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JEL codes:

C72, D71

STRATEGY PROOFNESS AND UNANIMITY IN PRIVATE GOOD ECONOMIES WITH SINGLE-PEAKED PREFERENCES

MOSTAPHA DISS* AHMED DOGHMI† ABDELMONAIM TLIDI‡

THIS VERSION: NOVEMBER 6, 2015

Abstract

In this paper we examine the relation between strategy-proofness and unanimity in a domain of private good economies with single-peaked preferences. We prove that, under a mild condition, a social choice function satisfies strategy-proofness if and only if it is unanimous. As implication, we show that when the property of citizen sovereignty holds, strategy-proofness and Maskin monotonicity become equivalent. We also give applications to implementation literature: We provide a full characterization for dominant strategy implementation, standard Nash implementation, and partially honest Nash implementation and we prove that these theories are equivalent.

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1 Introduction

Arrow [2] was the first who wondered about the issue of constructing non-dictatorial welfare functions by examining the combination of some desirable properties. These well-known prespecified properties are called unrestricted domain (U), non-dictatorship (ND), Pareto efficiency (PE), and independence of irrelevant alternatives (IIA). He showed that, if there are at least two individuals and at least three alternatives, any social welfare function satisfying U , PE , and IIA is dictatorial. Gibbard [13] and Satterthwaite [21] inspected the possibility of constructing non-dictatorial and non-manipulable (strategy-proof) social choice functions. They proved that, in a strategic voting, if there are at least three alternatives, any strategy-proof social choice function is dictatorial.

From these two well-known impossibility results, the literature on social welfare and social choice functions contains two main approaches in order to derive possibility results. The first one is based on the relaxation of some properties provided by Arrow [2] for social welfare functions and Gibbard [13] and Satterthwaite [21] for social choice functions. The second approach, that we are concerned in this paper, is based on the restriction of preference domains available to the individuals. The most commonly used domain restriction on individual preferences is single-peakedness which allows to have very nice and interesting results. This notion, introduced by Black [4], requires that each agent has a unique best alternative. In this domain, a large number of research papers have examined the class of strategy-proof rules in different types of economies and voting schemes. For instance, Moulin [19] showed that, when adding some fixed ballots to the agent's ballots, all strategy-proof anonymous and efficient voting rules can be derived from the Condorcet procedure.

In parallel to strategy-proof rules, Maskin monotonic rules are another important class of rules which attracted the attention of several authors. Muller and Satterthwaite [20] were among the first authors who characterized this type of rules. They proved that, when there are at least three alternatives, every voting procedure that satisfies Maskin monotonicity and citizen sovereignty is dictatorial. More recently, Aşan and Sanver [1] examined Maskin monotonic aggregation rules. They showed that, when the properties of anonymity and neutrality hold, an aggregation rule is Maskin monotonic if and only if it is an absolute qualified majority rule. Doğan and Koray [11] characterized Maskin monotonic scoring rules using a certain unanimity requirement.

In addition to these studies which examined separately each of the two properties of strategy-proofness and Maskin monotonicity, some literature has been interested in the connection between these two requirements. For instance, the first work returns to Muller and Satterthwaite [20] who showed that, in a model of public good economies, the two conditions are equivalent. Bochet and Storcken [5] defined several conditions to construct maximal domains for Maskin monotone and strategy-proof rules. They proved that a choice rule is strategy-proof if and only if it is Maskin monotone. Recently, Klaus and Bochet [17] generalized the model of Muller and Satterthwaite by covering private goods economies and they proved that there is a close link between the two conditions.

The characterization of strategy-proofness is not limited only to its relationship with Maskin monotonicity, but covers other properties; the well-known is the one of Pareto efficiency. The connection of this property with strategy-proofness was established in a wide range of literature and in various contexts. It begins with Hurwicz [16] who studied

the structure of strategy-proof and Pareto-efficient social choice functions in classical exchange economies. In these domains, Zhou [25] proved that there is no allocation mechanism that is efficient, non-dictatorial, and strategy-proof. Both Hurwicz [16] and Zhou [25] considered a classical domain where agents preferences are assumed to be continuous, strictly monotonic and strictly convex.

These studies are succeeded by many others on domain restriction problem in pure exchange economies. For example, Schummer [22] assumed that agents have homothetic and strictly convex preferences and examined the existence of rules that verify the properties of strategy-proofness and Pareto-efficiency. He proved that any rule satisfying these two properties is dictatorial. Hashimoto [15] considered a domain of Cobb-Douglas preferences and demonstrated that a SCF is strategy-proof and Pareto-efficient if and only if it is dictatorial. More recently, in a specific quasi-linear domain of pure exchange economies, Goswami et al [14] showed that if a SCF satisfies Pareto-efficiency, strategy-proofness, non-bossiness and a mild continuity property, then it is dictatorial.

In parallel to this literature in pure exchange economies, other studies have examined strategy-proof and Pareto-efficient SCFs in different economic domains such as public good economies. However, there are few studies that inspected strategy-proofness and Pareto-efficiency in private good economies. The well-known work in this subject is the one of Sprumont [23] who characterized the uniform rule of Benassy [3] on single-peaked domain. He established that this rule is the only rule that is strategy-proof, anonymous, and Pareto-efficient. Ching [7] reinforced this result by replacing anonymity by symmetry. He proved that the uniform rule is the only rule that satisfies strategy-proofness, symmetry, and Pareto-efficiency.

In this paper, we consider a domain of private good economies with single-peaked preferences and we introduce the property of Unanimity as a principle requirement to characterize strategy-proof social choice functions for fair allocation problem. Unanimity is a very mild efficiency requirement that tends to be quite compatible with strategy-proofness and also with other requirements, but not much attention has been paid to this property in previous literature. By introducing a weak condition on agents preferences, we prove that a SCF satisfies strategy-proofness if and only if it is unanimous. This result has an implication on the relationship between strategy-proofness and Maskin monotonicity. Thus, we show that if the requirement of citizen sovereignty holds, these two properties become equivalent. In addition and in connection with implementation literature, we give a full characterization for dominant strategy implementation, standard Nash implementation, and partially honest Nash implementation and we prove that these theories are equivalent.

The rest of this paper is organized as follows. In Section 2, we introduce notations and definitions in our framework of private good economies with single-peaked preferences. In Section 3, we present the main result of the paper that establishes the equivalence between strategy-proofness and unanimity. In Section 4, we study the implication of this result on the relationship between strategy-proofness and Maskin monotonicity. Section 5 examines the connection between these results and implementation literature. Section 6 concludes.

2 Notations and definitions

In this section, we provide the terminology and notations required for our results. We use the standard model of private good economies with single-peaked preferences. We consider an amount $\Omega \in \mathbb{R}_{++}$ of a certain infinitely divisible good that is to be allocated among a set $N = \{1, \dots, n\}$ of n agents. We represent the preference of each agent $i \in N$ by a continuous and single-peaked preference relation R_i over $[0, \Omega]$. For all $x_i, y_i \in [0, \Omega]$, $x_i R_i y_i$ means that, for the agent i , to consume a share x_i is as good as to consume the quantity y_i . The asymmetrical and symmetrical parts of the relation R_i are given by P_i and \sim_i , respectively. Single-peakedness means that there is a number $p(R_i)$, called the *peak* of R_i , such that for all $x_i, y_i \in [0, \Omega]$, if $y_i < x_i \leq p(R_i)$ or $p(R_i) \leq x_i < y_i$, then $x_i P_i y_i$.

The class of all single-peaked preference relations is represented by $\mathfrak{R}_{sp_i} \subseteq \mathfrak{R}_i$. Let $\mathfrak{R}_{sp} = \mathfrak{R}_{sp_1} \times \dots \times \mathfrak{R}_{sp_n}$ be the domain of single-peaked preferences. For $R \in \mathfrak{R}_{sp}$, let $p(R) = (p(R_1), \dots, p(R_n))$ be the profile of peaks (or of preferred consumptions). A single-peaked preference relation $R_i \in \mathfrak{R}_{sp_i}$ can be described by the function $r_i : [0, \Omega] \rightarrow [0, \Omega]$ which is defined as follows: $r_i(x_i)$ is the consumption of the agent i on the other side of the peak which is indifferent to x_i if it exists, or else, it is 0 or Ω . In other words, if $x_i \leq p(R_i)$, then, $r_i(x_i) \geq p(R_i)$ and $x_i \sim_i r_i(x_i)$ if such a number exists or $r_i(x_i) = \Omega$ otherwise. However, if $x_i \geq p(R_i)$, then, $r_i(x_i) \leq p(R_i)$ and $x_i \sim_i r_i(x_i)$ if such a number exists or $r_i(x_i) = 0$ otherwise.

For $R \in \mathfrak{R}_{sp}$, a feasible allocation for the economy (R, Ω) is a vector $x \equiv (x_i)_{i \in N} \in \mathbb{R}_+^n$ such that $\sum_{i \in N} x_i = \Omega$ and X is the set of the feasible allocations. For all $R_i \in \mathfrak{R}_{sp_i}$ and all $x \in X$, the lower contour set for agent i at allocation x is denoted by $L(x, R_i) = \{y \in X \mid x R_i y\}$. The strict lower contour set and the indifference lower contour set are denoted $LS(x, R_i) = \{y \in X \mid x P_i y\}$ and $LI(x, R_i) = \{y \in X \mid x \sim_i y\}$, respectively. We note that the feasible allocations set is $X \subseteq [0, \Omega] \times \dots \times [0, \Omega]$. Thus, $L(x, R_i) = X$ is equivalent to $L(x_i, R_i) = [0, \Omega]$. In addition, for all two feasible allocations $x \equiv (x_i)_{i \in N}$ and $y \equiv (y_i)_{i \in N}$ in the set of the feasible allocations X , the expression $x R_i y$ implies $x_i R_i y_i$. Finally, we note that the free disposability of the good is not assumed and we introduce the definitions that will be useful throughout the paper. Our first definition is the notion of *Social choice function*.

Definition 1. (*Social choice function*)

A *social choice function (SCF)* is a single-valued mapping from \mathfrak{R}_{sp} into X , that associates to every $R \in \mathfrak{R}_{sp}$ an element of X .

In other words, a SCF is a function that maps the individual preferences (here assumed to be single-peaked preferences) to a single collective choice in the set X . We now discuss in turn the properties that will play a central role throughout this essay.

The property of *Strategy-proofness* gives agents an incentive to bid their true preferences since it is a property which requires that no agent ever benefits from misrepresenting his preference relation. For agent $i \in N$ and a preference profile $R \in \mathfrak{R}_{sp}$, we obtain a preference profile $(R'_i, R_{-i}) \in \mathfrak{R}_{sp_i}$ by replacing the i 's true preference R_i by R'_i and keeping the preferences of other agents R_{-i} unchanged. Then, strategy-proofness is formally defined as follows:

Definition 2. (*Strategy-proofness*)

A SCF f satisfies the strategy-proofness property if for all $R \in \mathfrak{R}_{sp}$, all $i \in N$, and all $R'_i \in \mathfrak{R}_{sp_i}$, $f(R)R_i f(R'_i, R_{-i})$.

Next, the well-known *Unanimity* condition means that if everyone prefers a certain allocation to another one, then society should produce the same preference. In our context, it is equivalent to the requirement that if there is an allocation at which each agent receives his peak amount, then it should be chosen by the society. The formal statement of this property is given as follows:

Definition 3. (*Unanimity*)

A SCF f satisfies unanimity if for any $x \in X$, any $R \in \mathfrak{R}_{sp}$, and any $i \in N$, $L(x_i, R_i) = [0, \Omega]$ implies $f(R) = x$.

The next axiom is called *Citizen sovereignty* which requires that every possible ranking of allocations can be achieved from a set of individual preference allocations. In other terms, this condition implies that everyone must, without restriction, have a say in the allocation process. Formally, we have the following definition:

Definition 4. (*Citizen sovereignty*)

A SCF f satisfies the property of citizen sovereignty if for each $x \in X$, there is a profile $R \in \mathfrak{R}_{sp}$ such that $f(R) = x$.

We also consider the well-known *Maskin monotonicity* condition. Loosely speaking, consider some profile of preferences R and an allocation x chosen as a solution by the social choice function f : $f(R) = x$. Now, consider a second preference profile R' such that, for each agent, the set of allocations that he now finds at most as good as that allocation contains the corresponding set for his initial preferences: $L(x_i, R_i) \subseteq L(x_i, R'_i)$. Then, the allocation should still be chosen as a solution for the new profile: $f(R') = x$. Maskin monotonicity is formally defined as follows:

Definition 5. (*Maskin monotonicity*)

For all $R, R' \in \mathfrak{R}_{sp}$, a SCF f satisfies Maskin monotonicity if for any $f(R) = x$ and any $i \in N$, $L(x_i, R_i) \subseteq L(x_i, R'_i)$ implies $f(R') = x$.

3 The main result

In this section we study the relation between strategy-proofness and unanimity in private good economies with single-peaked preferences. More precisely, in this framework and under a certain condition, we prove in Theorem 1 that a SCF satisfies strategy-proofness if and only if it is unanimous. In this way, both Proposition 1 and Proposition 2 present a complete proof of this theorem.

Proposition 1. *In the private good economies with single-peaked preferences, any strategy-proof SCF satisfies unanimity.*

Proof. Assume that f satisfies strategy-proofness, but not unanimity. The unanimity is violated if, for any $x \in X$, any $R \in \mathfrak{R}_{sp}$, and any $i \in N$, we have $L(x_i, R_i) = [0, \Omega]$

(1), but $f(R) \neq x$. By strategy-proofness, we have for all $R \in \mathfrak{R}_{sp}$, all $i \in N$, and all $R'_i \in \mathfrak{R}_{sp_i}$, $f(R)R_i f(R'_i, R_{-i})$. Denote $f(R) = y \neq x$ and $f(R'_i, R_{-i}) = x$. Hence, for all $R \in \mathfrak{R}_{sp}$, all $i \in N$, $y_i R_i x_i$. From (1), we have for all $R \in \mathfrak{R}_{sp}$, all $i \in N$, $x_i \sim_i y_i$. By (1) and single-peakedness, $x_i = p(R_i)$. Since by definition the peak $p(R_i)$ must be a singleton, then $x_i = y_i$ for all $i \in N$, and thus $x = y$, a contradiction. $Q.E.D.$

To examine the converse, we need to introduce the following preference domain “richness” condition. The formal statement of this property is as follows:

Condition 1. *Let $R \in \mathfrak{R}_{sp}$, $i \in N$, and $x, y \in X$. If $x_i \in LS(y_i, R_i)$, then there exists $R' \in \mathfrak{R}_{sp}$ with $R'_i \neq R_i$ and $R'_{-i} = R_{-i}$ such that $LS(y_i, R_i) \subseteq L(x_i, R'_i)$, and $L(x_{-i}, R'_{-i}) = [0, \Omega]$.*

Roughly, Condition 1 means that if a share x_i is strictly dominated by a quantity y_i for an agent i at a profile R , then there exists a new profile R' , in which only the agent i change her/his preference, such that the share x_i is top-ranked for all agents at the new profile R' . Condition 1 is logically independent to the preference domain richness conditions developed in Klaus and Bochet [17]. Under our rich domain, we obtain the following proposition.

Proposition 2. *In the private good economies with single-peaked preferences, if Condition 1 holds, then any unanimous SCF satisfies strategy-proofness.*

Proof. Assume that f satisfies unanimity, but not strategy-proofness. The strategy-proofness is violated if there exist $R \in \mathfrak{R}_{sp}$, $i \in N$, and $R'_i \in \mathfrak{R}_{sp_i}$, such that $f(R'_i, R_{-i})P_i f(R)$ (1). Denote $f(R) = x$ and $f(R'_i, R_{-i}) = y$. Hence $x_i \in LS(y_i, R_i)$ and by Condition 1 then there exists $R' \in \mathfrak{R}_{sp}$ with $R'_i \neq R_i$ and $R'_{-i} = R_{-i}$ such that $LS(y_i, R_i) \subseteq L(x_i, R'_i)$, and $L(x_{-i}, R_{-i}) = [0, \Omega]$. By single-peakedness, $L(x_i, R'_i) = [0, \Omega]$ for all $i \in N$. By unanimity $f(R'_i, R_{-i}) = x$. Therefore, $x = y$ which contradicts (1). $Q.E.D.$

From Propositions 1 and 2, we complete the proof of the following theorem.

Theorem 1. *In the private good economies with single-peaked preferences, if Condition 1 holds, then a SCF satisfies strategy-proofness if and only if it is unanimous.*

4 Implications

As noticed in the Introduction, the investigation of the relation between Maskin monotonicity condition and strategy-proofness is not new and there is considerable literature dealing with this issue. In this section, we study the consequence(s) of introducing of the property of unanimity on the relationship between strategy-proofness and Maskin monotonicity. For this, we begin by examining the relation between unanimity and Maskin monotonicity. In this way, we give the following proposition.

Proposition 3. *In the private good economies with single-peaked preferences, if the property of citizen sovereignty holds, then any Maskin monotonic SCF satisfies unanimity.*

Proof. Suppose not. Let $x \in X$ and $\tilde{R} \in \mathfrak{R}_{sp}$ be such that for any $i \in N$, $[0, \Omega] = L(x_i, \tilde{R}_i)$, and $f(\tilde{R}) \neq x$. By the property of citizen sovereignty, for all $x \in X$, there is a profile $R \in \mathfrak{R}_{sp}$ such that $f(R) = x$ and so for all $i \in N$, $L(x_i, R_i) \subseteq [0, \Omega] = L(x_i, \tilde{R}_i)$. By Maskin monotonicity, $f(\tilde{R}) = x$, a contradiction. *Q.E.D.*

From Proposition 2 and Proposition 3, we give the following corollary.

Corollary 1. *In the private good economies with single-peaked preferences, if the requirements of citizen sovereignty and Condition 1 hold, then any Maskin monotonic SCF satisfies strategy-proofness.*

Now, by using Proposition 1, we prove that strategy-proofness implies Maskin monotonicity.

Proposition 4. *In the private good economies with single-peaked preferences, any strategy-proof SCF satisfies Maskin monotonicity.*

Proof. Assume that f satisfies strategy-proofness, but not Maskin monotonicity. Then, for any $R, R' \in \mathfrak{R}_{sp}$, any $f(R) = x$, and any $i \in N$, $L(x_i, R_i) \subseteq L(x_i, R'_i)$ (1), but $f(R') \neq x$ (2). In addition, f satisfies strategy-proofness, that is for all $R \in \mathfrak{R}_{sp}$, all $i \in N$, and all $R'_i \in \mathfrak{R}_{sp_i}$, $f(R)R_i f(R'_i, R_{-i})$. Denote $f(R'_i, R_{-i}) = y$, hence $x_i R_i y_i$ (3). From Proposition 1, f is unanimous. Hence, by (2), $f(R') \neq x$ implies that there exist $i \in N$ and $y \in X$ such that $y_i P'_i x_i$ and by (1) we have $y_i P_i x_i$, which contradicts assumption (3). *Q.E.D.*

Through Corollary 1 and Proposition 4, we complete the proof of the second main theorem of the paper.

Theorem 2. *In the private good economies with single-peaked preferences, if the requirements of citizen sovereignty and Condition 1 hold, a SCF satisfies strategy-proofness if and only if it is Maskin monotonic.*

5 Applications to Nash and dominant strategy implementation

In this section we present two applications of our findings on some results from implementation theory. We begin this section by recalling key ideas from implementation theory that are relevant to the topics of our applications.

Implementation theory provides a framework for situations where resources have to be allocated among agents but the information needed to make these allocation decisions is dispersed and privately held. In addition, the agents possessing the private information behave strategically. When a social designer want to maximize the welfare of a society, which is represented by a social choice rule that selects desired outcomes, she/he confronts some agents who state false preferences on alternatives in order to improve their payoff. To address this problem of the truth revelation, the social designer must conceive a mechanism (game form) which interacts individuals according to their strategic behavior. These individual strategic interactions can produce the predicted outcomes *via* a solution

concept (equilibrium) of the game. A social choice rule is said to be implementable if both the desired and predicted outcomes coincide. This is eventually the aim of implementation theory.

In this literature, the conditions of strategy-proofness and Maskin monotonicity are central. More precisely, strategy-proofness is a necessary condition for dominant strategy implementation, and Maskin monotonicity is a necessary condition for Nash implementation. Using the above results of Theorems 1 and 2, we provide in the next subsections a full characterization.

5.1 Strategy-proofness versus Nash and dominant strategy implementation with standard agents

A SCF f provides desired outcomes for a social designer. To implement this function, the social designer organizes a non-cooperative game (game form) among a set of agents. This game form is a pair $\Gamma = (S, g)$ with $S = S_1 \times \dots \times S_n$ and $g : S \rightarrow X$. For each agent $i \in N$, S_i is agent i 's strategy space, and g is the outcome function that associates an outcome with each profile of strategies. Let $R \in \mathfrak{R}_{sp}$ a profile of preferences, and let Nash equilibrium be a solution concept of the game (Γ, R) . The set of Nash equilibria at state R is denoted by $NE(S, g, R)$ and the set of Nash equilibria outcomes is $g(NE(S, g, R))$. A mechanism $\Gamma = (S, g)$ implements a SCF f in Nash equilibria if for all $R \in \mathfrak{R}_{sp}$, $f(R) = g(NE(S, g, R))$. Alternatively, let dominant strategy equilibrium be a solution concept of the game (Γ, R) . The set of dominant strategy equilibria at state R is denoted by $DSE(S, g, R)$ and the set of strategy dominant equilibria outcomes is $g(DSE(S, g, R))$. A mechanism $\Gamma = (S, g)$ implements a SCF f in dominant strategy equilibria if for all $R \in \mathfrak{R}_{sp}$, $f(R) = g(DSE(S, g, R))$. We say that a SCF f is implementable in Nash equilibria (respectively, in dominant strategy equilibria) if there is a mechanism which implements it.

Now, we define the strict version of Maskin monotonicity developed by Yao and Yi [24] that we call Y -monotonicity. This condition is formally defined as follows:

Definition 6. (*Y-monotonicity*)

A SCF f satisfies Y-monotonicity if for all $R, R' \in \mathfrak{R}_{sp}$, for any $f(R) = x$, if for any $i \in N$, $L(x_i, R_i) \setminus \{x_i\} \subseteq LS(x_i, R'_i)$, then $f(R') = x$.

In other terms, consider some profile of preferences $R = (R_1, \dots, R_n)$ and an allocation x such that the SCF f assigns alternative x when the preference profile is R , i.e., $f(R) = x$. If the preferences of each individual $i \in N$ change from R_i to R'_i in a monotonic way around x_i such that whenever $x_i R_i y_i$ one has that $x_i P'_i y_i$, then the alternative socially chosen with R should not change with R' : $f(R') = x$.

Proposition 5. *In the private good economies with single-peaked preferences, Y-monotonicity becomes equivalent to Maskin monotonicity.*

Proof. Let $R, R' \in \mathfrak{R}_{sp}$, $x, y, z \in X$, and $f(R) = x$. *i)* Y -monotonicity implies Maskin monotonicity; this first implication is immediate from the inclusions $L(x_i, R_i) \setminus \{x_i\} \subseteq L(x_i, R_i)$ and $LS(x_i, R'_i) \subseteq L(x_i, R'_i)$. *ii)* Maskin monotonicity implies Y -monotonicity; in this case, suppose that $L(x_i, R_i) \setminus \{x_i\} \subseteq LS(x_i, R'_i)$. From this, we have $L(x_i, R_i) =$

$L(x_i, R_i) \setminus \{x_i\} \cup \{x_i\} \subseteq LS(x_i, R'_i) \cup \{x_i\}$. Since $LS(x_i, R'_i) \cup \{x_i\} \subseteq L(x_i, R'_i)$, we obtain $L(x_i, R_i) \subseteq L(x_i, R'_i)$ for all $i \in N$. By Maskin monotonicity, $f(R) = x$. *Q.E.D.*

In addition, we know from Corollary 1 of Yao and Yi [24] that when there are at least three alternatives, every SCF that satisfies citizen sovereignty is fully implementable in dominant strategies if and only if it satisfies Maskin monotonicity and Y -monotonicity. As a consequence of this Corollary and From Proposition 5, we complete the proof of the following theorem.

Theorem 3. *Let $n \geq 3$. In the private good economies with single-peaked preferences, if the properties of citizen sovereignty holds, a SCF is implementable in dominant strategies if and only if it satisfies Maskin monotonicity.*

In addition, from Theorem 2 of Doghmi and Ziad [10], it follows that in the private good economies with single-plateaued preferences, when there are at least three alternatives and if the requirement of citizen sovereignty holds, any SCF is Nash implementable if and only if it satisfies Maskin monotonicity. Notice that this result remains true for the domain of single-peakedness which is a particular case of the large domain of single-plateauedness. This domain, which allows agents to be indifferent among several best elements, has been explored by several authors in social choice theory and games theory. The reader is referred to the more recent work of Bossert and Peters [6]. As a direct consequence of this result and Theorem 3 above, we have the following important result:

Corollary 2. *In the private good economies with single-peaked preferences, Nash and dominant strategy implementation are equivalent.*

The relation of strategy-proofness with Nash and dominant strategy implementation can be summarized from Theorems 2 and 3, Theorem 2 of Doghmi and Ziad [10], Corollary 1 of Yao and Yi [24] and Corollary 2 as follows.

Corollary 3. *Let $n \geq 3$. In the private good economies with single-peaked preferences, if the requirements of citizen sovereignty and Condition 1 hold, a SCF is Nash and dominant strategy implementable if and only if it satisfies strategy-proofness.*

5.2 Strategy-proofness and unanimity versus Nash implementation with partially honest agents

Here, we present an environment of partial honest agents. We consider the same well-known model as the one considered in Dutta and Sen [12], Lombardi and Yoshihara [18], and Doghmi and Ziad [9], among others. More precisely, we assume that there are some players who have a “small” intrinsic preference for honesty and each honest individual expresses her preferences in a *lexicographic* way. For a domain of single-peaked preferences \mathfrak{R}_{sp} , let C_i be the other components of the strategy space (which depends on individual preferences, social states, ...). The set $S_i = \mathfrak{R}_{sp} \times C_i$ represents the strategy profiles for a player i and $S = S_1 \times \dots \times S_n$ is a set of strategy profiles. The elements of S are denoted by $s = (s_1, \dots, s_n)$. A *domain* is a set $\mathcal{D}_{sp} \subset \mathfrak{R}_{sp}$. For each $i \in N$, and $R \in \mathcal{D}_{sp}$, let $\tau_i(R) = \{R\} \times C_i$ be the set of truthful messages of agent i . We denote

by $s_i \in \tau_i(R)$ a truthful strategy as player i is reporting the true preference profile. We extend a player's ordering over the set X to an ordering over strategy space S . This is because, the players' preference between being honest and dishonest depends on strategies that the others played and the outcomes which they obtain. Let \succeq_i^R be the preference of player i over S in preference profile R . The asymmetrical and symmetrical parts of \succeq_i^R are denoted respectively by \succ_i^R and \sim_i^R . Let Γ be a mechanism (game form) represented by the pair (S, g) , where $S_i = \mathcal{D}_{sp} \times C_i$ and $g : S \rightarrow A$ is a payoff function.

Definition 7. *A player i is partially honest if for all preference profile $R \in \mathcal{D}_{sp}$ and $(s_i, s_{-i}), (s'_i, s_{-i}) \in S$,*

- (i) *When $g(s_i, s_{-i})R_i g(s'_i, s_{-i})$ and $s_i \in \tau_i(R)$, $s'_i \notin \tau_i(R)$, then $(s_i, s_{-i}) \succ_i^R (s'_i, s_{-i})$.*
- (ii) *In all other cases, $(s_i, s_{-i}) \succeq_i^R (s'_i, s_{-i})$ iff $g(s_i, s_{-i})R_i g(s'_i, s_{-i})$.*

Let $NE(g, \succeq^R, S)$ be the set of Nash equilibria of the game (Γ, \succeq^R) . A mechanism $\Gamma = (S, g)$ implements a SCF f in Nash equilibria if for all $R \in \mathcal{D}_{sp}$, $f(R) = g(NE(g, \succeq^R, S))$. We say that a SCF f is partially honest implementable in Nash equilibria if there is a mechanism which implements it in these equilibria. In this framework, we recall the Assumption A of Dutta and Sen [12].

Assumption A. *There exists at least one partially honest individual and this fact is known to the planner. However, the identity of this individual is not known to her.*

According to Doghmi and Ziad [9], under Assumption A, the property of unanimity alone is a sufficient condition for partially honest Nash implementation in private good economies with single-peaked preferences. Doghmi [8] shows that, when the requirement of citizen sovereignty holds, unanimity also becomes a necessary property in this area. In connection with strategy-proofness, Proposition 7 and Theorem 2 of Doghmi [8] give together with Theorem 1 the following result.

Corollary 4. *Let $n \geq 3$. In the private good economies with single-peaked preferences, if the requirements of citizen sovereignty, Assumption A, and Condition 1 hold, a SCF is partially honest Nash implementable if and only if it satisfies strategy proofness.*

From Corollaries 3 and 4, we give the following important result.

Corollary 5. *Let $n \geq 3$. In the private good economies with single-peaked preferences, if the requirements of citizen sovereignty, Assumption A, and Condition 1 hold, then Nash implementation, dominant strategy implementation, and partially honest Nash implementation are equivalent.*

6 Conclusion

We have introduced the property of unanimity as a mild requirement of efficiency to characterize strategy-proof social choice functions. We have showed that, under a certain weak condition, a SCF satisfies strategy-proofness if and only if it is unanimous. We have proved that this result has an impact on the relation between strategy-proofness and Maskin monotonicity, which become equivalent. We have applied these results to

implementation setting and we have showed Nash implementation, dominant strategy implementation, and partially honest implementation are equivalent.

Finally, we would like to mention that our work is based on single-valued rules, thus, it seems very interesting to extend these results for multi-valued rules in a future research. Another important open question is whether our findings can be extended to other environments of individual preferences.

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