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Keywords:

Networks, Network defense, Network design, Attacks on links.

JEL codes:

D74, D85

Optimal design and defense of networks under link attacks*

Christophe Bravard[†], Liza Charroin[‡]

July 9, 2015

Abstract

Networks facilitate the exchange of goods and information and create benefits. We consider a network composed of complementary nodes, *i.e.*, nodes that need to be connected to generate a positive payoff. This network may face intelligent attacks on links. To study how the network should be designed, we develop a strategic model, inspired by Dziubiński and Goyal (2013), with two players: a Designer and an Adversary. The Designer has two potential ways to defend her network: forming destructible links among the given set of nodes to increase connectivity or protecting a group of nodes (with indestructible links). Links formation and protections (indestructible links) are costly. The Adversary then allocates her resources to attack links. We examine two situations which differ according to the number of protections available to the Designer. Our main findings are that if the number of protections is not limited, the Designer should either protect all the nodes, or create a large number of (destructible) links to absorb the Adversary's attack; if the available number of protections is limited, then a strategy that uses protections and links can be the equilibrium.

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1 Introduction

Networks can be seen as communication structures. They are composed of nodes and links, where links represent the flow of information. These networks can face threats, from natural disasters to intelligent attacks that can deteriorate the information flow. Our question is to design the network in an optimal way, such that no part of the network can be isolated (disconnected) from the network. More precisely, we deal with a situation where each node is necessary in order to make the network works. Concretely,

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nodes in the network are complementary: they need to be interlinked to produce a positive result. For instance, suppose a firm which consists of several production units (nodes of the network). Each production unit produces a part of the product and the parts are assembled by a given production unit. The links of the network allow the parts of the good to be transferred among the units. If one unit is not connected to the rest of the units, its part cannot be transferred and the product has no value. We study the optimal way to protect the network when attacks target the links. This assumption can reflect for instance some situations during the Second World War. The production units (the nodes) were buried, and attacks had to target the links (the roads) in order to destroy the production process of the enemy.

We consider a setting with two players: a Designer and an Adversary. The Designer moves first and her strategy can be divided into two parts.

1. She chooses a group of nodes to protect. More precisely, she connects them with indestructible links. Indestructible links cannot be removed by attacks. It is worth noting that in our setting x indestructible links allow to connect together a group of $x + 1$ nodes. Attacks cannot disconnect a protected group. In the following, indestructible links are called protections.
2. The Designer chooses destructible links between the nodes of the network.

Then, during the second stage, the Adversary attacks the network by allocating attack units to specific links. We assume that if a destructible link is attacked, then it is destroyed. Moreover, we assume that the Adversary can only remove destructible links. The number of attacks, k , available to the Adversary is given.

Creating destructible links and protections is costly for the Designer. The benefits obtained by the Designer at the end of the game depends on the connectivity of the residual network, that is the network obtained after the attacks of the Adversary. If the residual network is connected, *i.e.*, no set of nodes has been isolated from the rest of nodes, then the Designer wins the game: her benefit is equal to 1 and the benefit of the Adversary is 0. If the residual network is not connected, then the Adversary wins the game: her benefit is equal to 1 and the benefit of the Designer is 0. The payoff obtained by the Designer is equal to the difference between the benefits and the cost associated to her strategy. The payoff obtained by the Adversary is equal to its benefits.

We are interested in the Sub-game Perfect Equilibrium (SPE) of the two-stage game. We assume that the cost of protections (indestructible links) and the cost of forming destructible links are sufficiently low to allow the Designer to have some profitable strategies which allow the residual network to be connected. The Designer wants to minimize the sum of the cost of forming links and the protection cost given that the residual network is connected.¹

First, we assume that the number of protections (the number of indestructible links) available is

¹Recall that the residual network is connected if the Designer proposes a strategy that does not allow the Adversary to disconnect any part of the network when she plays in stage 2.

sufficient to protect all the nodes. We establish that the two polar strategies play a crucial role in equilibrium.

1. The strategy which consists in forming a minimal $(k + 1)$ -links-connected network without protections.²
2. The strategy which consists in forming no destructible links and protect all the nodes via indestructible links (protections).

The first strategy is the unique SPE when the cost of forming destructible links is sufficiently low relative to the the cost of protections. The second strategy is the unique SPE when the cost of forming destructible links is sufficiently high relative to the cost of protections. This result is slightly modified when the number of nodes is odd and the number of attacks is even. In that case, the strategy where the Designer protects a group of two nodes and form $(n - 1)(k + 1)/2$ destructible links may be a SPE. Second, we examine a situation where the defense of the Designer is imperfect. More precisely, the number of protections available to the Designer does not allow her to protect a group which contains more than $n - 3$ nodes. In this situation, we show that for intermediate relative costs (costs of each unit of protection/cost of each destructible link), the strategy where the Designer uses protections and formation of destructible links, is a SPE.

Our paper contributes to two rich strands of research: the theory of networks in economics and the theory of reliable networks.

Over the last twenty years, the literature on networks has become broader (*Jackson* [10], *Goyal* [7] and *Vega-Redondo* [15], ...). The two seminal papers on the formation of social and economic networks are the paper of *Bala and Goyal* [2] and the paper of *Jackson and Wolinsky* [11]. In this paper, we deal with the robustness of networks under an intelligent attack. Our question is to study how the network must be designed and protected given that links can be attacked. The contribution of our paper is to build on a classical problem in graph theory: how to design networks to retain connectivity in the face of link removal.

A growing literature on attacked networks studies situations where the Adversary attacks the nodes. The model we propose is close to the connectivity model of *Dziubiński and Goyal* (DG, [6]). DG study the optimal design and defense of networks under an intelligent attack on nodes. In the DG's framework and in our framework, the Designer chooses both the architecture of the network and the group of nodes she wants to protect. More precisely, in their setting, there are two players: the Designer and the Adversary; the Designer can form links between n nodes, and/or protect these nodes to ensure their survival. The major differences between the DG's framework and our framework are the following.

1. The Adversary attacks the nodes in the DG's framework while she attacks links in our framework;
2. In our framework, the Designer wins the game if every node of the population is able to communicate with each other node of the population in the residual network. In the DG's framework, the Designer

²A network g is a minimal $(k + 1)$ -links-connected network, if it is not possible to disconnect it by removing k links, but it is possible to disconnect it by removing $k + 1$ links.

wins the game even if a lot of nodes have been removed by the Adversary. Indeed, the Designer wins the game if the residual network is connected whatever the number of nodes removed by the Adversary.

In our setting, each node has to be connected with other nodes in order to generate a positive payoff: our setting is based on the complementarity of nodes. By contrast, in their model, DG assume that there is some substitutability between the nodes. Consequently, the connectivity of the network is the unique objective of the Designer, *i.e.*, the number of surviving nodes does not modify the payoff of the Designer. DG show that in a SPE, the Designer protects 0 or 1 node. If the Designer protects 0 node, then she may win the connectivity game by designing a minimal $(k + 1)$ -nodes-connected network.³ If the Designer protects 1 node, then she protects the central node and designs a star network. Our result differs since in our framework a strategy which uses more protections and destructible links may arise in equilibrium. This result follows the fact that there exists a discontinuity in the number of destructible links that each protection allows the Designer to save. More precisely, the number of destructible links that each protection allows the Designer to save is constant till a threshold, then this number brutally decreases. After this threshold, the number of destructible links that each protection allows the Designer to save increases. Moreover, in a SPE we obtain the same type of networks when the Designer uses no protection in the DG's framework and in our framework. At first sight, this result seems intriguing since the Adversary attacks nodes in DG's paper and links in our paper. However, a minimal $(k + 1)$ -nodes-connected network defined as in DG is also a network that contains the minimal number of links and resists to the Adversary who attacks links.

DG also propose a value function which is component⁴ additive and in which the value obtained by the Designer from each component is an increasing and convex function of the size of this component. Their analysis yields two insights. First, when the Designer uses protections, the nature of equilibrium networks is similar to what they found in the connectivity model. Second, when the Designer chooses 0 protection, a variety of networks which lie between the empty network and the $(k + 1)$ -nodes-connected network may arise in equilibrium.

A recent independent work of *Landwehr* [13] also extends the model of DG by introducing imperfect defense. He assumes that the protections used by the Designer can fail when they are attacked by the Adversary. More precisely, an attack on an unprotected target always destroys the target, and an attack on a protected target destroys the target with a positive probability. *Landwehr* shows that for a certain range of protection cost and costs of forming links, strategies that use both protections and several links are equilibria. In our setting, the limitation of the number of available protections to the Designer leads to a similar result. Consequently, weakening the possibility of protections of the Designer leads to the emergence of strategies that use both protection and formation of links as equilibria.

³A network g is a minimal $(k + 1)$ -nodes-connected network, if it is not possible to disconnect it by removing k nodes (and their links), but it is possible to disconnect it by removing $k + 1$ nodes.

⁴A component of a network g is a group X of nodes such that if $i, j \in X$, then i and j are connected in g and if $i \in X$ and $j \notin X$, then i and j are not connected in g .

There are other papers about protection of networks under attack. *Tambe* [14] studies network security under attack. He uses computational game theory. The main difference is that the Adversary does not know the allocation of defense units.

Moreover, a part of the literature on attacked networks examines the role played by the contagion of an attack. An attack does not only concern one node but spreads inside the network. *Cabrales, Gottardi and Vega-Redondo* [4] deal with the tradeoff between a better connectivity and a greater risk exposure due to contagion. *Goyal and Vigier* [8] also study the contagion of attacks (or threats) in a model with two players. Even if an attack can spread in the network with the additional problem of contagion, the star network with a protected center remains an optimal network under a contagious threat. *Cerdeiro, Dziubinski and Goyal* [5] also study contagion in networks. Their work is focused on the individual level to study the incentives to invest in security. The two players (the Designer and the Adversary) and the n nodes have an active role. The authors propose a three-stage game. First, the Designer chooses the network. Second, each node n observes the network formed and chooses independently and simultaneously if they invest in protection or not. Third, the Adversary observes the protected network and chooses the nodes to infect.

The literature on attacked networks can be used in a large variety of applications. For instance, a growing literature exists on criminal networks. In particular, the tradeoff between connectivity and risk vulnerability plays a crucial role in the paper of *Baccara and Bar-Isaac* [1]. Similarly, *Brown, Carlyle, Salmeron and Wood* [3] examine terrorist attacks and homeland security. Moreover, the field of computer science is a major literature in the design of survivable networks (see the survey of *Kerivin and Mahjoun* [12]).

The rest of the paper is organized as follows. In section 2, we present the model setup. In section 3, we present our main results. In section 4, we discuss our results.

2 Model setup

To simplify the notation, we set $\llbracket a, b \rrbracket = \{a, \dots, b\}$. Let \mathcal{E} be the set of even natural numbers and \mathcal{O} be the set of odd natural numbers.

Network. A network g is an ordered pair of disjoint sets (N, E) such that $E(g) \subset N \times N$. The set $N = \llbracket 1, n \rrbracket$, with $n \geq 3$, is the set of nodes and $E(g)$ is the set of destructible links. A network is called *simple* if it contains no loop and there exists at most one link between two nodes. Links are said destructible when the attacks considered in this paper remove them from the network; they are said indestructible when the attacks considered in this paper cannot remove them from the network. We denote by ij a destructible link between nodes i and j . Let $d_i(g)$ be the number of links incident to the node i in g , that is the degree of node i in g . A path between two nodes i_0 and i_L in g is a

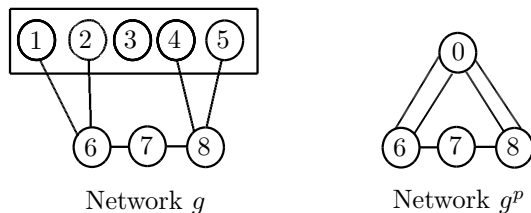


Figure 1: Networks g and g^p

sequence of distinct nodes and destructible links $(i_0, i_0i_1, i_1, i_1i_2, \dots, i_{L-1}i_L, i_L)$ such that $i_k, i_{k+1} \in g$ for $k \in \llbracket 0, L-1 \rrbracket$. We say that two nodes i_0 and i_L are connected in g if there is a path between i_0 and i_L in g . A network g is connected if for each pair of nodes (i, j) , there exists a path between them. A component $N^0 \subset N$ in g is a set of nodes which are connected in g and such that there is no $N^1 \supset N^0$, $N^1 \neq N^0$, which is connected in g . A component N^0 is minimal if the number of links that allows to connect the nodes in N^0 is equal to $|N^0| - 1$ links. A network g is a k -links-connected network if any network g' obtained from g by removing $k - 1$ links is connected, and there exists a network g' obtained from g by removing k links that is not connected.

Protected group. We denote by $N^p \subset N$, $|N^p| \geq 2$, a group of nodes that is protected through indestructible links, *i.e.*, links that cannot be removed by attacks. More precisely, this group forms a component. Since the links are indestructible, a minimal component is sufficient to obtain a protected group. Let p be the number of indestructible (protected) links formed to protect a group N^p . Clearly, p indestructible links allow to protect $p + 1$ nodes, *i.e.*, $p = |N^p| - 1$.⁵ The set of nodes which are not protected in g are denoted by $N^u = N \setminus N^p$. In the following, we define \mathcal{G}^p as the set of networks which have no destructible link between two nodes $i \in N^p$ and $j \in N^p$.

To simplify the analysis, we allow the possibility that there exist several links between two nodes.⁶ Consider network g , we build the *condensation network* g^p associated with g as follows. (O1) Nodes in N^p are identified to a node called 0 in g^p and other nodes in g are preserved. We denote by $N(g^p)$ the set of nodes of g^p . (O2) If there exists a link between node $i \in N^u$ and node $j \in N^u$ in g , then there is a link between node $i \neq 0$ and node $j \neq 0$ in g^p . (O3) If there exists a link between a node $i \in N^p$ and $j \in N^u$ in g , then there is a link between node 0 and node $j \neq 0$ in g^p . We illustrate the construction of g^p from g in Figure 1.

Remark 1. In network g^p , there is a unique node, 0, that may have several links with the same node $i \in N(g^p) \setminus \{0\}$.

Remark 2. By using operations that are the inverse of (O1), (O2) and (O3), we can build a network g from the network g^p .

Two player game: Strategies. We consider a two-stage game. The players are the Designer (D) and

⁵It is worth noting that it is not possible to protect only one node by using indestructible links.

⁶This type of graph is a multigraph.

the Adversary (A). In the first stage, D chooses a network and a subset of nodes to defend given the number of attacks k available to A . In the second stage, A observes the network and the protected group of nodes and then chooses a subset of (destructible) links to attack. To sum up, a strategy for D consists in a pair $\mathbf{g} = (E(g), N^p)$ where $g \in \mathcal{G}^p$ and $p \in \llbracket 0, n-1 \rrbracket$; the set of strategies of D is denoted by \mathbf{G} .

A strategy for A consists in a set of links $E(g^A) \subset E(g)$ that are attacked.⁷ The maximal number of attacks that A can do is denoted by k , with $k \in \llbracket 0, n-2 \rrbracket$,⁸ so $|E(g^A)| \leq k$. We do not take into account costs for the attacks of A . We assume that an attacked link is always removed. Consequently, an optimal choice for A is to attack k destructible links (when they exist).

Residual network and benefits. Given the strategy $\mathbf{g} = (E(g), N^p)$, $g \in \mathcal{G}^p$, $p \in \llbracket 0, n-1 \rrbracket$, formed at the first stage by D and the removal of the links $E(g^A) \subset E(g)$ at the second stage by A , we obtain a *residual network* g^R such that $E(g^R) = E(g) \setminus E(g^A)$. The benefits of D are given by:

$$\phi(g^R) = \begin{cases} 1, & \text{if } g^R \text{ is connected,} \\ 0, & \text{otherwise.} \end{cases} \quad (1)$$

Network and costs. Both links and protections (indestructible links) are costly. We assume linear costs: each link in a network has a strictly positive cost $c_L > 0$ and each protection has a strictly positive cost $c_P > 0$. The cost of a defended network is

$$c(\mathbf{g}) = c_L |E(g)| + p c_P.$$

To obtain non trivial results, we assume that the cost of protections and the cost of links formation are sufficiently low to allow D to form the number of protections and the number of links necessary to protect the network (given k). If the costs of links or protection are too high, then D cannot use one of her two strategies. It follows that in the rest of the paper, we suppose that $c_P < 1/(n-1)$ and $c_L < 2/(n(k+1))$.

Payoffs. The payoff of the Designer from choosing $\mathbf{g} = (E(g), N^p)$ when the Adversary chooses an attack $E(g^A) \subset E(g)$ is

$$\Pi^D(\mathbf{g}, E(g^A)) = \phi(g^R) - c(\mathbf{g}) \quad (2)$$

The payoff associated with g^R obtained by A is $1 - \phi(g^R)$.

To sum up, the objective of the Designer is to obtain a connected residual network at a minimal cost. The objective of the Adversary is to obtain a residual network that is disconnected. Hence, her goal is to isolate a part of the network, and she will never use her attack resources to target an indestructible link (links between nodes in N^p). Note that network g^p is k -links-connected if and only if g is k -links-

⁷Recall that A cannot successfully attack indestructible links.

⁸If $k = n-1$, then the Designer cannot use destructible links to protect the network.

connected.⁹

In the following, we are studying the Sub-game Perfect Equilibrium (SPE) of the game, *i.e.*, a situation that is a Nash equilibrium in each sub-game of the game. In a SPE, player D plays in stage 1 a best response (response which maximizes her payoff) given the best response that player A will play in stage 2.

Specific architectures. A network g is κ -regular if the degree of each node is κ . A κ -regular network without any protected link is denoted by g^κ . A network g is almost κ -regular if one node is adjacent to $\kappa + 1$ links and all other nodes are adjacent to k links. The *empty network* is the network which contains no links. We now define a class of networks which plays a crucial role in our analysis. Let $\lceil x \rceil$ be the smallest integer higher than x . It is easy to see that every node of a κ -links-connected network has a degree at least κ , as otherwise it could be separated by removing all its neighbors. Thus the minimal number of links in such a network is $\lceil n\kappa/2 \rceil$. As was shown by *Harary* ([9]), this number is also sufficient. The proof of this result is constructive – *Harary* describes how to obtain the desired family of graphs which are now called κ -*Harary-networks*. These networks are simple. *Harary* shows that the set of minimal κ -links-connected networks, that is κ -links-connected networks with the minimal number of links, consists in networks that are κ -regular if $n\kappa$ is even or almost κ -regular if $n\kappa$ is odd. To give the reader some idea of how κ -*Harary-networks* look like, we provide some examples in Figure 2 with 5 nodes.

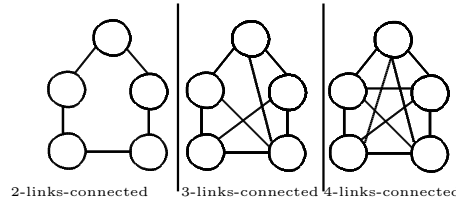


Figure 2: Example of κ -*Harary-networks*

For full description of the construction the interested reader is referred to [9]. In the following, we call κ -*Harary-networks*, the κ -links-connected networks which are built according to the process described by *Harary*.

We now define the main architecture of the paper. Given the number of attacks $k < n - 2$, we consider for each p a network $\hat{g}^p(k + 1)$ which contains $n - p$ nodes, with $0 \in N(\hat{g}^p(k + 1))$:

1. for $p \in \llbracket 0, n - k - 2 \rrbracket$, $\hat{g}^p(k + 1)$ is a $(k + 1)$ -*Harary-network*;
2. for $p \in \llbracket n - k - 1, n - 2 \rrbracket$, $\hat{g}^p(k + 1)$ contains a link between each node $i \neq 0$ and each other node $j \neq 0$, and there are $k + 1 - (n - p - 2)$ links between each node i and the node 0.

We illustrate the construction of $\hat{g}^p(k + 1)$ in Figure 3, when $n = 7$ and $p = 3$.

Specific strategies. Let us consider that the number of links that A may attack is k . We define the strategy \mathbf{g}^{k+1} as the strategy of D where she uses no protection and designs a network $\hat{g}^0(k + 1)$,

⁹This result follows the fact that nodes in N^P are linked with indestructible links.

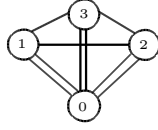


Figure 3: $\hat{g}^3(4)$ when $n = 7$.

i.e., a $(k + 1)$ -Harary-network. Strategy \mathbf{g}_{n-k-2}^{k+1} is the strategy where D uses $n - k - 2$ protections and designs the network $\hat{g}^{n-k-2}(k + 1)$, this network is complete and contains $k + 2$ nodes in the condensation network. Strategy \mathbf{g}^F is the strategy where D uses $n - 1$ protections and forms no destructible links. Strategy \mathbf{g}_1^{k+1} is the strategy where D uses 1 protection, and designs a network $\hat{g}^1(k + 1)$.

3 Model Analysis

3.1 Optimal number of links in a network

In this section, we are interested in the minimal number of links of the $(k + 1)$ -links-connected networks. This problem has been solved by Harary when the degrees of nodes $k + 1$ is strictly smaller than the number of nodes in a network. We extend the analysis of Harary in the situation where $k + 1$ is greater than the number of nodes in a network. Recall that, due to the condensation networks, we deal with networks where a node may be involved in several links with each other node.

Lemma 1 *Suppose $p \in \llbracket n - k - 1, n - 2 \rrbracket$. Then $\hat{g}^p(k + 1)$ is a minimal $(k + 1)$ -links-connected network.*

Proof Suppose that $p \in \llbracket n - k - 1, n - 2 \rrbracket$. First, we establish that $\hat{g}^p(k + 1) := \hat{g}^p$ is $(k + 1)$ -links-connected. If $p = n - 2$, then $|N(\hat{g}^p)| = 2$, and by construction the network \hat{g}^p is a minimal $(k + 1)$ -links-connected network. Suppose that k links are removed in \hat{g}^p , with $|N(\hat{g}^p)| \geq 3$. It is sufficient to show that there is no sub-set $X \subset N(\hat{g}^p)$ that can be disconnected from the sub-set $N(\hat{g}^p) \setminus X$ in \hat{g}^p by removing k links. There are two cases: either X does not contain node 0, or X contains node 0. If X contains node 0, then by construction it has more links with $N \setminus X$ than if X does not contain node 0. Therefore, it is sufficient to show that if X does not contain 0, then it is not possible to disconnect it from $N \setminus X$. If $|X| = 1$, say $X = \{i\}$, then by construction it is not possible to disconnect i by removing k links: $d_i(\hat{g}^p) = k + 1$. We now establish that it is not possible to isolate a sub-set X , with $|X| > 1$. Each node in X has $|X| - 1$ links with other nodes in X in \hat{g}^p . Consequently, by construction, each node in X has $k + 1 - (|X| - 1)$ links with nodes that does not belong to X in \hat{g}^p . It follows that it is necessary to remove $|X|(k + 1 - (|X| - 1))$ links to disconnect nodes in X from the rest of the nodes in \hat{g}^p . By construction of \hat{g}^p , $k + 1 - |X| > 0$, so $(|X| - 1)(k + 1 - |X|) > 0$ and we have $|X|(k + 1 - (|X| - 1)) > k + 1 > k$. It follows that it is not possible to disconnect X from the rest of the network \hat{g}^p .

Second, in a network g' with a number of links smaller than in $\hat{g}^p(k + 1)$ there exists a node i such that $d_i(g') < k + 1$. Consequently, network g' is not $(k + 1)$ -links-connected. \square

We note that network $\hat{g}^{n-k-2}(k+1)$ plays a specific role in our analysis since for all networks $\hat{g}^p(k+1)$, with $p \in \llbracket 0, n-k-2 \rrbracket$, node 0 is involved in at most one link with other nodes and for all networks $\hat{g}^p(k+1)$, with $p \in \llbracket n-k-1, n-2 \rrbracket$, node 0 is involved in several links with other nodes in $N(\hat{g}^{n-k-2}(k+1))$.

The following lemma uses the Harary result and Lemma 1 in order to provide the number of links in a minimal $(k+1)$ -links-connected network $\hat{g}^p(k+1)$, for $p \in \llbracket 0, n-1 \rrbracket$.

Lemma 2 (1) Suppose $p \in \llbracket 0, n-k-2 \rrbracket$. If $(n-p)(k+1) \in \mathcal{E}$, then $\hat{g}^p(k+1)$ contains $(1/2)(n-p)(k+1)$ links. If $(n-p)(k+1) \in \mathcal{O}$, then $\hat{g}^p(k+1)$ contains $(1/2)((n-p)(k+1) + 1)$ links.
(2) Suppose $p \in \llbracket n-k-1, n-1 \rrbracket$. Then $\hat{g}^p(k+1)$ contains $(n-p-1)(k+2 - \frac{n-p}{2})$ links.

Proof The proof of Lemma 2 follows the construction of $\hat{g}^p(k+1)$ for $p \in \llbracket 0, n-2 \rrbracket$. We now examine $\hat{g}^{n-1}(k+1)$. By construction, if $p = n-1$, then $|N(\hat{g}^p)| = 1$ so the network cannot be disconnected and it contains no links: $(n - (n-1) - 1) \left(k + 2 - \frac{n-(n-1)}{2} \right) = 0$. \square

3.2 Optimal cost function in a SPE

By using Lemma 2, we now determine the optimal costs that D incurs to protect her network. In this section, we provide the optimal cost function of D given the number of attacks available to A , say k , and the number of indestructible links that D forms, p . Lemma 2 establishes that we have to take into account two types of situation according to the size of the protected group. We define for (p, k) , with $p \in \llbracket 0, n-k-2 \rrbracket$ and $k \in \llbracket 1, n-2 \rrbracket$, the following function:

$$C_1(p, k) = \begin{cases} \frac{c_L}{2} [(n-p)(k+1)] + pc_P, & \text{if } (n-p)(k+1) \in \mathcal{E}, \\ \frac{c_L}{2} [(n-p)(k+1) + 1] + pc_P, & \text{if } (n-p)(k+1) \in \mathcal{O}. \end{cases}$$

Moreover, we define for (p, k) , with $p \in \llbracket n-k-1, n-1 \rrbracket$ and $k \in \llbracket 1, n-2 \rrbracket$, the following function:

$$C_2(p, k) = (n-p-1) \left(k + 1 - \frac{n-p-2}{2} \right) c_L + pc_P$$

Finally, we define the optimal cost function as follows:

$$C^*(p, k) = \begin{cases} C_1(p, k), & \text{if } p \in \llbracket 0, n-k-2 \rrbracket, \\ C_2(p, k), & p \in \llbracket n-k-1, n-1 \rrbracket. \end{cases} \quad (3)$$

In other words, for each pair (p, k) , the minimal cost associated with a $(k+1)$ -links-connected network is $C^*(p, k)$ given in Eq. 3. It is worth noting that $C_1(n-k-2, k) = C_2(n-k-2, k)$. We now provide the optimal cost C^* for some pairs (p, k) which play a specific role in the rest of the paper:

$$C^*(0, k) = \frac{n(k+1)c_L}{2}, \text{ if } n(k+1) \text{ is even, and } C^*(0, k) = \frac{c_L}{2} [n(k+1) + 1], \text{ if } n(k+1) \text{ is odd,}$$

$$C^*(1, k) = \frac{(n-1)(k+1)c_L}{2} + c_P, \text{ if } n(k+1) \text{ is odd,}$$

$$C^*(n-k-2, k) = \frac{(k+2)(k+1)c_L}{2} + (n-k-2)c_P, \text{ and } C^*(n-1, k) = (n-1)c_P.$$

We now compare the costs associated with \mathbf{g}^F , \mathbf{g}_{n-k-2}^{k+1} and \mathbf{g}^{k+1} . First, we consider that n or $k+1$ is even. Moreover, we restrict our attention to the situations where $k < n-2$ to avoid the possibility of multiple SPE.

Lemma 3 *Suppose that n or $k+1$ is even and $k < n-2$.*

1. *We have $C^*(n-k-2, k) - C^*(0, k) < 0$ iff $c_P/c_L < (k+1)/2 = \mathcal{A}_1$; $C^*(n-1, k) - C^*(0, k) < 0$ iff $c_P/c_L < (n/n-1)(k+1)/2 = \mathcal{A}_2$; $C^*(n-1, k) - C^*(n-k-2, k) < 0$ iff $c_P/c_L < (k+2)/2 = \mathcal{A}_3$.*
2. *We have $\mathcal{A}_3 > \mathcal{A}_2 > \mathcal{A}_1$.*

Proof Part 1 of the lemma follows straightforward calculations. Part 2 of the lemma follows the facts that $n/(n-1) > 1$ and $k < n-2$. \square

Second, we consider that n and $k+1$ are odd. We restrict our attention to situations where $k < n-3$ to simplify the analysis. The next lemma follows straightforward calculations. We omit its proof.

Lemma 4 *Suppose that n and $k+1$ are odd and $k < n-3$.*

1. *We have $C^*(n-k-2, k) - C^*(0, k) < 0$ iff $c_P/c_L < (1/2)[k+1 + [1/(n-k-2)]] = \mathcal{A}'_1$; $C^*(n-1, k) - C^*(0, k) < 0$ iff $c_P/c_L < (1/2)([n/(n-1)][(k+1)] + [1/(n-1)]) = \mathcal{A}'_2$; $C^*(n-1, k) - C^*(n-k-2, k) < 0$ iff $c_P/c_L < \mathcal{A}_3$*
2. *We have $C^*(n-k-2, k) - C^*(1, k) < 0$ iff $c_P/c_L < \mathcal{A}_1$; $C^*(n-1, k) - C^*(1, k) < 0$ iff $c_P/c_L < [(k+1)/2][(n-1)/(n-2)] = \mathcal{A}'_2$; $C^*(1, k) - C^*(0, k) < 0$ iff $c_P/c_L < \mathcal{A}_3$.*
3. *We have $\mathcal{A}_3 > \mathcal{A}'_2$, $\mathcal{A}'_2 > \mathcal{A}'_1 > \mathcal{A}_1$, and $\mathcal{A}'_2 > \mathcal{A}'_1 > \mathcal{A}_1$.*

3.3 Network and protections in a SPE

We begin the analysis with two lemmas that allow us to restrict the number of candidates for being a SPE. Recall that $c_P < 1/(n-1)$ and $c_L < 2/[n(k+1) + 1]$.

Lemma 5 *Suppose $p \in \llbracket 0, n-k-2 \rrbracket$. (1) If $k+1$ is even, then $|\hat{g}^p(k+1)| - |\hat{g}^{p+1}(k+1)| = (k+1)/2$. (2) If $k+1$ and $n-p$ are odd, then $|\hat{g}^p(k+1)| - |\hat{g}^{p+1}(k+1)| = (k/2) + 1$. (3) If $k+1$ is odd and $n-p$ is even, then $|\hat{g}^p(k+1)| - |\hat{g}^{p+1}(k+1)| = k/2$.*

By using the fact that $C_1(n-k-2, k) = C_2(n-k-2, k)$, we obtain the following lemma.

Lemma 6 *Suppose $p \in \llbracket n-k-2, n-1 \rrbracket$. Then, we have $|\hat{g}^p(k+1)| - |\hat{g}^{p+1}(k+1)| = k+2 - (n-p)$.*

Lemma 5 part (1) means that if $k+1$ is even and $p \in \llbracket 0, n-k-2 \rrbracket$, then the number of links saved due to an additional protection is constant. It follows that if $c_P/c_L \neq \mathcal{A}_1$, then there is no SPE associated with a number of protections $p \in \llbracket 1, n-k-3 \rrbracket$. In Lemma 5 parts (2) and (3), we assume that $k+1$ is odd. Lemma 5 parts (2) and (3) mean that the number of links saved, when a protection is added when $(n-p)$ is odd, is higher than the number of links saved, when $(n-p)$ is even. Therefore, the linearity of the costs c_L and c_P implies that only p such that $n-p$ is even should be a SPE when $c_P/c_L \neq \mathcal{A}_1$. Note that if $p = n-k-2$, then $n-p = k+2$, and $n-p$ is even since $k+1$ is odd. By the linearity of the costs if $c_P/c_L \neq \mathcal{A}_1$, then there is no SPE associated with a number of protections $p \in \llbracket 2, n-k-3 \rrbracket$. It is worth noting that $n-p$ is even either for $p=0$ or for $p=1$. The situation where $n-p$ is even for $p=1$ is interesting since D may have an incentive to use $p=1$ protection instead of $p=0$.

Lemma 6 means that the number of links saved due to an additional protection is increasing with p , when $p \in \llbracket n-k-2, n-1 \rrbracket$. Due to the linearity of the cost of protections and the cost of links, if $c_P/c_L \neq \mathcal{A}_1$, then there is no SPE associated with $p \in \llbracket n-k, n-2 \rrbracket$. We sum up these observations in the following lemma.

Lemma 7 *Suppose $c_P/c_L \neq \mathcal{A}_1$. (1) If n or $k+1$ are even, then there exist three strategies candidates for being SPE: \mathbf{g}^{k+1} , \mathbf{g}^F and \mathbf{g}_{n-k-2}^{k+1} . (2) If n and $k+1$ are odd, then there exist four strategies candidates for being SPE: \mathbf{g}^{k+1} , \mathbf{g}^F , \mathbf{g}_1^{k+1} and \mathbf{g}_{n-k-2}^{k+1} .*

In the following proposition, we provide a full characterization of the SPE. We assume that $c_P/c_L \in \mathbb{R} \setminus \{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_2'', \mathcal{A}_3\}$ in order to avoid that D is indifferent between several strategies. We show that there is no situation where \mathbf{g}_{n-k-2}^{k+1} is a SPE. Moreover, we provide conditions under which the other candidates are the unique SPE.

Proposition 1 *We assume that $c_P/c_L \in \mathbb{R} \setminus \{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_2'', \mathcal{A}_3\}$.*

1. *Suppose that n or $k+1$ is even and $k < n-2$.*
 - (a) *If $c_P/c_L > \mathcal{A}_2$, then strategies \mathbf{g}^{k+1} are the unique SPE.*
 - (b) *If $c_P/c_L < \mathcal{A}_2$, then the strategy \mathbf{g}^F is the unique SPE.*
2. *Suppose that n and $k+1$ are odd and $k < n-3$.*
 - (a) *If $c_P/c_L < \mathcal{A}_2''$, then the strategy \mathbf{g}^F is the unique SPE.*
 - (b) *If $\mathcal{A}_3 > c_P/c_L > \mathcal{A}_2''$, then strategies \mathbf{g}_1^{k+1} are the unique a SPE.*
 - (c) *If $\mathcal{A}_3 < c_P/c_L$, then strategies \mathbf{g}^{k+1} are the unique SPE.*

Proof We prove successively the two parts of the proposition.

1. Suppose that $k+1$ or n is even. By Lemma 7, we know that there are only three strategies candidates for being SPE: \mathbf{g}^{k+1} , \mathbf{g}^F and \mathbf{g}_{n-k-2}^{k+1} . Moreover, by Lemma 3 part 2, we know that if $c_P/c_L < \mathcal{A}_2$, then $c_P/c_L < \mathcal{A}_3$, i.e. if \mathbf{g}^F is preferred by D to \mathbf{g}^{k+1} , then \mathbf{g}^F is preferred by D to \mathbf{g}_{n-k-2}^{k+1} . Similarly, if $c_P/c_L > \mathcal{A}_2$, then $c_P/c_L > \mathcal{A}_1$, i.e. if \mathbf{g}^{k+1} is preferred by D to \mathbf{g}^F , then \mathbf{g}^{k+1} is preferred by D to \mathbf{g}_{n-k-2}^{k+1} . The result follows.

2. Suppose that $k+1$ and n are odd and $k < n-3$. By Lemma 7, we know that there exist four strategies candidates for being SPE: \mathbf{g}^{k+1} , \mathbf{g}^F , \mathbf{g}_1^{k+1} and \mathbf{g}_{n-k-2}^{k+1} . We divide the proof into three parts.

(i) Suppose $c_P/c_L < \mathcal{A}_2''$, that is the strategy \mathbf{g}^F is less costly than strategy \mathbf{g}_1^{k+1} . Then $c_P/c_L < \mathcal{A}_2'$ and $c_P/c_L < \mathcal{A}_3$; that is \mathbf{g}^F is less costly than strategies \mathbf{g}^{k+1} and \mathbf{g}_{n-k-2}^{k+1} . It follows that \mathbf{g}^F is the unique SPE.

(ii) Suppose that $\mathcal{A}_3 > c_P/c_L > \mathcal{A}_2''$. Since $c_P/c_L > \mathcal{A}_2''$, we have $c_P/c_L > \mathcal{A}_1$; that is the strategy \mathbf{g}^F and \mathbf{g}_{n-k-2}^{k+1} are more costly than strategy \mathbf{g}_1^{k+1} . Since $c_P/c_L < \mathcal{A}_3$ then \mathbf{g}_1^{k+1} is less costly than \mathbf{g}^{k+1} . It follows that \mathbf{g}_1^{k+1} is the unique SPE.

(iii) Suppose that $\mathcal{A}_3 < c_P/c_L$, then strategy \mathbf{g}^F is less costly than strategies \mathbf{g}_{n-k-2}^{k+1} , \mathbf{g}_1^{k+1} and \mathbf{g}^{k+1} . It follows that \mathbf{g}^{k+1} is the unique SPE. □

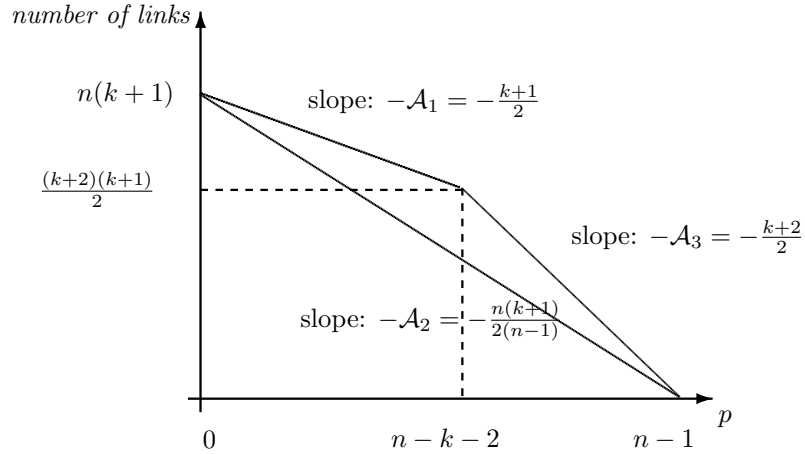


Figure 4: Proposition 1, n or $k+1$ even. Protection and *average* number of links saved.

First, we provide the intuition of Part 1 of Proposition 1: n or $k+1$ is even. By Lemmas 5 and 6, we know that there are only three strategies candidates for being SPE. In Figure 4, we draw the relation between the number of links and the number of protections for the three strategies candidates for being SPE; the slopes of each line can be interpreted as the *average number of links* that each protection allows to save. We observe that if D chooses \mathbf{g}^{k+1} instead of \mathbf{g}_{n-k-2}^{k+1} , then each additional protection allows her to save $(k+1)/2$ links. Similarly, if D chooses \mathbf{g}^F instead of \mathbf{g}_{n-k-2}^{k+1} , then each additional protection allows her to save an *average* of $(k+2)/2$ links. Finally, if D chooses \mathbf{g}^F instead of \mathbf{g}^{k+1} , then each additional protection allows her to save an *average* of \mathcal{A}_2 links, with $(k+2)/2 < \mathcal{A}_2 < (k+1)/2$. Consequently, if the ratio c_P/c_L is smaller than \mathcal{A}_2 , then the cost associated with \mathbf{g}^F is lower than the cost associated with \mathbf{g}^{k+1} . In this case, the ratio c_P/c_L is smaller than $(k+2)/2$ and the cost associated with \mathbf{g}^F is lower than the cost associated with \mathbf{g}_{n-k-2}^{k+1} . The intuition is similar if c_P/c_L is higher than

\mathcal{A}_2 .

Second, we provide the intuition of Part 2 of Proposition 1: n and $k + 1$ are odd. By observing the slopes in Figure 5 we conclude that if $c_P/c_L < \mathcal{A}_2''$, then $c_P/c_L < \mathcal{A}'_2$ and $c_P/c_L < \mathcal{A}_3$. In other words, if the strategy \mathbf{g}^F is less costly than strategy \mathbf{g}_1^{k+1} , then \mathbf{g}^F is less costly than strategies \mathbf{g}^{k+1} and \mathbf{g}_{n-k-2}^{k+1} . Similarly, if $c_P/c_L > \mathcal{A}_3$, then by observing the slopes in Figure 5 we conclude that $c_P/c_L > \mathcal{A}'_1$ and $c_P/c_L > \mathcal{A}'_2$. In other words, if the strategy \mathbf{g}^{k+1} is less costly than strategy \mathbf{g}_1^{k+1} , then \mathbf{g}^{k+1} is less costly than strategies \mathbf{g}_{n-k-2}^{k+1} and \mathbf{g}^F . We use similar arguments to conclude when $c_P/c_L > \mathcal{A}'_2$ and $c_P/c_L < \mathcal{A}_3$.

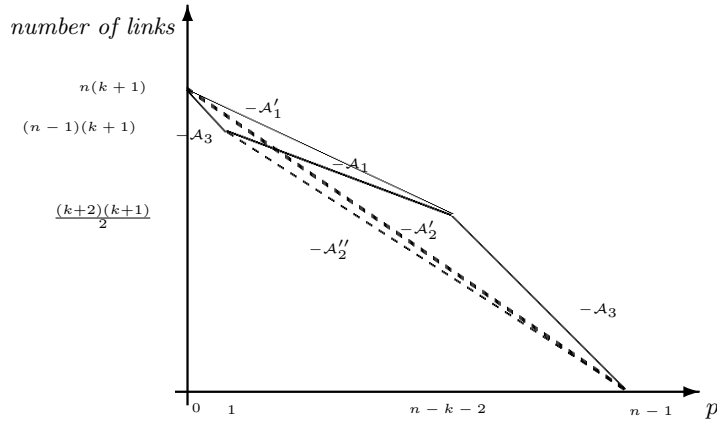


Figure 5: Proposition 1. Intuition when n and $k + 1$ are odd

We now compare the results obtained in our framework where A attacks the links and the results obtained in DG's framework where A attacks the nodes (Proposition 1, [6]). Recall that in DG, in a SPE D forms either a star¹⁰ where the central node is protected, or she forms an Harary network without any protection: she plays the strategy \mathbf{g}^{k+1} . In other words, D uses at most one protection in a SPE, when A attacks nodes. We observe that the strategy \mathbf{g}^{k+1} is a SPE when the cost of links is sufficiently low relative to the cost of protection in both frameworks. The role played by the protection is different since in DG one protection is sufficient in the star network to protect the network. In our framework, when D uses protections in a SPE, then she uses more protections and less destructible links than in DG. Protecting the links of the network is more expensive than protecting its nodes. This is due to the fact that we require the survival of every node in our framework, this requirement is not true in the DG's framework.

¹⁰A star g is a network where a node i , called the central node, is linked with each other node $j \neq i$ and there is no other link in g .

3.4 Limited number of protections and SPE

Proposition 1 establishes that the strategy \mathbf{g}_{n-k-2}^{k+1} cannot arise in equilibrium. In particular, in a SPE, it is not possible to obtain a situation where D uses both links and protections in order to protect the network against A when n or $k+1$ is even.

We now examine a situation where the maximal number of protections, \bar{p} , that D can form is strictly smaller than $n-1$. In the following, we establish that there exist some situations where the intermediate strategy \mathbf{g}_{n-k-2}^{k+1} is the unique SPE.

We restrict our attention to situations where n or $k+1$ is even. We have for $p \in \llbracket n-k-1, n-1 \rrbracket$, $\Delta^*(p, k) = C^*(p, k) - C^*(n-k-2, k) = (p - (n-k-2))[c_P - c_L[(k+3) - (n-p)]]/2$, and we note that $\Delta^*(n-2, k) = 0$ if $c_P/c_L = (k+1)/2 = \mathcal{A}_1$.

Proposition 2 *Suppose that $\bar{p} \in \llbracket n-k-1, n-3 \rrbracket$. There exists $\epsilon > 0$ such that if $c_P/c_L < \mathcal{A}_1$ and $c_P/c_L > \mathcal{A}_1 - \epsilon$, then \mathbf{g}_{n-k-2}^{k+1} is the unique SPE.*

Proof By Lemma 6, we know that the number of links saved due to an additional protection is increasing with p , when $p \in \llbracket n-k-1, \bar{p} \rrbracket$. Consequently, for $p \in \llbracket n-k-1, \bar{p} \rrbracket$ we have $\Delta^*(p, k) = 0$ if $c_P/c_L = \mathcal{B}$, with $\mathcal{B} < (k+1)/2 = \mathcal{A}_1$. We set $\epsilon = (\mathcal{A}_1 - \mathcal{B})/2$.

We now assume that $c_P/c_L < \mathcal{A}_1$ and $c_P/c_L > \mathcal{A}_1 - \epsilon$. By Lemma 5, we know that there is no SPE associated with $p \in \llbracket 2, n-k-3 \rrbracket$. Moreover, since $c_P/c_L < \mathcal{A}_1$, we have $C^*(n-k-2, k) - C^*(0, k) < 0$. It follows that \mathbf{g}^{k+1} is not a SPE. By Lemma 6, we know that there is no SPE associated with $p \in \llbracket n-k-1, \bar{p}-1 \rrbracket$. Since $c_P/c_L > \mathcal{A}_1 - \epsilon$, $\Delta^*(\bar{p}, k) > 0$ and there is no SPE associated with \bar{p} . Consequently, the unique SPE is \mathbf{g}_{n-k-2}^{k+1} . \square

Corollary 1 *Suppose that $\bar{p} \in \llbracket n-k-1, n-3 \rrbracket$. If $c_P/c_L < \mathcal{A}_1$ and $c_P/c_L > [(k+3) - (n-\bar{p})]/2$, then the strategy \mathbf{g}_{n-k-2}^{k+1} is the unique SPE.*

Proof The proof is based on similar arguments as those used in the proof of Proposition 2. \square

We now provide the intuition of Proposition 2. In Figure 6, we draw the relation between the number of links and the number of protections for different strategies, that is the number of links that each protection allows the Designer to save.¹¹ In particular, if D uses an optimal strategy with p' protections, then she saves in *average* \mathcal{B} links for each protection added to the strategy \mathbf{g}_{n-k-2}^{k+1} . Suppose now that $\bar{p} = p'$. If $c_P/c_L < \mathcal{A}_1$ and $c_P/c_L > \mathcal{B}$. Then, \mathbf{g}_{n-k-2}^{k+1} is less costly than \mathbf{g}^{k+1} and \mathbf{g}_{n-k-2}^{k+1} is less costly than the optimal strategy associated with p' .

DG show that when A attacks nodes, there exist situations where the SPE consists in a star and the central node is protected. In this case D uses both protections and links to protect her network. D also uses both protections and links to protect her network in our framework if the number of protection available to D is smaller than $n-2$, and relative costs c_P/c_L are intermediate. More precisely, in our framework,

¹¹Note that we draw a continuous graph in order to simplify the reading.

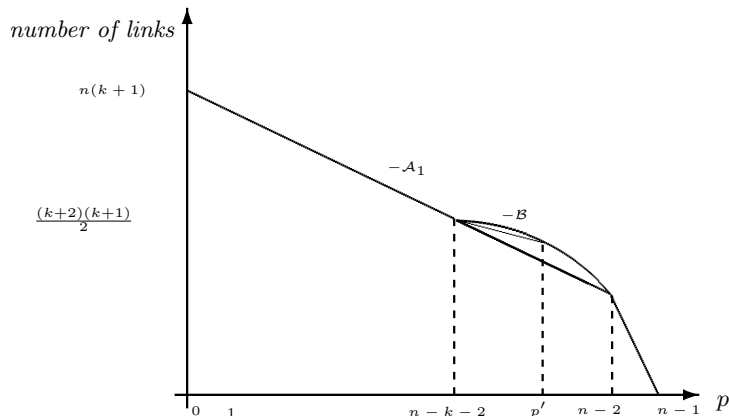


Figure 6: Number of protections is limited. Intuition

D uses more protections and links than in DG's framework. We find this result because of the existence of a discontinuity in the number of destructible links that each protection allows the Designer to save. Moreover, D mixes protections and destructible links because each protection allows D to save a number of links that is constant and equal to $(k+1)/2$ till $p = n - k - 2$; then this number brutally decreases: this number is equal to 1 between $p = n - k - 2$ and $p = n - k - 1$. After this threshold, the number of destructible links that each protection allows D to save increases. Consequently, there is a range of value for the maximal number of protections such that D has no incentive to increase protections when the relative cost c_P/c_L is sufficiently high.

4 Conclusion

Based on the model of DG, this paper studies the optimal way to design and protect a network under attack. The first difference with the model of DG is that the Adversary attacks the nodes in the DG's framework while she attacks links in our setting. The second difference is that in DG the objective of the Designer is to guarantee the connectivity of the network after the attack of the Adversary, but the number of surviving nodes does not influence her payoff in their benchmark model. In our setting, nodes are complementary and can only create a positive result if they are all interlinked in the residual network. To summarize, our objective is to design and protect the connections inside a network which consists of separate complementary units.

Our main findings are the following. If the Designer has no constraint on the number of protections, she has two potential strategies that are polar strategies. If the cost of protection is sufficiently lower than the linking cost, the optimal strategy is to protect all the nodes via indestructible links. In this case, the Adversary cannot destroy any link. By contrast, if the linking cost is sufficiently lower than

the protection cost, the optimal strategy is to increase the connectivity of the network by creating many links. More precisely, each node needs a degree of $k + 1$, such that the k attack units of the Adversary are not sufficient to isolate a part of the network. This result is slightly modified when the number of nodes is odd and the number of attacks is even: the strategy where the Designer protects a group of two nodes and forms $(n - 1)(k + 1)/2$ destructible links may be a SPE. We also introduce situations where the Designer is limited in the number of available protections. In this case, for intermediate relative costs (cost of protection/cost of forming links), a strategy that uses both protections and destructible links is the unique SPE.

In this paper, we make three assumptions: (i) the budget of the Adversary is exogenous, (ii) the protection is perfect (protected links are indestructible by definition), and (iii) costs of destructible links and protection are linear.

(i) Let us start with the assumption that Adversary budget is exogenous. If the cost of each attack is a constant, then the result does not change qualitatively. Indeed, the Adversary will choose an optimal amount of attacks given that she will earn 1 if the residual network is disconnected. In other words, in this situation the number of attacks is endogenously obtained, but the problem the Designer has to deal with is not modified.

(ii) Suppose that protections are not perfect. First, it implies that the protected group will not be always a minimal component. In other words, the Designer may have an incentive to form more than p “indestructible” links in order to protect a group that contains $p + 1$ nodes. Second, the strategy \mathbf{g}^{k+1} would become more attractive since the “indestructible” links are less efficient relative to destructible ones.

(iii) Suppose that the costs are not linear. More precisely, suppose for instance that the cost of protections (indestructible links) is convex. In this case, there exist situations where an optimal strategy associated with $p \in \llbracket 1, n - k - 1 \rrbracket$ is a SPE. In other words, the Designer will use both protections and destructive links in a SPE even if the number of protections is not restricted.

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