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Hourya Benis Sinaceur, Marco Panza, Gabriel Sandu

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Frege's and Dedekind's Logicisms

Hourya Benis-Sinaceur

CNRS, IHPST (CNRS and Univ. of Paris 1 Panthéon-Sorbonne)

sinaceur@canoe.ens.fr

Marco Panza

CNRS, IHPST (CNRS and Univ. of Paris 1 Panthéon-Sorbonne)

marco.panza@univ-paris1.fr

Gabriel Sandu

Department of Philosophy, University of Helsinki

sandu@mappi.helsinki.fi

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Logicism is usually presented as “the thesis that mathematics is reducible to logic”, and is, then, “nothing but a part of logic”. This is, at least, the way Carnap describes it in his influential 1931 paper ([Carnap, 1931], p. 91; [Benacerraf and Putnam, 1983], p. 41). Though ascribing to Russell the role of “chief proponent” of it, Carnap also adds that “Frege was the first to espouse this view” (*ibid.*).

Still, strictly speaking, Frege never argued for such a thesis. At most, he argued that arithmetic and real analysis are part of logic. But, also if it is so restricted, this thesis renders his view only very roughly. For what makes Frege's view distinctive is the way the inclusion relation between these mathematical theories and logic is conceived. And once this way is made clear, it becomes also clear that this relation does not depend, for him, on the mere possibility of a reduction of the former to the latter. Frege's point is, indeed, less that of showing how coming back from arithmetic and real analysis to logic, than that of developing logic enough so as to find natural and real numbers within it, and, then, show that what arithmetic and real analysis deal with is logical in nature.

Let us begin with arithmetic. In the *Vorwort* of the *Grundgesetze*, he mentions the claim that “arithmetic is merely logic further developed further [*weiter entwickelte Logik*]” ([Frege, 1903], *Vorwort*, p. VII; [Frege, 2013], p. VII₁), as the claim which he aims to argue for. This is only a rephrasing of the claim that Frege had taken himself to have hopefully established, though only informally, some years earlier, in the *Grundlagen*, namely that “Arithmetic is nothing but further pursued logic [*weiter ausgebildete Logik*], and every arithmetical statement a law of logic, albeit a derived one” ([Frege, 1884], § 87; [Frege, 1953],

p. 99)¹. Frege’s point seems, then, that “arithmetic is a branch [*Zweig*] of pure logic” ([Frege, 1903], *Einleitung*, pp. 1 and 3; [Frege, 2013], pp. 1₁ and 3₁) because the former results from an appropriate development (but not an extension), of the latter, that is, because “the simplest laws of cardinal number [*Anzahl*]² [...] are] derived by logical means alone” ([Frege, 1903], *Einleitung*, p. 1; [Frege, 2013], p. 1₁).

The crucial point of Frege’s arithmetical logicism consists in fixing these logical means. This, in turn, entails identifying appropriate logical laws (or axioms, in modern terminology), deductive rules, and definitions from and according to which arithmetical truths follow. The purpose of the first and second parts of the *Grundgesetze* ([Frege, 1903], §§ I.1-II.54)³ is precisely that of fixing these means and using them for deriving these truths.

In Frege’s mind, what ensures the logical nature of these laws, rules and definitions is that the laws and rules are appropriately general, while the definitions are explicit and have recourse, in their *definiendum*, only to linguistic tools already introduced by previous analogous definitions, or directly belonging to the language in which the laws are stated. The appropriate generality of the laws and rules is, in turn, ensured by the fact that all they concern is (the values of) a small number of basic functions, defined by merely appealing to two basic objects, the True and the False (whose existence is taken for granted), and to the totalities of objects and of first- and second-level one-argument functions (so as to avoid to appeal to each of these totalities of functions for defining a function belonging to it). In other terms, these laws and rules are general because they merely pertain to (the values) of some basic functions, which are defined by relying on no device used for selecting some specific portions or elements of these totalities other than the True and the False. Now, defining these basic functions results in fixing a language to be used to form either names of values of these functions or of whatever other functions resulting from appropriately composing them, or general marks apt to “indeterminately indicate” [*unbestimmt andeuten*]” ([Frege, 1903], §§ I.1, I.8, I.17; [Frege, 2013], pp. 5₁, 11₁, 31-32₁) these values. It follows, that, for Frege, the boundaries of logic are established by fixing a functional formal language, a small number of basic truths stated in this language, and a

¹Here and later, time to time, both in the present foreword and in the three following chapters, we feel free to slightly modify the English translations we quote, for sake of faithfulness to the original.

²We agree with Ebert and Rossberg in translating Frege’s term ‘Anzahl’, when used in a technical context, with ‘cardinal number’, by conserving ‘number’ for his term ‘Zahl’ (cf. [Frege, 2013], “Translators’ Introduction”, p. xvi). A reason for using ‘cardinal number’, rather than ‘natural number’ is that Frege explicitly distinguishes (both in *Grundlagen* and in *Grundgesetze*) *endlich Anzahlen* from *unendliche* ones, namely finite cardinal numbers from infinite ones, among the latter of which he pays particular attention to the *Anzahl Endlos*, the cardinal number belonging to the concept ‘*endliche Anzahl*’ (cf. [Frege, 1884], §§ 84-86 and [Frege, 1903], *Vorwort*, p. 5, and §§ I.122-157). Notice, moreover, that, in *Grundlagen*, Frege also uses twice (§§ 19 and 43) the term ‘natürliche Zahl’ (to be mandatorily translated with ‘natural number’)—in the latter case, merely in a quote from Schröder, but in the former by speaking on his own behalf—and many times (§§ 76-79, 81-84, 104, and 108) the term ‘natürlichen Zahlenreihe’ (to be mandatorily translated with ‘series of natural numbers’ or ‘natural numbers series’). Though in *Grundgesetze* (§§ I.43-46, I.66, I.88, I.100, I.104, etc.), this last term is replaced with ‘Anzahlenreihe’ (to be translated with ‘series of cardinal numbers’ or ‘cardinal numbers series’), it seems, then, that Frege takes a natural number to be a finite cardinal one.

³We shall come back later on the third part.

small number of rules used to draw truths stated in this language from other such truths. To put it shortly, when he speaks of logic, Frege is referring to a well-identified and (in his mind) appropriately established formal system, and when he claims that arithmetic is a branch of logic, he is implying that arithmetical truths are nothing but theorems of this system.

As we shall see pretty soon, this is not as trivial as it could appear at first glance. But it is still compatible with a conception of arithmetical logicism as a reductionistic program. To see what makes Frege's arithmetical logicism much more than that, one has to consider another distinctive and essential aspect of it. This depends on Frege's considering that values of functions (of whatever level) are objects, and that "objects stand opposed to functions", to the effect that "everything that is not a function" is an object ([Frege, 1903], §§ I.2; [Frege, 2013], p. 7₁). Insofar as functions are, for him, unsaturated, this entails that cardinal, and, *a fortiori*, natural numbers could not but be objects, for him. It follows, that Frege's arithmetical logicism involves the thesis that natural numbers are objects, namely logical objects—objects whose intrinsic nature is made manifest by explicit definitions stated in the language of logic—, and arithmetical truths are truths about these objects. But, insofar as it seems quite clear that natural numbers cannot be the True and the False, arguing for this thesis requires admitting that the language of logic is enough for defining some objects other than the True and the False.

The problem arises, then: how can such other objects be defined through this language, provided that it merely results from defining the basic functions of logic, and this is done by merely appealing to the True, the False, and to the totalities of objects and first- and second-level one-argument functions? The answer depends (and could not but depend) on Frege's countenance, among his basic functions, of a function having values other than the True and the False. This is the case of the value-ranges function: a second-level one-argument function taking first-level one-argument functions (without any restriction), and giving value-ranges. Still, given the mentioned restrictions on the way basic functions are defined, taking such a function as a basic one entails renouncing to definite it explicitly, and, then, admitting of an implicit definition for it. Frege's infamous Basic Law V provides such a non-explicit definition: it implicitly defines value-ranges by stating an identity condition for value-ranges of first-level one-argument functions, that is, by asserting, as it is well-known, that the value-range of a first-level one-argument function $\Phi(\xi)$ is the same as that of a first-level one-argument function $\Psi(\xi)$ if and only if the value of $\Phi(\xi)$ is the same as that of $\Psi(\xi)$ for whatever argument, which in Frege's formal language is expressed thus: $(\dot{\epsilon}f(\epsilon) = \dot{\alpha}g(\alpha)) = (\text{---}\text{---} f(\mathbf{a}) = g(\mathbf{a}))$, where ' f ' and ' g ' are marks used to indeterminately indicate first-level one-argument functions.

Frege was perfectly aware that, by admitting of such an implicit definition, he was derogating from the strict criterion of logicity that any other ingredient of his system meets. In the *Vorwort* of the *Grundgesetze* he recognises, indeed, that "a dispute" concerning the logical nature of this system "can arise [...] only concerning [...] Basic Law of value-ranges (V)" ([Frege, 1903], *Vorwort*, p. VII; [Frege, 2013], p. VII₁). Still, according to Frege, without this Law, and without value-ranges, there could not be other logical objects but the True and the False, and arithmetical logicism would, then, not be viable. This is what he

openly claims in his tentative reply to Russell’s paradox: “[...] even now I do not see how arithmetic can be founded scientifically, how the numbers can be apprehended as logical objects and brought under consideration, if it is not—at least conditionally—permissible to pass from a concept to its extension” ([Frege, 1903], *Nachwort*, p. 253; [Frege, 2013], p. 253₂)⁴. Hence, for Frege, calling Basic Law V in question was not just his “approach to a foundation in particular, but rather the very possibility of any a logical foundation of arithmetic” (*ibid.*). For, Frege seems to argue, if the value-range function is to be dismissed, what other logical function having other values than the True and the False is permissible? And if no such function might be permissible, how can natural numbers be logical objects? And if natural numbers are not logical objects, how can arithmetic be a branch of logic?

We know today that an alternative route for arithmetical logicism—allegedly understood, if not in the same way, at least in a way close to Frege’s—has been suggested ([Wright, 1983], [Hale and Wright, 2001]). Still, it is clear that also this route depends on the admission of a basic function, namely the cardinal-number function, which, while being taken to be a logical function, is required to have as its possible values some particular objects whose existence is not a necessary condition for the admissibility of the relevant system of logic.

This is, in Frege’s original terminology, a second-level one-argument function, like Frege’s value-range one. And it is, like this latter function again, defined by a principle, namely Hume’s principle, working as an axiom of the relevant system, and taken as an implicit definition. But, differently from Frege’s value-range function, the cardinal-number function is not second-level insofar as its arguments are taken to be first-level functions. These arguments are rather taken to be concepts no more intended as functions from the totality of objects to the True and the False, but rather as the items designated by monadic first-order predicates⁵. The cardinal-number functions is, thus, a total function, like the value-range one, only insofar as a previous restriction is, so to say, incorporated in the logical system that its definition depends on: a restriction that makes the predicate variables of this system range only over concepts, rather than over items so generally conceived as to render the larger variety of Frege’s first-level one-argument functions. This goes together with the fact that the values of the cardinal-number function are *ipso facto* cardinal numbers, rather than more general items among which cardinal, and, more specifically, natural numbers, are selected with the help of appropriate explicit definitions (which might suggest that this function is not general enough to count as logical in Frege’s sense).

It is not our purpose, here, to discuss neologicism. Touching on it is only a way to emphasise the main difficulty of Frege’s logicism, by showing that, *mutatis mutandis*, it is still a crucial difficulty for its modern consistent version. This is the difficulty of fixing objects to be identified with natural numbers by having recourse only to means recognised as logical.

The way we have presented this difficulty hides a decisive aspect of it, however. This

⁴Remember that a concept is, in Frege’s terminology, a one-argument function whose values are either the True or the False, and its extension is nothing but its value-range.

⁵This entails that taking the cardinal-number function as a second-level function is imprecise, strictly speaking: this is, rather, a second-order function.

aspect only appears when it is made clear that, for Frege (as well as for the neologicists), nothing could be taken to be an object if it were not also taken to exist (in the only rightful sense in which anything can be taken to exist, both for him and for them), and no statement could be taken to be a truth if the singular terms and the first-order quantified variable included in it (if any) were not respectively taken to be names of, or to vary over existing individuals. The difficulty does not only consist, then, in defining natural numbers by having recourse only to means recognised as logical, but in doing it so as to ensure that these numbers exist, that is, that the (non-atomic) term that provides the *definiendum* of the explicit definition of each of them denotes an existing individual, and the (non-atomic) formula that provides the *definiendum* of the explicit definition of the property of being a natural number is satisfied by some (namely a countable infinity) of existing individuals (which means, in Frege’s formalism, that the explicit definition of the first-level concept ‘natural number’ designates a function whose value is the True for some, namely for countably many, arguments).

There is no room here for discussing the reason for the neologicists to claim that their definitions comply with this condition. What is relevant is that, for Frege, no independent existence proof is needed for this purpose, since, for him, the relevant explicit definitions are so shaped as to ensure by themselves that this condition obtains. In other words, according to Frege, his explicit definition of each natural number directly exhibits an object to be identified with this number, while his definition of the property of being a natural number makes directly manifest that there are these numbers and which objects they are. This means that, according to Frege, these explicit definitions make directly manifest that “there are logical objects” and that “the objects of arithmetic [i. e. the natural numbers] are such” ([Frege, 1903], § II.147; [Frege, 2013], p. 149₂).

For real numbers, Frege does not seem to have thought that something like this would have been achievable, instead. Since, though he closed his informal exposition of the way he was planning to define these numbers by claiming that in this way he would have succeeded “in defining the real number purely arithmetically or logically as a ratio of magnitudes that are demonstrably there” ([Frege, 1903], § II.164; [Frege, 2013], p. 162₂), his plan explicitly calls for an existence proof of domains of magnitudes going far beyond the simple inspection of the definition of these domains, and consisting, rather, in the independent exhibition of a particular domain of magnitudes generated from natural numbers. Frege actually realised only a part of his plan: in the third part of the *Grundgesetze* ([Frege, 1903], §§ II.55-II.245), after having discussed and questioned several (informal) definitions of natural numbers (*ibid.* II.55-II.155) and having exposed his plan (*ibid.* II.155-II.164), he proceeds to formally defining domains of magnitudes (*ibid.* II.165-II.245) and to prove some crucial properties of them, by leaving to a never appeared third volume of his treatise the accomplishment of the remaining part of the plan, including the existence proof of such domains, and the definition of real numbers as ratios over them.

Let $D(\xi)$ be the first-level concept of domains of magnitudes, namely the concept under which an object falls if and only if it is a domain of magnitudes, which means that $D(s)$ is the True if and only if s is such a domain. Frege’s formal definition of domains of magnitudes consists in stating an identity like ‘ $D(s) = \mathcal{D}(s)$ ’, where ‘ $\mathcal{D}(s)$ ’ stands for an

appropriate (non-atomic) formula (*ibid.* II.173–174 and II.197)⁶. In modern terminology, this means that domains of magnitudes are explicitly defined as the objects that satisfy this formula. And this formula is such that s satisfies it (which means, in Frege’s terminology, that $\mathcal{D}(s)$ is the True) if and only if s is the extension of another first-level concept $\mathcal{M}(\xi)$, under which an object falls, in turn, if and only if it is the extension of a first-level binary relation that, if taken together with all the other extensions of a first-level binary relation that fall under this very concept, forms a certain structure⁷. This means that domains of magnitudes are explicitly defined as extensions of first-level concepts under which the extensions of some first-level binary relations that form, when taken all together, a certain structure fall.

This definition is stated within the same functional formal language in which natural numbers are defined. Still, Frege openly claims (*ibid.* II.164) that it does not ensure that there are objects that stand to each other in some binary relations whose extensions, when taken all together, meet the relevant structural condition, and are many enough for the ratios over them could be identified with the real numbers. And, he argues, if there were not such objects, real numbers could not be defined as ratios over domains of magnitudes. The existence proof of domains of magnitudes envisaged by Frege should have consisted in showing how, by starting from natural numbers and by appropriately operating on them, one can get enough—i.e. continuous many—other suitable objects. It is not necessary to enter the details of the way Frege planned to conduct this proof, in order to understand that he could not have imagined that the relevant objects could be directly exhibited by explicit definitions, as he held to have done for natural numbers. This, together with the fact that he held that his definition of domains of magnitudes does not secure, by itself, the existence of appropriate such domains, is enough for concluding that real numbers could not have been taken by Frege as logical objects in the same sense as natural numbers were taken to be logical objects by him. Hence, his logicism about real numbers, once completely expounded in agreement with his plan, could not have appeared similar in nature to his arithmetical logicism.

These short and quite general remarks should be enough to make clear that Frege’s logicism is quite complex a thesis, or better that it consists of two distinct quite complex theses, respectively pertaining to natural and real numbers, that are only very partially and broadly rendered by the simple claim that arithmetic and real analysis are part of logic. It follows that it is not enough for someone to be credited with the same foundational program as Frege’s that he made this same claim. Only a careful comparative scrutiny of way this claim is justified and explained could allow one to evaluate whether this claim is an expression of logicism in the same sense as Frege’s.

A case in point is that of Dedekind. From the very beginning of the *Vorwort* to the first edition of *Was sind und was sollen die Zahlen?*, he explicitly identifies the “simplest

⁶As a matter of fact, this formula is not openly written by Frege, but it is easily deducible by other formulas which he openly writes.

⁷Remember that for Frege a binary first-level relation is a first-level two-arguments function whose values are either the True or the False.

science” with “that part [*Theil*] of logic which deals with the theory of numbers [*Lehre von den Zahlen*]”, and refers to it as to “arithmetic (algebra, analysis) [*Arithmetik (Algebra, Analysis)*]” ([Dedekind, 1888], p. VII; [Dedekind, 1901], p. 14). There is little doubt that what Dedekind is meaning here with ‘theory of numbers’ is much more than the theory of natural numbers, and also includes real analysis. This is not only suggested by the parenthesis following the term ‘Arithmetik’, but also by the possessive pronouns ‘its [*ihr*]’ in what one finds some lines below: “it is only through the purely logical construction of the science of numbers and in its acquiring the continuous number-realm that we are prepared accurately to investigate our notions of space and time” (*ibid.*)⁸. It seems then quite clear that Dedekind here is endorsing the claim that both arithmetic and real analysis are parts of logic. Still, the way this claim is justified with respect to arithmetic, as well as and the tacit extension of it to real analysis, delineate a quite different conception than Frege’s.

Dedekind’s main point is that “the number-concept [*Zahlbegriff*] [is] entirely independent of the conceptions or intuitions of space and time”, being rather “an immediate result from the laws of thought”, since “what is done in [looking for] the number of a set [*Zahl der Menge*] or the number of some things [*Anzahl von Dingen*]” depends on “the ability of the mind to relate [*beziehen*] things to things, to let a thing correspond [*entsprechen*] to a thing, or to represent [*abzubilden*] a thing by a thing, an ability without which no thinking is possible” ([Dedekind, 1888], p. VIII; [Dedekind, 1901], p. 14).

The reference, here, is to the crucial role played, in Dedekind’s definition of natural numbers, by the notion of a “mapping [*Abbildung*]. In his terminology: a “thing [*Ding*]” is “any object of our thought [*Gegenstand unseres Denkens*]” ([Dedekind, 1888], §. 1; [Dedekind, 1901], p. 21); a “system [*System*]” is that which “different things [...] constitute [*bilden*]” when they are “considered from a common point of view [*unter einem gemeinsamen Gesichtspunkte aufgefasst*]” and are, then, “associated in the mind [*im Geiste zusammengestellt*]” ([Dedekind, 1888], §. 2; [Dedekind, 1901], p. 21); the “elements [*Elemente*]” of a system are the things that constitute such a system (*ibid.*); and a “mapping [*Abbildung*][...] of a system *S* [...] [is] a law [*Gesetz*] according to which a determinate thing pertains to [*gehört zu*] every determinate element [...] of *S*” ([Dedekind, 1888], §. 21; [Dedekind, 1901], p. 24).

His point is, then, that if these notions are appropriately used together, they provide enough conceptual tools for defining the natural numbers and setting up a theory of them. The way this is done in Dedekind’s treatise is, however, essentially informal. The foregoing explanations are everything Dedekind’s exposition relies on in order to make these notions operate and carry out the required definition of natural numbers and the corresponding theorems. Hence, Dedekind’s theory of natural numbers is in no way embedded, like Frege’s, within a well-identified formal system. If it is logic, or a part of logic, it is, then, not because it is part of such a system, but because of the intellectual abilities these notions and our handling them hinge on. In other terms, what is taken to be logical,

⁸This is also confirmed by the reference, which Dedekind makes few lines below, to his supplement XI to Lejeune Dirichlet’s *Vorlesungen über Zahlentheorie*, which is devoted to the theory of finite algebraic numbers ([Dirichlet, 1879], pp. 434-626, esp. p. 470, footnote).

in Dedekind's arithmetical logicism, is not a formal system and its ingredients, but some notions, informally explained, and our intellectual ability to handle them. Moreover, the relation of being part of, which relates arithmetic to logic, is not conceived as a relation of inclusion of a system into a system, but rather as a relation depending on the sufficiency of this ability for realising the relevant task.

Concerning real numbers, things are even clearer. Since, Dedekind's extension of his arithmetical logicism to real analysis merely depends on his remark that the "creation [*Schöpfungen*] of [...] negative, fractional, irrational [...] numbers is always accomplished by reduction to the earlier concepts [...] without the introduction of foreign conceptions", as thoroughly shown, for irrational numbers, in *Stetigkeit und Irrationale Zahlen*, and suggested in section III of this very treatise, for the other numbers ([Dedekind, 1888], p. X; [Dedekind, 1901], p.15; [Dedekind, 1872]). The point, here, is not only that the definition of irrational numbers offered in *Stetigkeit* is as informal as that of natural numbers offered in *Was sind*, and that in the former treatise Dedekind advances no thesis assimilable to some sort of logicism, but also, and above all, that Dedekind seems to consider useless to show explicitly how the logical intellectual ability the theory of natural numbers depends on is also enough to pass from these numbers to real ones. All that is relevant, for him, is a generic appeal to the possibility of a conceptual reduction. There is nothing, then, like what justifies Frege's logicism for real numbers, namely a further development of the same formal system in which the definition of natural numbers is embedded, resulting in an independent formal definition of the former numbers.

But as essential as these differences might appear, they are far from being the only ones. Another, possibly even more essential one, already shines through Dedekind's speaking of creation of negative, fractional and irrational numbers by reduction to earlier concepts. It depends on Frege's and Dedekind's respective conceptions of the very nature of natural and real numbers. For Frege, they are objects, that is, individuals existing as such, and they are logical not insofar as their existence depends on logic, but rather insofar logic is enough for defining them (and, at least in the case of natural numbers, for ensuring their existence). This is not at all Dedekind's view. For him, they are, rather, "free creations [*frei Schöpfungen*] of the human mind" to be used for "apprehending more easily and more sharply the difference of things" ([Dedekind, 1888], pp. VII-VIII; [Dedekind, 1901], p.14), and these creations are logical insofar as logic, understood as we have said above, is enough for fixing their concept, which is all what is needed to create them. Since, for Dedekind, the only sense in which a number can be said to be an object depends of its being a thing, namely, as we have already said, an object of our thought, and "a thing is completely determined [*vollständig bestimmt*] by all that can be affirmed or thought concerning it" ([Dedekind, 1888], §. 1; [Dedekind, 1901], p. 21). The more difficult task of Frege's logicism, consisting in offering a justification of the existence of natural and real numbers is, then, simply dismissed by Dedekind, which merely implies that it is nonsensical.

Though the first edition of *Was sind* appeared four years after Frege's *Grundlagen*, no mention of the latter is made in the former. A short mention is made, instead, in the *Vorwort* of the second edition ([Dedekind, 1893], p. XVII; [Dedekind, 1901], p. 19). Though declaring of having become acquainted with Frege's treatise only after the publication of

his own, and bringing to the reader’s mind the difference of his and Frege’s “view on the essence of number [*Wesen der Zahl*]”, Dedekind emphasises Frege’s standing “upon the same ground” with him. Had he been aware of the way Frege’s view is spelt out in the *Grundgesetze*, whose first volume just appeared in the same year as the second edition of Dedekind’s treatise, the latter would have probably advanced a different judgement, since the main differences among the two approaches are much more evident when *Was sind* is compared with *Grundgesetze* (as we have done above), rather than with the *Grundlagen*, where Frege’s logicism is, indeed, merely sketched out informally.

It was, then, up to Frege to emphasise the differences of his and Dedekind’s approaches ([Frege, 1903], *Vorwort*, p. VII-VIII; [Frege, 2013], pp. VII₁-VIII₁):

My purpose demands some divergences from what is common in mathematics. This will be especially striking if one compares Mr Dedekind’s essay, *Was sind und was sollen die Zahlen?*, the most thorough study I have seen in recent times concerning the foundations of arithmetic. It pursues, in much less space, the laws of arithmetic to a much higher level than here. This concision is achieved, of course, only because much is not in fact proven at all. Often, Mr Dedekind merely states that a proof follows from such and such statement; he uses dots, as in ‘ $\mathfrak{M}(A, B, C \dots)$ ’; nowhere in his essay do we find a list of the logical or other laws he takes as basic; and even if it were there, one would have no chance to verify whether in fact no other laws were used, since, for this, the proofs would have to be not merely indicated but carried out gaplessly. Mr Dedekind too is of the opinion that the theory of numbers is a part of logic; but his essay barely contributes to the confirmation of this opinion since his use of the expressions ‘system’, ‘a thing belongs to a thing’ are neither customary in logic nor reducible to something acknowledged as logical.

If the most part of this quote keeps our attention to the first difference we have remarked above, namely the fact that Dedekind’s presentation is informal, to the effect that his logicism is in no way the thesis that the relevant truths are theorems of a formal system taken as logic, the last remark points in a different direction: what makes Dedekind’s alleged logic not be logic at all is its involving set-theoretic notions.

As Frege makes clear in the *Einleitung* of his treatise, by coming back to the same point, the complaint, here, is not merely with Dedekind’s using the term ‘system’ with the “same intention [*Absicht*]” with which others were using, in the same years, the term ‘set [*Menge*]’ ([Frege, 1903], *Einleitung*, p. 1; [Frege, 2013], p. 1₁), or with his considering systems to be things to which other things belong⁹. After all, Frege himself will largely use a set-theoretic term as ‘class [*Klasse*]’ in the second volume of the *Grundgesetze*, and will have no reticence to say that a class “contains [*umfassen*]” objects, and that an object “belongs [*angehört*]” to a class ([Frege, 1903], § II.164 and II.173, for example). Still, for

⁹As a matter of fact, in *Was sind*, one finds only one occurrence, in § 34, of the verb ‘to belong to [*gehören zu*]’ or its cognates used in this sense; possibly Frege’s was here referring to *Stetigkeit*, where this use is much more frequent, namely in §§ IV-V.

Frege, a class is the “extension of a concept” ([Frege, 1903], § II.99 and II.161), that is, a value-range. This is exactly the point. For Dedekind, he argues, “every element [...] of a system [...] can itself be regarded as a system”, and, “since in this case element and system coincide, it here becomes very perspicuous that, according to Dedekind, the elements are what properly makes out the system [*die Elemente den eigentlichen Bestand des Systemes ausmachen*]” ([Frege, 1903], *Einleitung*, p. 2; [Frege, 2013], p. 2₁), which is, in fact, inappropriate. Since, “every time a system has to be specified”, even Dedekind, despite his “lack of insight”, cannot but mention “the properties a thing must have in order to belong to the system, i. e. he defines a concept in virtue of its characteristic marks”, and so makes it appear that “it is the characteristic marks that make out the concept, rather than the objects falling under it” ([Frege, 1903], *Einleitung*, p. 3; [Frege, 2013], p. 3₁; cf. also the quote from Frege’s undated letter to Peano quoted in § ?? of chapter ??, below). In other terms: if specifying a system reduces to defining a concept, what makes out the system cannot but be the same as what makes out the concept, and this is not given by the extension (or value-range) of this concept, but by what makes an object be part of such an extension. So, it is not Dedekind’s use of set-theoretic notions that Frege is questioning, but his taking them as primitive notions, his not clearly submitting them to the more general notion of a concept, or, even, to the still more general notion of a function. Logic, Frege seems to mean, can include consideration of sets, but only insofar this is part of a more general study of the objects-concepts relations, of the falling of objects under concepts: dealing with the notions of a set as a primitive notion, and merely conceiving of a set as an aggregate of things is *ipso facto* departing from logic.

This is, possibly, the deeper difference between Frege’s and Dedekind’s logicism, and it sheds new light also on the differences between Dedekind’s notion of a mapping and Frege’s notion of a function. For the former, a mapping is something that “relates things to things” or makes “a thing correspond to a thing”, or a thing “represent” a thing, and only appears, then, when appropriate things, which we independently take to be there as such, are related. For the latter, a function is that which an object is a value of, and it appears any time we refer to a certain object, through the way we refer to it, with the only exception of the primitive reference to the True and the False, which is necessary for defining the basic functions of logic. This exception, however, does not make our reference to the True and the False is always independent of any function. Logic is, rather, for Frege, the study of our referring to the True and the False as values of particular sorts of functions, namely concepts and relations, and it is our referring to the True or to the False as the value of a certain concept or relation, for a certain object or a certain pair of objects as arguments, that makes this object fall or not fall under this concept, or the objects composing this pair are or are not related by this relation. The notions of a concept and a relation, rather than those of a system and its mappings are then, as Frege explicitly says, “the foundation stones [*Grundsteine*] on which [...] [he] build [...] [his] construction” ([Frege, 1903], *Einleitung*, p. 3; [Frege, 2013], p. p. 3₁).

We have, thus, arrived at the two basic, strictly connected, topics the present book is devoted to, namely the way Dedekind’s logicism differs from Frege’s, and the conception

of a function Frege’s logicism is grounded on.

The first chapter, by Hourya Benis Sinaceur, is specifically devoted to the former topic, and goes much further than we could have done here in accounting for the specific features of Dedekind’s logicism, and of his conception of logic, in connection, of course, with his conception of systems and mappings.

The second chapter, by Marco Panza, provides a historically situated account of Frege’s notion of a function, by insisting on the intensional nature of this notion, and then questioning a widespread tendency to ascribe to Frege a Platonist view about functions, and, more specifically, concepts and relations.

This question is related to a lively discussion that took place after the publication, in 1992, of a paper by Jakko Hintikka and Gabriel Sandu ([Hintikka and Sandu, 1992]), where the ascription to Frege of a notion assimilable to the modern, essentially extensional and set-theoretic notion of an arbitrary function is questioned. In the third chapter, Gabriel Sandu comes back on this discussion, by offering new arguments for his views, based on the consideration of Ramsey’s reaction to Russell’s logicism.

The intimate connection among these topics should be clear from the previous considerations. But it becomes even more evident when it is observed that the two features of Frege’s logicism that are mainly responsible for the difference between this logicism and Dedekind’s, namely Frege’s conceiving of logic as a formal system, and his grounding it on a general notion of a function—as opposed, respectively, to Dedekind’s taking as logical some informal notions and our intellectual ability to handle them, and to his conceiving of systems as aggregates of things that we take to be there as such—, are strictly connected, in turns. For Frege’s notion of a function is fashioned as to provide the ground on which this logical system is erected, that is, it is, essentially, a linguistic notion. While Frege conceives of objects as existing individuals, he does not assign existence to functions, as such. He rather conceives them as matrices to be used to form objects-names, which is precisely what, for him, makes a function appear through the way we refer to an object, as we have said above.

This has a crucial consequence on the way the totalities of objects and functions, and, then, an arbitrary function, are conceived by him. The question is largely discussed in Panza’s and Sandu’s chapters, but a short remark is in order, here as well.

True, Frege was aware of Cantor’s distinction among different sorts of infinity. He entitled §§ 84-86 of *Grundlagen* “Infinite cardinal numbers [*Unendliche Anzahlen*]”¹⁰ (in the plural), by explicitly referring to Cantor’s very recently appeared *Grundlagen einer allgemeinen Mannigfaltigkeitslehre* ([Cantor, 1883]). Though, in them, he specifically deals, in fact, only with “the cardinal number which belongs to the concept ‘finite cardinal number’”, he significantly denotes it with ‘ ∞_1 ’, by suggesting that other infinite cardinal numbers follow, namely ∞_2 , ∞_3 , etc. After having come back to this same number in the first volume of *Grundgesetze*, this time by calling it ‘Endlos’¹¹ and denoting it with

¹⁰Cf. footnote 2, above.

¹¹Cf. again footnote 2, above.

‘ ∞ ’ ([Frege, 1903], *Vorwort*, p. 5, and §§.122-157), he remarks, in the second volume (*ibid.*, II.164), that his envisaged definition of real numbers as ratios of magnitudes could not be carried out without having available an infinite “class of objects” having an infinity greater than *Endlos*, namely that of the cardinal number of the concept ‘class of finite cardinal numbers’. Still, he seems to have realised neither that the objects that such a class would contain could not all have a (finite) name (even in a language including countably infinite atomic names), nor that they would have allowed to form enough subclasses as to assign an extension to an even greater infinity of unnameable concepts, and, more in general, a value-range to an even greater infinity of unnameable functions. In other terms, he does not seem to have realised that allowing for uncountable classes of objects goes together with admitting unnameable objects, including the possible value-ranges of unnameable functions, and, then, also unnameable functions, in extension.

A symptom of his unawareness is his using Latin letters to form universal statements. Within Frege’s system, a particular statement is formed by the name of a value of a concept or relation (which is a name of a truth-value), preceded by a special sign of assertion, namely ‘ \vdash ’. Such a statement is taken to assert, then, that the truth-value named by this name is the True. A general statement is formed in the same way, except for the replacement of the name of a value of a concept or relation with a Roman object-marker, namely an expression involving Roman letters ([Frege, 1903], §. 26; [Frege, 2013], p. 44). And it is taken to assert that the object-name that is obtained from such an object-marker, by replacing within it each Roman letter for objects with whatever object-name (which actually refers to an object), and any Roman letter for functions with whatever function-name (of the appropriate sort), is a name of the True ([Frege, 1903], §§ I.5, I.8, I.17 and I.19). So, to give a simple example, the general statement ‘ $\vdash f(b)$ ’, namely theorem IIIc

$$\begin{array}{l} \vdash f(b) \\ \quad \vdash f(a) \\ \quad \quad a = b \end{array}$$

of Frege’s system ([Frege, 1903], § I.50), is taken to assert that the truth-value named by the object-name got from ‘ $\vdash f(b)$ ’, by replacing ‘ a ’ and ‘ b ’ with whatever object-names

$$\begin{array}{l} \vdash f(b) \\ \quad \vdash f(a) \\ \quad \quad a = b \end{array}$$

(which actually refer to an object), and ‘ $f(\xi)$ ’ with whatever name of a one-argument first-level function, is the True. It is clear that Frege could have not conceived of a general statement in this way if he had not admitted that any possible object and function can somehow be named¹².

There is no doubt that a notion of a function leaving no room for the idea of an extensionally arbitrary function, and a notion of a set based on such a notion of function could not have provided a basis for the development of mathematics and mathematical logic that followed the pioneering works of Frege, Dedekind, and Cantor, among others. Hence, despite the lack of intelligibility that the idea of an extensionally arbitrary function

¹²Unless he were admitting, instead, that the relevant generality is merely a linguistic one, that is, one that concerns only objects and function that can be somehow named, which is quite implausible, indeed.

brings with it, and despite the promises of clarification that alternative notions, intensional, in nature, could generate or have generated, it is a fact that a foundational program based on these last notions could hardly have been suitable for accounting for this development, even if it had presented no other internal difficulty. Still, the question that the present book is devoted to is not whether Frege's or Dedekind's different forms of logicism were based on suitable conceptions. Our aim is rather that of keeping the reader's attention on some crucial aspects of these forms of logicism, so as to contribute to a better understanding of their nature, their motivations, and their mutual differences.

References

- [Benacerraf and Putnam, 1964] Benacerraf, P. and Putnam, H., editors. *Philosophy of mathematics*. Prentice-Hall, Inc., Englewood Cliffs, New Jersey.
- [Benacerraf and Putnam, 1983] Benacerraf, P. and Putnam, H., editors. *Philosophy of mathematics*. Cambridge Univ. Press, Cambridge.
- [Cantor, 1883] Cantor, G. *Grundlagen einer allgemeinen Mannigfaltigkeitslehre in mathematisch-philosophischer Versuch in der Lehre des Unendlichen*,. Teubner, Leipzig.
- [Carnap, 1931] Carnap, R. Die logizistische Grundlagen der Mathematik. *Erkenntnis*, 2:91–105. An English Translation is included in [Benacerraf and Putnam, 1964], pp. 31-41 and [Benacerraf and Putnam, 1983], pp. 41-52.
- [Dedekind, 1872] Dedekind, R. *Stetigkeit und irrationale Zahlen*. F. Vieweg und Sohn, Braunschweig.
- [Dedekind, 1888] Dedekind, R. *Was sind und was sollen die Zahlen?* F. Vieweg und Sohn, Braunschweig.
- [Dedekind, 1893] Dedekind, R. *Was sind und was sollen die Zahlen?* F. Vieweg und Sohn, Braunschweig. Second Edition.
- [Dedekind, 1901] Dedekind, R. *Essays on the Theory of Numbers*. The Open Court P. C., Chicago. Translated by W. W. Beman.
- [Dirichlet, 1879] Dirichlet, P. G. L. *Vorlesungen über Zahlentheorie*. F. Vieweg und Sohn, Braunschweig. Third ed. Herausgegeben und mit Zusätzen versehen von R. Dedekind.
- [Frege, 1884] Frege, G. *Die Grundlagen der Arithmetik*. W. Köbner, Breslau.
- [Frege, 1903] Frege, G. *Grundgesetze der Arithmetik*. H. Pohle, Jena. 2 volumes.
- [Frege, 1953] Frege, G. *The Foundations of Arithmetic*. Blackwell, Oxford. Translated by J. L. Austin.

- [Frege, 2013] Frege, G. *Basic Laws of Arithmetic*. Oxford University Press, Oxford. Translated and edited by P. A. Ebert and M. Rossberg, with C. Wright.
- [Hale and Wright, 2001] Hale, B. and Wright, C. *The Reason's Proper Study*. Clarendon Press, Oxford.
- [Hintikka and Sandu, 1992] Hintikka, J. and Sandu, G. The skeleton in Frege's cupboard: The standard versus nonstandard distinction. *The Journal of Philosophy*, 89:290–315.
- [Wright, 1983] Wright, C. *Frege's Conception of Numbers as Objects*. Aberdeen Univ. Press, Aberdeen.