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Abstract

In response to the Great Recession, Central Banks around the world adopted "unconventional" monetary policies. In particular, the Fed, and more recently the ECB, launched massive debt monetization programs. In this paper, we develop a formal analysis of the short- and long-run consequences of deficit and debt monetization, through an endogenous growth model in which economic growth interacts with productive public expenditure. This interaction can generate two positive balanced growth paths (BGP) in the long-run: a high BGP and a low BGP, and further, depending on the form of the CIA constraint, possible multiplicity and indeterminacy. Thus, monetizing deficits is found to be remarkably powerful. First, a large dose of monetization might allow avoiding, whenever present, BGP indeterminacy. Second, monetization always allows increasing growth and welfare along the (high) BGP, by weakening the debt burden in the long-run. Third, with a CIA on consumption only, monetization provides a rationale for deficits in the long-run: for high degrees of monetization, the impact of deficits and debts on economic growth and welfare becomes positive in the steady-state.

1. Introduction

The Great Recession shaped the conduct of monetary and fiscal policies in an unprecedented way. On the monetary side, on October the 8$^{\text{th}}$ 2008 several major Central Banks (Canada, England, ECB, Fed, Sweden, and Switzerland) implemented a coordinated reduction in policy interest rates, in an attempt for providing a global monetary easing. On the fiscal side, governments of most developed countries launched massive debt-financed spending programs, which might have generated explosive debt paths: for example, the average debt-to-GDP ratio in the United States rose from 65% in 2007 to around 102% in 2014, and from 66% to 91% in the Eurozone.

However, these efforts failed to restore economic growth, which remains mild in most developed countries, and particularly in the Eurozone. In a context of high public debt burden, the space for further increases in public spending financed by fiscal deficits is limited. Besides, given the decline of interest rates close to the zero bound, interest-rate-based policies are ineffective. Thus, policymakers need to draw upon alternative policies. So far, only monetary policy seems to have met this challenge: given the presence of historically low rates, many Central Banks recently launched "unconventional" policies, particularly in the form quantitative easing. For example, while the Fed bought an unprecedented amount of public and private debt to keep rates low,[1] the ECB collected the equivalent of 210 billion euros (around 2.2% of the Eurozone GDP) in bonds, mainly as collateral in refinancing facilities. Moreover, the statement of Mario Draghi (President of the ECB, on July 26th, 2012) that *"the ECB is ready to do whatever it takes to preserve the euro"*, suggested at that time that the ECB could buy Eurozone sovereign debt, despite its institutional arrangement prohibiting monetizing debt or directly buying Government bonds. Such a decision was indeed adopted on January 23rd 2015, as the ECB decided for 1.1 trillion bond-buying.

Consequently, the age of Central Banks "independence" (from fiscal policy) and of monetary policy isolationism seems to be over (see, e.g. Taylor, 2012). Even if the question of monetizing public debt and deficits never has disappeared from the theoretical perspective,[2] this question comes back into the spotlight of the policy debate, fueled by several discussions.

First, as regards current economic conditions, several Eurozone countries experience very high public debt ratios (Greece and Italy) or rapidly growing debt paths (Ireland and Spain, and, to some extent, France). This raised concerns about their capacity to continue servicing their debts, particularly in an environment of low economic growth.[3] As such, Paris & Wyplosz (2014) concluded that the only realistic way for fiscally-sunk (i.e. debt-to-GDP ratios above 100%) Eurozone countries to escape default is to sell monetized debt to the ECB. According to these authors, the ECB should *"bury the debt forever"*, by restructuring 50% of sovereign debts, a plan that they qualify to be *"politically acceptable"*.[4] Effectively, in history, the liquidation of public debt mainly occurred through defaults or monetization (Reinhart & Rogoff, 2008).

Second, beyond the issue of restructuring the existing stocks of public debt, a related question concerns the monetization of new sovereign debts, namely public deficits. In this respect, as regards the institutional design of the Eurozone, monetization is advocated within institutional reforms packages. According to De Grauwe (2013) *"Ideally, the euro zone would combine a symmetrical budget policy with debt monetization by the ECB"*, in such a manner that low-deficit countries like Germany would run a more expansionary

[1] Between 2008 and 2013, the Fed diverted around 1 trillion USD for buying US government bonds (however, the extent to which this could be considered as debt monetization is subject to debate, see Thornton, 2010, or Andolfatto & Li, 2013).

[2] The proponents of the "fiscal theory of the price level", for example, show that, if current public debt is fully covered by money creation, namely, by seigniorage revenues, the general level of prices is not independent of the stock of public debt (see Aiyagari & Gertler, 1985, Leeper, 1991, or Woodford, 1994). In this paper, the price level will be determined by the quantity of money, but the inflation rate will closely be related to debt emissions, as in Sargent & Wallace (1981).

[3] From a wider perspective, the crisis fractured the Eurozone between "North" and "South" countries, and observers now insist on the need for debt monetization, not only in periphery countries but also in larger ones. Indeed, even if Greece could be bailed out by other Eurozone countries, this would not be feasible for the much larger public debts of Italy, Spain, or France.

[4] In addition, Giavazzi & Tabellini (2014) state that *"fiscal expansion without monetary easing would be almost impossible, because public debt in circulation is already too high in many countries"*, and defend the issuance of 30-years maturity public debt to be financed by the ECB. Relatedly, several recent contributions revisit debt monetization from the perspective of the famous "helicopter money" parabola, as coined by Friedman (1948, 1969). Reichlin et al. (2013), Turner (2013) and Buiter (2014) all make the case for debt monetization.

fiscal policy and share the burden of the adjustment in the periphery of the Eurozone.

Third, the main argument against deficit monetization, namely the additional inflation it may create, (following Sargent & Wallace, 1981's "unpleasant monetarist arithmetic") does not seem to be so relevant in depressed economies and in the presence of liquidity traps that disconnect inflation from the money stock. As such, deficit monetization could affect inflation only marginally,[5] and would even allow avoiding deflation and its possibly harmful effects on growth (see Muellbauer & Aron, 2008, or Blanchard, 2014).

However, to the best of our knowledge, there is no theoretical work for assessing the impact of such a public deficit monetization on economic growth in the short- and in the long-run. For example, in a DSGE model, Gali (2014) finds that a money-financed fiscal stimulus improves output in the short-run, if rigidities are at work. In this case, money-financed public spending might dominate debt-financed public spending as regards their short-run growth and welfare effects. Nevertheless, his analysis is limited to wasteful exogenous public spending and focuses on the short-run, without formally studying deficit monetization. Another example is Buiter (2014), who suggests in an OLG model that in high-debt contexts monetizing a fiscal stimulus might be the preferred strategy, but without precisely address the question of deficit monetization.

To establish more formal results on the long- and short-run effects of monetization, we study in this paper the impact of monetizing public deficits in an endogenous growth model with permanent public indebtedness. In order to give a role for public expenditure, we model endogenous growth from the canonical model of Barro (1990) with public spending entering the production function as a flow of "productive" services. In addition, we introduce a general budget constraint for the Government, in which public expenditures can be financed both by taxes and public indebtedness. To introduce money, we rest on a standard "cash-in-advance" (hereafter CIA) specification, in which the demand for money is generated by the need of a liquid asset to finance either consumption goods only or both consumption and investment goods. Contrary to the usual assumption, in which money supply is modeled as exogenous, we suppose that money creation is proportional to fiscal deficits. This allows analyzing the impact of the degree of deficit-monetization, which in the long-run corresponds to monetizing a share of public debt.

Our findings are twofold. First, as regards the balanced growth path (hereafter BGP), our model exhibits a multiplicity of steady-states, namely, a high BGP and a low BGP, with the following intuitive explanation. The rate of economic growth positively depends on public expenditure, which increases the marginal productivity of private capital. In addition, public expenditure is an increasing function of economic growth in the Government budget constraint, because growth allows reducing the debt burden in the long-run. This dual positive interaction between economic growth and public expenditure generates multiplicity. High economic growth allows reducing the impact of the debt burden, and boosts growth-enhancing productive expenditure. However, low economic growth gives rise to an increase in public debt, with an associated crowding-out effect on productive public spending, which, in turn, has an adverse effect on growth. Moreover, we show that the low BGP positively depends on deficits, whatever the degree of monetization. On the contrary, the high BGP is negatively affected by public deficits if monetization is "low",

[5] Turner (2013) defends debt monetization on the grounds of current non-inflationary growth perspectives.

but can be positively affected if monetization is "high". Furthermore, independently of the effect of deficits, monetization is shown to increase economic growth and welfare in the long-run along the high BGP.

Second, as regards transitional dynamics, results change dramatically depending on the nature of the CIA constraint. If only consumption is subject to the constraint, the high BGP is saddle-point stable and the low BGP is unstable, so that multiplicity can be removed, regardless the degree of monetization. However, if the CIA holds for both consumption and investment, the low BGP becomes saddle-point stable, and the high BGP is undetermined for "low" levels of pmonetization, but saddle-point stable for "high" levels of monetization. In this case, the model conducts to multiplicity *and* indetermination of BGPs. Consequently, in terms of economic policy, our model calls for a strong monetization of public debt and deficits, both to increase economic growth and welfare along the BGP, and to avoid short-run indeterminacy of economic growth paths.

Our model can be seen as unifying two strands of literature. On the one hand, in the current context of large and increasingly debt ratios, a considerable amount of theoretical and empirical work focused on analyzing growth and welfare effects of debt and deficits. In endogenous growth settings, but without productive services of public expenditure, Saint-Paul (1992) and Futagami & Shibata (1998) find that higher debt and deficits are harmful to economic growth. On the theoretical side, these findings have been extended by Minea & Villieu (2010, 2012), who show that, even if deficits are devoted to productive expenditure, long-run economic growth is worsened by the presence of public debt, because the crowding-out effect of the debt burden always outmatches the increase in public spending authorized by the deficits along the BGP. Nevertheless, as shows our present results, monetization of public debt or fiscal deficits significantly modifies this result. On the empirical side, previous work established a negative relationship between public debt and economic growth.[6] However, subsequent studies, especially since Reinhart & Rogoff's (2010) controversial findings, suggested that the effect of debt on growth might be subject to threshold effects.[7] Our setup suggests revisiting the debt-growth relationship, which could be crucially altered by fiscal and monetary policies interactions through debt and deficits monetization.

On the other hand, an important strand of literature explores the effects of the form of the money demand on economic growth, based on the classical contributions of Tobin (1965), Clower (1967), Sidrauski (1967), Stockman (1981), and Abel (1985). Capitalizing on the setup of Wang & Yip (1992), Palivos et al. (1993), and Palivos & Yip (1995), who develop CIA endogenous growth models, a large literature emphasized several mechanisms through which the CIA constraint may give rise to multiplicity and/or indeterminacy of BGPs.[8] For example, in an "Ak" model with a CIA money demand, Suen & Yip (2005) show that indeterminacy is caused by a strong intertemporal substitution effect on capital accumulation. Next, Itaya & Mino (2007) emphasize the crucial

[6] See, e.g. Patillo et al. (2002) and Clements et al. (2003) for developing countries, and Kumar & Woo (2010) and Chudik et al. (2013) for samples mixing advanced and developing countries.

[7] Such threshold effects are defended by Minea & Parent (2012), Baum et al. (2013), Pescatori et al. (2014), and Egert (2015) in developed countries, and by Eberhardt & Presbitero (2013) and Kourtellos et al. (2013) in developing countries.

[8] The possibility of indeterminacy in dynamic equilibrium models has been largely explored in the literature (for a survey of the mechanisms that can give rise to indeterminacy, see, e.g. Benhabib & Farmer, 1999).

role of technology and preferences for multiplicity. Moreover, when the CIA only partially affects consumption, Bosi & Magris (2003), in a one-sector, and Bosi et al. (2010), in a two-sector economy, show that indeterminacy and multiplicity can occur. Finally, Bosi & Dufourt (2008) and Chen & Guo (2008) highlight that the form of the CIA, and specifically the extent to which it affects investment, is the key factor in generating indeterminacy. By studying debt and deficits monetization, our model extends these works in several dimensions. First, we introduce a role for government spending, through productive services of public expenditures. Second, we relax an important assumption of this literature, namely the presence of a balanced budget rule.[9] Third, by accounting for the possibility of debt monetization, we go beyond the recent analysis of Futagami et al. (2008) and Minea & Villieu (2012), who discuss the issue of multiplicity and indeterminacy of the BGP in models with public investment and debt.

Section 2 presents the model, Section 3 describes the long-run solution and presents some preliminary results on the effect of deficit and monetization. Section 4 studies the model with a CIA constraint on consumption only. Section 5 discusses the way indeterminacy can occur in the model with a CIA constraint on consumption and investment. Section 6 concludes the paper.

2. The model

We consider a continuous-time endogenous growth model describing a closed economy populated with a private sector and a Government.

2.1. The private sector

The private sector consists of a producer-consumer infinitely-lived representative agent, who maximizes the present value of a discounted sum of instantaneous utility functions based on consumption $c_t > 0$, with $\rho > 0$ the discount rate and $S \equiv -u_{cc}c_t/u_c > 0$ (with $u_c \equiv du(c_t)/dc_t$) the consumption elasticity of substitution

$$U = \int_0^\infty u(c_t)\exp(-\rho t)dt, u(c_t) = \begin{cases} \frac{S}{S-1}\left\{(c_t)^{\frac{S-1}{S}} - 1\right\}, & for\ S \neq 1 \\ \log(c_t), & for\ \ S = 1 \end{cases}. \tag{1}$$

For U to be bounded, we need to ensure that $(S-1)\gamma_c < S\rho$, where γ_x is the long-run growth rate of variable x.[10]

Output is produced with private capital and productive public expenditure g_t

$$y_t = Ak_t^\alpha g_t^{1-\alpha}. \tag{2}$$

All variables are per capita and population is normalized to unity. $0 < \alpha < 1$ is the elasticity of output to private capital. Public expenditure provides "productive services", and as such enters the production function, with an elasticity $1-\alpha$ (as in Barro, 1990).

[9] In a setup with no debt nor seigniorage, Palivos et al. (2003) and Park & Philippopoulos (2004) show that endogenous public investment can lead to both multiplicity and indeterminacy of BGPs.

[10] This condition corresponds to a no-Ponzi game constraint $\gamma_c < r$, with r the real interest rate to be defined below.

Household's budget constraint is (we define $\dot{x}_t \equiv dx_t/dt, \forall x_t$)

$$\dot{k}_t + \dot{b}_t + \dot{m}_t = r_t b_t + (1-\tau)\, y_t - c_t - \delta k_t - \pi_t m_t. \tag{3}$$

The Household uses her net income ($(1-\tau)\, y_t$, with τ a flat tax rate on output) to consume (c_t) and to invest ($\dot{k}_t + \delta k_t$, with δ the rate of private capital depreciation). In addition, the Household can buy government bonds (b_t), which return the real interest rate r_t, and hold money. All variables are defined in real terms (i.e. deflated by the price level) and $\pi_t m_t$ represents the "inflation tax" on real money holdings.

To motivate a demand for real balances, we also suppose that the Household is subject to the following cash-in-advance (CIA) constraint[11]

$$\phi^c c_t + \phi^k \left(\dot{k}_t + \delta k_t\right) = m_t, \tag{4}$$

where $\phi^c > 0$ and $\phi^k \geq 0$ are parameters reflecting the transaction technology. If $\phi^k = 0$ the CIA constraints holds for consumption only, while it also affects investment if $\phi^k > 0$ (see Stockman, 1981).

2.2. Monetary and fiscal authorities

The Government provides productive public expenditure, levies taxes on output, and borrows from the Household. He also collects the inflation tax on real balances;[12] hence the following budget constraint in real terms

$$\dot{b}_t + \frac{\dot{M}_t}{P_t} = \dot{b}_t + \dot{m}_t + \pi_t m_t = r_t b_t + g_t - \tau y_t \equiv d_t. \tag{5}$$

The budget constraint (5) is an extension of those in Barro (1990) and in Minea & Villieu (2012). Barro (1990) considers only balanced-budget-rules ($g_t = \tau y_t$), while Minea & Villieu (2012) introduce public debt, but without money $\left(\dot{b}_t = r_t b_t + g_t - \tau y_t\right)$. In our model, by using public debt and seigniorage, the Government can make productive expenditure eventually higher than fiscal revenues τy_t. Thus, we define the deficit as d_t. This deficit can be financed either by issuing debt $\left(\dot{b}_t\right)$ or by issuing money $\left(\dot{M}_t/P_t\right)$, with M_t and P_t the money stock and the price level, respectively.

To close the model, we have to specify the instruments available for public finance. First, it must be emphasized that, to obtain an endogenous growth solution, productive public expenditure must be endogenous in the Government budget constraint. In what follows, we suppose that the Government adopts a deficit rule, which specifies a gradual adjustment path of the deficit-to-output ratio to a long-run target. Let $d_{yt} \equiv d_t/y_t$ be the deficit-to-GDP ratio and $\theta \equiv d^*/y^*$ its long-run target, where a star denotes steady-state values. At each period, the deficit ratio evolves according to

$$\dot{d}_{yt} = -\mu \left(d_{yt} - \theta\right). \tag{6}$$

[11] We assume that the nominal interest rate is strictly positive, so the CIA constraint holds as a strict equality.

[12] In our model, high-powered money is the only form of money, so that the Central Bank collects the inflation tax and transfers it to the Government. We ignore possible developments related to the presence of banking and financial sectors.

Thus, the fiscal policy instruments are the flat tax rate (τ), the target for the deficit-to-GDP ratio in the long-run (θ), and the speed of adjustment of current deficit to this target (μ). A low value of the latter parameter describes a "gradualist" strategy (i.e. the speed of adjustment of the deficit ratio is small), and a high value accounts for a "shock therapy" strategy, which gives rise to a faster reduction in the deficit ratio. In addition, monetary authorities must decide the deficit share they accept to monetize. For simplicity, we assume that a fraction $\eta \in [0,1]$ of the deficit is monetized at each instant, namely

$$\dot{m}_t + \pi_t m_t = \eta d_t. \tag{7}$$

It follows that the Government must cover the remaining part of deficit by issuing public debt

$$\dot{b}_t = (1-\eta) d_t. \tag{8}$$

2.3. Equilibrium

By solving Household's program (see Appendix A) we obtain the two following relations

$$\frac{\dot{c}_t}{c_t} = S\left[r_t - \rho - \frac{\phi^c \dot{R}_t}{1+\phi^c R_t}\right], \tag{9}$$

$$\frac{(1-\tau)\,\alpha A\,(g_t/k_t)^{1-\alpha}}{1+\phi^k R_t} - \delta = r_t - \frac{\phi^k \dot{R}_t}{1+\phi^k R_t}. \tag{10}$$

Relation (9) is the usual Keynes-Ramsey rule obtained in standard optimal growth problems. With a CIA constraint on consumption goods ($\phi^c > 0$), the consumption path is affected by the path of the nominal interest rate, which represents a part of the effective cost of consumption. Thus, in periods with increasing (decreasing) nominal interest rates, the growth rate of consumption will be lower (higher) than under the usual Keynes-Ramsey rule. Relation (10) defines the real return of capital. In the absence of the CIA constraint on capital goods, this return is simply the real interest rate (the rate of return of Government bonds). With a CIA requirement on investment $\left(\phi^k > 0\right)$, the return of capital is lower, since it must be deflated by the financing cost $\left(1+\phi^k R_t\right)$, as shown by first term in the LHS of (10). In addition, the nominal interest rate path introduces a wedge between the return of bonds and the return of capital: with a growing nominal interest rate, the return of capital will be lower, as shown by the second term of the RHS of (10).

Since we are interested in an endogenous growth solution, we transform variables into long-run stationary ratios. To do this, we deflate all steady-state growing variables by the capital stock, namely $x_k \equiv x_t/k_t$ (and we henceforth remove time indexes). Thus, the CIA constraint (4) becomes

$$m_k = \phi^c c_k + \phi^k \left(\frac{\dot{k}}{k} + \delta\right), \tag{11}$$

where the path of the capital stock is obtained from the goods market equilibrium

$$\frac{\dot{k}}{k} = y_k - c_k - g_k - \delta, \tag{12}$$

with the production function defined as

$$y_k = A g_k^{1-\alpha}. \tag{13}$$

We extract the deficit-to-capital ratio from the Government budget constraint (5)

$$d_k = r b_k + g_k - \tau y_k, \tag{14}$$

and the behavior of monetary and fiscal authorities (7)-(8) leads to

$$\frac{\dot{m}}{m} = \eta \left(\frac{d_k}{m_k} \right) - \pi, \tag{15}$$

$$\frac{\dot{b}}{b} = (1-\eta) \left(\frac{d_k}{b_k} \right). \tag{16}$$

Assuming Fisher's equation $R \equiv r + \pi$, relations (9)-(16), together with the deficit rule (6), fully characterize the equilibrium of the model.

3. The long-run endogenous growth solution

In our endogenous growth setting, we define a BGP as a path in which consumption, capital, public spending, money, output, public debt, and deficit grow at a common (endogenous) rate $\left(\gamma^* = \dot{c}/c = \dot{k}/k = \dot{m}/m = \dot{b}/b = \dot{d}/d \right)$, while the real ($r^*$) and nominal ($R^*$) interest rates (and, as a consequence, the inflation rate π^*) are constant. Thus, in the steady-state, the real interest rate is defined by

$$r^* = \frac{(1-\tau)\, \alpha A g_k^{*\,1-\alpha}}{1 + \phi^k R^*} - \delta, \tag{17}$$

and the rate of economic growth is simply

$$\gamma^* = S\,(r^* - \rho)\,. \tag{18}$$

In addition, since $d_k^* = \theta y_k^*$, we obtain from (16)

$$(1-\eta)\,\theta A g_k^{*\,1-\alpha} = \gamma^* b_k^*, \tag{19}$$

and, from the definition of deficit in Government's budget constraint (5)

$$r^* b_k^* = (\theta + \tau)\, A g_k^{*\,1-\alpha} - g_k^*. \tag{20}$$

3.1. The effect of deficit and monetization in the long-run: some intuition

Proposition 1. *(Deficits and monetization in the long-run) For a given long-run economic growth (γ^*):*

(i) *any increase in the degree of deficit monetization increases the public expenditure to capital ratio in the long-run;*

(ii) *any increase in the deficit target reduces the public expenditure to capital ratio in the long-run if monetization is small (namely $\eta < \bar{\eta}$), but rises it if monetization is large ($\eta > \bar{\eta}$), where: $\bar{\eta} \equiv (r^* - \gamma^*)/r^* = 1 - \frac{\gamma^*}{\rho+\gamma^*/S} \in]0,1]$.*

Proof.
From (18), (19) and (20), we obtain the public-spending-to-capital ratio in steady-state:

$$g_k^* = \left[(\theta+\tau)A - (1-\eta)\theta A\left(\frac{r^*}{\gamma^*}\right)\right]^{1/\alpha}. \tag{21}$$

Thus, by (21), it is clear that:

(i) $\left.\frac{\partial g_k^*}{\partial \eta}\right|_{\gamma^*} > 0$ and,

(ii) $\left.\frac{\partial g_k^*}{\partial \theta}\right|_{\gamma^*} \geq 0 \Leftrightarrow 1 \geq (1-\eta)\left(\frac{r^*}{\gamma^*}\right) \Leftrightarrow \eta \geq \bar{\eta}$, where $\overline{\eta} \equiv (r^* - \gamma^*)/r^*$.

$\square$

From (21) without deficit ($\theta = 0$), we find the solution of Barro (1990), namely: $g_k^* = (\tau A)^{1/\alpha} \equiv g_k^B$
$.With deficit but no monetization($ $\theta > 0$ and $\eta = 0$), the public spending ratio is lower $\left(g_k^* < g_k^B\right)$, namely $g_k^* = \left[\tau A - \theta A\left(r^* - \gamma^*\right)/\gamma^*\right]^{1/\alpha}$. Since the standard transversality condition ensures that $r^* > \gamma^*$, for the consumption path to be bounded, the public spending ratio is lower with deficits (and no monetization) than under a balanced budget rule. The basic mechanism driving this crowding-out effect is the following. On the one hand, deficits generate a permanent flow of new resources $\left(\dot{b}\right)$. On the other hand, debt generates a permanent flow of new unproductive expenditures (the debt burden rb). In steady-state, the standard transversality condition $\left(r^* > \gamma^* = \dot{b}/b\right)$ means that the latter dominates the former $\left(rb > \dot{b}\right)$, so that any rule that authorizes permanent deficits involves net costs for public finance in the long-run, irrespective of the precise nature of this rule.

But this configuration radically changes if deficit-monetization is authorized. Effectively, in such circumstances, the debt burden can be accommodated by money creation. Suppose, for example, that the deficit is fully monetized ($\eta = 1$ in equation (21)); compared to the balanced-budget-rule used in Barro (1990), taxes are now supplemented by the deficit: $g_k^* = \left[(\theta+\tau)A\right]^{1/\alpha} > g_k^B$ if $\theta > 0$. Intuitively, this is because new resources provided by the deficit are devoted to productive spending, while the (additional) interest burden is financed by issuing new money.

More generally, Proposition 1 shows that (i) the monetization of fiscal deficits allows reducing their crowding-out effect on productive expenditure, and (ii) can even, if large enough, increase the latter. As economic growth positively depends on public expenditure, deficits will impede economic growth in the long-run for small values of monetization, but they will increase it for high values of monetization. However, this result is only preliminary; indeed, in equilibrium $\bar{\eta}$ depends on η (and on other parameters of the model), since γ^* is an endogenous function of parameters, including η. In the following, we present the long-run solution of the model.

3.2. The steady-state

From (17) and (18) we compute the long-run solution of the model

$$\gamma^* = S\left[\frac{(1-\tau)\,\alpha A g_k^{*1-\alpha}}{1+\phi^k R^*} - \delta - \rho\right]. \tag{22}$$

Using (15) with $\pi^* = R^* - r^*$, we obtain in steady-state

$$\eta\theta A g_k^{*1-\alpha} - (\gamma^* - r^* + R^*)\, m_k^* = 0, \tag{23}$$

which combined with (18) yields: $R^* = \rho + \eta\theta A g_k^{*1-\alpha}/m_k^* + \gamma^*(1-S)/S$. By introducing this relation into (22), we get a first implicit relation between γ^* and g_k^*

$$g_k^* = \left\{\left[1+\phi^k\left(\frac{\eta\theta A g_k^{*1-\alpha}}{m_k^*(\gamma^*, g_k^*)} + \rho + \frac{\gamma^*(1-S)}{S}\right)\right]\frac{\left(\frac{\gamma^*}{S}+\rho+\delta\right)}{\alpha A(1-\tau)}\right\}^{\frac{1}{1-\alpha}} \equiv \mathcal{F}(\gamma^*), \tag{24}$$

where the money-to-capital ratio comes from (11) in steady-state

$$m_k^*(\gamma^*, g_k^*) = \left(\phi^c - \phi^k\right)\left[A g_k^{*1-\alpha} - g_k^* - \gamma^* - \delta\right] + \phi^k\left(A g_k^{*1-\alpha} - g_k^*\right).$$

In addition, (18) and (21) provide a second implicit relation between γ^* and g_k^*

$$g_k^* = \left[(\theta+\tau)\,A - (1-\eta)\,\theta A\left(\frac{1}{S}+\frac{\rho}{\gamma^*}\right)\right]^{1/\alpha} \equiv \mathcal{G}(\gamma^*). \tag{25}$$

Relations (24) and (25) allow computing γ^* and g_k^*, and the remaining endogenous variables in the steady-state: $y_k^* = A g_k^{*1-\alpha}$, $b_k^* = (1-\eta)\,\theta y_k^*/\gamma^*$, $c_k^* = y_k^* - \gamma^* - g_k^* - \delta$, $r^* = \rho + \gamma^*/S$, $m_k^* = \left(\phi^c - \phi^k\right) c_k^* + \phi^k\left(y_k^* - g_k^*\right)$, $R^* = \rho + \eta\theta y_k^*/m_k^* + \gamma^*(1-S)/S$, and $\pi^* = R^* - r^*$.

Before studying the effect of deficit and monetization, it can be useful to describes the long-run solution of the model without deficit (and consequently without monetization).

Definition 1. (*The steady-state solutions without public deficit*) *Without public deficit* ($\theta = 0$) *the model gives rise to two solutions: a no-growth solution (that we call the "Solow" solution* $\gamma^S = 0$*) and a positive growth solution (that we call the "Barro" solution* $\gamma^B > 0$*).*

We find the "Solow" solution by putting $\theta = \gamma^* = 0$ in (24), namely: $g_k^S = \left[\left(1+\rho\phi^k\right)(\rho+\delta)/\alpha A(1-\tau)\right]^{\frac{1}{1-\alpha}}$ and $\gamma^S = 0$ (point S in Figure 1 below), and the "Barro" solution by putting $\theta = 0$ in (25), namely: $g_k^B = (\tau A)^{1/\alpha}$, and, from (24), γ^B comes from the following implicit relation[13]

$$\left(1+\phi^k\left[\rho+\frac{\gamma^B(1-S)}{S}\right]\right)\left(\frac{\gamma^B}{S}+\rho+\delta\right) = \alpha A(1-\tau)(\tau A)^{\frac{1-\alpha}{\alpha}}. \tag{26}$$

This solution corresponds to a zero stock of public debt in the steady state $\left(b_k^B = 0\right)$ and is depicted by point B in Figure 1 below.

Intuitively, the no-growth solution comes from the fact that public debt is very high in the long-run.[14] Effectively, with $\theta = 0$, public debt must be constant but not necessarily zero in the long-run. From (20) with $\gamma^B = 0 \Rightarrow r^* = \rho$, we get: $b_k^S = \left(g_k^S\right)^{1-\alpha}\left[\tau A - \left(g_k^S\right)^\alpha\right]/\rho > 0$. The Solow solution results from the fact that the constant level of public debt forces private capital to be constant in the long-run to achieve a constant steady-state b_k^S ratio. Thus, economic growth disappears in steady-state $\left(\gamma^S = 0\right)$.

In what follows we reintroduce deficit and monetization, and we study two configurations of the model, with a CIA constraint on consumption only $\left(\phi^k = 0\right)$ and with a CIA constraint both on consumption and investment $\left(\phi^k > 0\right)$.

4. The model with a CIA constraint on consumption only

We first establish the multiplicity of BGPs in the long-run, before studying their local dynamics.

4.1. Deficits and monetization in the long-run

Proposition 2. *(Multiplicity of BGPs) For $\theta > 0$ and $0 \leq \eta < 1$, two BGPs characterize the long-run solution of the model: a high BGP $\left(\gamma^{*h}\right)$ and a low BGP $\left(0 < \gamma^{*l} < \gamma^{*h}\right)$.*

Proof.
On the one hand, with a CIA constraint on consumption only $\left(\phi^k = 0\right)$, relation (24) is much simpler and becomes independent of the deficit ratio (θ), since

$$g_k^* = \left\{\frac{\left(\frac{\gamma^*}{S}+\rho+\delta\right)}{\alpha A(1-\tau)}\right\}^{\frac{1}{1-\alpha}} \equiv \mathcal{F}(\gamma^*). \tag{24a}$$

Therefore, it is clear that $\mathcal{F} \in C^\infty(\mathbb{R}^*)$, and that $\mathcal{F}$ is an increasing strictly convex function, since $\mathcal{F}'(\gamma^*) = 1/\{(1-\alpha)\mathcal{F}(\gamma^*)\} > 0$, and $\mathcal{F}''(\gamma^*) = \alpha/\{(1-\alpha)\mathcal{F}(\gamma^*)\}^2 > 0$. In addition, $\mathcal{F}(0) = \left[\frac{\rho+\delta}{\alpha A(1-\tau)}\right]^{\frac{1}{1-\alpha}} \equiv g_k^S > 0$, and $\lim_{\gamma^*\to+\infty}\mathcal{F}(\gamma^*) = +\infty$.

[13] We assume a unique solution (this is straightforward if $S = 1$).

[14] Notice that, even with a balanced-budget rule ($r_t b_t + g_t - \tau y_t = 0$ in the long-run), public debt is not necessarily zero in steady-state.

On the other hand, by inverting the relation (25), the second implicit relation ($\mathcal{G}(\cdot)$) between γ^* and g_k^* becomes

$$\gamma^* = \frac{\rho(1-\eta)\theta A}{S[(\theta+\tau)A - g_k^{*\alpha}] - (1-\eta)\theta A} \equiv \tilde{\mathcal{G}}(g_k^*). \tag{25a}$$

By (25a), we notice that $\tilde{\mathcal{G}} \in C^2(]-\infty, \overline{\overline{g}}_k^*[)$, where $\overline{\overline{g}}_k^* \equiv [(\theta+\tau)A - (1-\eta)\theta A/S]^{1/\alpha} > g_k^S = \mathcal{F}(0) \geq 0$,[15] hence, $\overline{\overline{g}}_k^* < (>)\, g_k^B$ if $\eta < (>)\, 1-S$. Besides, $\tilde{\mathcal{G}}$ is a monotonically increasing function on $]-\infty, \overline{\overline{g}}_k^*[$, since $\tilde{\mathcal{G}}'(g_k^*) = \frac{\rho(1-\eta)\theta A\alpha S}{g_k^{1-\alpha}\{S[(\theta+\tau)A - g_k^{*\alpha}] - (1-\eta)\theta A\}^2} > 0$. At last, $\lim\limits_{g_k^* \to -\infty} \tilde{\mathcal{G}}(\gamma^*) = 0$, and $\lim\limits_{g_k^* \to \overline{\overline{g}}_k^*} \tilde{\mathcal{G}}(\gamma^*) = +\infty$.

Finally, as $\overline{\overline{g}}_k^* > g_k^S$, according to Bolzano's theorem, there are two values of γ^*, denoted by γ_1^* and γ_2^*, such as: $\gamma_i^* > 0$, and $(\tilde{\mathcal{G}} \circ \mathcal{F})(\gamma_i^*) = \gamma_i^*$, $i = 1, 2$. As BGPs are obtained at the intersection of (24a) and (25a), we define by $\gamma^{*h} \equiv \max(\gamma_1^*, \gamma_2^*)$ the high BGP solution, and by $\gamma^{*l} \equiv \min(\gamma_1^*, \gamma_2^*)$ the low BGP solution (see Figure 1).

□

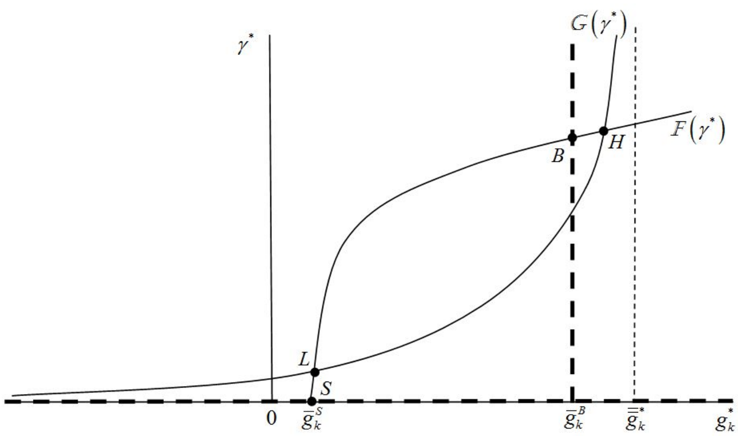


Figure 1: Multiplicity of the BGPs

The intuitive explanation of this multiplicity is the following. Economic growth positively depends on the public-expenditure-to-capital ratio, which increases the marginal productivity of private capital in the Keynes-Ramsey rule (24a). In addition, public expenditure is an increasing function of economic growth in the Government budget constraint (25), because growth allows reducing public debt in the long-run (in the steady-state $b_k^* = (1-\eta)\,\theta y_k^*/\gamma^*$). Consequently, the higher economic growth, the lower the public debt, with an unchanged deficit target (θ). This dual interaction between economic growth and public expenditure generates multiplicity: for the same set of parameters, a high BGP (H) and a low BGP (L) coexist. Effectively, a high growth, by

[15] The case $\overline{\overline{g}}_k^* \leq g_k^S$ is not relevant because there is no equilibrium in this situation.

reducing the debt burden, allows increasing public expenditure, which further enhances growth, while low growth magnifies the crowding-out effect of debt on productive public spending, which, in turn, decreases growth.

Proposition 3. *(The effect of deficits and monetization on BGPs) Any upwards shift in the deficit target (θ):*

(i) *increases the low BGP $\left(\gamma^{*l}\right)$,*

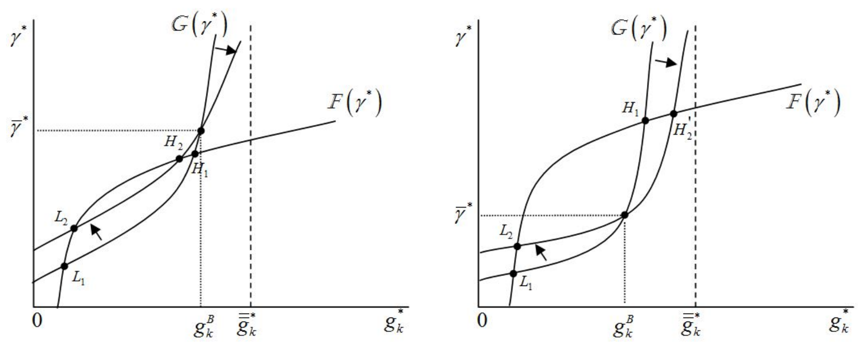
(ii) *reduces the high BGP $\left(\gamma^{*h}\right)$ if monetization is small ($\eta < \overline{\eta}$), but increases the high BGP if monetization is large ($\eta > \overline{\eta}$).*

Proof.
A shift in the deficit target (θ) leaves the $\mathcal{F}(\gamma^*)$ curve unchanged, but, in accordance with Proposition 1, moves the $\mathcal{G}(\gamma^*)$ curve. From (25) we obtain

$$\left.\frac{dg_k^*}{d\theta}\right|_{\gamma^*} = \frac{A}{\alpha}\left[1-(1-\eta)\left(\frac{1}{S}+\frac{\rho}{\gamma^*}\right)\right]g_k^{*1-\alpha} \equiv \left[\frac{\eta-\overline{\eta}}{1-\overline{\eta}}\right]\frac{A}{\alpha}g_k^{*1-\alpha}, \tag{27}$$

where $\overline{\eta} \equiv (r^*-\gamma^*)/r^*$.

Clearly, the $\mathcal{G}(\gamma^*)$ curve pivots around point: $(\bar{\gamma}^*, \bar{g}_k^*)$, with $\bar{\gamma}^* \equiv \frac{S(1-\eta)\rho}{S-1+\eta}$ and $\bar{g}_k^* = g_k^B$, which corresponds to $\eta = \overline{\eta}$, and we have: $\mathrm{Sgn}\{\left.\frac{dg_k^*}{d\theta}\right|_{\gamma^*}\} = \mathrm{Sgn}\left\{(S-1+\eta)(\gamma^*-\bar{\gamma}^*)\right\}$. First, if $\eta \leq 1-S$, since $\bar{\gamma}^* < 0$, $\left.\frac{dg_k^*}{d\theta}\right|_{\gamma^*} < 0$ for any positive value of γ^*. In this case, the $\mathcal{G}(\gamma^*)$ curve moves towards the left for any positive value of γ^*; hence the low BGP increases and the high BGP decreases, since $\mathcal{G}$ is an increasing convex function. Second, if $\eta > 1-S$, $\mathrm{Sgn}\{\left.\frac{dg_k^*}{d\theta}\right|_{\gamma^*}\} = \mathrm{Sgn}\left\{\gamma^*-\bar{\gamma}^*\right\} = \mathrm{Sgn}\left\{\eta-\overline{\eta}\right\}$. Yet, $\mathcal{F}(\gamma^*) = g_k^B$ precisely when $\eta = \overline{\eta}$. Therefore, if monetization is low ($\eta < \overline{\eta}$), the pivot point is above the $\mathcal{F}(\gamma^*)$ curve, as in Figure 2a; whereas if monetization is high ($\eta > \overline{\eta}$), on the contrary, the pivot point is below the $\mathcal{F}(\gamma^*)$ curve, as in Figure 2b. In both cases, an increase in the deficit target raises the low BGP (from L_1 to L_2 in Figure 2) but it reduces the high BGP in the former case (from H_1 to H_2 in Figure 2a) and increases it (from H_1 to $H_2^{'}$ in Figure 2b) in the latter. □



(a): $\eta < \overline{\eta}$ (b): $\eta > \overline{\eta}$

Figure 2: Effect of an increase in the deficit target

However, as we will show below, the low BGP is unstable; thus, we can exclude multiplicity on the basis of dynamic analysis. Consequently, if we consider only the stable high BGP, the optimal degree of deficit-monetization, from the point of view of economic growth, is one (i.e. full monetization).

For small values of the deficit target, we can extract the precise value of monetization that allows a deficit-increase to be growth-enhancing along the high BGP. Effectively, since by construction γ^B is independent from η in Definition 1, we have, for $\theta \to 0$: $\frac{d\gamma^{*h}}{d\theta} > (<) \, 0$ if $\eta > (<) \, \bar{\eta} = 1 - \frac{\gamma^B}{\rho + \gamma^B / S}$.

In addition, simulation-based results show that the same holds from a long-run welfare perspective. On the BGP, Household's welfare (1) becomes[16]

$$U = \begin{cases} (\rho \left[Log\,(c^*) + Log\,(K_0) \right] + \gamma^*)/\rho^2 & if \quad S = 1, \\ S \left\{ \frac{(c^* K_0)^{\frac{S-1}{S}}}{\rho - \gamma^*(S-1)/S} - \frac{1}{\rho} \right\} / (S-1) & if \quad S \neq 1, \end{cases} \tag{28}$$

where K_0 is the initial capital stock (that we normalize to be one in our simulations).

Since a change in the degree of monetization impacts economic growth and consumption in steady-state, namely $c_k^* = y_k^* - \gamma^* - g_k^* - \delta$, its effect on welfare might differ from its effect on growth. However, simulations show that a higher degree of monetization enhances Household's welfare on the high BGP (see Figure 3).[17]

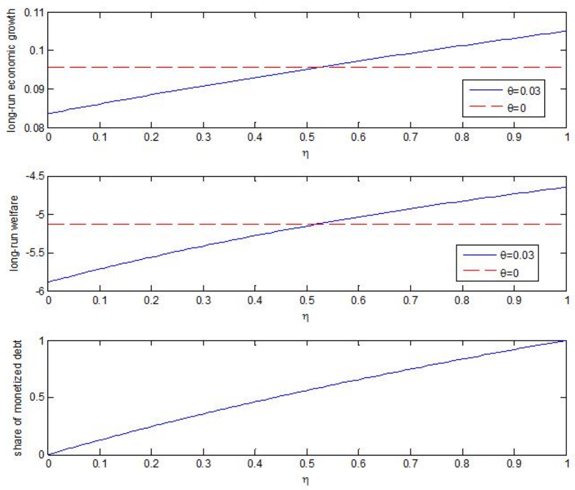


Figure 3: Monetization, Economic Growth and Welfare in the long-run (high BGP)

[16]We focus on steady-state welfare effects, namely we compare different BGPs associated with different values of parameters. In other words, we are not interested in the transition from a steady-state to another, and we do not study transitional dynamics following a change of parameters; thus, we perform comparative statics among different BGPs.

[17]We use standard values of parameters in our benchmark calibration: $\phi^c = S = A = 1$, $\rho = 0.1$ and $\delta = 0$. In particular, the tax rate equals its optimal value in the Barro (1990) model, and the productivity of public capital is the one estimated by Aschauer (1989), namely $\tau = 1 - \alpha = 0.4$.

As shows Figure 3, an increase in the deficit target (say, a target of 3% of GDP, relative to the balanced-budget rule $\theta = 0$) generates welfare losses when monetization is low, but welfare gains if the degree of monetization is high. Thus, the effect of deficit on long-run welfare corroborates our findings for the impact of deficits on economic growth.

4.2. Transitional dynamics

To study transitional dynamics, we compute a reduced form of the model. From (9) with $\phi^k = 0$, the real interest rate equals the net marginal return of capital

$$r = (1-\tau)\,\alpha A g_k^{1-\alpha} - \delta \equiv r(g_k)\,. \tag{29}$$

Then, from the definition of the deficit in Government's budget constraint (5), it comes, with $d_k = d_y y_k = d_y A g_k^{1-\alpha}$

$$b_k = \frac{(d_y+\tau)\,A g_k^{1-\alpha} - g_k}{r(g_k)} \equiv b(d_y, g_k)\,. \tag{30}$$

Finally, by defining

$$\gamma_k = A g_k^{1-\alpha} - c_k - g_k - \delta \equiv \gamma(c_k, g_k)\,, \tag{31}$$

the reduced form is the following (see Appendix B)

$$\left\{\begin{array}{l} \dot{d}_y = -\mu\,(d_y - \theta) \\ \frac{\dot{g}_k}{g_k} = \frac{r(g_k)\left[(1-\eta)d_y A g_k^{1-\alpha} - \gamma(c_k,g_k)b(d_y,g_k)\right] + \mu A g_k^{1-\alpha}(d_y-\theta)}{[\tau-\alpha(1-\tau)b(d_y,g_k)](r(g_k)+\delta) - g_k + (1-\alpha)d_y A g_k^{1-\alpha}} \\ \dot{R} = \left(\frac{1+\phi^c R}{\phi^c}\right)\left[r(g_k) - \rho - \frac{1}{S}\left(\frac{\eta d_y A g_k^{1-\alpha}}{\phi^c c_k} + r(g_k) - R\right)\right] \\ \dot{c}_k = \frac{\eta d_y A g_k^{1-\alpha}}{\phi^c} - [\gamma(c_k,g_k) - r(g_k) + R]\,c_k \end{array}\right. . \tag{32}$$

In this reduced form, there are two jump variables (R and c_k) and two predetermined variables (d_y and g_k). Effectively, the deficit-to-output ratio (d_y) cannot jump at any time, because it is defined by the smooth adjustment dynamics (6). Moreover, the debt-to-output ratio $b_k = b/k$ cannot jump, because the stocks of public debt (b) and of capital (k) are predetermined at each instant. Thus, from (30), it comes that g_k is not a free jump variable.

Proposition 4. *(Transitional dynamics) For small values of the deficit target (θ), the high BGP is saddle-path stable, while the low BGP is unstable.*

Proof.
See Appendix D. To study local dynamics of steady-states, we linearize system (32) in the neighborhood of BGPs, namely, using the index $h\,(l)$ to design the high (low) BGP

$$\begin{pmatrix} \dot{d}_y \\ \dot{g}_k \\ \dot{R} \\ \dot{c}_k \end{pmatrix} = \mathbf{J}_1^i \begin{pmatrix} d_y - d_y^{*i} \\ g_k - g_k^{*i} \\ R - R^{*i} \\ c_k - c_k^{*i} \end{pmatrix}, \quad i \in \{h, l\},$$

where $\mathbf{J}_1^i$ is the Jacobian matrix in the neighborhood of BGP $i = h, l$. For Blanchard-Kahn conditions to be fulfilled, $\mathbf{J}_1^i$ must contain 2 negative eigenvalues and 2 positive eigenvalues. In addition, the autonomous dynamics of the deficit-to-output ratio (equation (6)) shows that one eigenvalue equals $-\mu < 0$. Appendix D shows that, as $\theta \to 0$, $\mathbf{J}_1^h$ contains two positive and two negative eigenvalues, equal respectively to: $\lambda_1 = \left(1 + \phi^c R^{*h}\right) / S\phi^c > 0$ and $\lambda_2 = c_k^{*h} > 0$, $\lambda_3 = -\mu < 0$, $\lambda_4 = -\gamma^B < 0$, while $\mathbf{J}_1^l$ has one negative and three positive eigenvalues.

By continuity on the right of S point and on the left of B point, the low BGP is unstable and the high BGP is saddle-point stable, as confirmed by simulations of the reduced form in Figures 4a-b. $\square$

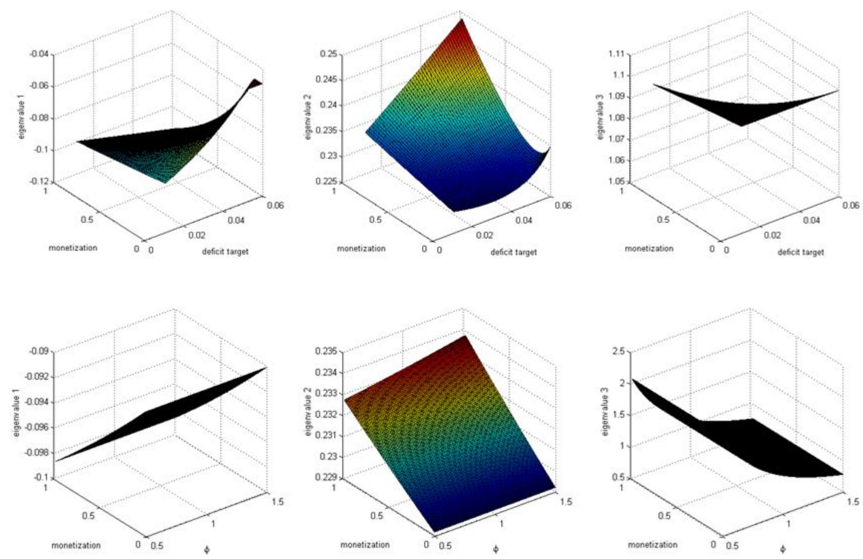


Figure 4a: Simulations of eigenvalues (high BGP): eigenvalue four is equal to -0.05

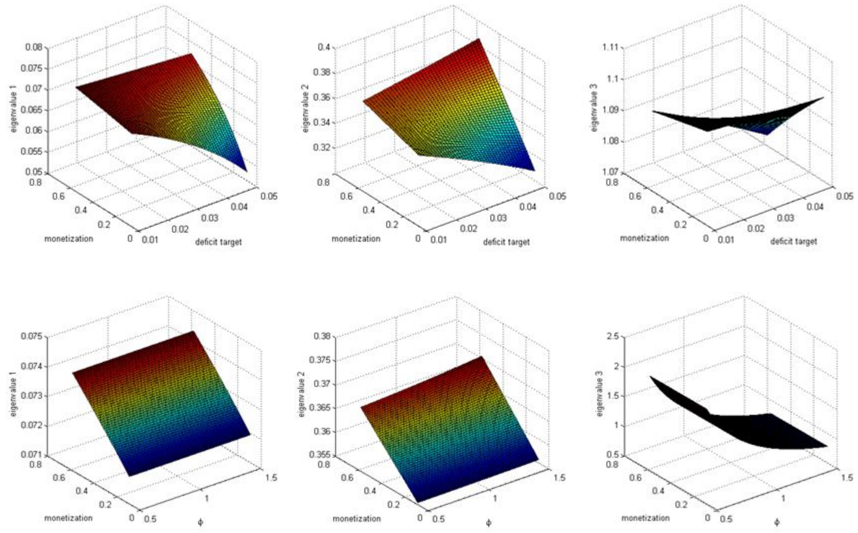


Figure 4b: Simulations of eigenvalues (low BGP): eigenvalue four is equal to -0.05

Consequently, we can exclude multiplicity on the basis of the analysis of local dynamics: only the high BGP is relevant in equilibrium.

5. The model with a CIA constraint on consumption and investment

As in the preceding Section, we first describe the long-run solution of the model, before studying local dynamics.

5.1. Deficits and monetization in the long-run

With a CIA constraint on consumption and investment $\left(\phi^k > 0\right)$, multiplicity still characterizes the long-run steady-state solution. The following Proposition establishes the properties of these solutions relative to public deficits and monetization.

Proposition 5. *(The effect of deficits and monetization on BGPs) For "small" deficit targets* (θ)*, any upwards shift in* θ

(i) *increases the low BGP* $\left(\gamma^{*l}\right)$ *and,*

(ii) *reduces the high BGP* $\left(\gamma^{*h}\right)$ *if monetization is "small"* $\left(\eta < \overline{\overline{\eta}}\right)$ *but increases the high BGP if monetization is "large"* $\left(\eta > \overline{\overline{\eta}}\right)$*, with* $\overline{\overline{\eta}} > \bar{\eta}$ *defined below.*

Proof.
With a CIA on consumption and investment, relation (25) is unchanged, but (24) becomes

$$g_k^* = \left\{ \left[1 + \phi^k \left[\frac{\eta\theta A}{\phi^k \left(A - g_k^{*\alpha}\right)} + \rho + \frac{\gamma^* (1-S)}{S} \right] \right] \frac{\left(\frac{\gamma^*}{S} + \rho + \delta\right)}{\alpha A (1-\tau)} \right\}^{\frac{1}{1-\alpha}}, \tag{24b}$$

and, by substituting $g_k^{*\alpha}$ from (25), we find

$$g_k^* = \left\{ \left[1 + \frac{\eta\theta A}{\left(A - \left(\mathcal{G}\left(\gamma^*\right)\right)^{\alpha}\right)} + \phi^k \rho + \frac{\phi^k \gamma^* \left(1-S\right)}{S} \right] \frac{\left(\frac{\gamma^*}{S} + \rho + \delta\right)}{\alpha A \left(1-\tau\right)} \right\}^{\frac{1}{1-\alpha}} \equiv \mathcal{F}\left(\gamma^*\right). \tag{34}$$

In this configuration, both $\mathcal{F}\left(\gamma^*\right)$ and $\mathcal{G}\left(\gamma^*\right)$ curves depend on the deficit ratio (θ): following an increase in θ, $\mathcal{G}\left(\gamma^*\right)$ moves as in the previous Section, while $\mathcal{F}\left(\gamma^*\right)$ moves downwards; effectively, from (24b): $\left.\frac{dg_k^*}{d\theta}\right|_{\gamma^*} = \frac{\eta\tau}{\alpha(1-\tau)^2}\left(\frac{\gamma^*}{S} + \rho + \delta\right) > 0$ for "small" deficit values ($\theta \to 0$).

Since the low BGP is located on the left of the pivot point of $\mathcal{G}\left(\gamma^*\right)$, both moves are favorable for economic growth along this BGP. To establish the effect of deficits along the high BGP, first notice that, in equilibrium: $\mathcal{F}\left(\gamma^{*h}\right) = \mathcal{G}\left(\gamma^{*h}\right)$; thus, γ^{*h} is defined by the implicit relation $\mathcal{K}(\gamma^{*h}, \theta) = 0$, where

$$\mathcal{K}\left(\gamma^{*h}, \theta\right) \equiv \left[1 + \frac{\eta\theta A}{\left(A - \left(\mathcal{G}\left(\gamma^{*h}, \theta\right)\right)^{\alpha}\right)} + \phi^k \rho + \frac{\phi^k \gamma^{*h} \left(1-S\right)}{S} \right] h\left(\gamma^{*h}\right) - \left(\mathcal{G}\left(\gamma^{*h}, \theta\right)\right)^{1-\alpha}, \tag{35}$$

with $h(\gamma^{*h}) \equiv (\rho + \delta + \gamma^{*h}/S)/\alpha A(1-\tau)$.

Therefore, for usual values of parameters (in particular, as soon as $S \leq 1$) we can establish

$$\frac{\partial \mathcal{K}\left(\gamma^{*h}, \theta\right)}{\partial \gamma^{*h}} = \left[1 + \phi^k \rho + \frac{\phi^k \gamma^{*h} \left(1-S\right)}{S} \right] h'\left(\gamma^{*h}\right) + \frac{\phi^k \left(1-S\right)}{S} h\left(\gamma^{*h}\right) > 0,$$

and,

$$-\frac{\partial \mathcal{K}\left(\gamma^{*h}, \theta\right)}{\partial \theta} = \frac{\left(1-\alpha\right)\mathcal{G}_\theta}{\left(\mathcal{G}\left(\gamma^{*h}, \theta\right)\right)^{\alpha}} - h\left(\gamma^{*h}\right) \frac{\eta A}{\left(A - \left(\mathcal{G}\left(\gamma^{*h}, \theta\right)\right)^{\alpha}\right)}. \tag{36}$$

Consequently, as $\mathcal{K} \in C^1(\mathbb{R}^2)$ and since $\mathcal{G}(\gamma^{*h}, \theta) < A^{1/\alpha}$, using the implicit function theorem we obtain, for "small" values of deficit ($\theta \to 0$)[18]

$$\frac{d\gamma^{*h}}{d\theta} = -\frac{\partial \mathcal{K}\left(\gamma^{*h}, \theta\right)/\partial\theta}{\partial \mathcal{K}\left(\gamma^{*h}, \theta\right)/\partial\gamma^{*h}}. \tag{37}$$

Thus, when $\theta \to 0$, (36) becomes

$$-\frac{\partial \mathcal{K}\left(\gamma^{*h}, \theta\right)}{\partial \theta} = \frac{\left(1-\alpha\right)\mathcal{G}_\theta}{A\tau} - h\left(\gamma^{*h}\right) \frac{\eta}{\left(1-\tau\right)}. \tag{38}$$

We can remark in particular that, for $\eta = \bar{\eta}$, $\mathcal{G}_\theta = 0$, thus: $-\partial\mathcal{K}\left(\gamma^{*h}, \theta\right)/\partial\theta < 0$ and, by (37), $d\gamma^{*h}/d\theta < 0$. We find the value of $\bar{\bar{\eta}}$ by setting $\partial\mathcal{K}\left(\gamma^{*h}, \theta\right)/\partial\theta = 0$ in (38), namely, using (27): $\left(\frac{1-\alpha}{\alpha}\right)\left(\tau A\right)^{\frac{1-\alpha}{\alpha}} \frac{\left(\bar{\bar{\eta}}-\bar{\eta}\right)}{\left(1-\bar{\eta}\right)} = h\left(\gamma^{*h}\right)\frac{\bar{\bar{\eta}}\tau}{\left(1-\tau\right)} > 0$. Thus: $\bar{\bar{\eta}} > \bar{\eta}$, which

[18]Remark that $\mathcal{G}_{\gamma^*} \to 0$ as $\theta \to 0$.

proves Proposition 5. It follows that $d\gamma^{*h}/d\theta > 0$ if $\eta > \overline{\overline{\eta}}$, where $\overline{\overline{\eta}}$ is defined by the following implicit relation[19]

$$\overline{\overline{\eta}} \equiv \left(\frac{1-\alpha}{\alpha}\right)\left(\frac{1-\tau}{\tau}\right)\left(1-\left(1-\overline{\overline{\eta}}\right)\left(\frac{1}{S}+\frac{\rho}{\gamma^B}\right)\right)\left[1+\phi^k\left[\rho+\frac{\gamma^B\left(1-S\right)}{S}\right]\right]. \quad (39)$$

□

(a) Low Monetization: $\eta = 10\%$ (b) High Monetization: $\eta = 60\%$

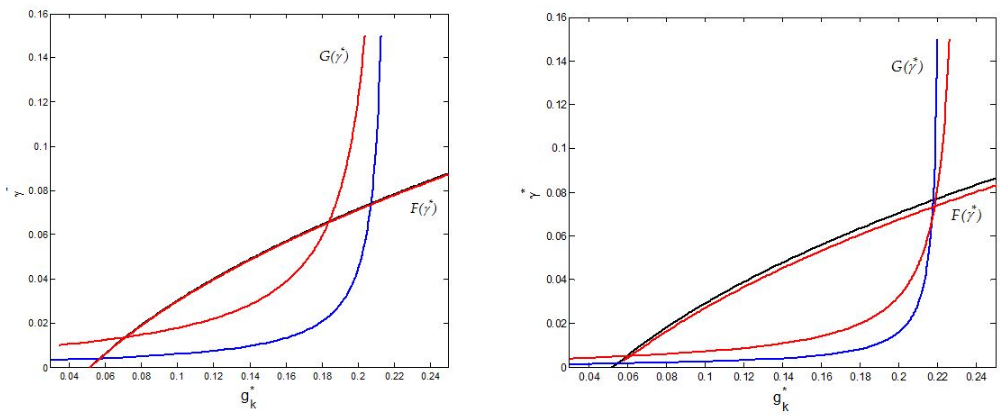


Figure 5: The BGPs (blue line: $\theta = 1\%$, red line $\theta = 5\%$)

Figure 5 presents a graphical confirmation of result (ii) in Proposition 5. Based on benchmark values of parameters, an increase in the deficit decreases the high BGP if monetization is "small" ($\eta = 10\%$, in Figure 5a), but increases the high BGP if monetization is "large" ($\eta = 60\%$, in Figure 5b).

The fact that the effect of monetization on the high BGP is weakened when the CIA constraint affects investment comes from the inflation tax on capital accumulation: by monetizing the deficit, monetary authorities can avoid the devastating effect of the debt burden on public spending, but at the cost of a reduction in private investment, because of higher funding costs (due to the increase in the inflationary tax associated with higher monetization).

Figures 6a-b show that, for our baseline calibration, economic growth and Household's welfare increase with the degree of monetization on the high BGP, but decrease on the low BGP.[20] Therefore, along the low BGP, increasing deficit-monetization is undesirable, especially since the debt-to-GDP ratio rises as deficit monetization grows. Effectively, economic growth becomes so weak, that the reduction in public debt accumulation due to monetizing deficits does not suffice to generate a decrease in the debt ratio.

[19]Remark that, in the neighborhood of the high BGP: $\gamma^{*h} \to \gamma^B$ as $\theta \to 0$. From Definition 1, γ^B is independent from η.

[20]In our baseline calibration we set $\phi^k = 0.5$.

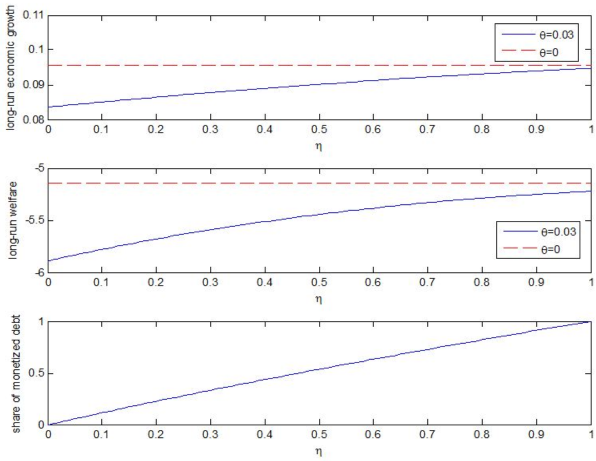


Figure 6a: Monetization, Economic Growth and Welfare in the long-run (high BGP)

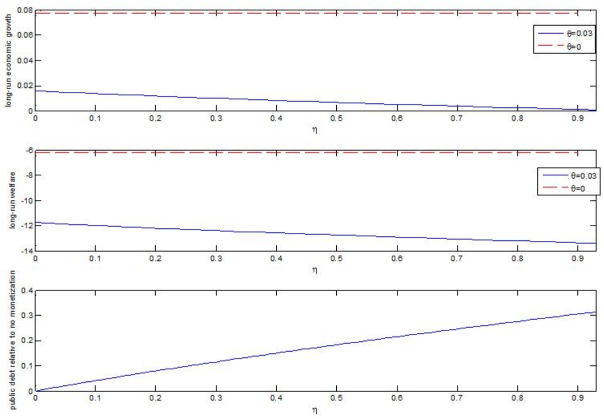


Figure 6b: Monetization, Economic Growth and Welfare in the long-run (low BGP)

5.2. Transitional dynamics

The model with a CIA on consumption and investment gives rise to a five-variable reduced form, which can be solved recursively (see Appendix B)

$$\left\{\begin{array}{l} \dot{d}_y = -\mu\left(d_y - \theta\right) \\ \dot{b}_k = \left(1-\eta\right) d_y A g_k^{1-\alpha} - \gamma\left(c_k, g_k\right) b_k \\ \dot{R} = \left[\left(r\left(d_y, g_k, b_k\right) + \delta\right)\left(1 + \phi^k R\right) - \left(1-\tau\right)\alpha A g_k^{1-\alpha}\right]/\phi^k \\ \dot{c}_k = S\left[r\left(d_y, g_k, b_k\right) - \rho - \phi^c \dot{R}/(1+\phi^c R)\right] c_k - \gamma\left(c_k, g_k\right) c_k \\ \dot{g}_k = \left\{\eta d_y A g_k^{1-\alpha} - \left[\gamma\left(c_k, g_k\right) - r\left(d_y, g_k, b_k\right) + R\right] m\left(c_k, g_k\right) - \left(\phi^c - \phi^k\right)\dot{c}_k\right\}/\phi^k z\left(g_k\right) \end{array}\right. , \tag{40}$$

where $\gamma(c_k, g_k)$ is defined by (31), $z(g_k) \equiv (1-\alpha) A g_k^{-\alpha} - 1$ and

$$m_k = \left(\phi^c - \phi^k\right) c_k + \phi^k \left(A g_k^{1-\alpha} - g_k\right) \equiv m(c_k, g_k), \tag{41}$$

$$r = \left((d_y + \tau) A g_k^{1-\alpha} - g_k\right)/b_k \equiv r(d_y, g_k, b_k). \tag{42}$$

As in the previous case, there are 2 predetermined variables in this system. Effectively the public debt ratio b_k and the deficit to output ratio d_y cannot jump.

The linearization of (40) in the neighborhood of BGPs provides the following system

$$\begin{pmatrix} \dot{d}_y \\ \dot{b}_k \\ \dot{R} \\ \dot{c}_k \\ \dot{g}_k \end{pmatrix} = \mathbf{J}_2^i \begin{pmatrix} d_y - d_y^{*i} \\ b_k - b_k^{*i} \\ R - R^{*i} \\ c_k - c_k^{*i} \\ g_k - g_k^{*i} \end{pmatrix}, \quad i = h, l, \tag{43}$$

where $\mathbf{J}_2^i$ stands for the Jacobian matrix in the neighborhood of BGP $i = h, l$. For Blanchard-Kahn conditions to be fulfilled, $\mathbf{J}_2^i$ must contain 2 negative eigenvalues (with one eigenvalue equal to $-\mu$) and 3 positive eigenvalues.

Proposition 6. *(Multiplicity and indeterminacy) Based on our simulation results*

(i) *the low BGP is saddle-path stable and,*
(ii) *the high BGP is locally undetermined or saddle-path depending on parameters.*

Proof. Simulation-based Proof.
The inspection of the reduced form (40) reveals that the dynamics are much more complex than in the model with a CIA only on consumption. In particular, we can remark that the dynamics fundamentally shift with the value of the term $z(g_k)$ in system (40). Effectively, the system is not defined for $z(g_k) = 0$, and the determinant of the Jacobian matrix $\mathbf{J}_2^i$ changes sign whenever $z\left(g_k^{*i}\right)$ changes sign. Noticeably, the case $z\left(g_k^{*i}\right) = 0$, namely $g_k^{*i} = A^{1/\alpha}(1-\alpha)^{1/\alpha} \equiv \hat{g}_k^*$, corresponds to the first-best public-spending-to-capital ratio in a model without deficit or money. From (25), we obtain, with $\bar{\eta} \equiv 1 - \frac{\gamma^*}{\rho + \gamma^*/S}$

$$g_k^* = \left[(1-\alpha) A + A\theta \left[1 - (1-\eta)\left(\frac{1}{S} + \frac{\rho}{\gamma^*}\right)\right]\right]^{1/\alpha} = A^{1/\alpha}\left[1 - \alpha + \varepsilon(\gamma^*)\right]^{1/\alpha}, \tag{44}$$

where $\varepsilon(\gamma^*) \equiv \tau + \alpha - 1 + \theta \frac{(\eta - \bar{\eta})}{(1-\bar{\eta})}$. Therefore, if $\varepsilon\left(\gamma^{*i}\right) > (<) 0$, then $g_k^{*i} > (<) \hat{g}_k^*$ and $z\left(g_k^{*i}\right) < (>) 0$. Along the low BGP, $\gamma^{*l} \to 0$, so that $\bar{\eta} \to 1$ and $\varepsilon\left(\gamma^{*l}\right) < 0$ independently of parameters. Thus: $z\left(g_k^{*l}\right) > 0$. Since the public-spending-to-capital ratio is very small along the low BGP, it always lies below the "first-best" level. Along the high BGP this is no longer the case. Effectively, we can find ranges of parameters such that $z\left(g_k^{*h}\right) > 0$ or $z\left(g_k^{*h}\right) < 0$ alternately. This means that the determinant of the Jacobian matrix changes sign depending on parameters. Yet, to be fully determined, the high BGP must be associated to exactly 2 negative eigenvalues. As the determinant of the Jacobian matrix is the product of the 5 eigenvalues, such a configuration is possible only if it takes a positive value. Consequently, the change of sign of the determinant is not consistent with determinacy of the high BGP.

More precisely, Appendix E and simulations of system (40) in Figures 7a-b show that multiplicity can no longer be removed on the basis of the local dynamics of steady-states. Effectively, the low BGP is now saddle-path stable (the Jacobian matrix presents 2 negative and 3 positive eigenvalues). Moreover, the high BGP is either saddle-path stable or locally indeterminate, depending on parameters, and in particular on the degree of monetization. □

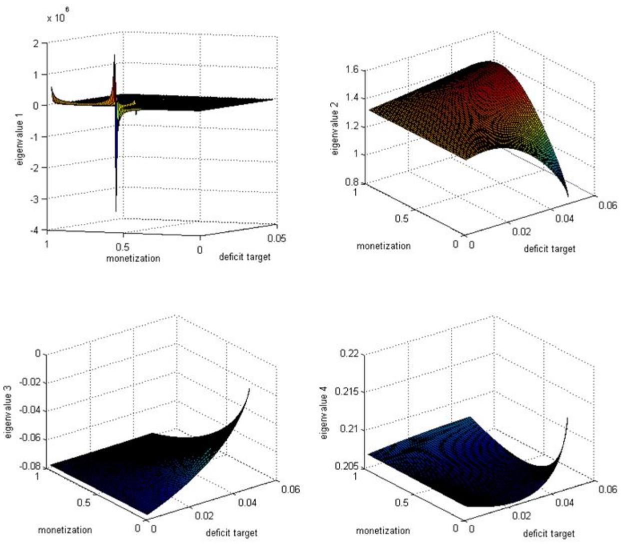


Figure 7a: Simulations of eigenvalues (high BGP): eigenvalue five is equal to -0.05

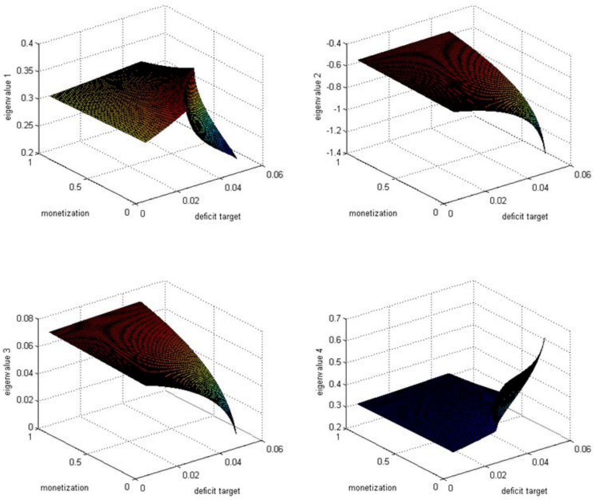


Figure 7b: Simulations of eigenvalues (low BGP): eigenvalue five is equal to -0.05

The high BGP is locally determined if $z\left(g_k^{*h}\right) < 0$, namely if $g_k^{*h} > \hat{g}_k^*$.[21] Proposition 6 shows that this configuration corresponds to the following condition

$$\varepsilon\left(\gamma^{*h}\right) = \tau + \alpha - 1 + \theta\frac{(\eta - \bar{\eta})}{(1 - \bar{\eta})} > 0. \tag{45}$$

Determinacy is all the more likely to occur that (i) the degree of monetization is high,[22] and (ii) that the tax rate is high relative to the first best rate $(1-\alpha)$ (if its positive direct impact in (45) dominates the negative indirect one coming from the rate of economic growth in $\bar{\eta}$). In our baseline simulation, we choose $\tau = 1 - \alpha$, so that determinacy is ensured for high degrees of monetization $(\eta > \bar{\eta})$. Figure 8 below synthesizes these results. Determinacy of the high BGP is ensured above the AA line, namely for high values of monetization. Isogrowth curves show that, consistent with our previous findings, long-run economic growth is negatively associated with public deficit, for a given level of monetization, but positively associated with monetization, for a given deficit target. Thus, by monetizing deficits, monetary authorities can generate productive public expenditure that ensure both a sizable and determined BGP.

[21]Therefore, in terms of first-best analysis, the case of local indeterminacy of the high BGP corresponds to a case of over-accumulation of public expenditure. This is no longer the case in a "second-best" analysis, in which Policymakers can only choose their instruments $(\tau, \eta, \theta, \mu)$ to maximize welfare, without being able to directly choose the public spending ratio g_k. Effectively, in a second-best analysis, a high degree of monetization improves Household's welfare along the high BGP, as revealed in the previous section.

[22]Notice that the rate of economic growth γ^{*h} positively depends on η; thus any increase in η reduces the value of $\bar{\eta}$.

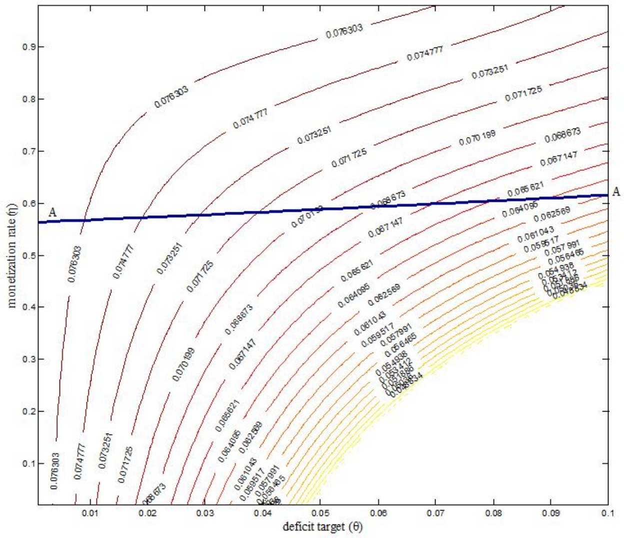


Figure 8: Isogrowth curves, Determinacy and Indeterminacy Loci of high BGP

5.3. Discussion

Our general result is that the high BGP is in some sense "more stable" than the low BGP, except for the case of a CIA constraint on consumption and investment with a high degree of monetization. The reduced form of our model shows that the dynamic properties of steady-states change due to the behavior of the public debt ratio in Government's budget constraint. Effectively, the dynamics of the public debt ratio are driven by the difference between the debt burden and economic growth. As usual in the analysis of Government's budget constraint, a sufficiently high economic growth rate allows circumventing the inherent instable dynamics of public debt, thus stabilizing the public debt ratio. This is the case in the neighborhood of the high BGP. On the contrary, along the low BGP, economic growth is very low and cannot stabilize the evolution of the public debt ratio. As the public debt ratio cannot jump, in the model with a CIA constraint on consumption the high BGP is locally saddle-point stable, while the low BGP is locally unstable. In the latter case, there is only one negative eigenvalue (associated to the deficit rule) for two predetermined variables (the public debt ratio and the public spending ratio).

When the CIA constraint covers both consumption and investment, the same reasoning applies, but now the model has one additional "stable" equation. Effectively, the CIA constraint (4) gives rise to a new determination of the dynamic of the capital stock, namely $\dot{k}_t = \frac{1}{\phi^k}(m_t - \phi^c c_t) - \delta k_t$, thus generating one additional stable eigenvalue. It follows that the low BGP becomes saddle-point stable and the high BGP becomes undetermined.

However there is one exception to this general result. With a CIA constraint on consumption only, the dynamic of $\dot{g}_k$ in (32) is qualitatively unaffected by parameters (and in particular by the degree of monetization); instead, this does not hold when the

CIA affects both consumption and investment. Indeed, in the reduced form (40), the nature of the dynamics of $\dot{g}_k$ fundamentally shifts with respect to $z\,(g_k)$.

The intuitive explanation of this shift is the following. The term $z\,(g_k) \equiv \frac{d(y_k - g_k)}{dg_k}$ is the response of the difference between output and public spending following an increase in public spending, or, in other words, the net impact of an additional unit of productive public expenditure on the goods equilibrium. Thus any rise in g_k increases (decreases) private demand if $z(g_k) > 0$ $(z(g_k) < 0)$. In money market equilibrium, money supply is defined by the difference between the monetization of public deficit and the inflation tax $(\dot{m}_k = \eta d_k - (\pi + \gamma) m_k)$, while money demand comes from the private demand (consumption plus investment: $c + \dot{k} + \delta k \equiv y^d$), which positively $(z(g_k) > 0)$ or negatively $(z(g_k) < 0)$ depends on productive public expenditure

$$A g_k^{1-\alpha} - g_k = y_k^d \Rightarrow z(g_k)\dot{g}_k = \dot{y}_k^d.$$

Suppose that $g_k > g_k^*$, *ceteris paribus.* As, by (42), the real interest rate strongly decreases, the inflation rate $(\pi = R - r)$ jumps up, and the seignorage collection $((\pi + \gamma) m_k))$ becomes higher than the monetization of deficit (ηd_k), which is negligible in steady-state because $\theta \cong 0$. Thus, the money supply decreases $(\eta d_k < (\pi + \gamma) m_k \Rightarrow \dot{m}_k < 0)$. In equilibrium, money demand must decline, thus private demand must decrease $(\dot{y}_k^d < 0)$, which implies: $\dot{g}_k < 0$ if $z(g_k) > 0,$ or $\dot{g}_k > 0$ if $z(g_k) < 0$.

In the first case, the law of motion of g_k is stable, leading to indeterminacy of the BGP, while in the latter, the law of motion of g_k is unstable, leading to determinacy of the BGP. Precisely, along the high BGP, for a "high" degree of monetization, the public spending ratio becomes so large that the derivative $z\,(g_k)$ becomes negative. As a result of this novel source of instability, the high BGP loses its undesirable property of being stable and undetermined, and becomes saddle-path stable.

6. Conclusion

To study long- and short-run effects of monetization, we built in this paper a model in which economic growth interacts with productive public expenditure. This interaction can generate two positive balanced growth paths in the long-run: a high BGP and a low BGP.

From a long-run perspective, our results support the monetization of deficits. In particular, our model provides a relatively new motivation for monetizing. Usually, monetization is defended, for example, on the basis of providing seigniorage revenues, or because Policymakers can reduce the real interest rate by generating inflation surprises. In our framework, monetization is useful because it avoids (or limits) the crowding-out effect of public indebtedness on productive public expenditure in the long-run. The latter motivation for monetization might be stronger than the former two ones. Indeed, on the one hand, seigniorage revenues are fairly small for public finance, namely less than 1% of GDP for moderate inflation rates, and up to 5% in high-inflation countries (see, e.g. Dornbusch & Fischer, 1993, or, Easterly et al., 1995). On the other hand, inflation surprises cannot be perpetuated in the long-run in rational expectation equilibria. On the contrary, in our model, money issuance increases economic growth on the long-run perfect-foresight BGP, owing to a composition effect in public finance, namely the substitution of a non-interest-bearing asset (money) to public debt in Government's budget constraint. This change in the composition of Government's liabilities generates a less distortive way of finance for productive public expenditure.

Furthermore, transitional dynamics crucially depend on the form of the CIA constraint. When only consumption is money-constrained, multiplicity can be removed, and monetizing deficits enhances economic growth and welfare in the long-run. When the CIA constraint affects both consumption and investment, the local dynamics of the BGPs depend on the degree of monetization of the deficit. For small monetization rates, the low BGP is locally determined (saddle-path), but the high BGP becomes locally undetermined. However, for sufficiently high monetization rates, both BGPs are characterized by the saddle-path property and are locally determined. Thus, if investment is money-constrained, multiplicity cannot be rejected.

Our findings match numerous results in the literature, emphasizing the importance of the transaction technology in generating long-run multiplicity and/or indeterminacy of perfect-foresight equilibria. However, in our model, the fundamental non-linearity governing the determination of the equilibrium growth rate comes from Government's budget constraint. In this constraint, public debt crowds out productive expenditure in the long-run, thus generating two BGPs: one with a strong economic growth, which gives rise to a small debt burden relative to income and a large share of productive public expenditure, with a positive feedback effect on economic growth. The other is associated with low growth and weak productive expenditure, due to the magnitude of the debt burden, which in turn does not support economic growth.

In our model, monetizing deficits is found to be remarkably powerful. First, a large dose of monetization might allow avoiding, whenever present, BGP indeterminacy. Second, monetization always allows increasing economic growth and welfare along the (high) BGP, by weakening the debt burden in the long-run. Third, monetization provides rationale for deficits in the long-run: for high degrees of monetization, the impact of deficits and debts on economic growth and welfare becomes positive in the steady-state. Although at odds with most findings in the theoretical literature that ignores money financing, this result is consistent with empirical work exhibiting a non-linear relationship between debt or deficits, and economic growth.

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Appendix A: Solution of Household's program.

The representative Household maximizes (1) subject to (2)-(3)-(4), k_0 and b_0 given, and the transversality condition: $\lim_{t\to\infty}\left(\exp\left(-\int_0^\infty r_s ds\right)(k_t+b_t+m_t)\right)=0$. Since investment is subject to transaction costs, it is convenient to replace the budget constraint (3) by two constraints on two state variables: $a_t \equiv m_t + b_t$ and k_t, using the definition of net investment: $\dot{k}_t = z_t - \delta k_t$. Thus, we can write the current Hamiltonian as

$$\begin{aligned} H_c = u(c_t) + \lambda_{1t}\left[r_t b_t + (1-\tau) y_t - c_t - z_t - \pi_t m_t\right] + \lambda_{2t}(z_t - \delta k_t)\, a \\ +\mu_t (a_t - b_t - m_t) + \chi_t\left(m_t - \phi^c c_t - \phi^k z_t\right), \end{aligned}$$

where λ_{1t} and λ_{2t} are the co-state variables associated with a_t and k_t, respectively, and χ_t is the co-state variable associated with the static constraint. First-order conditions (hereafter FOC) are

$$/b_t \qquad \mu_t/\lambda_{1t} = r_t, \tag{A.1}$$

$$/c_t \qquad u_c(c_t) = \lambda_{1t} + \phi^c \chi_t = \lambda_{1t}(1+\phi^c R_t), \tag{A.2}$$

$$/m_t \qquad -\lambda_{1t}\pi_t - \mu_t + \chi_t = 0 \Rightarrow \chi_t/\lambda_{1t} = \pi_t + \mu_t/\lambda_{1t} = R_t, \tag{A.3}$$

$$/z_t \qquad -\lambda_{1t} + \lambda_{2t} - \phi^k \chi_t = 0 \Leftrightarrow \frac{\lambda_{2t}}{\lambda_{1t}} = 1 + \frac{\chi_t}{\lambda_{1t}}\phi^k = 1 + \phi^k R_t, \tag{A.4}$$

$$/a_t \qquad \dot{\lambda}_{1t}/\lambda_{1t} = \rho - r_t, \tag{A.5}$$

$$/k_t \qquad \dot{\lambda}_{2t}/\lambda_{2t} = \rho + \delta - \frac{(1-\tau) f_k \lambda_{1t}}{\lambda_{2t}} = \rho + \delta - \frac{(1-\tau) f_k}{1+\phi^k R_t}. \tag{A.6}$$

FOCs have a standard interpretation. λ_1 is the shadow price (i.e. the opportunity cost) of financial wealth (a_t), which differs from the shadow price of capital (λ_2) in (A.4), if investment expenditures are subject to the CIA constraint (namely if $\phi^k > 0$). Effectively, in this case, wealth cannot directly buy capital, because the latter must be acquired with money: the opportunity cost of capital is higher than the opportunity cost of wealth, as soon as $\phi^k > 0$. If capital is not subject to the CIA constraint, this feature disappears, and $\lambda_{1t} = \lambda_{2t}$. Similarly, in (A.2), the marginal utility of consumption has to be distinguished from the shadow price of financial wealth, since wealth cannot directly buy consumption goods. The opportunity cost of money for consumption ($\phi^c R_t$) introduces a wedge between the marginal utility of consumption and the marginal value of wealth. Equation (A.3) states that the marginal cost of money (the nominal interest rate R_t) must equalize its marginal return (the marginal value of a unit of money in the CIA constraint relative to the marginal value of wealth χ_t/λ_{1t}). Finally, equations (A.5) and (A.6) describe the evolution of shadow prices of wealth and capital, respectively. They show, in particular, that the marginal return of capital (its marginal productivity net from taxes and depreciation $(1-\tau) f_k(.) - \delta$) differs from the marginal return of bonds (the real interest rate r_t), as soon as the CIA constraint affects capital goods.

By differentiating (A.2) and (A.4) and after some simple manipulations we obtain equations (9) and (10) of the main text.

Appendix B: The reduced form of the model with a CIA constraint on consumption only.

With a CIA constraint on consumption goods, we have: $m = \phi^c c$, which implies $\dot{m} = \phi^c \dot{c}$, and, since money emissions are proportional to the deficit ($\dot{m} + \pi m = \phi^c(\dot{c} + \pi c) = \eta d$), we obtain: $\dot{c}/c = \frac{\eta d}{\phi c} - \pi$. Besides, $\dot{c}_k/c_k = \dot{c}/c - \gamma_k$, hence, with $\pi = R - r$

$$\frac{\dot{c}_k}{c_k} = \frac{\eta d_k}{\phi^c c_k} - (\gamma_k - r + R), \tag{B.1}$$

which is the second equation of the reduced-form (32). From the Keynes-Ramsey rule (9), we have

$$\frac{\dot{c}_k}{c_k} = S\left[r - \rho - \frac{\phi^c \dot{R}}{1 + \phi^c R}\right] - \gamma_k. \tag{B.2}$$

Thus, we obtain the path of the nominal interest rate from (B.1) and (B.2)

$$\dot{R} = \left(\frac{1 + \phi^c R}{\phi^c}\right)\left[r(g_k) - \rho - \frac{1}{S}\left(\frac{\eta d_k}{\phi^c c_k} + r(g_k) - R\right)\right], \tag{B.3}$$

which is the first equation of the reduced-form (32). The definition of the deficit rule (6) provides the fourth equation of the reduced form

$$\dot{d}_y = -\mu(d_y - \theta). \tag{B.4}$$

Finally, in Government's budget constraint (5), the deficit is defined as: $d_t = r_t b_t + g_t - \tau y_t$. Deflating both sides by the capital stock, and using the definition of the real interest rate (29), it comes

$$b_k = \frac{d_k + \tau A g_k^{1-\alpha} - g_k}{(1-\tau)\alpha A g_k^{1-\alpha} - \delta}.$$

By differentiating this relation, we have

$$\dot{b}_k = \frac{\dot{d}_k + \left[[\tau - \alpha(1-\tau) b_k](1-\alpha) A g_k^{-\alpha} - 1\right]\dot{g}_k}{r}. \tag{B.5}$$

Since, from (8)

$$\dot{b}_k = (1-\eta) d_k - \gamma_k b_k, \tag{B.6}$$

and using (B.5) and (B.6), we obtain the third equation of the reduced form (32)

$$\dot{g}_k = \left\{ \frac{r\left[(1-\eta) d_k - \gamma_k b_k\right] + \mu\left(d_k - \theta A g_k^{1-\alpha}\right)}{\left[\tau - \alpha(1-\tau) b_k\right](1-\alpha) A g_k^{1-\alpha} - g_k + (1-\alpha) d_k} \right\} g_k. \tag{B.7}$$

We get (32) by replacing d_k with $d_y A g_k^{1-\alpha}$.

Appendix C: The reduced form with a CIA constraint on consumption and investment.

When the CIA constraint affects both consumption and investment $\left(\phi^k > 0\right)$, the behavior of the nominal interest rate directly results from (10)

$$\dot{R} = \frac{1}{\phi^k}\left[(r+\delta)\left(1+\phi^k R\right) - (1-\tau)\alpha A g_k^{1-\alpha}\right], \tag{C.1}$$

which constitutes the third equation of the reduced form (40).

Then, the path of the consumption ratio (the fourth equation of the reduced-form (40)), is still defined by the Keynes-Ramsey rule (B.2), and the law of motion of the deficit ratio (the first equation of (40)) is still obtained in equation (B.4). The second equation of the reduced form is the evolution of the public debt ratio (B.6), and we just have to find the path of the public spending ratio g_k.

To this end, we rewrite the CIA constraint (4) as: $m_k = \left(\phi^c - \phi^k\right) c_k + \phi^k\left(A g_k^{1-\alpha} - g_k\right)$. It follows that: $\dot{m}_k = \left(\phi^c - \phi^k\right)\dot{c}_k + \phi^k\left[A(1-\alpha) g_k^{-\alpha} - 1\right]\dot{g}_k$ and, since $\frac{\dot{m}_k}{m_k} = \eta\frac{d_k}{m_k} - \pi - \gamma_k$,

$$\dot{g}_k = \frac{\eta d_k - (\gamma_k + \pi) m_k - \left(\phi^c - \phi^k\right)\dot{c}_k}{\phi^k z(g_k)}, \tag{C.2}$$

which corresponds to the last equation of the reduced form (40).

Appendix D: Local stability with the CIA constraint on consumption only.

Define $\hat{g}^i \equiv (1-\alpha) A\left(g_k^{*i}\right)^{-\alpha} > 0$, $\hat{x}^i \equiv \left[\theta + \tau - \alpha(1-\tau) b_k^{*i}\right]\hat{g}^i - 1 = -\left[\alpha + \frac{(1-\alpha)\delta b_k^{*i}}{g_k^{*i}}\right] < 0$ and $\hat{\phi}^i \equiv \left(1+\phi^c R^{*i}\right)/S\phi^c > 0$, and notice that, from (30): $\left.\frac{\partial b_k}{\partial d_y}\right|_{*i} = y_k^{*i}/r^{*i}$ and $\left.\frac{\partial b_k}{\partial g_k}\right|_{*i} = \hat{x}^i/r^{*i}$. The Jacobian matrix of system (32) is

$$\mathbf{J}_1^i = \begin{bmatrix} -\mu & 0 & 0 & 0 \\ GD^i & GG^i & 0 & r^{*i} b_k^{*i}/\hat{x}^i \\ -\frac{\eta\hat{\phi}^i y_k^{*i}}{\phi^c c_k^{*i}} & RG^i & \hat{\phi}^i & \frac{\eta\theta\hat{\phi}^i y_k^{*i}}{\phi^c\left(c_k^{*i}\right)^2} \\ \frac{\eta y_k^{*i}}{\phi^c} & CG^i & -c_k^{*i} & c_k^{*i} - \gamma^{*i} + r^{*i} - R^{*i} \end{bmatrix}, \tag{D.1}$$

where

$$GG^i \equiv \left.\frac{\partial \dot{g}_k}{\partial g_k}\right|_{*i} = -\gamma^{*i} + \left\{(1-\tau)\alpha\hat{g}^i\left[(1-\eta)\theta y_k^{i*} - \gamma^{i*} b_k^{*i}\right] - r^{*i}\left[\left(\hat{g}^i - 1\right) b_k^{*i} + (1-\eta)\theta\hat{g}^i\right]\right\}/\hat{x}^i,$$

$$GD^i \equiv \left.\frac{\partial \dot{g}_k}{\partial d_y}\right|_{*i} = \left[(1-\eta)\, r^{*i} + \mu - \gamma^{*i}\right] y_k^{*i} / \hat{x}^i,$$

$$RG^i \equiv \left.\frac{\partial \dot{R}}{\partial g_k}\right|_{*i} = \hat{\phi}^i \hat{g}^i \left[\alpha\,(1-\tau)\,(S-1) - \frac{\eta\theta}{\phi^c c_k^{*i}}\right],$$

$$CG^i \equiv \left.\frac{\partial \dot{c}_k}{\partial g_k}\right|_{*i} = \frac{\eta\theta\hat{g}^i}{\phi^c} + \left[1 - (1-\alpha\,(1-\tau))\,\hat{g}^i\right] c_k^{*i}.$$

It follows that, for "small" deficit values ($\theta \to 0$)

$$\mathbf{J}_1^h = \begin{bmatrix} -\mu & 0 & 0 & 0 \\ GD^h & -\gamma^B & 0 & 0 \\ -\frac{\eta\hat{\phi}^h y_k^{*h}}{\phi^c c_k^{*h}} & RG^h & \hat{\phi}^h & 0 \\ \frac{\eta y_k^{*h}}{\phi^c} & CG^h & -c_k^{*h} & c_k^{*h} \end{bmatrix}. \tag{D.2}$$

Thus the "Barro" BGP is saddle-path stable, with the associated eigenvalues equal to: $-\mu < 0$, $-\gamma^B < 0$, $\hat{\phi}^h > 0$ and $c_k^{*h} > 0$. Regarding the low BGP, the Jacobian matrix is

$$\mathbf{J}_1^l = \begin{bmatrix} -\mu & 0 & 0 & 0 \\ \left[(1-\eta)\,\rho + \mu\right] y_k^{*l}/\hat{x}^l & \left(1-\hat{g}^l\right) \rho b_k^{*l}/\hat{x}^l & 0 & \rho b_k^{*l}/\hat{x}^l \\ -\frac{\eta\hat{\phi}^l y_k^{*l}}{\phi^c c_k^{*l}} & \alpha\,(1-\tau)\,(S-1)\,\hat{\phi}^l\hat{g}^l & \hat{\phi}^l & 0 \\ \frac{\eta y_k^{*l}}{\phi^c} & \left[1-(1-\alpha\,(1-\tau))\,\hat{g}^l\right] c_k^{*l} & -c_k^{*l} & c_k^{*l} \end{bmatrix}, \tag{D.3}$$

and

$$\det\left(\mathbf{J}_1^l\right) = -\frac{\mu\rho\hat{\phi}^l c_k^{*l} b_k^{*l}}{\hat{x}^l} \det \begin{bmatrix} 1-\hat{g}^l & 0 & 1 \\ (1-\tau)\,\alpha\,(S-1)\,\hat{g}^l & 1 & 0 \\ 1-(1-\alpha\,(1-\tau))\,\hat{g}^l & -1 & 1 \end{bmatrix} = \frac{\alpha\mu\,(1-\tau)\,S\rho\hat{\phi}^l\hat{g}^l}{\hat{x}^l} c_k^{*l} b_k^{*l}.$$

Since $\hat{x}^l < 0$, $\det\left(\mathbf{J}_1^l\right) < 0$. Yet, one eigenvalue is equal to $-\mu$. Thus, among the three remaining eigenvalues, we can have either 0 or 2 negative eigenvalues. Let us prove that the latter case is impossible. The characteristic equation of $\mathbf{J}_1^l$ is $\det\left(\mathbf{J}_1^l - \lambda\mathbf{I}\right) = (\lambda+\mu)\,P(\lambda) = 0$, where $P(\lambda)$ is the following polynomial of degree three: $P(\lambda) = \lambda^3 - B\lambda^2 + C\lambda + D$, where $D \equiv \frac{\alpha(1-\tau)S\rho b_k^{*l}\hat{\phi}^l\hat{g}^l c_k^{*l}}{\hat{x}^l} < 0$, $B \equiv \left(1-\hat{g}^l\right)\frac{\rho b_k^{*l}}{\hat{x}^l} + c_k^{*l} + \hat{\phi}^l > 0$ and $C \equiv \hat{\phi}^l\left[\left(1-\hat{g}^l\right)\frac{\rho b_k^{*l}}{\hat{x}^l} + c_k^{*l}\right] - \alpha\,(1-\tau)\,\hat{g}^l c_k^{*l}\frac{\rho b_k^{*l}}{\hat{x}^l} > 0$, on the sufficient (unnecessary) condition that $\hat{g}^l > 1$, namely that $g_k^{*l} < \left[(1-\alpha)\,A\right]^{1/\alpha}$.

Notice first that $P'(\lambda) = 0$ for $\hat{\lambda}_{1,2} = \frac{1}{6}\left[2B \pm \sqrt{4B^2 - 12C}\right]$, namely for at least one positive value of λ (say $\hat{\lambda}_1 = \frac{1}{6}\left[2B + \sqrt{4B^2-12C}\right] > 0$). Therefore, the case of two negative eigenvalues can arise only in two configurations, as show the following Figures: if $D < 0$ and $P'(0) = C < 0$ or if $D > 0$ and $P'(0) = C > 0$. Consequently, the case $D < 0$ and $C > 0$ corresponds to zero negative eigenvalues.

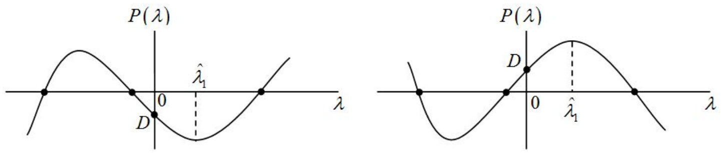

Figure D.1: Sign of eigenvalues

Appendix E: Local stability with the CIA constraint on consumption and investment.

The Jacobian matrix of system (40) is

$$\mathbf{J}_2^i = \begin{bmatrix} -\mu & 0 & 0 & 0 & 0 \\ (1-\eta)\, y_k^{i*} & -\gamma^{*i} & 0 & b_k^{i*} & BG^i \\ RD^i & RB^i & r^{*i}+\delta & 0 & RG^i \\ CD^i & CB^i & CR^i & c_k^{i*} & CG^i \\ GD^i & GB^i & GR^i & GC^i & GG^i \end{bmatrix}.$$

We first define: $z^i \equiv \hat{g}^i - 1$, and the derivatives of r in (42) as: $rd^i \equiv \left.\frac{\partial r}{\partial d_y}\right|_{*i} = y_k^{*i}/b_k^{*i}$, $rg^i \equiv \left.\frac{\partial r}{\partial g_k}\right|_{*i} = \left[(\theta+\tau)\,\hat{g}^i - 1\right]/b_k^{*i}$ and $rb^i \equiv \left.\frac{\partial r}{\partial b_k}\right|_{*i} = -r^{*i}/b_k^{*i} = \left[g_k^{*i} - (\theta+\tau)\, y_k^{*i}\right] / \left(b_k^{*i}\right)^2$.

It follows that:

$$\begin{aligned}
BG^i &\equiv \left.\tfrac{\partial \dot{b}_k}{\partial g_k}\right|_{*i} = (1-\eta)\,\theta\hat{g}^i - z^i b_k^{*i}, \\
RD^i &\equiv \left.\tfrac{\partial \dot{R}}{\partial d_y}\right|_{*i} = rd^i\left(\tfrac{1+\phi^k R^{*i}}{\phi^k}\right),\ RB^i \equiv \left.\tfrac{\partial \dot{R}}{\partial b_k^{*i}}\right|_{*i} = rb^i\left(\tfrac{1+\phi^k R^{*i}}{\phi^k}\right), \\
RG^i &\equiv \left.\tfrac{\partial \dot{R}}{\partial g_k^{*i}}\right|_{*i} = \left[rg^i\left(1+\phi R^{*i}\right) - \alpha\,(1-\tau)\,\hat{g}^i\right]/\phi^k, \\
CD^i &\equiv \left.\tfrac{\partial \dot{c}_k}{\partial d_y}\right|_{*i} = \left[rd^i - \left(\tfrac{\phi^c RD^i}{1+\phi^c R^{*i}}\right)\right] Sc_k^{*i},\ CB^i \equiv \left.\tfrac{\partial \dot{c}_k}{\partial b_k^{*i}}\right|_{*i} = \left[rb^i - \left(\tfrac{\phi^c RB^i}{1+\phi^c R^{*i}}\right)\right] Sc_k^{*i}, \\
CR^i &\equiv \left.\tfrac{\partial \dot{c}_k}{\partial R}\right|_{*i} = -\tfrac{Sc_k^{*i}\left(r^{*i}+\delta\right)\phi^c}{\left(1+\phi^c R^{*i}\right)}, \\
CG^i &\equiv \left.\tfrac{\partial \dot{c}_k}{\partial g_k}\right|_{*i} = \left[rg^i - \tfrac{\phi^c RG^i}{1+\phi^c R^{*i}}\right] Sc_k^{*i} - z^i c_k^{*i}, \\
GD^i &\equiv \left.\tfrac{\partial \dot{g}_k}{\partial d_y}\right|_{*i} = \left(\eta y_k^{*i} + rd^i m_k^{*i} - \left(\phi^c - \phi^k\right) CD^i\right)/\phi^k z^i, \\
GB^i &\equiv \left.\tfrac{\partial \dot{g}_k}{\partial b_k}\right|_{*i} = \left(rb^i m_k^{*i} - \left(\phi^c - \phi^k\right) CB^i\right)/\phi^k z^i, \\
GR^i &\equiv \left.\tfrac{\partial \dot{g}_k}{\partial R}\right|_{*i} = -\left(m_k^{*i} + \left(\phi^c - \phi^k\right) CR^i\right)/\phi^k z^i, \\
GC^i &\equiv \left.\tfrac{\partial \dot{g}_k}{\partial c_k}\right|_{*i} = \left(m_k^{*i} + \left(r^{*i} - \gamma^{*i} - R^{*i} - CC^i\right)\left(\phi^c - \phi^k\right)\right)/\phi^k z^i, \\
GG^i &\equiv \left.\tfrac{\partial \dot{g}_k}{\partial g_k}\right|_{*i} = r^{*i} - \gamma^{*i} - R^{*i} + \left(\eta\theta\hat{g}^i - \left(z^i - rg^i\right) m_k^{*i} - \left(\phi^c - \phi^k\right) CG^i\right)/\phi^k z^i.
\end{aligned}$$

To establish formal results, we study the local dynamics of the two BGPs for "small" deficit values ($\theta \to 0$), namely in the neighborhood of the "Barro" BGP and of the

"Solow" BGP, respectively. In addition, to simplify calculations, we consider that $\phi^k = \phi^c = \phi > 0$.

Let us first analyze the dynamics in the neighborhood of the high BGP. For "small" deficit values ($\theta \to 0$), the Jacobian matrix becomes

$$\mathbf{J}_2^h = \begin{bmatrix} -\mu & 0 & 0 & 0 & 0 \\ (1-\eta)\, y_k^{h*} & -\gamma^B & 0 & 0 & 0 \\ RD^h & RB^h & r^{*h}+\delta & 0 & RG^h \\ 0 & 0 & CR^h & c_k^{h*} & CG^h \\ GD^h & GB^h & GR^h & GC^h & GG^h \end{bmatrix}.$$

Thus, $\mathbf{J}_2^h$ has two negative eigenvalues, namely $\lambda_1 = -\mu$ and $\lambda_2 = -\gamma^B$. Furthermore: $\det\left(\mathbf{J}_2^h\right) = -\frac{\mu\gamma^B\left(y_k^{*h} - g_k^{*h}\right)}{z^h} \det \begin{bmatrix} r^{*h}+\delta & 0 & RG^h \\ CR^h & c_k^{h*} & CG^h \\ 1 & -1 & z^h - rg^h \end{bmatrix}$, thus it changes sign as z^h changes sign. Consequently, $\mathbf{J}_2^h$ has at least 3 negative eigenvalues, when $\det\left(\mathbf{J}_2^h\right) < 0$, and the high BGP is undetermined. By continuity, this property is verified for positive (but low) deficit values. This proves point (ii) of Proposition 6.

Now, let us focus on the dynamics in the neighborhood of the low BGP. For "small" deficit values ($\theta \to 0$), the Jacobian matrix becomes

$$\mathbf{J}_2^l = \begin{bmatrix} -\mu & 0 & 0 & 0 & 0 \\ (1-\eta)\, y_k^{l*} & 0 & 0 & b_k^S & -z^l b_k^S \\ RD^l & \frac{-\rho}{b_k^S}\left(\frac{1+\phi\rho}{\phi}\right) & \rho+\delta & 0 & RG^l \\ 0 & 0 & CR^l & c_k^{l*} & CG^l \\ GD^l & \frac{\rho}{b_k^S}\left(\frac{y_k^{*l} - g_k^{*l}}{z^i}\right) & GR^l & GC^l & GG^l \end{bmatrix}.$$

The characteristic equation of this matrix is

$$Det\left(\mathbf{J}_2^l - \lambda \mathbf{I}\right) = -(\lambda+\mu)\, Det\left(\mathbf{A} - \lambda\mathbf{I}\right) = 0,$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & b_k^S & -z^l b_k^S \\ \frac{-\rho}{b_k^S}\left(\frac{1+\phi\rho}{\phi}\right) & \rho+\delta & 0 & RG^l \\ 0 & CR^l & c_k^{l*} & CG^l \\ \frac{\rho}{b_k^S}\left(\frac{y_k^{*l} - g_k^{*l}}{z^i}\right) & GR^l & GC^l & GG^l \end{bmatrix}.$$

Thus, one eigenvalue is equal to $\lambda_1 = -\mu < 0$. In addition, observe that

$$Det\left(\mathbf{A} - \lambda\mathbf{I}\right) = -Det \begin{bmatrix} \frac{\rho}{b_k^S}\left(\frac{1+\phi\rho}{\phi}\right) - \lambda & \lambda - \rho - \delta & \tilde{b}_k^S & -RG^l - z^l b_k^S \\ \frac{-\rho}{b_k^S}\left(\frac{1+\phi\rho}{\phi}\right) & \rho+\delta-\lambda & 0 & RG^l \\ 0 & CR^l & -\lambda + c_k^{l*} & CG^l \\ \frac{\rho}{b_k^S}\left(\frac{y_k^{*l} - g_k^{*l}}{z^i}\right) & GR^l & GC^l & -\lambda + GG^l \end{bmatrix}.$$

Since $\rho/b_k^{*l} \cong 0$, we can rewrite

$$Det\left(\mathbf{A}-\lambda\mathbf{I}\right) \cong \left(\lambda - \frac{\rho}{b_k^S}\left(\frac{1+\phi\rho}{\phi}\right)\right) Det\left(\mathbf{B}-\lambda\mathbf{I}\right),$$

where

$$\mathbf{B} = \begin{bmatrix} \rho+\delta & 0 & RG^l \\ CR^l & c_k^{*l} & CG^l \\ GR^l & GC^l & GG^l \end{bmatrix}.$$

Therefore, another eigenvalue is equal to $\lambda_2 \cong \frac{\rho}{b_k^S}\left(\frac{1+\phi\rho}{\phi}\right) \to 0^+$. Furthermore

$$Det\left(\mathbf{B}\right) = \frac{\left(y_k^{*l} - g_k^{*l}\right)}{z^l} c_k^{*l} \left\{\left(\rho+\delta\right)\left(1-S\right) rg^l + RG^l\right\}$$

and

$$Tr\left(\mathbf{B}\right) = \rho + \delta + c_k^{*l} - \left(1 + rg^l/z^l\right)\left(y_k^{*l} - g_k^{*l}\right) = \rho + rg^l\left(y_k^{*l} - g_k^{*l}\right)/z^l.$$

First, notice that $rg^l = \left(\tau\hat{g}^l - 1\right)/b_k^S > 0$, since $g_k^{*l} \ll \left[\tau\left(1-\alpha\right)A\right]^\alpha$ in the neighborhood of the low BGP. It follows that $z^i \equiv \hat{g}^i - 1 > 0$. Consequently: $Tr\left(\mathbf{B}\right) > 0$. Second, it can be readily verified that $RG^l < 0$; thus $Det\left(\mathbf{B}\right) < 0$ on the (unnecessary) sufficient condition that $S \cong 1$, which we suppose. It follows that $\mathbf{B}$ has exactly one negative and two positives eigenvalues, which proves point (i) of Proposition 6.