

Abstract Economies with Endogenous Sharing Rules

Philippe Bich, Rida Laraki

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Philippe Bich, Rida Laraki. Abstract Economies with Endogenous Sharing Rules. 2015. halshs-01179102

HAL Id: halshs-01179102
https://shs.hal.science/halshs-01179102

Submitted on 21 Jul 2015

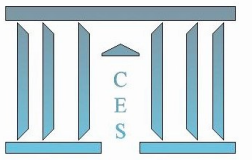
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Philippe BICH, Rida LARAKI

2015.58



Maison des Sciences Économiques, 106-112 boulevard de L'Hôpital, 75647 Paris Cedex 13
http://centredeconomiesorbonne.univ-paris1.fr/
ISSN : 1955-611X

Abstract Economies with Endogenous Sharing Rules

Philippe Bich[*], Rida Laraki[†]

June 27, 2015

Abstract

Endogenous sharing rules was introduced by Simon and Zame [15] to model payoff indeterminacy in discontinuous games. Their main result concerns the existence of a *solution*, i.e., a mixed Nash equilibrium and an associated sharing rule. This note extends their result to abstract economies [1] where, by definition, players are restricted to pure strategies, and provide an interpretation of Simon and Zame's model in terms of preference incompleteness.

Keywords: abstract economies, endogenous sharing rules, competitive equilibrium, incomplete and discontinuous preferences, better-reply security
JEL Classification: C02, C62, C72, D50.

[*]Paris School of Economics, Centre d'Economie de la Sorbonne UMR 8174, Université Paris I Panthéon/ Sorbonne.

[†]CNRS, LAMSADE (Université Paris Dauphine) and Economics Department, Ecole Polytechnique, France.

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1 Introduction

The model of games with endogenous sharing rules was introduced by Simon and Zame [15]. Formally, it is a $(N+1)$-tuple $\mathcal{G} = ((X_i)_{i\in N}, \mathcal{U})$, where X_i is the strategy set[1] of player $i \in N$, and $\mathcal{U}$ is a multivalued function from $X := \prod_{i\in N} X_i$ to $\mathbf{R}^N$ with nonempty values. The set $\mathcal{U}(x) \subset \mathbf{R}^N$ can be interpreted as the universe of payoff possibilities, given the strategy profile $x \in X$. When $\mathcal{U}(x) = \{(u_i(x))_{i\in N}\}$ is a singleton for every $x \in X$, $\mathcal{G}$ reduces to a standard strategic game, u_i being the payoff function of player i. Simon and Zame [15] provide conditions that guarantee the existence of a *solution* for $\mathcal{G}$, i.e., existence of a selection $u = (u_i)_{i\in N}$ of $\mathcal{U}$ (a *sharing rule* of $\mathcal{U}$), together with a mixed Nash equilibrium of the game $G = ((X_i)_{i\in N}, (u_i)_{i\in N})$.

The concept of sharing rules gives rise to many interpretations. Imagine a designer who must determine who wins an indivisible object in some auction including tie-breaking rules. In that case, selections of $\mathcal{U}$ represent admissible auction rules, and a solution can be seen as a mechanism and a Nash equilibrium of the induced game. Another motivation comes from the payoff indeterminacy that many economic models exhibit: for example, several producers have to choose, each, a location in an area where a continuum of consumers are uniformly distributed. Assume each consumer goes to the closest location. Then payoffs are not well defined when some producers choose the same location: indeed, any division of consumers between the producers is plausible. Simon and Zame's result guarantees the existence of a market sharing rule under which the discontinuous game played by the producers admits a mixed Nash equilibrium.

In this note, we prove existence of a Simon and Zame "solution" in abstract economies. This is an equilibrium model, introduced by Arrow and Debreu [1], in which the strategies of a player are constrained by the strategies of his opponents. For example, in exchange economies, consumers are limited by their budget constraint, which depends on the price vector, itself depending on consumers' demands. Observe that mixed equilibria are not relevant in this context: if consumers choose random consumption bundles, then equality

[1] For simplicity, we use the same letter N for the set of players or the number of players.

of supply and demand is, in general, not compatible with the independence of players' strategies.

Our second contribution is an interpretation of sharing rules indeterminacy in terms of *preference incompleteness*. As Aumann [2] argues: "*of all axioms of utility theory, the completeness axiom is perhaps the most questionable*". Following this seminal paper, many extensions of equilibrium models to incomplete preferences have been investigated, either for continuous preferences [8, 13], or discontinuous ones [5, 12, 16]. In this note, we will assume that the ambiguity generated by the indeterminacy of payoffs creates incompleteness in the preferences. This permits to associate to every abstract economy with endogenous sharing rules an abstract economy with incomplete and discontinuous preferences. We prove that, in general, this economy does not possess a Nash equilibrium, but it is possible to complete the preferences in a weak sense to restore the existence of an equilibrium.

2 Abstract Economies with Endogenous Sharing Rules

An abstract economy with endogenous sharing rules $\mathcal{E}$ is a couple $\mathcal{E} = (\mathcal{G}, \mathcal{B} = (\mathcal{B}_i)_{i\in N})$ where $\mathcal{G} = ((X_i)_{i\in N}, \mathcal{U})$ is a game with endogenous sharing rules, and $\mathcal{B}_i$ is a multivalued mapping from X_{-i} to X_i with a closed graph and nonempty convex values (i.e., a Kakutani-type mapping).

Definition 1 *A solution of $\mathcal{E}$ is a couple (x, u), where $u = (u_i)_{i\in N}$ is a selection of $\mathcal{U}$, and $x \in X$ is a generalized Nash equilibrium of $((X_i)_{i\in N}, (u_i)_{i\in N}, \mathcal{B})$, i.e.:*
(i) For every $i \in N$, $x_i \in \mathcal{B}_i(x_{-i})$.
(ii) For every $d_i \in \mathcal{B}_i(x_{-i})$, $u_i(d_i, x_{-i}) \leq u_i(x)$.

Consider the following assumptions:
$A1$: X is a convex and compact subset of a Hausdorff and locally convex topological vector space;

$A2$: $\mathcal{U}$ is bounded;

$A3$: The graph of $\mathcal{E}$, defined by $\Gamma := \{(x, v) : v \in \mathcal{U}(x) \text{ and } x_i \in \mathcal{B}_i(x_{-i}) \text{ for every } i \in N\}$, is closed;

$A4$: $\mathcal{U}$ admits a selection $u = (u_i)_{i \in N}$, such that each u_i is quasiconcave in player i's strategy.

Remark 2 *Simon and Zame proved existence of a solution in mixed strategies in strategic games under $A1$, $A2$, $A3$ and convexity of $\mathcal{U}(x)$ for every $x \in X$.*

Theorem 3 *Any abstract economy with endogenous sharing rules satisfying $A1$ to $A4$ admits a solution*[2]*.*

3 Applications

3.1 Incomplete Preferences

Let us give an interpretation of Theorem 3 in terms of incomplete preferences. If $\mathcal{G} = ((X_i)_{i \in N}, \mathcal{U})$ is a game with endogenous sharing rules, then we can define the following preorders[3] on X.

Definition 4 *We say that $y \in X$ is $\mathcal{U}$-preferable to $x \in X$ for player i, denoted $x \lesssim_i y$, if and only if $u_i(x) \leq u_i(y)$ for every selection*[4] *u of $\mathcal{U}$.*

When x and y are distinct, $x \lesssim_i y$ is equivalent to $\sup \mathcal{U}_i(x) \leq \inf \mathcal{U}_i(y)$, where $\mathcal{U}_i(x)$ denotes the projection of $\mathcal{U}(x) \subset \mathbf{R}^N$ on the i-th component. In short, $x \lesssim_i y$ if and

[2]When $\mathcal{B}_i(x_{-i}) = X_i$ for every $x_{-i} \in X_{-i}$ and every $i \in I$, we get the existence of a solution à la Simon-Zame in pure strategies. This was an open question in Jackson et al. [9] and was solved recently in Bich and Laraki [4].

[3]A preorder is a reflexive and transitive binary relation.

[4]Every preorder $\lesssim$ on X admits a multi-utility representation (see [10]), that is there exists a family $(v_j)_{j \in J}$ of real-valued functions defined on X such that: $x \lesssim y \Leftrightarrow$ for every $j \in J$, $v_j(x) \leq v_j(y)$. Thus, there is no loss of generality in working with a cardinal multi-representation.

only if y is at least as good as x, whatever the indeterminacy of payoffs modelized by $\mathcal{U}$. Formally, to every abstract economy $(\mathcal{G}, (\mathcal{B}_i)_{i\in N})$, one can associate an abstract economy with incomplete preferences $\mathcal{E} = ((X_i)_{i\in N}, (\lesssim_i)_{i\in N}, (\mathcal{B}_i)_{i\in N})$, where the preorders $\lesssim_i$ are derived from $\mathcal{U}$ as described above.

It is then standard to define a generalized Nash equilibrium of $\mathcal{E}$ as a profile $x \in \Pi_{i\in N}\mathcal{B}_i(x_{-i})$ such that there is no player $i \in N$ and no deviation $y_i \in \mathcal{B}_i(x_{-i})$ with[5] $x \lesssim_i (y_i, x_{-i})$. The following example proves that, in general, $\mathcal{E}$ fails to have a generalized Nash equilibrium, even if the initial game $\mathcal{G}$ satisfies assumptions $A1$ to $A4$.

Example 5 *Consider a strategic game with endogenous sharing rules and two players. The strategy spaces are* $X_1 = X_2 = [0,1]$. *The endogenous sharing rules are defined by* $\mathcal{U}(x_1, x_2) = (1 - x_1(1 - x_2), 1 - (1 - x_1 - x_2)^2)$ *if* $(x_1, x_2) \neq (0, 1)$ *and* $\mathcal{U}(0, 1) = \{(-1, 1), (1, 1)\}$. *This satisfies assumption* $A1$ *to* $A4$. *In particular, any selection* u *of* $\mathcal{U}$ *satisfies the quasiconcavity requirement* $A4$. *As described above, this defines a game with incomplete preferences* $\mathcal{E} = ((X_i)_{i=1,2}, (\lesssim_i)_{i=1,2})$. *Clearly, for player 2, the unique best-response to* x_1 *is* $x_2 = 1 - x_1$. *Thus, for every* $x_1 > 0$, (x_1, x_2) *is not a Nash equilibrium of* $\mathcal{E}$, *since it would imply* $x_2 = 1 - x_1 < 1$, *but then the only best-response of player 1 is* $x_1 = 0$, *a contradiction. Thus, the only candidate to be a Nash equilibrium is* $(0, 1)$, *but it is not, since* $(0, 1) \lesssim_1 (\varepsilon, 1)$ *for every* $\varepsilon \in]0, 1]$. *Indeed,* $1 = \sup \mathcal{U}_1(0, 1) \leq \inf \mathcal{U}_1(\varepsilon, 1) = 1$ *and* $1 = \sup \mathcal{U}_1(\varepsilon, 1) > \inf \mathcal{U}_1(0, 1) = -1$. *Hence,* $\mathcal{E}$ *has no Nash equilibria. In particular, it is not generalized correspondence secure (see [7]), a condition that would imply the existence of a Nash equilibrium of* $\mathcal{E}$.

Thus, one cannot apply recent generalized Nash existence results to $\mathcal{E}$ (e.g., Yannelis, He [16] or Carmona and Podzeck [7]) simply because the game may fail to have a Nash equilibrium. We now study the possibility of restoring existence after some completion of the preferences. Recall that a completion of the preorder $\lesssim_i$ defined on X is a total order $\lesssim'_i$ on X such that:

[5]Here, $\lesssim_i$ denotes the strict preorder associated to $\lesssim_i$, that is: for every $(x, y) \in X^2$, $x \lesssim_i y$ if and only if $x \lesssim_i y$ and not $(y \lesssim_i x)$.

(i) $\forall (x,y) \in X^2$, $x \lesssim_i y \Rightarrow x \lesssim'_i y$;

(ii) $\forall (x,y) \in X^2$, $x \lesssim\!\!\!\!\!\!\sim_i y \Rightarrow x \lesssim\!\!\!\!\!\!\sim'_i y$.

When the preorders $\lesssim_i$, $i \in N$, are defined from $\mathcal{U}$ as above, then for every selection u of $\mathcal{U}$, one can define a u-completion of $\lesssim_i$ as the total order $\lesssim_i^u$ on X such that: $x \lesssim_i^u y \Leftrightarrow u_i(x) \leq u_i(y)$. This is a weak completion of $\lesssim_i$, in the sense that it satisfies property (i) but not property (ii). This is because $x \lesssim\!\!\!\!\!\!\sim_i y$ is defined by: $v_i(x) \leq v_i(y)$ for every selection v of $\mathcal{U}$, and $w_i(x) < w_i(y)$ for at least one selection of $\mathcal{U}$, and this may not imply[6] $u_i(x) < u_i(y)$.

Corollary 6 *Consider an abstract economy with endogenous sharing rule $(\mathcal{G}, (\mathcal{B}_i)_{i\in N})$ which satisfies assumptions A1 to A4, and let $\lesssim_i$ be the preorders associated to $\mathcal{U}$ as described above. Then there exists u-completions $\lesssim_i^u$ of the preorders $\lesssim_i$ $(i \in N)$ for some selection u of $\mathcal{U}$, such that $((X_i)_{i\in N}, (\lesssim_i^u)_{i\in N}, (\mathcal{B}_i)_{i\in N})$ has a generalized Nash equilibrium $\bar{x} \in X$.*

3.2 Discontinuous Abstract Economies

To every discontinuous abstract economy, we can associate endogenous sharing rules, and thus incomplete preferences, to which we can apply Theorem 3. Indeed, consider an

[6]To avoid this problem, we could strengthen the definition of the preorders $\lesssim\!\!\!\!\!\!\sim_i$ as follows: say that $y \in X$ is $\mathcal{U}$-strongly preferable to $x \in X$ for player i, denoted $x \ll_i y$, if and only if $u_i(x) < u_i(y)$ for every selection u of $\mathcal{U}$. This leads to a more restrictive notion of profitable deviation, thus to a weaker notion of Nash equilibrium. Under assumptions A1 to A4, the existence of a Nash equilibrium is a direct consequence of Shafer-Sonnenschein's Theorem (see [14]). Indeed, denote by $P_i(x)$ the set of strategies of player i strictly preferred to x_i given the strategies of the other players. If $y_i \in P_i(x)$, then $\sup \mathcal{U}_i(x) < \inf \mathcal{U}_i(y_i, x_{-i})$. Since $\mathcal{U}$ has a closed graph, we get $\sup \mathcal{U}_i(x') < \inf \mathcal{U}_i(y'_i, x'_{-i})$, that is $x' \ll (y'_i, x'_{-i})$, for every (x', y'_i) in some neighborhood of (x, y_i). This proves that P_i has an open graph. Moreover, $x_i \notin co P_i(x)$ for every $i \in I$. Indeed, otherwise, x_i is in the convex hull of a finite family of strategies $(x_i(k))_{k=1}^K$ with $x_i(k) \in P_i(x)$, that is, $u_i(x_i(k), x_{-i}) > u_i(x)$ for every selection u of $\mathcal{U}$. In particular, no selection u of $\mathcal{U}$ can be quasiconcave with respect to x_i, which contradicts assumption $A4$. Consequently, we can apply Shafer-Sonnenschein's Theorem to get the existence of a Nash equilibrium in the game defined by the preorders $\ll_i$.

abstract economy $(G = ((X_i)_{i\in N}, (u_i)_{i\in N}), \mathcal{B})$ with bounded payoff functions u_i, quasiconcave with respect to x_i for all $i \in N$. For every profile $y \in X$, define $\mathcal{U}(y)$ to be the set of limits of $(u(y^n))_{n\in\mathbb{N}}$ for all possible sequences $(y^n)_{n\in\mathbb{N}}$ converging to y and such that $y_i^n \in \mathcal{B}_i(y_{-i}^n)$ for all $i \in N$. Following the previous subsection, a strategy profile $y \in X$ is $\mathcal{U}$-preferable to another strategy profile $x \in X$ for player i if $u_i(x) \leq u_i(y)$ for every selection u of $\mathcal{U}$: in short, it means that player i prefers y to x, even after some small modifications of the strategy profiles x and y. Example 5 proves that these incomplete preferences are too restrictive to permit the existence of a generalized Nash equilibrium. But, from Corollary 6 (and since $\mathcal{U}$ satisfies all the assumptions $A1$ to $A4$), there is a q-completion of these preferences for which the new economy $(G' = ((X_i)_{i\in N}, (q_i)_{i\in N}), \mathcal{B})$ has a generalized Nash equilibrium. Formally, q satisfies: for every $y \in X$, there is a sequence $(y^n)_{n\in\mathbb{N}}$ converging to y such that $y_i^n \in \mathcal{B}_i(y_{-i}^n)$ for every $i \in N$ and such that $q(y) = \lim_{n\to+\infty} u(y^n)$. This extends Theorem 2 in [4].

3.3 Exchange Economies

Theorem 3 can be applied to exchange economies as follows. Consider n consumers and m commodities. Consumer i's consumption set X_i is a nonempty convex and compact subset of $\mathbf{R}_+^m$. The initial endowment e_i of consumer i is assumed to be in the interior of X_i. Following the interpretation of subsection 3.1, consumer's incomplete preferences are assumed to be represented by a multivalued function[7] $\mathcal{U}_i$ from X_i to $\mathbf{R}_+$ with a closed graph, nonempty bounded values and a quasiconcave selection[8]. Under these assumptions, theorem 3 implies that there exists[9] a selection u_i of $\mathcal{U}_i$ $(i \in N)$ such that the economy $\{X_i, u_i, e_i\}_{i\in N}$ admits a Walrasian equilibrium $(x, p) \in \Pi_{i\in N} X_i \times \Delta(\mathbf{R}_+^m)$, that is

[7]Here, to simply the exposition, we do not allow externalities, that is $\mathcal{U}_i$ depends only of player i's strategies.

[8]For example, let us assume that for every $i \in N$, for every $v_i \in \mathcal{U}_i(x)$, where x is the convex combination of a finite number of points $x^1, ..., x^K$ of X, there is $u^k \in \mathcal{U}_i(x^k)$ $(k = 1, ..., K)$ such that $v \geq \min\{u^1, ..., u^K\}$. Then, each $u_i(x) = \min \mathcal{U}_i(x)$ defines a selection of $\mathcal{U}_i$, and u_i is quasiconcave in player i's strategy.

[9]In the theorem, $\Delta(\mathbf{R}_+^m)$ denotes the unit simplex of $\mathbf{R}_+^m$.

(1) $\sum_{i\in N} x_i = \sum_{i\in N} e_i$, and
(2) x_i maximizes the utility function u_i of agent i on his budget set $B_i(p) = \{y \in X_i : p \cdot (y_i - e_i) \leq 0\}$.

The proof is as follows. Consider the following $(N+1)$-player abstract economy with endogenous sharing rules: for $i = 1, ..., N$, player i's strategy space is X_i. The strategy space of player $(N+1)$ (called the auctioneer) is $X_{N+1} = \Delta(\mathbf{R}^m_+)$, and his payoff function is $v_{N+1}(x,p) = p.\sum_{i\in N}(x_i - e_i)$. The payoff correspondence is defined by $\mathcal{V}(x,p) = \prod_{i\in N}\mathcal{U}_i(x_i) \times \{v_{N+1}(x,p)\}$. Last for every $i \in N$, $\mathcal{B}_i(x_{-i},p) = \{x_i \in X_i : p \cdot x_i \leq p \cdot e_i\}$, and finally $\mathcal{B}_{N+1}(x_{-i},p) = X_{N+1}$. This defines an abstract economy $(\mathcal{G},\mathcal{B})$. From Theorem 3, it has a solution (x,p,u). By definition of a solution, (x,p) is a Walrasian equilibrium of $\{X_i,u_i,e_i\}_{i\in N}$.

4 Appendix: Proof of Theorem 3

By assumption, $\mathcal{U}$ admits a single-valued selection $v = (v_i)_{i\in I}$ where each v_i is quasiconcave in player i's strategy. Following an idea of Reny [11], we associate to the abstract economy $(G = ((X_i)_{i\in N}, v), \mathcal{B})$ a strategic game G' as follows. Because $\mathcal{U}$ is bounded, there exists $\Lambda \in \mathbb{R}$ such that $v_i(x) \geq \Lambda + 1$ for every $i \in N$ and every profile $x \in X$. The game G' has N players. For every $i \in N$, strategy set of player i is X_i, and his payoff is

$$u_i(x) = \begin{cases} v_i(x) & \text{if } x_i \in \mathcal{B}_i(x_{-i}), \\ \Lambda & \text{otherwise.} \end{cases}$$

The proof of Theorem 3 rests on the claim below. Throughout this proof, for every $i \in N$, $x \in X$, and U in $\mathcal{V}(x_{-i})$ (the set of open subsets of X_{-i}), denote by $W_U(x_i,x_{-i})$ the set of Kakutani-type[10] multivalued mappings d_i from U to X_i such that $x_i \in d_i(x_{-i})$. Let $\underline{\underline{u_i}} : X \to \mathbf{R}$ be the following regularization[11] of the utility function u_i

(1) $\quad \underline{\underline{u_i}}(x) := \sup_{U\in\mathcal{V}(x_{-i})} \sup_{d_i\in W_U(x)} \inf_{x'_{-i}\in U, x'_i\in d_i(x'_{-i})} u_i(x')$.

[10] A Kakutani-type multivalued mapping is a multivalued mapping with nonempty convex values and a closed graph.

[11] This function was introduced by Carmona (see [6]).

Remark that $\underline{\underline{u_i}}(x) \leq u_i(x)$ for every $x \in X$, since in the infimum above one can take $x' = x$.

Claim. There exists some new payoff functions $(q_i)_{i\in I}$ and a pure Nash equilibrium $x \in X$ of $G'' = ((X_i)_{i\in N}, (q_i)_{i\in N})$, with the additional properties:

(i) for every i and $d_i \in X_i$, $q_i(d_i, x_{-i}) \geq \underline{\underline{u_i}}(d_i, x_{-i})$.

(ii) For every $i \in N$ and every $y \in X$, there exists some sequence (y^n) converging to y such that $u(y^n)$ converges to $q(y)$.

Proof of the Claim. Let $\Gamma := \{(x, u(x)) : x \in X\}$. Recall that G' is generalized better-reply secure (Barelli and Meneghel [3]) if whenever $(x, v) \in \overline{\Gamma}$ and x is not a Nash equilibrium, there exists a player i and a triple (ϕ_i, V_x, α_i), where V_x is an open neighborhood of x, ϕ_i is a Kakutani-type multivalued function from V_x to X_i and $\alpha_i > u_i$ is a real, such that for every x' in V_x such that $x'_i \in \phi_i(x')$, one has $u_i(x'_i, x'_{-i}) \geq \alpha_i$.

Let us prove that there exists a couple $(x, v) \in \overline{\Gamma}$ such that:

$$(2) \qquad \forall i \in N, \sup_{d_i \in X_i} \underline{\underline{u_i}}(d_i, x_{-i}) \leq v_i.$$

When u is continuous, this implies $(x, v) = (x, u(x))$, and it is simply the definition of a Nash equilibrium. By contradiction, assume that there is no such couple, and let us prove that G' is generalized better-reply secure. For, consider $(x, v) \in \overline{\Gamma}$ such that x is not a Nash equilibrium. By assumption, (x, v) does not satisfy Equation 2, thus there exists some player $i \in N$ such that $\sup_{d_i \in X_i} \underline{\underline{u_i}}(d_i, x_{-i}) > v_i$. From the definition of $\underline{\underline{u_i}}$, there is $\varepsilon > 0$, $U \in \mathcal{V}(x_{-i})$, $\phi_i \in W_U(x)$ such that for every $x'_{-i} \in U$ and every $x'_i \in \phi_i(x'_{-i})$, $u_i(x'_i, x'_{-i}) \geq v_i + \varepsilon$: this implies generalized better-reply security. Consequently, from Barelli and Meneghel [3], since G' is generalized better-reply secure, it admits a Nash equilibrium. But this is a contradiction, since if $x \in X$ is a Nash equilibrium, $(x, u(x))$ satisfies Equation 2 (because $\underline{\underline{u_i}}(x) \leq u_i(x)$ for every $x \in X$). This proves the existence of $(x, v) \in \overline{\Gamma}$ satisfying Equation 2.

Now, for every $i \in N$, denote by $\underline{\underline{\mathcal{S}_i}}(x)$ the space of sequences $(x^n)_{n\in\mathbf{N}}$ of X converging

to x such that $\lim_{n\to+\infty} u_i(x^n) = \underline{\underline{u_i}}(x)$. Then, define $q : X \to \mathbf{R}^N$ by

$$q(y) = \begin{cases} v & \text{if } y = x, \\ \text{any limit point of } u(x^n)_{n\in\mathbf{N}} & \text{if } y = (d_i, x_{-i}) \text{ for some } i \in N, d_i \neq x_i, (x^n)_{n\in\mathbf{N}} \in \underline{\underline{\mathcal{S}_i}}(d_i, x_{-i}), \\ q(y) = u(y) & \text{otherwise.} \end{cases}$$

Since $(x, v) \in \overline{\Gamma}$, and by definition of q, condition (ii) of the claim above is satisfied at x. Clearly, by definition, it is also satisfied at every y different from x for at least two components, and finally also at every (d_i, x_{-i}) with $d_i \neq x_i$ (for some $i \in N$), from the definition of $q(d_i, x_{-i})$ in this case. Condition (i) is true at x or at every y different from x for at least two components (from $\underline{\underline{u_i}} \leq u_i$), is true at every (d_i, x_{-i}) with $d_i \neq x_i$ by definition, and is finally true at x from Equation 2. This ends the proof of the claim.
Now, we finish the proof of Theorem 3. Take $d_i \in \mathcal{B}_i(x_{-i}) \neq \emptyset$. For every x'_{-i} in some neighborhood of x_{-i} and every $x'_i \in \mathcal{B}_i(x'_{-i}),$, we have $u_i(x'_i, x'_{-i}) = v_i(x'_i, x'_{-i}) \geq \Lambda + 1$. since $\mathcal{B}_i$ is a Kakutani-type mapping, this implies, by definition, $\underline{\underline{v_i}}(d_i, x_{-i}) \geq \Lambda+1$. Thus, from condition (i) of the claim, we get

$$(3) \quad \forall d_i \in \mathcal{B}_i(x_{-i}),\ q_i(d_i, x_{-i}) \geq \underline{\underline{v_i}}(d_i, x_{-i}) \geq \Lambda + 1$$

Since x is a Nash equilibrium of G'', we have:

$$\forall i \in N,\ q_i(x) \geq \sup_{d_i\in X_i} q_i(d_i, x_{-i}) \geq \Lambda + 1.$$

From condition (ii) of the claim, there is a sequence (x^n) converging to x such that $u(x^n)$ converges to $q(x)$. Since $q_i(x) \geq \Lambda + 1$ for every $i \in N$, we cannot have $u_i(x^n) = \Lambda$ for n large enough. Consequently, $u_i(x^n) = v_i(x^n)$ and $x_i^n \in \mathcal{B}_i(x_{-i}^n)$ for n large enough. Passing to the limit, we get $x_i \in \mathcal{B}_i(x_{-i})$ for every $i \in I$ (because $\mathcal{B}_i$ has a closed graph), and also $v_i(x) = u_i(x)$ (from the definition of v_i). A similar argument can be applied to any $(y_i, x_{-i}) \in X$ for which $y_i \in \mathcal{B}_i(x_{-i})$: there is a sequence (x^n) converging to (y_i, x_{-i}) such that $u(x^n)$ converges to $q(y_i, x_{-i})$. Since $q_i(y_i, x_{-i}) \geq \Lambda + 1$ (from Equation 3), we cannot have $u_i(x^n) = \Lambda$ for n large enough. Consequently, $u_i(x^n) = v_i(x^n)$ and $x_i^n \in \mathcal{B}_i(x_{-i}^n)$ for n large enough. In particular, since v is a selection of U and since U has

a closed graph, we get

$$(4) \quad \forall y_i \in \mathcal{B}_i(x_{-i}),\ q(y_i, x_{-i}) \in U(y_i, x_{-i}).$$

Now, define $\tilde{q}(y_i, x_{-i}) = q(y_i, x_{-i})$ whenever $y_i \in \mathcal{B}_i(x_{-i})$ for some $i \in N$, and $\tilde{q}(y) = v(y)$ elsewhere. The proof that x is a equilibrium of $((X_i)_{i\in N}, (\tilde{q}_i)_{i\in N}, B)$ is straightforward. Last, we have to prove that $\tilde{q}(y) \in U(y)$ for every $y \in X$. This is clear at $y = (y_i, x_{-i})$ for $y_i \in \mathcal{B}_i(x_{-i})$, because of Equation 4. Otherwise, $\tilde{q}(y) = v(y) \in U(y)$ by definition of v. This ends the proof of Theorem 3.

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