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Immanent Reasoning and the Dialogical Genealogy of Equality

Interaction and the notion of Equality in Constructive Type Theory

Shahid Rahman* and Nicolás Clerbout♦

I. Introduction

The question about the nature of the notion of identity is an old and venerable one and, in the western tradition the history of its written sources takes us from Parmenides' famous poem and its challenge by Heraclitus, to the discussions of Plato and Aristotle, up to the puzzles of Frege and Wittgenstein,¹ and the introduction of the notation “ = “ for it by Robert Recorde in 1557:

*And to avoide the tedious repetition of these woordes : is equalle to : I will sette as I doe often in woорke use, a paire of paralleles, or Gemowe lines of one lengthe, thus : =, bicause noe 2 thynges, can be moare equalle.*²

From the very start different pairs of concepts were linked to identity and puzzled the finest minds, such as numerical (or extensional) identity – qualitative identity (or intensional), ontological principle – logical principle, real-definition – nominal definition and on top of these pairs the relation between sign and object. The following puzzling lines of Plato's *Parmenides* contain already the core of many of the discussions that took place long after him:

*If the one exists, the one cannot be many, can it? No, of course not [...]. Then in both cases the one would be many, not one.” “True.” “Yet it must be not many, but one.” “Yes.” (Plato, *Parmenides*, 137c-d)*

Hegel takes the tension between the one and many mentioned by Plato as constitutive of the notion of identity. Moreover, Hegel defends the idea that the concept of identity, conceived as the fundamental law of thought, if it should express more than a tautology, must be understood as a principle that comprehends both the idea of identical (that expresses reflexive cases of the principle) and the idea of different (that expresses non-reflexive cases). Hegel points out that expressions such as $A = A$ have a “static” character empty of meaning – presumably in contrast to expressions such as $A = B$:

*In its positive formulation [as the **first law of thought**], $A = A$, this proposition is at first no more than the expression of empty tautology. It is rightly said, therefore, that this law of thought is without content and that it leads nowhere. It is thus to an empty identity that they cling, those who take it to be something true, insisting that identity is not difference but that the two are different. They do not see that in saying, “Identity is different from difference,” they have thereby already said that identity is something different. And since this must also be conceded as the nature of identity, the implication is that to be different belongs to identity not externally, but within it, in its nature. – But, further, inasmuch*

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¹ Quite often Plato's dialogue *Theaetetus* (185a) is mentioned as one of the earliest explicit uses of the principle.

² Recorde (1577). There are no page numbers in this work, but the quoted passage stands under the heading “The rule of equation, commonly called Algebers Rule” which occurs about three quarters into the work. The quote has been overtaken from Granström (2011), p. 33.

as these same individuals hold firm to their unmoved identity, of which the opposite is difference, they do not see that they have thereby reduced it to a one-sided determinateness which, as such, has no truth. They are conceding that the principle of identity only expresses a one-sided determinateness, that it only contains formal truth, truth abstract and incomplete. – Immediately implied in this correct judgment, however, is that the truth is complete only in the unity of identity and difference, and, consequently, that it only consists in this unity . (Hegel (2010), 1813, Book 2, Vol. 2, II.258, 2nd remark, p. 358).³

What Hegel is going after, is that the clue for grasping a conceptually non-empty notion of identity lies in the understanding the links of the reflexive with the non-reflexive form and vice-versa.

The history of studies involving this interplay, before and after Hegel, is complex and rich. Let me briefly mention in the next section the well-known “linguistic” approach to the issue that followed from the work of Gottlob Frege and Ludwig Wittgenstein, that had a decisive impact in the logical approach to identity.

I.1 Intensional and extensional equality at the propositional level:

One of the most influential studies of the relation between sign and object as involving the (dyadic) equality-predicate expressed at the propositional level was the one formulated in 1892 by Gottlob Frege in his celebrated paper *Über Sinn und Bedeutung*. The paper starts by asking the question: Is identity a relation? If it is a relation, is it a relation between objects, or between signs of objects. To take the notorious example of planet Venus, *the morning star = the morning star* is a statement very different in cognitive value from *the morning star = the evening star*. The former is analytically true, while the second records an astronomical discovery. If we were to regard identity as a relation between a sign and what the sign stands for it would seem that if $a = b$ is true, then $a = a$ would not differ from $a = b$. *A relation would thereby be expressed of a thing to itself, and indeed one in which each thing stands to itself but to no other thing.* (Frege, *Über Sinn und Bedeutung*, pp. 40-42). On the other hand if every sentence of the form $a = b$ really signified a relationship between symbols, it would not express any knowledge about the extra-linguistic world. The equality *morning star = the evening star* would record a lexical fact rather than an astronomical fact. Frege’s solution to this dilemma is the famous difference between the way of presentation of an object, called its *sense (Sinn)* and the *reference (Bedeutung)* of that object. In the equality *the morning star = the evening star* the reference of the two expressions at each side of the relation is the same, namely the planet Venus, but the sense of each is different. This distinction entitles Frege the following move: a statement of identity can be informative only if the difference between

³ *Der Satz der Identität [als das erste Denkgesetz] in seinem positiven Ausdrucke $A=A$, ist zunächst nichts weiter, als der Ausdruck der leeren Tautologie. Es ist daher richtig bemerkt worden, daß dieses Denkgesetz ohne Inhalt sey und nicht weiter führe. So ist die leere Identität, an welcher diejenigen festhängen bleiben, welche sie als solche für etwas Wahres nehmen und immer vorzubringen pflegen, die Identität sey nicht die Verschiedenheit, sondern die Identität und die Verschiedenheit seyen verschieden. Sie sehen nicht, daß sie schon hierin selbst sagen, daß die Identität ein Verschiedenes ist; denn sie sagen, die Identität sey verschieden von der Verschiedenheit; indem dieß zugleich als die Natur der Identität zugegeben werden muß, so liegt darin, daß die Identität nicht äußerlich, sondern an ihr selbst, in ihrer Natur dieß sey, verschieden zu seyn. - Ferner aber indem sie an dieser unbewegten Identität festhalten, welche ihren Gegensatz an der Verschiedenheit hat, so sehen sie nicht, daß sie hiermit dieselbe zu einer einseitigen Bestimmtheit machen, die als solche keine Wahrheit hat. Es wird zugegeben, daß der Satz der Identität nur eine einseitige Bestimmtheit ausdrücke, daß er nur die formelle eine abstrakte, unvollständige Wahrheit enthalte. - In diesem richtigen Urtheil liegt aber unmittelbar, daß die Wahrheit nur in der Einheit der Identität mit der Verschiedenheit vollständig ist, und somit nur in dieser Einheit bestehe.* (Hegel (1999), 1813, Teil 2, Buch II; II.258, pp. 29-30).

signs corresponds to a difference in the mode of presentation of the object designated (Frege, *Über Sinn und Bedeutung*, p. 65): that is why, according to Frege, $a = a$ is not informative but $a = b$ is.

In the *Tractatus Logico-Philosophicus* Ludwig Wittgenstein, who could be seen as addressing Hegel's remark quoted above, adds another twist to Frege's analysis:

- 5.53 Identity of object I express by identity of sign; and not by using a sign for identity. Difference of objects I express by difference of signs.
 - 5.5301 Obviously, identity is not a relation between objects [...].
 - 5.5303 By the way, to say of two things that they are identical is nonsense, and to say of one thing that it is identical with itself is to say nothing at all.
- (L. Wittgenstein, *Tractatus Logico-Philosophicus*)

Wittgenstein's proposal is certainly too extreme: a language that provides a different sign to every different object will make any expression of equality false and thus the use of equations, such as arithmetical ones, will be impossible.

Unsurprisingly, Wittgenstein's proposal was not followed, particularly not by either logicians or mathematicians. In fact, in standard first-order logic, it is usual to introduce an equality-predicate for building propositions that express numerical equality. Moreover, numerical equality is seen as a special case of qualitative equality. Indeed, qualitative identities or equivalences are relations which are reflexive, symmetric and transitive and structure the domain into disjoint subsets whose members are regarded as indiscernible with respect to that relation. *Identity* or numerical identity is the smallest equivalence relation, so that each of the equivalence classes is a singleton, i.e., each contains one element

However, when introducing equations in the way we are used to in mathematics there are two main different notions at stake. On the one hand we use equality when introducing both *nominal definitions* (that establish a relation between linguistic expressions – such a relation yields abbreviations) and *real definitions* (that establish a relation between objects within a type – this relation yields equivalences in the type). But definitions are neither true nor false, though real definitions can make propositions true. For example, the following equalities are not propositions but certainly constitute an assertion:

$a + 0$ and a are equal objects in the set of numbers

Which we can write – using the notation of chapter 2 – as:

$a + 0 = a : \textit{number}$

Since it is an assertion we can formulate the following inference rule:

$$\frac{a : \textit{number}}{\text{-----}} \\ a + 0 = a : \textit{number}$$

Once more, a real definitional equality is a relation between objects, it does not express a proposition. In other words, it is not the dyadic-predicate as found in the usual presentation of first-order logic. However in mathematics, we do have, and even need, an equality predicate. For example when we assert that $a + b = b + a$. In fact, we can prove it: we prove it by

induction. It is proving the proposition that expresses the commutativity of equality. Thus equality expresses here a dyadic predicate.⁴

- Since we do not have much to add to the subject of nominal definitions, in the following, when we speak of *definitional equality* we mean those equalities that express a *real definition*.

It is the Constructive Type Theory of Per Martin-Löf that enabled us to express these different forms of equality in the object language. On our view, these distinctions can be seen as the result of the different forms that a specific kind of dynamic process can take when (what we call) *immanent reasoning* is deployed. Immanent reasoning is the reasoning where the speaker endorses his responsibility of grounding the conclusion by rooting it in the assertion of the relevant premises made by the antagonist. In fact the point of such a kind of reasoning is that the speaker accepts the assertions of the premises brought forward by the antagonist and he has now the duty to develop his reasoning towards the conclusion based on this acceptance. We call this kind of reasoning *immanent* since there is no other authority that links premises and conclusion beyond the intertwining of acceptance and responsibility during the interaction. Göran Sundholm (1997) called such premises *epistemic assumptions*, since with them we assume that the proposition involved is known, though no demonstration backing the assumption has been (yet) produced. In a recent talk in Paris, Martin-Löf (2015) provided a dialogical interpretation for them:

[...] *the speaker is under an obligation, he is undertaking a certain duty when he makes the assertion, whereas the hearer has the right to trust that he can fulfil his obligation. So the speaker has a duty, whereas the hearer has a right, and right I take to be the same as permission. So, you see that from this dialogical perspective these deontic notions of obligation and permission come in, and they are of course central notions of normative ethics [...].*

[...] *I have thought about them in this way, because I have been plagued, since six years ago in connection with a meeting organized by Maria van der Schaar, called Days of Judgement, in Leiden, that was September 2009, six years ago when preparing that lecture I became acutely aware of a circularity problem which I had not seen before [...].*

[...] *When you are giving an account of the notion of immediate inference, the notion of demonstration is not yet at your disposal. So, to say, Assume that J_1, \dots, J_n have already been demonstrated makes you accusable of trying to explain things in a circle. The solution to this circularity problem, it seems to me now, comes naturally out of this dialogical analysis – once you have seen it, you can go to the normal logical situation and explain things properly there also, but at least I have seen it via the dialogical analysis. The solution is that the premisses here should not be assumed to be known in the qualified sense, that is, to be demonstrated, but we should simply assume that they have been asserted, which is to say that others have taken responsibility for them, and then the question for me is whether I can take responsibility for the conclusion. So, the assumption is merely that they have been asserted, not that they have been demonstrated. That seems to me to be the appropriate definition of epistemic assumption in Sundholm's sense.*

These paragraphs, deploy in a deontic language⁵ one of the main features of the dialogical framework: the proponent is entitled to use the opponent's moves in order to develop the defence of his own thesis.⁶ According to this perspective the proponent takes the assertions of the opponent as epistemic assumptions (to put it into Sundholm's happy terminology), and

⁴ For a thorough discussion on this issue see Granström (2011, pp. 30-36, and pp. 63-69)

⁵ Let us point out that one of the main philosophical assumptions of the constructivist school of Erlangen was precisely the tight interconnection between logic and ethics, see among others: Lorenzen/Schwemmer (1975) and Lorenzen (1984).

⁶ In fact, Martin-Löf's discussion is a further development of Sundholm's remark on the links between some pragmatist tenets and inferentialism, that emerge from the following insight of J. L. Austin (1946): "When I say *therefore*, I give others my authority for asserting the conclusion, given theirs for asserting the premises."

this means that the proponent trusts them only because of its force, just because she claims that she has some grounds for them.⁷

The main aim of the present study is to show that the Π - and Σ -rules of definitional equality can be seen as highlighting the dialogical interaction between entitlements and duties mentioned above. Under this perspective the standard monological presentation of these rules for definitional equality encodes implicitly an underlying process – by the means of which the proponent “copies” some of the opponent’s choices – that provides its dialogical and normative roots. Moreover, this can be extended to the dialogical interpretation of the equality-predicate. We are tracing back, in other words, the systematic origins of the dialogical interpretation recently stressed by Göran Sundholm and Per Martin-Löf. This journey to the origins also engages us to study the whole process at the level of plays, that is, the material which winning-strategies (the dialogical notion of demonstration) are made of. Our study centers in what we called immanent reasoning, where, as explained, the claims of the opponent are taken to be grounded, without requiring a defence for them. However, (as will be discussed at the end of the present study), we could also develop a similar kind of analysis for the case of the so-called *material dialogues*: here the claims of the opponent engage her to put under dialogical scrutiny that what supports her claim. This should provide the most basic form of definitional equality, and not only of the definitional equality displayed by the Π - and Σ -rules. However, this development will be left for future work. More generally, the conceptual links between equality and the formal rule, nowadays called by Marion/Rückert (2015) more aptly the *Socratic rule*, is one of the many lessons Plato and Aristotle left us concerning the meaning of expressions taking place during an argumentative process. Unfortunately we cannot discuss here the historical source which must also be left for future studies. What we will deploy here are the systematic aspects of the interaction that links equality and immanent reasoning.

Before we start our journey towards a dynamic perspective on identity, let briefly mention the take of *Constructive Type Theory*, on equality.

⁷ The recent terminology in dialogical logic, following a suggestion of Sundholm, uses the denomination “posit” rather than epistemic assumption, but this presumably is only a terminological variant.