'(LS_n - \Psi_{\nu_k,\lambda_k}(S))] \to E_{\Pi_\nu}[\Phi(S_n).H'(LS_n - \Psi_{\nu,\lambda}(S_n))]$$

But this is exactly our claim 2 in previous lemma and formula (19) follows.

According to the Definition 17 we get $\alpha_{\nu_k,\lambda_k} = \frac{1}{E_{\nu_k}[Y_{\nu_k,\lambda_k}]}$, but with formula (19) for the particular $\Theta \equiv 1$, we get that $E_{\nu_k}[Y_{\nu_k,\lambda_k}] \to E_\nu[Y_{\nu,\lambda}]$. Since $Y_{\nu,\lambda}$ is the lower bounded by $\epsilon > 0$ (assumption **A2** on H), we conclude then that $\alpha_{\nu_k,\lambda_k} \to \alpha_{\nu,\lambda}$. Finally combining this result with formula (19), we get the convergence announced in formula (18) and the corollary is proved. □

According to Lemma 21 and 25, to prove the existence of an equilibrium in $\overline{G}_n(\mu)$, we have to show that there exists $\nu_n \in \Delta(\mathbb{R})$ such that $T_{\overline{\lambda}_n}(\nu_n) = \overline{\lambda}_n$. Remember that $\overline{\lambda}_n \in \Delta_f(\mathbb{R})$ where $\Delta_f(\mathbb{R})$ is the set of probability measures on $\mathbb{R}$ with finite support. Observe next that $T_\lambda(\nu)$ is defined by a density function with respect to ν. In particular $T_\lambda(\nu) \ll \nu$ and therefore $T_\lambda(\nu) \in \Delta_f(\mathbb{R})$ if $\nu \in \Delta_f(\mathbb{R})$.

The next theorem can then be applied to $T_{\overline{\lambda}_n}$ to conclude the existence of equilibrium.

Theorem 36. *A map* $T : \Delta_f(\mathbb{R}) \to \Delta_f(\mathbb{R})$ *that is continuous for the* W_2 *metric and satisfies* $T(\nu) \ll \nu$ *for all* ν *is necessarily onto.*

Proof. Let λ be a measure in $\Delta_f(\mathbb{R})$ and denote K its support. If $T(\nu) \ll \nu$, then necessarily the support of $T(\nu)$ is included in the support of ν. Therefore T maps $\Delta(K)$ to $\Delta(K)$. $\Delta(K)$ can be identify with the $|K|$-dimensional simplex hereafter denoted Δ and the restriction of T to Δ is a continuous map. It further preserves the faces $F_i := \{x \in \Delta | x_i = 0\}$. It follows for an argument used in a proof in Gale [1984] that T is onto. Indeed, let $\lambda \in \Delta$ and define $C_i := \{x \in \Delta | T(x)_i \leq \lambda_i\}$. Since T is continuous, C_i is clearly a closed subset of Δ. Furthermore, if $x \in F_i$ then $x_i = 0$ and thus $T_i(x_i) = 0 \leq \lambda_i$. We conclude therefore that for all i, $F_i \subset C_i$. We next argue that $\Delta \subset \cup_i C_i$. Indeed, for all $x \in \Delta$, $T(x) \in \Delta$. There must exists i such that $T(x)_i \leq \lambda_i$. Indeed otherwise we would have for all i, $T(x)_i > \lambda_i$, and summing all those inequalities we would get $1 > 1$. Therefore there exists i such that $x \in C_i$. According to KKM theorem (see Mertens et al. [1994]) there exists $x \in \cap_i C_i$: for all i, $T(x)_i \leq \lambda_i$. Since the sum over i of both sides equal to 1, we infer that these inequalities are in fact equalities, and thus $T(x) = \lambda$. □

Corollary 37. *For all n, there exists ν_n such that $T_{\overline{\lambda}_n}(\nu_n) = \overline{\lambda}_n$. The corresponding pair $(\Pi_{\nu_n}, \Psi_{\nu_n, \overline{\lambda}_n})$ satisfies **(C1)**, **(C2)**, **(C3')** and **(C4)**. There exists therefore a reduced equilibrium in $G_n(\mu)$.*

7 Convergence of ν_n

In this section we analyze the asymptotics of any sequence (ν_n) that satisfies for all n: $T_{\overline{\lambda}_n}(\nu_n) = \overline{\lambda}_n$. From now on, ν_n denotes any such sequence.

First observe that $\overline{\lambda}_n$ is the law of $S_n = \frac{\sum_{i=1}^n u_i}{\sqrt{n}}$ when $(u_1, \ldots, u_n)$ are independent and centred. It follows from the central limit theorem that $\overline{\lambda}_n$ converges in law to $\overline{\lambda}_\infty := \mathscr{N}(0, 1)$. Observing that the second order moments $E_{\overline{\lambda}_n}[S_n^2] = 1$ for all n, this weak convergence in $\Delta(\mathbb{R})$ implies (see 30) the W_2-convergence of $\overline{\lambda}_n$ to $\overline{\lambda}_\infty$.

In the first subsection we use a compactness argument to prove that any subsequence of (ν_n) admits an accumulation point ν which further satifies $T_{\overline{\lambda}_\infty}(\nu) = \overline{\lambda}_\infty$.

In the second subsection we prove that if ν is a solution of that equation, the pair $(\Psi_{\nu,\lambda}, \frac{1}{\alpha_{\nu,\lambda}})$ is a smooth solution of a differential problem $\mathscr{D}$.

The third subsection is devoted to the proof of the uniqueness of the solution to this differential problem which in turn will imply the uniqueness of the solution $T_{\overline{\lambda}_\infty}(\nu) = \overline{\lambda}_\infty$.

In the last subsection, we infer from these results that ν_n converges to this unique solution.

7.1 The accumulation point of (ν_n).

Lemma 38. *The sequence (ν_n) is relatively compact: any subsequence of (ν_n) has an accumulation point in $P_2(\mathbb{R})$.*

Proof. We just have to prove that $E_{\nu_n}[s^2]$ is bounded by some constant M. Indeed, if this is the case, the sequence ν_n is tight, as it follow from Markov inequality that: $[-\sqrt{\frac{M}{\eta}}, \sqrt{\frac{M}{\eta}}]$ is a compact set such that $\nu_n([-\sqrt{\frac{M}{\eta}}, \sqrt{\frac{M}{\eta}}]) \geq 1 - \eta$ for all n. It admits therefore a subsequence that converges weakly in $\Delta(\mathbb{R})$. This subsequence having bounded moment of order 2, one can select a sub-subsequence $\nu_{n(k)}$ with converging moment of order 2. According to Proposition 30, ν_{n_k} converges weakly in $P_2(\mathbb{R})$, and thus also in the sense of W_2 metric.

We now prove that $E_{\nu_n}[s^2]$ is bounded. It follows immediately from the assumptions **A2** on H as well as the definition of $Y_{\nu,\lambda}$ and $\alpha_{\nu,\lambda}$ (see equation (17)), that $\epsilon < Y_{\nu,\lambda} < K$, and $\frac{1}{K} < \alpha_{\nu,\lambda} < \frac{1}{\epsilon}$. Therefore: $\frac{\epsilon}{K} < \alpha_{\nu,\lambda}.Y_{\nu,\lambda} < \frac{K}{\epsilon}$. According to the definition of $\overline{\lambda}_n$, we have $E_{\overline{\lambda}_n}(S_n^2) = 1$. And thus:

$$1 = E_{\overline{\lambda}_n}(s^2) = E_{T_{\overline{\lambda}_n}(\nu_n)}[s^2] = E_{\nu_n}[\alpha_{\nu_n,\overline{\lambda}_n} Y_{\nu_n,\overline{\lambda}_n} s^2] \geq \frac{\epsilon}{K} E_{\nu_n}[s^2]$$

Which leads to $E_{\nu_n}(s^2) \leq \frac{K}{\epsilon}$. □

Corollary 39. *Any accumulation point ν of the sequence (ν_n) satisfies $T_{\overline{\lambda}_\infty}(\nu) = \overline{\lambda}_\infty$ where $\overline{\lambda}_\infty = \mathscr{N}(0,1)$.*

Proof. Take a subsequence $\nu_{n(k)}$ converging to ν in W_2. Since we also have $\overline{\lambda}_{n(k)} \to \overline{\lambda}_\infty$ in W_2, we may apply our continuity result on T (see Corollary 35) to conclude $T_{\overline{\lambda}_\infty}(\nu) = \overline{\lambda}_\infty$. □

7.2 Equivalence between equation $T_{\overline{\lambda}_\infty}(\nu) = \overline{\lambda}_\infty$ and a differential problem.

Proposition 40. *Suppose that ν is a probability measure such that $T_{\overline{\lambda}_\infty}(\nu) = \overline{\lambda}_\infty$ with $\overline{\lambda}_\infty = \mathscr{N}(0,1)$, then:*

1/ The function $\Psi_{\nu,\overline{\lambda}_\infty}$ (see Definition 13) is C^2.

2/ The pair $(\psi, c) := (\Psi_{\nu,\overline{\lambda}_\infty}, \frac{1}{\alpha_{\nu,\overline{\lambda}_\infty}})$ is a solution of the following differential system $\mathscr{D}$:

$$(\mathscr{D}) \begin{cases} (1) & \forall s \in \mathbb{R}, f_\mu(\psi'(s))\psi''(s)H'(s\psi'(s) - \psi(s)) = c\mathscr{N}(s) \\ (2) & lim_{s\to-\infty}\psi'(s) = 0 \\ (3) & lim_{s\to+\infty}\psi'(s) = 1 \\ (4) & \int_{-\infty}^{+\infty} \psi(z)\mathscr{N}(z)dz = 0 \end{cases}$$

where $\mathscr{N}(z) := \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}}$

Proof. Let ν satisfy the equation $T_{\overline{\lambda}_\infty}(\nu) = \overline{\lambda}_\infty$. This implies that $\overline{\lambda}_\infty$ has a strictly positive density with respect to ν, and therefore ν has a density with respect to $\overline{\lambda}_\infty$. In turn this implies that it has also a density f_ν with respect to the Lebesgue measure.

We first deal with the smoothness of $\Psi_{\nu,\overline{\lambda}_\infty}$. Remember that $\Psi_{\nu,\overline{\lambda}_\infty}$ differs from Γ_ν just by a constant. Γ_ν was defined as an integral of $\gamma_\nu(s) = F_\mu^{-1}(F_\nu(s))$. Since F_μ is a strictly increasing and continuous map from $[0,1]$ to $[0,1]$, its inverse is itself continuous. Since F_ν is also continuous, it follows that Γ_ν and $\Psi_{\nu,\overline{\lambda}_\infty}$ are C^1 and $\Psi'_{\nu,\overline{\lambda}_\infty} = \gamma_\nu$.

We next prove that γ_ν is absolutely continuous[6]. This will imply on one hand (see theorem 7.18 in Rudin [1987]) the existence of a function g integrable with respect to the Lebesgue measure such that $\gamma_\nu(s) = \int_{-\infty}^s g(t)dt$ and on the other hand (by the Lebesgue differentiation theorem) that γ_ν is almost surely differentiable and for almost every s: $\gamma'_\nu(s) = g(s)$. The first claim of the proposition will then be proved by establishing that g, which is only defined up to negligeable set, admits a continuous version.

[6] A function $f : \mathbb{R} \to \mathbb{R}$ is absolutely continuous (see definition 7.17 in Rudin [1987]) if for all $\epsilon > 0$, and for all sequences of disjoint real intervals $[a_n, b_n]$, there exists δ such that:

$$\sum_{n\geq 0} |b_n - a_n| < \delta \Rightarrow \sum_{n\geq 0} |f(a_n) - f(b_n)| < \epsilon$$

ν is absolutely continuous:

Since ν is absolutely continuous with respect to the Lebesgue measure, its cumulative distribution function F_ν is an absolutely continuous function [7].

We next observe that F_μ^{-1} is Lipschitz continuous. According to **A1**, f_ν is C^1 on $[0,1]$ and strictly positive. Let then $\kappa > 0$ be such that $\kappa < f_\mu$. For $\tilde{s} < s$, we set $b = F_\mu^{-1}(s)$ and $\tilde{b} = F_\mu^{-1}(\tilde{s})$ then:

$$|s - \tilde{s}| = s - \tilde{s} = F_\mu(b) - F_\mu(\tilde{b}) = \int_{\tilde{b}}^{b} f_\mu(x)dx \geq \kappa(b - \tilde{b})$$

Therefore, we have:

$$|F_\mu^{-1}(s) - F_\mu^{-1}(\tilde{s})| \leq \frac{1}{\kappa}|s - \tilde{s}|.$$

The function γ_ν introduced in Definition 13 is therefore absolutely continuous. Indeed, since F_ν is absolutely continuous, for $\epsilon > 0$, and $[a_n, b_n]$ disjoint real intervals there exists δ such that:

$$\sum_{n\geq 0} |b_n - a_n| < \delta \Rightarrow \sum_{n\geq 0} |(F_\nu(a_n) - F_\nu(b_n)| < \kappa\epsilon$$

$\gamma_\nu(s) = F_\mu^{-1}(F_\nu(s))$.
Suppose that $\sum_{n\geq 0} |b_n - a_n| < \delta$. Then:

$$\sum_{n\geq 0} |F_\mu^{-1}(F_\nu(a_n)) - F_\mu^{-1}(F_\nu(b_n))| \leq \sum_{n\geq 0} \frac{1}{\kappa}|(F_\nu(a_n) - F_\nu(b_n)| \leq \epsilon.$$

Since f_μ is C^1 (see conditions **A1**) and positive, F_μ^{-1} is itself C^1 and $F_\mu^{-1\prime}(u) = \frac{1}{f_\mu(F^{-1}(u))}$. Since F_ν is absolutely continuous, it is almost surely differentiable and $F_\nu'(s) = f_\nu(s)$. Therefore, by the composition rule:

$$g(s) = \gamma_\nu'(s) = \frac{f_\nu(s)}{f_\mu(F_\mu^{-1}(F_\nu(s)))} = \frac{f_\nu(s)}{f_\mu(\gamma_\nu(s))} \tag{20}$$

Since $\Psi_{\nu,\overline{\lambda}_\infty}$ is C^1, and $\overline{\Pi}_\nu$ satisfies **(C4)**: $\overline{\Pi}_\nu(L \in \partial\Psi_{\nu,\overline{\lambda}_\infty}(S)) = 1$, we conclude that L is almost surely equals to $\Psi'_{\nu,\overline{\lambda}_\infty}(S)$ under $\overline{\Pi}_\nu$ and thus:

$$E_{\overline{\Pi}_\nu}[H'(LS - \Psi_{\nu,\overline{\lambda}_\infty}(S))|S] = H'(\Psi'_{\nu,\overline{\lambda}_\infty}(S)S - \Psi_{\nu,\overline{\lambda}_\infty}(S))$$

Our equation $T_{\overline{\lambda}_\infty}(\nu) = \overline{\lambda}_\infty$ becomes then:

$$\frac{\partial\overline{\lambda}_\infty}{\partial\nu} = \alpha_{\nu,\overline{\lambda}_\infty} E_{\overline{\Pi}_\nu}[H'(LS - \Psi_{\nu,\overline{\lambda}_\infty}(S))|S] = \alpha_{\nu,\overline{\lambda}_\infty}.H'(\Psi'_{\nu,\overline{\lambda}_\infty}(S)S - \Psi_{\nu,\overline{\lambda}_\infty}(S))$$

Finally $\frac{\partial\overline{\lambda}_\infty}{\partial\nu}$ is also the quotient $\frac{\mathcal{N}}{f_\nu}$ of the densities with respect to the Lebesgue measure. We get therefore almost surely:

[7] If ν is absolutely continuous with respect to the Lebesgue measure, we have $F_\nu(x) - F_\nu(a) = \int_a^x f_\nu(t)dt$. Theorem 7.18 in Rudin [1987] implies that F_ν is absolutely continuous.

$$\mathcal{N}(s) = \alpha_{\nu,\overline{\lambda}_\infty}.H'(\Psi'_{\nu,\overline{\lambda}_\infty}(s)s - \Psi_{\nu,\overline{\lambda}_\infty}(s))f_\nu(s)$$

Combining this equation with equation (20), we get almost surely:

$$\mathcal{N}(s) = \alpha_{\nu,\overline{\lambda}_\infty}.H'(\Psi'_{\nu,\overline{\lambda}_\infty}(s)s - \Psi_{\nu,\overline{\lambda}_\infty}(s))f_\mu(\gamma_\nu(s))g(s)$$

From this equation we get $g(s) = \frac{\alpha_{\nu,\overline{\lambda}_\infty}.H'(\Psi'_{\nu,\overline{\lambda}_\infty}(s)s-\Psi_{\nu,\overline{\lambda}_\infty}(s))f_\mu(\gamma_\nu(s))}{\mathcal{N}(s)}$ almost surely.

Since the right hand side of this equality is continuous, it is the continuous version of g we were seeking. $\Psi_{\nu,\overline{\lambda}_\infty}$ is thus C^2 which is the first claim of our proposition and for every s:

$$\Psi''_{\nu,\overline{\lambda}_\infty}(s) = \frac{\alpha_{\nu,\overline{\lambda}_\infty}.H'(\Psi'_{\nu,\overline{\lambda}_\infty}(s)s - \Psi_{\nu,\overline{\lambda}_\infty}(s))f_\mu(\Psi'_{\nu,\overline{\lambda}_\infty}(s))}{\mathcal{N}(s)}$$

The pair $(\psi, c) := (\Psi_{\nu,\overline{\lambda}_\infty}, \frac{1}{\alpha_{\nu,\overline{\lambda}_\infty}})$ is then a solution of the first equation of the system $\mathscr{D}$.

It also satisfies the following ones. Indeed, since $[0,1]$ is the support of μ we get:

$$\begin{cases} lim_{s\to-\infty}\Psi'_{\nu,\overline{\lambda}_\infty}(s) = lim_{s\to-\infty}F_\mu^{-1}(F_\nu(s)) = 0 \\ lim_{s\to+\infty}\Psi'_{\nu,\overline{\lambda}_\infty}(s) = lim_{s\to+\infty}F_\nu^{-1}(F_\mu(s)) = 1. \end{cases}$$

Furthermore we have $\Psi_{\nu,\overline{\lambda}_\infty} = \Gamma_\nu - E_{\overline{\lambda}_\infty}[\Gamma_\nu]$ and thus:

$$\int_{-\infty}^{+\infty} \Psi_{\nu,\overline{\lambda}_\infty}(z)\mathcal{N}(z)dz = E_{\overline{\lambda}_\infty}[\Psi_{\nu,\overline{\lambda}_\infty}] = 0$$

□

7.3 The unique solution of the differential system $\mathscr{D}$.

Theorem 41. *There exists at most one pair (ψ, c) solution to the system $\mathscr{D}$.*

The proof of this theorem is made of several lemma that are presented in the remaining part of this section.

Let (ψ_1, c_1) and (ψ_2, c_2) be two solutions of the problem $\mathscr{D}$. Without loss of generality we may assume that $c_1 \geq c_2 > 0$.

Indeed $\mathscr{D}$-2 and $\mathscr{D}$-3 imply that ψ''_i must be strictly positive at some point s, and from $\mathscr{D}$-1 that $c_i > 0$. Define then $\theta := \psi_1 - \psi_2$. θ is a C^2 function.

By $\mathscr{D}$-4: $\int_{-\infty}^{+\infty} \theta(z)\mathcal{N}(z)dz = 0$. Since θ is a continuous function and $\mathcal{N}(z) > 0$ for all z, there exists s_0 such that $\theta(s_0) = 0$.

Let $\Gamma^+ := \{s > s_0 \mid \theta'(s) = 0\}$ and $\Gamma^- := \{s < s_0 \mid \theta'(s) = 0\}$. We also define $s^+ := \inf \Gamma^+$ and $s^- := \sup \Gamma^-$

We denote

$$\Lambda(s) := s\psi'_i(s) - \psi_i(s) \tag{21}$$

Observe that $\theta'(s)$ can not vanish on $]s_0, s^+[$ nor on $=]s^-, s_0[$. θ' has thus a constant sign on each interval.

Lemma 42. *Suppose that $s_0 < s^+$. Then $\theta' < 0$ on $]s_0, s^+[$.*

Proof.

Assume on the contrary that $s_0 < s^+$ and $\theta' > 0$ on $]s_0, s^+[$.

First case: suppose that $s^+ < +\infty$.
Since θ is increasing on $]s_0, s^+[$:

$$\theta(s^+) > \theta(s_0) = 0 \tag{22}$$

and also, by definition of s^+,

$$\theta'(s^+) = 0, \text{ so } \alpha := \psi_1'(s^+) = \psi_2'(s^+) \tag{23}$$

From $\mathscr{D}$-1 we get:

$$\begin{aligned}
\theta''(s^+) \quad & = \frac{\mathscr{N}(s^+)}{f_\mu(\alpha)}\left(\frac{c_1}{H'(s^+\alpha - \psi_1(s^+))} - \frac{c_2}{H'(s^+\alpha - \psi_2(s^+))}\right) \\
& \geq \frac{\mathscr{N}(s^+)c_2}{f_\mu(\alpha)}\left(\frac{1}{H'(s^+\alpha - \psi_1(s^+))} - \frac{1}{H'(s^+\alpha - \psi_2(s^+))}\right) > 0
\end{aligned}$$

since $\psi_1(s^+) > \psi_2(s^+)$, as indicates equation (22) and since H' is strictly increasing and $c_2 > 0$. But this is not possible since $\theta'(s^+) = 0$ and $\theta' > 0$ on $]s_0, s^+[$ (which implies that $\theta''(s^+) \leq 0$).

Second case: suppose now that $s^+ = +\infty$. It is convenient in this case to introduce the function R on $[0, 1]$: $R(u) = \theta'(F_{\mathscr{N}}^{-1}(u))$ where $F_{\mathscr{N}}$ is the cumulative function of the normal law $\mathscr{N}(0, 1)$. We first prove that $lim_{u\to 1} R'(u) > 0$. Indeed:

$$\begin{aligned}
lim_{u\to 1} R'(u) = & \quad lim_{s\to+\infty} \frac{\theta''(s)}{\mathscr{N}(s)} \\
= & \quad lim_{s\to+\infty} \frac{c_1}{f_\nu(1)H'(\Lambda_1(s))} - \frac{c_2}{f_\nu(1)H'(\Lambda_2(s))} \\
\geq & \quad lim_{s\to+\infty} \frac{c_2}{f_\nu(1)}\left(\frac{1}{H'(\Lambda_1(s))} - \frac{1}{H'(\Lambda_2(s))}\right)
\end{aligned} \tag{24}$$

where Λ was defined in 21.

We now claim that $lim_{+\infty}\Lambda_1(s) < lim_{+\infty}\Lambda_2(s)$. Indeed, since $lim_{s\to+\infty}\theta'(s) = 0$ according to $\mathscr{D}$-2, we get $\theta'(s) = -\int_s^\infty \theta''(u)du$.

By assumptions **A1** and **A2**, there exists k_1 be such that for all s: $0 < k_1 < H'(s)$ and k_2 be such that for all s: $0 < k_2 < f_\mu(s)$.

$$|s\theta'(s)| = |s\int_s^\infty \theta''(u)du| \leq \frac{c_1 - c_2}{k_1 k_2}|s| \int_s^\infty \mathscr{N}(u)du \to 0$$

$$lim_{+\infty}\Lambda_1(s) - \Lambda_2(s) = lim_{+\infty}s\theta'(s) - \theta(s) = 0 - lim_{+\infty}\theta(s).$$

But $lim_{+\infty}\theta(s) > 0$. Indeed: $\theta(s_0) = 0$ and $\forall s \in [s_0, +\infty]$, we have $\theta'(s) > 0$.

And thus as announced $lim_{+\infty}\Lambda_1(s) < lim_{+\infty}\Lambda_2(s)$, this implies with equation (24) that:

$$lim_{u\to 1}R'(u) > 0 \tag{25}$$

Note that according to the definition of R and the fact that $\theta' > 0$ on $]s_0, +\infty[$ we get:

$$R(x) > 0 \text{ for } x \in]F_{\mathscr{N}}(s_0), 1[\tag{26}$$

Finally,

$$lim_{u\to 1}R(u) = lim_{u\to 1}\theta'(F_{\mathscr{N}}^{-1}(u)) = lim_{u\to +\infty}\theta'(u) = 0 \tag{27}$$

but relations (26) and (27) are in contradiction with (25). This conclude the proof of the lemma. □

A similar argument leads to a dual result on the left side of s_0:

Lemma 43. *Suppose that $s^- < s_0$. Then $\theta' > 0$ on $]s^-, s_0[$.*

Lemma 44. $\theta(s_0) = \theta'(s_0) = \theta''(s_0) = 0$.

Proof. Suppose $\theta'(s_0) > 0$. There must exist $\delta > 0$ such that $\theta'(s) > 0$ for s $\in]s_0, s_0+\delta[$. The definition of s^+ implies therefore $s^+ \geq s_0 + \delta > s_0$. Furthermore, θ' is strictly positive on $]s_0, s^+[$. But this is in contradiction with Lemma 42. Similarly, the assumption $\theta'(s_0) < 0$ is in contradiction with the dual result Lemma 43. And we must therefore have $\theta'(s_0) = 0$.

Suppose now that $\theta''(s_0) > 0$. Then there exists $\epsilon > 0$ such that $\theta' > 0$ on $]s_0, s_0 + \epsilon[$ in contradiction with Lemma 42. With the same arguments, it is impossible that $\theta''(s_0) < 0$ and the lemma is proved. □

Lemma 45. $c_1 = c_2$.

Proof. Indeed, equation $\mathscr{D}$-1 gives, for $i = 1, 2$:

$$c_i = \frac{f_\nu(\psi_i'(s_0))\psi_i''(s_0)H'(s_0\psi_i'(s_0) - \psi_i(s_0))}{\mathscr{N}(s_0)}$$

But, according to Lemma 44 the right hand side does not depend on i. □

Proof of Theorem 41.

Let c denote the common value $c := c_1 = c_2$. Our two functions ψ_1 and ψ_2 are now solutions to the same differential equation:

$$\psi''(s) = F(s, \psi(s), \psi'(s))$$

where

$$F(s, x, y) := \frac{c\mathcal{N}(s)}{H'(sy - x) f_\mu(y)}$$

Due to our assumptions **A1**, **A2** on f_μ and H, F is C^1 with respect to (s, x, y). Therefore, according to Cauchy-Lipschitz theorem, ψ_1 and ψ_2 must coincide since they are both solution of the same differential equation and have the same initial conditions $\psi(s_0), \psi'(s_0)$ at $s = s_0$. □

7.4 Convergence of ν_n.

We are now ready to prove that $T_{\overline{\lambda}_\infty}(\nu) = \overline{\lambda}_\infty$ has a unique solution and that ν_n must converge.

Corollary 46. *There exists a unique measure ν such that $T_{\overline{\lambda}_\infty}(\nu) = \overline{\lambda}_\infty$ where $\overline{\lambda}_\infty = \mathcal{N}(0, 1)$*

Proof. If ν_1 and ν_2 are two solutions of $T_{\overline{\lambda}_\infty}(\nu) = \overline{\lambda}_\infty$, then the pairs $(\Psi_{\nu_i, \overline{\lambda}_\infty}, \frac{1}{\alpha_{\nu_i, \overline{\lambda}_\infty}})$ for $i = 1, 2$ would be solutions of the system $\mathcal{D}$ according to Proposition 40. As a result of Theorem 41: $\Psi_{\nu_1, \overline{\lambda}_\infty} = \Psi_{\nu_2, \overline{\lambda}_\infty}$. Thus the derivatives of these functions also coincide: $\gamma_{\nu_1} = \gamma_{\nu_2}$ where γ_{ν_i} are defined in Definition 12. Since F_μ is one-to-one, this implies that $F_{\nu_1} = F_{\nu_2}$ and thus $\nu_1 = \nu_2$. □

Corollary 47. *The sequence (ν_n) is convergent to the unique solution ν of of $T_{\overline{\lambda}_\infty}(\nu) = \overline{\lambda}_\infty$.*

Proof. Otherwise there would exists a subsequence $(\nu_{n(k)})$ that does not admit ν has accumulation point. This is impossible since this sequence has an accumulation $\tilde{\nu}$ according to Lemma 38 which is solution to the equation $T_{\overline{\lambda}_\infty}(\tilde{\nu}) = \overline{\lambda}_\infty$. But it follows from Lemma 46 that $\tilde{\nu} = \nu$. □

8 Convergence of the price process to a CMMV

Our analysis in this section applies to any sequence (Π_n, X_n) of reduced equilibria in $G_n(\mu)$. We will focus on the price process $(p^n_q)_{q=1,\dots,n}$ posted by player 2 in these equilibria. In a reduced equilibrium, the strategy $\bar{p}_n$ of player 2 is pure (non random) but his moves depend on the past actions $\omega = (u_1, \dots, u_n)$ of player 1 which are random. The process $(p^n_q)_{q=1,\dots,n}$ is then a stochastic process. Its law when ω is $\Pi_{n|\omega}$-distributed is called the historical law and is denoted P_n. When Ω is endowed with the probability λ_n the law of the price process is denoted Q_n. We have seen in section 5.3 that under Q_n, the process $(p^n_q)_{q=1,\dots,n}$ is a martingale. Furthermore Q_n is the unique martingale equivalent measure as stated in Theorem 20. Our purpose on this section is to analyze the asymptotics of (Q_n) in subsection 8.1 and (P_n) is subsection 8.2.

8.1 Convergence under the martingale equivalent probability

Let (Π_n, X_n) be a sequence of reduced equilibria in $G_n(\mu)$. We already know that $X_n = \Psi_{\nu_n,\lambda_n}(S_n)$ and that $\overline{\Pi}_n = \Pi_{\nu_n}$ for a measure ν_n satisfying $T_{\lambda_n}(\nu_n) = \lambda_n$. According to formula (10), the price posted a period q is:

$$p_q^n = \sqrt{n} E_{\lambda_n}[u_n \Psi_{\nu_n,\lambda_n}(S_n) \mid u_1, \dots, u_{q-1}] \tag{28}$$

It is convenient to analyze this discrete time process through the continuous time process $Z^n : t \in [0,1] \to Z_t^n := p^n_{\lfloor nt \rfloor}$ where $\lfloor x \rfloor$ is the largest integer less or equal to x.

We analyze in this section the asymptotics of the law Q_n of Z^n when $(u_1, \dots, u_n)$ are endowed with the probability λ_n.

Let us introduce the notation $S_q^n = \frac{\sum_{i=1}^q u_i}{\sqrt{n}}$. The formula (28) can be written as:

$$\begin{aligned} p_q^n =& \sqrt{n} E_{\lambda_n}[u_n \Psi_{\nu_n,\lambda_n}(S_{n-1}^n + \frac{u_n}{\sqrt{n}}) \mid u_1, \dots, u_{q-1}] \\ =& \frac{\sqrt{n}}{2} E_{\lambda_n}[\Psi_{\nu_n,\lambda_n}(S_{n-1}^n + \frac{1}{\sqrt{n}}) - \Psi_{\nu_n,\lambda_n}(S_{n-1}^n - \frac{1}{\sqrt{n}}) \mid u_1, \dots, u_{q-1}] \\ =& \frac{\sqrt{n}}{2} E_{\lambda_n}[\Psi_{\nu_n,\lambda_n}(S_{n-1}^n + \frac{1}{\sqrt{n}}) - \Psi_{\nu_n,\lambda_n}(S_{n-1}^n - \frac{1}{\sqrt{n}}) \mid S_{q-1}^n] \end{aligned} \tag{29}$$

Heuristically we have that $p_q^n \approx E_{\lambda_n}[\Psi'_{\nu_n,\lambda_n}(S_{n-1}^n) \mid S_{q-1}^n]$. From corollary 47, we have that ν_n converges to ν. Furthermore, according to Donkster theorem, $S^n_{\lfloor tn \rfloor}$ converges in law to B_t where B is a standard Brownian motion. We can heuristically expect therefore that Z_t^n converges in law to $Z_t := E[\Psi'_{\nu,\lambda}(B_1)|B_t]$. Since $\Psi'_{\nu,\lambda}$ is an increasing function, it results from Remark 4 that this asymptotic process Z is a CMMV. This is the result we will establish formally in this section.

Let us remind here the definition of the weak convergence in finite distributions of a sequence of stochastic processes:

Definition 48. *A sequence (Z^n) of processes converges in finite dimensional distribution to a process Z if and only if for all finite family J of times $(t_1 < \cdots < t_k)$, the random vectors $(Z_t^n)_{t \in J}$ converge in law to the random vector $(Z_t)_{t \in J}$.*

Our main theorem is then:

Theorem 49. *Under the equivalent martingale measure, (Z^n) converges in finite dimensional distribution to the CMMV Z where $Z_t := E[\Psi'_{\nu,\lambda}(B_1)|B_t]$*

Proof. We will prove this convergence by proving that the $W_2(\rho_n, \rho) \to 0$ when ρ_n and ρ are respectively the laws of the vectors $(Z_t^n)_{t \in J}$ and $(Z_t)_{t \in J}$. We use "Skorokhod representation" technics to get that result. Let $(\tilde{\Omega}, \mathscr{A}, \tilde{P})$ be a probability space on which B is a Brownian motion. In this section, unless otherwise stated, all expectations on $\tilde{\Omega}$ are taken with respect to $\tilde{P}$. $Z_t = E[\Psi'_{\nu,\lambda}(B_1)|B_t]$ can be considered as a process on that space.

We will introduce hereafter a sequence of processes $\tilde{Z}^n$ defined on $\tilde{\Omega}$ such that:

$$\begin{aligned} &1/ \quad \tilde{Z}^n \text{ and } Z^n \text{ have the same laws.} \\ &2/ \quad \sup_t \|\tilde{Z}^n_t - Z_t\|_{L^2} \to 0. \end{aligned} \tag{30}$$

Theorem will then be proved. Indeed, $(\tilde{Z}^n, Z)$ is a pair of processes on the same probability space $(\tilde{\Omega}, \mathscr{A}, \tilde{P})$. The joint joint law of $(\tilde{Z}^n_t, Z_t)_{t\in J}$ is a probability distribution on $\mathbb{R}^{2|J|}$ with respective marginals ρ_n and ρ. Therefore:

$$W_2(\rho_n, \rho) \leq E[\sqrt{\sum_{t\in J} |\tilde{Z}^n_t - Z_t|^2}] \leq \sqrt{|J|} \sup_t \|\tilde{Z}^n_t - Z_t\|_{L^2} \to 0$$

In order to construct those random variables $\tilde{Z}^n$, it is convenient to apply the embedding techniques already used in De Meyer [2010]. Let $\mathscr{F}_t$ denote the natural filtration of the Brownian motion B. Define $\tau^n_0 = 0$ and, recursively, τ^n_{q+1} as the first time $t > \tau^n_q$ such that $|B_t - B_{\tau^n_q}| = \frac{1}{\sqrt{n}}$. Since the one-dimensional Brownian motion is a recurrent process $\tau^n_q < \infty$ almost surely and clearly $\tilde{u}_q := \sqrt{n}(B_{\tau^n_q} - B_{\tau^n_{q-1}})$ has the same distribution as u_q under λ_n. Indeed $\tilde{u}_q \in \{-1, +1\}$ and $E[\tilde{u}_q] = 0$. They are furthermore independent since the increments $B_{\tau^n_q} - B_{\tau^n_{q-1}}$ are independent of $\mathscr{F}_{\tau^n_{q-1}}$.

Therefore, $B_{\tau^n_q} = \sum_{j=1}^q (B_{\tau^n_j} - B_{\tau^n_{j-1}}) = \frac{1}{\sqrt{n}} \sum_{j=1}^q \tilde{u}_j$ has the same distribution as S^n_q under λ_n. We set:

$$\tilde{z}^n_n := \frac{\sqrt{n}}{2} \left(\Psi_n(B_{\tau^n_{n-1}} + \frac{1}{\sqrt{n}}) - \Psi_n(B_{\tau^n_{n-1}} - \frac{1}{\sqrt{n}}) \right) \tag{31}$$

$\tilde{z}_n$ has then the same distribution as p^n_n. Furthermore, if we define:

$$\tilde{z}^n_q := E[\tilde{z}_n | \tilde{u}_1, \ldots, \tilde{u}_{q-1}] = E[\tilde{z}_n | \mathscr{F}_{\tau^n_{q-1}}]$$

the process $(\tilde{z}^n_q)_{q=1,\ldots,n}$ has the same distribution as the process $(p^n_q)_{q=1,\ldots,n}$ under λ_n, as it follows from equations (29) and (31). We next define:

$$\tilde{Z}^n_t := \tilde{z}^n_{\lfloor nt \rfloor}$$

It is then clear that $\tilde{Z}^n$ and Z^n have the same laws which claim 1 in (30). We next prove claim 2:

$$\begin{aligned} \|\tilde{Z}^n_t - Z_t\|_{L^2} =& \|E[\tilde{z}^n_n | \mathscr{F}_{\tau^n_{\lfloor nt \rfloor -1}}] - Z_t\|_{L^2} \\ \leq& \|E[\tilde{z}^n_n | \mathscr{F}_{\tau^n_{\lfloor nt \rfloor -1}}] - Z_{\tau^n_{\lfloor nt \rfloor -1}}\|_{L^2} + \|Z_{\tau^n_{\lfloor nt \rfloor -1}} - Z_t\|_{L^2} \\ =& \|E[\tilde{z}^n_n - Z_1 | \mathscr{F}_{\tau^n_{\lfloor nt \rfloor -1}}]\|_{L^2} + \|Z_{\tau^n_{\lfloor nt \rfloor -1}} - Z_t\|_{L^2} \\ \leq& \|\tilde{z}^n_n - Z_1]\|_{L^2} + \|Z_{\tau^n_{\lfloor nt \rfloor -1}} - Z_t\|_{L^2} \end{aligned}$$

We next argue that both terms of the right hand side go to zero as n goes to ∞.

Let us start with the second one. First observe that all the martingales on the Brownian filtration are continuous (see Revuz and Yor [1999], theorem V.3.5), and $Z_t = E[Z_1 | \mathscr{F}_t]$ in particular. If $\|Z_{\tau^n_{\lfloor nt\rfloor -1}} - Z_t\|_{L^2}$ was not converging to zero, there would exists a subsequence $n(k)$ such that $(Z_{\tau^{n(k)}_{\lfloor n(k)t\rfloor -1}})$ does not admit Z_t as accumulation point in L^2. We prove in Lemma 50 that $\tau^n_{\lfloor nt\rfloor -1} \to t$ in L^2. The sequence $n(k)$ can thus be selected such that $\tau^{n(k)}_{\lfloor n(k)t\rfloor -1} \to t$ almost surely. By continuity we get then that $(Z_{\tau^{n(k)}_{\lfloor n(k)t\rfloor -1}})$ converges almost surely to Z_1 and the convergence also holds in L^2 since (Z_t) is uniformly integrable (Z_1 is bounded). This contradicts the definition of the subsequence $n(k)$.

Assume now that the first term does not converge to zero. There would exist a subsequence $n(k)$ such that $\tilde{z}^{n(k)}_{n(k)}$ does not have Z_1 as accumulation point in L^2.

Setting $a := B_{\tau^{n(k)}_{n(k)-1}} - \frac{1}{\sqrt{n(k)}}$ and $b := B_{\tau^{n(k)}_{n(k)-1}} + \frac{1}{\sqrt{n(k)}}$, equation 31 becomes $\tilde{z}^{n(k)}_{n(k)} = \frac{\Psi_{n(k)}(b) - \Psi_{n(k)}(a)}{b-a}$. With the mean value theorem, we conclude that there exists $x_{n(k)} \in [a,b]$ such that $\tilde{z}^{n(k)}_{n(k)} \in \partial\Psi_{n(k)}(x_{n(k)})$.

But it follows from Lemma 50 here below that $B_{\tau^{n(k)}_{n(k)-1}}$ converges in L^2 to B_1. The subsequence $n(k)$ can thus be selected in such a way that $B_{\tau^{n(k)}_{n(k)-1}}$ converges to B_1 almost surely and so does does $x_{n(k)}$. Since $\Psi_n = \Psi_{\nu_n, \lambda_{n|S}}$ uniformly converges to $\Psi_{\nu,\lambda}$ which is C^2, we may apply the forecoming Lemma 51 to conclude that $\tilde{z}^n_n$ converges almost surely to $\Psi'_n(B_1) = Z_1$. Since $\tilde{z}^n_n$ belongs to $\partial\Psi_n(k)(x_{n(k)}) \subset [0,1]$, it follows from the Lebesgue dominated convergence theorem that $\tilde{z}^{n(k)}_{n(k)}$ converges to Z_1 in L^2, in contradiction with the definition of the subsequence $n(k)$. Hence, as announced both terms go to zero. Therefore both claims in (30) are satisfied by the process $\tilde{Z}^n$ and the Theorem is thus proved. □

We next prove the announced lemma.

Lemma 50.
Claim 1: $\tau^n_{\lfloor nt\rfloor} \xrightarrow[L^2]{} t$
Claim 2: $B_{\tau^n_{n-1}} \xrightarrow[L^2]{} B_1$

Proof. As well known:

$$E(\tau^n_q) = E(B^2_{\tau^n_q}) = E_{\lambda_n}((S^n_q)^2) = \frac{q}{n}$$

On the other hand, $\tau^n_{q+1} - \tau^n_q$ is independent of $\mathscr{F}_{\tau^n_q}$. Therefore, $\tau^n_q = \sum_{i=0}^{q-1} \tau^n_{i+1} - \tau^n_i$ is a sum of independent random variables with expectation $\frac{1}{n}$.

Moreover we have $Var(\tau^n_{\lfloor nt\rfloor}) \to 0$ when $n \to \infty$.

Indeed:

$$Var(\tau^n_{q+1} - \tau^n_q) \le E((\tau^n_{q+1} - \tau^n_q)^2) \le CE[|B_{\tau^n_{q+1}} - B_{\tau^n_q}|^4] = C(\frac{1}{\sqrt{n}})^4 = \frac{C}{n^2}$$

where C is the Burkholder's constant for $p = 4$ (see Theorem IV.4.1 in Revuz and Yor [1999]).

Therefore:

$$Var(\tau^n_q) = \sum_{i=0}^{q-1} Var(\tau^n_{i+1} - \tau^n_i) \le \frac{qC}{n^2} \le \frac{C}{n}$$

And:
$$\|\tau_q^n - \frac{q}{n}\|_{L^2}^2 = \|\tau_q^n - E[\tau_q^n]\|_{L^2}^2 = Var(\tau_q^n) \le \frac{C}{n} \tag{32}$$

Replacing q by $\lfloor nt \rfloor$, we get claim 1 as announced.

It is also well known that $E[(B_{\tau_{n-1}^n} - B_1)^2] = E[|\tau_{n-1}^n - 1|]$. With equation (32) we get:

$\|B_{\tau_{n-1}^n} - B_1\|_{L^2}^2 = \|\tau_{n-1}^n - 1\|_{L^1} \le \|\tau_{n-1}^n - \frac{n-1}{n}\|_{L^2} + \frac{1}{n} \le \frac{C+1}{n} \to 0$. Claim 2 is thus also proved. □

Lemma 51. *Let (Ψ_n) be a sequence of convex functions that converges uniformly to a C^1 function Ψ. Let (x_n) and (z_n) be two real sequences such that:*
(1) x_n converges to x.
(2) for all n: $z_n \in \partial\Psi_n(x_n)$.
Then z_n converges to $\Psi'(x)$.

Proof. Since $z_n \in \partial\Psi_n(x_n)$, we get with $u \in \{-1, +1\}$ that:

$$\Psi(x_n + u) + \|\Psi - \Psi_n\|_\infty \ge \Psi_n(x_n + u) \ge \Psi_n(x_n) + uz_n \ge \Psi(x_n) - \|\Psi_n - \Psi\|_\infty + uz_n$$

Therefore $uz_n \le \Psi(x_n + u) - \Psi(x_n) + 2\|\Psi_n - \Psi\|_\infty$ and thus:

$$|z_n| \le \max_{u\in\{-1,+1\}} \Psi(x_n + u) - \Psi(x_n) + 2\|\Psi_n - \Psi\|_\infty$$

Since the right hand side is bounded, any subsequence of (z_n) has an accumulation point. All these accumulation points must be in $\partial\Psi(x)$. Indeed, if a subsequence $(z_{n(k)})$ converges to z, we have for all y: $\Psi_{n(k)}(y) \ge \Psi_{n(k)}(x_{n(k)}) + z_{n(k)}(y - x_{n(k)})$. Letting k go to infinity, we get then for all y: $\Psi(y) \ge \Psi(x) + z(y - x)$ and therefore $z \in \partial\Psi(x) = \{\Psi'(x)\}$ since Ψ is C^1. All subsequence of (z_n) has $\Psi'(x)$ as accumulation point, this is only possible if z_n converges to $\Psi'(x)$. □

8.2 Convergence under the historical probability

Let (Π_n, X_n) be a sequence of reduced equilibria in $G_n(\mu)$. We already know that $X_n = \Psi_{\nu_n,\overline{\lambda}_n}(S_n^n)$ and that the marginal $\overline{\Pi}_n$ of Π_n on (L, S_n^n) coincides with Π_{ν_n} for a measure ν_n satisfying $T_{\overline{\lambda}_n}(\nu_n) = \overline{\lambda}_n$. We further know that ν_n converges to the unique solution ν of $T_{\overline{\lambda}_\infty}(\nu) = \overline{\lambda}_\infty$. Therefore, $\overline{\Pi}_n$ converges to Π_ν. Our aim in this section is to analyze the asymptotics of the law Π_n of $(u_1, \ldots, u_n, L)$.

Let $y_n(\omega)$ denote the density of $\frac{\partial\Pi_{n|\omega}}{\partial\lambda_n}$. So y_n is a function of $\omega = (u_1, \ldots, u_n)$. In the previous subsection, we created sequences $\tilde{S_q^n} = B_{\tau_q^n}$ and $\tilde{u}$ of random variables on $(\tilde{\Omega}, \mathscr{A}, \tilde{P})$ a probability space on which B is a Brownian motion in such a way that $\tilde{S_q^n}$ and $\tilde{u}$ have the same distribution as S_n and u under λ_n.

Setting $\tilde{y}_n := y_n(\tilde{u}_1, \ldots, \tilde{u}_n)$, we infer that $\tilde{y}_n$ is a probability density on $(\tilde{\Omega}, \mathscr{A}, \tilde{P})$, and under the probability $\tilde{P}_n := \tilde{y}_n.\tilde{P}$, the process $(\tilde{u}_1, \ldots, \tilde{u}_n)$ is $\Pi_{n|\omega}$-distributed.

We first prove the following lemma:

Lemma 52. *$\tilde{y}_n$ converges in $L^1(\tilde{P})$ to $\tilde{y} := \frac{\beta}{\tilde{Y}}$ where $\tilde{Y} := H'(\Psi'_\nu(B_1)B_1 - \Psi_\nu(B_1))$ and $\beta = \frac{1}{E_{\tilde{P}}[\frac{1}{\tilde{Y}}]}$*

Proof. Our first task will be to define a variable $\tilde{L}_n$ on the space $(\tilde{\Omega}, \mathscr{A}, \tilde{P})$ such that the process $(\tilde{u}_1, \ldots, \tilde{u}_n, \tilde{L}_n)$ is Π_n-distributed under $\tilde{P}_n$.

This can be done as follow: $\tilde{\omega} := (\tilde{u}_1, \ldots, \tilde{u}_n)$ is $\mathscr{F}_{\tau_n^n}$ measurable. Let $V_n := B_{\tau_n^n+1} - B_{\tau_n^n}$. Under $\tilde{P}$, $V_n \sim \mathscr{N}(0,1)$ and is independent of $\mathscr{F}_{\tau_n^n}$. Since $\tilde{y}_n = y_n(\tilde{\omega})$, V_n will have the same law $\mathscr{N}(0,1)$ and will still be independent of $\mathscr{F}_{\tau_n^n}$ under $\tilde{P}_n$. Let F_ω denote the cumulative distribution function of the conditional law of L conditionally on ω under Π_n. We then set $\tilde{L}_n := F_{\tilde{\omega}}^{-1}(F_{\mathscr{N}(0,1)}(V_n))$. $\tilde{L}_n$ has the same conditional law given $\tilde{\omega}$ as L given ω under Π_n. Therefore $(\tilde{\omega}, \tilde{L}_n)$ under $\tilde{P}_n$ has the same law as (ω, L) under Π_n.

We now prove that, under $\tilde{P}$, $\tilde{L}_n$ converges to $\Psi'_\nu(B_1)$ almost surely. Indeed, since L_n belongs Π_n-almost surely to $\partial\Psi_n(S_n^n)$, we infer that $\tilde{L}_n$ belongs $\tilde{P}_n$-almost surely to $\partial\Psi_n(\tilde{S}_n^n)$. Since $\tilde{P}_n$ is equivalent to $\tilde{P}$, we conclude that $\tilde{L}_n \in \partial\Psi_n(\tilde{S}_n^n)$ $\tilde{P}$-almost surely.

Since Ψ_n converges uniformly to $\Psi_\nu \in C^2$, and since $\tilde{S}_n^n$ converges almost surely to B_1, we apply Lemma 51 to conclude that $\tilde{L}_n$ converges $\tilde{P}$ almost surely to $\Psi'_\nu(B_1)$.

We define $Y_n := E_{\Pi_n}[H'(LS_n^n - \Psi_n(S_n^n))|\omega]$. Y_n is then a function $Y_n(\omega)$. It follows from Corollary 17 that $\frac{\partial\lambda_n}{\partial\Pi_{n|\omega}} = \frac{Y_n}{E_{\Pi_n}[Y_n]}$.

We set $\tilde{Y}_n := Y_n(\tilde{\omega})$. We clearly have $\tilde{P}_n$-, and thus $\tilde{P}$-almost surely that $\tilde{Y}_n = E_{\tilde{P}_n}[H'(\tilde{L}_n\tilde{S}_n^n - \Psi_n(\tilde{S}_n^n))|\tilde{\omega}]$. Note that $\tilde{S}_n^n$ is $\tilde{\omega}$-measurable, and the law of $\tilde{L}_n$ conditionally to $\tilde{\omega}$ is the same under $\tilde{P}_n$ and $\tilde{P}$. Therefore, $\tilde{Y}_n = E_{\tilde{P}}[H'(\tilde{L}_n\tilde{S}_n^n - \Psi_n(\tilde{S}_n^n))|\tilde{\omega}] = E_{\tilde{P}}[H'(\tilde{L}_n\tilde{S}_n^n - \Psi_n(\tilde{S}_n^n))|\mathscr{F}_{\tau_n^n}]$. We claim that $\tilde{Y}_n$ converge in L^1 to $\tilde{Y} = H'(\Psi'_\nu(B_1)B_1 - \Psi_\nu(B_1))$. Indeed:

$$\begin{aligned}\|\tilde{Y}_n - \tilde{Y}\|_{L^1} =& \|E_{\tilde{P}}[H'(\tilde{L}_n\tilde{S}_n^n - \Psi_n(\tilde{S}_n^n))|\mathscr{F}_{\tau_n^n}] - \tilde{Y}\|_{L^1}\\ \leq& \|E_{\tilde{P}}[H'(\tilde{L}_n\tilde{S}_n^n - \Psi_n(\tilde{S}_n^n)) - \tilde{Y}|\mathscr{F}_{\tau_n^n}]\|_{L^1} + \|E_{\tilde{P}}[\tilde{Y}|\mathscr{F}_{\tau_n^n}] - \tilde{Y}\|_{L^1}\end{aligned}$$

We clearly have that $H'(\tilde{L}_n\tilde{S}_n^n - \Psi_n(\tilde{S}_n^n))$ converges almost surely to $\tilde{Y}$. Indeed $(\tilde{L}_n, \tilde{S}_n^n)$ converges almost surely to $(\Psi_\nu(B_1), B_1)$, and Ψ_n converges uniformly to Ψ_ν (remember that H' is continuous). Since H' is bounded, this convergence holds also in L^1 and thus:

$$\|E_{\tilde{P}}[H'(\tilde{L}_n\tilde{S}_n^n - \Psi_n(\tilde{S}_n^n)) - \tilde{Y}|\mathscr{F}_{\tau_n^n}]\|_{L^1} \leq \|H'(\tilde{L}_n\tilde{S}_n^n - \Psi_n(\tilde{S}_n^n)) - \tilde{Y}\|_{L^1} \to 0$$

We next claim that $E_{\tilde{P}}[\tilde{Y}|\mathscr{F}_{\tau_n^n}]$ converges to $\tilde{Y}$ in L^1. On the contrary one would have a subsequence $n(k)$ such that $E_{\tilde{P}}[\tilde{Y}|\mathscr{F}_{\tau_{n(k)}^{n(k)}}]$ does not admit $\tilde{Y}$ as accumulation point in L^1.

Since $\mathscr{F}_t$ is the natural filtration of a Brownian motion, it results from theorem V.3.5 in Revuz and Yor [1999] that the martingale $r_t := E[\tilde{Y}|\mathscr{F}_t]$ is continuous and uniformly integrable. Therefore, due to the optional stopping theorem, we have $E_{\tilde{P}}[\tilde{Y}|\mathscr{F}_{\tau_{n(k)}^{n(k)}}] = r_{\tau_{n(k)}^{n(k)}}$.

Since τ_n^n converges in L^2 to 1, there is no loss of generality to assume, possibly after selection of a smaller subsequence, that $n(k)$ further satisfies that $\tau_{n(k)}^{n(k)}$ converges almost surely to 1. But then $r_{\tau_{n(k)}^{n(k)}}$ converges almost surely to $r_1 = E_{\tilde{P}}[\tilde{Y}|\mathscr{F}_1] = \tilde{Y}$. But due to the uniform integrability of the martingale r_t, this convergence also holds in L^1, in contradiction with the definition of the subsequence $n(k)$. Therefore, as announced, $E_{\tilde{P}}[\tilde{Y}|\mathscr{F}_{\tau_n^n}]$ converges to $\tilde{Y}$ in L^1.

According to Corollary 17, $\frac{\partial \lambda_n}{\partial \Pi_{n|\omega}} = \alpha_n.Y_n$, and thus $\frac{\partial \Pi_{n|\omega}}{\partial \lambda_n} = \frac{\beta_n}{Y_n}$ for a constant β_n. Therefore for all ω, $y_n(\omega) = \frac{\beta_n}{Y_n(\omega)}$ and $\tilde{y}_n = \frac{\beta_n}{\tilde{Y}_n}$. Since $\tilde{Y}_n$ is a probability density under $\tilde{P}$ we get $\beta_n = \frac{1}{E_{\tilde{P}}[\frac{1}{\tilde{Y}_n}]}$.

Since $0 < \epsilon < \tilde{Y} < K$ (assumptions **A2** on H), $\frac{1}{\tilde{Y}_n}$ converges in L^1 to $\frac{1}{\tilde{Y}}$ and it results as announced that $\tilde{y}_n$ converges in L^1 to $\tilde{y} = \frac{\beta}{\tilde{Y}}$ where $\beta = \frac{1}{E_{\tilde{P}}[\frac{1}{\tilde{Y}}]}$. □

Theorem 20 claims that the martingale equivalent distribution Q_n converges to a limit distribution Q. The next theorem is the counterpart of this result for the historical distribution. It claims that P_n converges to a limit distribution P which is the law of the process Z when $\tilde{\Omega}$ is endowed with the probability measure $\tilde{y}\tilde{P}$. Therefore the limit distributions P and Q are equivalent.

This result is the main result of this paper. It claims that the asymptotics of the historical price process is a CMMV under an appropriate martingale equivalent measure Q.

Theorem 53. *The price process Z^n_t under the historical probability Π_n converges in finite dimensional distribution to the process Z when $\tilde{\omega}$ is endowed with the probability $\tilde{y}\tilde{P}$ where $\tilde{y} = \frac{1}{E_{\tilde{P}}(\tilde{Y})\tilde{Y}} > 0$.*

Proof. Let J a finite family of times. Let ϕ be a continuous and bounded function: $\mathbb{R}^{|J|} \to \mathbb{R}$. It is convenient to introduce the notations $\tilde{Z}^n_J := (\tilde{Z}^n_t)_{t\in J}$ and $Z^n_J := (Z^n_t)_{t\in J}$. Then observe that $E_{\Pi_n}[\phi(Z^n_J)] = E_{\tilde{P}_n}[\phi(\tilde{Z}^n_J)] = E_{\tilde{P}}[\tilde{y}_n\phi(\tilde{Z}^n_J)]$. We next claim that $E_{\tilde{P}}[\tilde{y}_n\phi(\tilde{Z}^n_J)]$ converges to $E_{\tilde{P}}[\tilde{y}\phi(Z_J)]$. Indeed, on the contrary there would exist a subsequence $n(k)$ that $E_{\tilde{P}}[\tilde{y}_{n(k)}\phi(\tilde{Z}^{n(k)}_J)]$ does not admit $E_{\tilde{P}}[\tilde{y}\phi(Z_J)]$ as accumulation point. However, as it results from equation (30) and Lemma 52, we have that $\tilde{Z}^{n(k)}_t \to Z_t$ in L^2 for all t and that $\tilde{y}_{n(k)} \to \tilde{y}$ in L^1. Possibly after selection of a smaller subsequence, we may assume without loss of generality that the sequence $n(k)$ is further such that $\tilde{Z}^{n(k)}_J \to Z_J$ and that $\tilde{y}_{n(k)} \to \tilde{y}$ almost surely. Due to the continuity of ϕ, we get that $\tilde{y}_{n(k)}\phi(\tilde{Z}^{n(k)}_J)$ converges almost surely to $\tilde{y}\phi(Z_J)$. Since both ϕ and $\tilde{y}_n$ are bounded, we have with Lebesgue dominated convergence theorem that $E_{\tilde{P}}[\tilde{y}_{n(k)}\phi(\widetilde{Z^{n(k)}_J})]$ converges to $E_{\tilde{P}}[\tilde{y}\phi(Z_J)]$, which is in contradiction with the definition of $n(k)$. Therefore, as announced, $E_{\Pi_n}[\phi(Z^n_J)] \to E_{\tilde{P}}[\tilde{y}\phi(Z_J)]$ for all J: the law of Z^n_J converges weakly in $\Delta(\mathbb{R}^{|J|})$ to the law of Z_J under $\tilde{y}\tilde{P}$ and the theorem is proved. □

9 Conclusion

To conclude this paper we would like to make some remarks on the obtained results.

The first one is about the dual game. In a previous unpublished version of the paper, we were using duality techniques to analyze the game. The dual game $G^\star_n(\phi)$ is in fact the reduced game where player 1 is allowed to select privately the value of L but his payoff is reduced by a penalty $\phi(L)$. Strategies and payoffs are the same for player 2. A strategy Π for the player 1 is a joint probability on (ω, L) but there is no constraint on the marginal $\Pi_{|L}$. It can be easily proved that if $(\Pi^\star, \overline{p})$ is an equilibrium in $G^\star_n(\phi)$ and if $\mu = \Pi^\star_{|L}$ then

$(\Pi^\star, \overline{p})$ is an equilibrium in $G_n(\mu)$. It can then be proved that there exists a function ϕ_n and an equilibrium $(\Pi_n^\star, \overline{p}_n)$ in $G_n^\star(\phi_n)$ such that $\Pi^\star_{|L} = \mu$. Therefore $(\Pi_n^\star, \overline{p}_n)$ is a sequence of equilibria in $G_n(\mu)$. One of the reason for introducing the dual game was that the asymptotics of the reduced equilibria in $G_n^\star(\phi)$ was quiet easy to analyze (with ϕ independent of n). However, to analyze the asymptotics of the equilibria in $G_n(\mu)$ using the dual game, we would have to analyze a sequence of equilibria in $G_n^\star(\phi_n)$ for an appropriate sequence of ϕ_n. This makes the analysis more involved and explains why we decided to limit our paper to the game $G_n(\mu)$.

The second remark we want to make is about the generality of our results. The results obtained in [De Meyer, 2010] were somehow more general than those obtained in the present paper: in the risk neutral case, if the mechanism satisfies the hypothesis **(H)**, then the price process at equilibrium converge to a CMMV for all sequences of equilibria in $G_n(\mu)$. The current paper is only concerned with one particular price based mechanism satisfying **(H)**. For this mechanism we do not analyze the asymptotic of any sequence of equilibria, but only of reduced equilibria: we prove that the price processes at a reduced equilibrium converges to a CMMV under the risk neutral probability. This naturally raises two questions: do we have the same asymptotic for any sequence of equilibria in our game? and will this dynamic appear for more general price based mechanism? We conjecture a positive answer to both questions but are presently unable to prove it.

Finally, we just want to mention an alternative approach to our results. It would indeed be possible to introduce continuous time games quite similar to the Brownian games introduced in De Meyer [1999] : a strategy Π_n in the reduced game can be viewed as a pair (y_n, ρ_n) where ρ_n is a conditional law of L given ω and y_n is the density $\frac{\partial \Pi_{n|\omega}}{\partial \lambda_n}$. Player 1's payoff is given by $E_{\lambda_n}[y_n(LS_n - \sum_{q=1}^n p_q(S_q - S_{q-1})]$. Heuristically this converges to $E[y(LB_1 - \int_0^1 p_t dB_t)]$. Similarly player 2 payoff would be $E[yH(LB_1 - \int_0^1 p_t dB_t)]$.

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