



HAL
open science

Optimal Growth and Debt Dynamics under GDP-Based Collaterals

Daria Onori

► **To cite this version:**

Daria Onori. Optimal Growth and Debt Dynamics under GDP-Based Collaterals. 2015. halshs-01251352

HAL Id: halshs-01251352

<https://shs.hal.science/halshs-01251352>

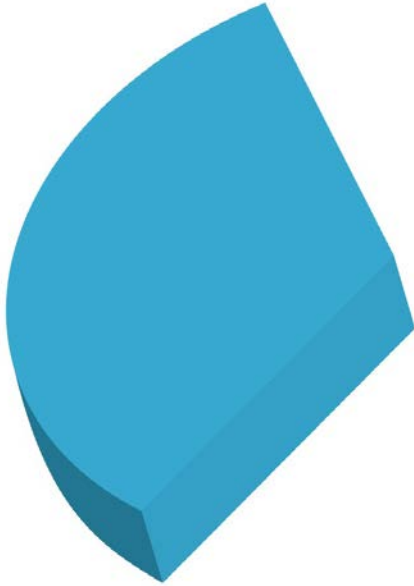
Preprint submitted on 6 Jan 2016

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

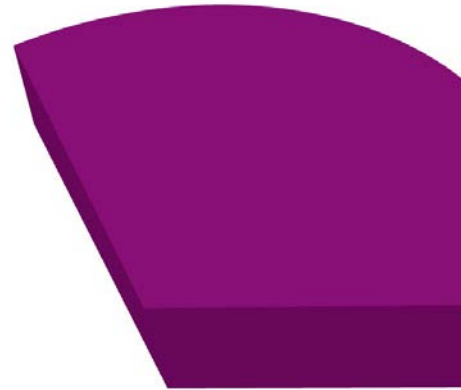
L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Document de Recherche du Laboratoire d'Économie d'Orléans

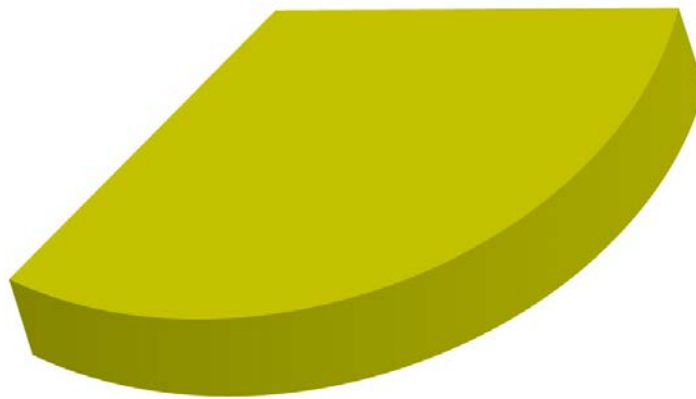
DR LEO 2015-04



**Optimal Growth and Debt Dynamics
under GDP-Based Collaterals**



Daria ONORI



Optimal growth and debt dynamics under GDP-based collaterals

Croissance optimale, dynamique de la dette et contraintes d'endettement basées sur le PIB

Daria Onori*

February 20, 2015

Abstract This paper analyzes the consequences of external debt collaterals on the optimal growth path of a country. To this end we develop a small open economy model of endogenous growth where public spending can be financed by borrowing on imperfect international financial markets, where the country's borrowing capacity is limited. In contrast to the existing literature, which assumes that debt is constrained by the stock of capital, we investigate the consequences and policy implications of GDP-based collaterals. First, we show that the economy may converge in a finite time to the regime with binding collateral constraint. Second, in such regime the steady state public expenditures-to-GDP ratio is greater than that of the existing literature's models. Finally, if the economy is not sufficiently developed, in financial and economic terms, the country will stay in autarky forever.

Keywords: open economy; two-stage growth; external debt; GDP-based collaterals; imperfect financial markets; multi-stage optimal control.

JEL Classification: CO2, C61, F34, F43, H63, O16, O4.

Résumé Cet article étudie les conséquences des contraintes à la dette externe sur la croissance optimale d'un pays. Nous développons un modèle de croissance endogène en économie ouverte, où les dépenses publiques peuvent être financées en s'endettant sur les marchés financiers internationaux. Ces derniers sont imparfaits, puisque la capacité d'endettement est limitée par le PIB. Nous montrons que l'économie peut converger, à une date finie, au régime où la contrainte est saturée. Dans ce régime un sentier de croissance équilibrée existe et le ratio entre dépenses publique et PIB est majeur de celui de la littérature. Enfin, pour les pays qui ne sont pas suffisamment développés du point de vue économique et financier, il est optimal de rester en autarcie.

Mots Clés : économie ouverte ; dette extérieure ; marchés financiers imparfaits ; contraintes d'endettement basées sur le PIB ; control optimal à régimes multiples.

*Univ. Orléans, CNRS, LEO, UMR 7322, F45067. E-mail: daria.onori@univ-orleans.fr

1 Introduction

The rapid development of financial globalization started in the mid-1980s allowed developing countries to open to capital inflows and many of them accumulated high levels of external debt. This exposed them to credit risk and solvency problems, so as international financial markets reacted by limiting external borrowing.

This paper is a contribution to the growth literature in open economy and studies the effects of external debt collaterals on the optimal growth dynamics of a country. We extend Barro (1990) and Turnovsky's (1996) models of productive public expenditures. Notably, in Barro, public spending is enhanced by the government only by levying income taxes, while in Turnovsky the country can also borrow without limits from the rest of the world. In our model, the country's external debt is constrained by GDP which, being the tax base, also sets an upper bound to the governments fiscal capacity. In this context, financial openness means that the country is allowed to borrow on international financial markets. The literature on growth, collateral constraints and financial markets is very limited and assumes that a country's debt is constrained by physical capital.¹ Our main assumption on the borrowing constraint is that the country's collateral is GDP, rather than (private or public) physical capital. This assumption is as appropriate as that of physical capital since public capital usually takes the form of public patrimony, portions of territory, monuments, infrastructures, etc., which are of difficult appropriability for foreign creditors. Similarly to what is assumed and found in Djankov et al. (2008), we suppose that the presence of enforcement costs implies that creditors can seize just a fraction of country's collaterals. One can conjecture that enforcement costs will be lower in such a scenario (consider, for example, the pre-existing transfer mechanisms within the European Union).

We focus on the policy implications of a collateral constraint (based on GDP, a proxy of fiscal capacity) for the optimal growth path of a country which opens to international borrowing. In this perspective, an increase in the economy's growth rate corresponds to a slackening of the fiscal constraint, as the borrowing capacity of the country increases.²

We also introduce some technical novelties. In order to verify if the debt stock actually reaches the constraint's upper bound once the country moves from financial autarky to openness, and to endogenously determine the date at which it can happen, we solve the model by using the multi-stage optimal control technique. Here, the standard optimal control problem is constructed as a dynamic programming problem, which allows to find the instant at which the economy changes regime. Optimal switching problems that can be found in the literature typically involves regimes which are characterized by different technological or ecological states and each regime is defined as a certain set of parameters values (see Tomiyama 1985 and Boucekkine et al. 2004). A more recent branch of research uses this technique to analyze the problem of irreversibility: when the state variable reaches

¹See: e.g. Eaton and Gersovitz (1981); Cohen and Sachs (1986); Brock (2009) and Boucekkine and Pintus (2012). Turnovsky (1997) studied the economy's dynamics when limited borrowing is represented by an upward-sloping supply of debt, where the cost of borrowing increases with the level of foreign debt, relative to the stock of capital.

²Domestic revenue is clearly non seizable. One should think that, in case of default, a government can seize whatever he wants in the country. Hence the collateral condition should be interpreted as a signal of solvency of the debtor that can be compared to the situation of an individual filing for a loan with a bank.

a certain threshold, the economy switches to a different regime in which the economy's parameters are different (see Boucekkine et al. 2013a, Boucekkine et al. 2013b, Boucekkine et al. 2013c, Prieur et al. 2011). In this paper we focus on another type of optimal switching time problem. The fact that at the initial date the constraint is not binding and then, if switching is optimal, it becomes binding modifies one of the state equation, thus defining two possible states of the economy.

We obtain a number of novel results and some broad policy implications. In the regime where the constraint is not binding (non binding regime) the optimal public expenditures-to-GDP ratio is equal to the income share of public expenditures as in Barro and Turnovsky's models. However, this phase may be transitory: under certain conditions, the economy will switch to the regime in which the constraint is binding (binding regime). Here, we show that the economy admits a balanced growth path (BGP hereafter), which is not immediately reached because of the presence of investment adjustment costs and we determine analytically the conditions for the BGP's stability. This state of the economy displays an important difference with respect to the Barro-Turnovsky model: the public expenditures-to-GDP ratio is greater than the income share of public expenditures. This happens because a higher investment in public goods relaxes the borrowing constraint. The increased borrowing capacity allows to finance greater public expenditures, which in turns boosts growth.

Finally, we determine the endogenous switching date. In the papers previously mentioned, the endogenous switching date can be found because all of the regimes admit closed form solutions and the date can consequently be determined by replacing these solutions in the matching conditions between the regimes. Our binding dynamic system is non linear and cannot be solved in closed form. Nevertheless, we are able to give analytical conditions for corner solutions and then simulate the inter-regime transition. First, we find that if there are no adjustment costs of capital, the economy immediately jumps to the regime where the constraint is binding. In other words, the presence of investment adjustment costs makes the inter-regime transition slower. Second, there exist some threshold conditions that must be satisfied by the relevant parameters, for the economy to switch to the binding regime and to fully benefit from financial openness. This happens when the interest rate and the adjustment costs parameter are low and the productivity is high. If, on the opposite, the economy is not sufficiently developed, in financial and economic terms, the country will stay in autarky forever. We also perform a comparative statics analysis of the effect of the relevant parameters on the switching date. This date turns out to be an increasing function of the degree of financial openness, the international interest rate, the adjustment costs and it is a decreasing function of the productivity parameter.³

The paper is structured as follow. In Section 2 we present and solve our model. Section 3 analyzes the balanced growth path. Section 4 discusses the inter-regime transition and the determination of the optimal switching date. Finally, Section 5 concludes. The Appendix contains technical details.

³Aizenman et al. (2007) build a similar model to ours: countries are characterized by limited tax and debt capacity. However they do not tackle the problem of collaterals and imperfect financial markets. Furthermore, the results and the technical analysis are different. First, the economy is closed. Second, they do not take into account any problem of endogenous switching.

2 The model

In this section we first describe the economy's framework under the assumption of unlimited borrowing capacity, as in Turnovsky's (1996) model,⁴ and subsequently discuss the case of GDP-based collateral constraints.

The production function is the same as in Barro's (1990): government's purchases of goods and services enter the production function as pure public goods. The technology is Cobb-Douglas:

$$Y = AG^\alpha K^{1-\alpha}, A > 0, 0 < \alpha < 1 \quad (2.1)$$

where K is the firm's stock of physical capital, A is the technology parameter and G is the flow of productive government expenditures. α represents the income share of public services. For a given G the economy faces diminishing returns to scale in the stock of capital and constant returns to scale in G and K . We normalize population to one throughout the paper: labor is supplied inelastically and population growth rate is exogenous. Capital can be interpreted in a broad sense by including human capital.

We define public expenditures as a flow variable. This assumption is usually adopted in the literature (see, e.g., Futagami et al. 2008).⁵

The economy's resource constraint (current account) can be expressed as

$$\dot{B} = C + I + G + \frac{h}{2} \frac{I^2}{K} + \hat{r}B - Y \quad (2.2)$$

which states that the economy accumulates debt to finance its total expenditures on consumption, C , public expenditures, private capital, interest payments on debt B (\hat{r} is the exogenous international interest) net of produced output. We assume here that the country is a net borrower on international financial markets, hence $B \geq 0$. Capital accumulation involves convex investment adjustment costs of the form $\Phi(I, K) = I(1 + \frac{h}{2} \frac{I}{K})$, $h > 0$. This is the traditional Hayashi's (1982) formulation, where the adjustment costs is proportional to the rate of investment in terms of units of installed capital. Their presence is a necessary assumption of small open economy models for two reasons. First, in a world where the country is not constrained by internal savings and has access to international financial markets, the investment rate would be infinity in the absence of adjustment costs. Second, without adjustment costs, this model would display the strong knife-edge condition according to which the internal marginal productivity of capital would be constrained to be equal to the exogenous interest rate (see Turnovsky 2002).

The agent's intertemporal utility is given by $U(C) = \int_0^{+\infty} \ln C e^{-\rho t} dt$ where $0 < \rho < 1$ is the discount factor.⁶ Finally we assume that capital does not depreciate for the sake of simplicity,⁷ so as

$$\dot{K} = I \quad (2.3)$$

⁴The results of this model will out turn to be useful to study the emergence of corner solutions in our case. See Section 4.

⁵The consideration of the public stock of capital adds further difficulties to the solution of the model.

⁶We assume logarithmic utility throughout the paper in order to make the model tractable.

⁷We will maintain this assumption throughout the paper, but the results do not change if the depreciation rate is taken positive.

2.1 Unlimited borrowing capacity: Turnovsky's model

Here the economy has unconstrained access to international financial markets. For coherency, we also assume that the country is a net borrower on world capital markets.⁸

The planner has to choose consumption, public expenditures, investment, debt and capital in order to maximize the intertemporal utility subject to (2.1), (2.2) and (2.3). The optimal public expenditures-to-GDP ratio is equal to the income share of public goods α , as in Barro's model. The growth rate of capital is $\dot{K}/K = (q - 1)/h$, where $q \equiv -\lambda_2/\lambda_1$ is the market value of capital in terms of the price of bonds (Tobin's q),⁹ and that of consumption is

$$\Psi = \hat{r} - \rho \quad (2.4)$$

The growth rate of capital is determined by q , whose dynamics is given by $\dot{q}/q + (q - 1)^2 / (2hq) + (1 - \alpha) / qA^{\frac{1}{1-\alpha}} \alpha^{\frac{\alpha}{1-\alpha}} = \hat{r}$. By setting $\dot{q} = 0$ we obtain two solutions

$$q_1 = (1 + \hat{r}h) - \sqrt{(1 + \hat{r}h)^2 - \left[1 + 2h(1 - \alpha) A^{\frac{1}{1-\alpha}} \alpha^{\frac{\alpha}{1-\alpha}}\right]} \quad (2.5)$$

$$q_2 = (1 + \hat{r}h) + \sqrt{(1 + \hat{r}h)^2 - \left[1 + 2h(1 - \alpha) A^{\frac{1}{1-\alpha}} \alpha^{\frac{\alpha}{1-\alpha}}\right]}$$

As it is shown in Turnovsky (1996), the smaller equilibrium value, q_1 , is an unstable root, while q_2 is stable. However any stable arm converging to q_2 violates the transversality condition, $\lim_{t \rightarrow \infty} \lambda_2(t) K(t) = \lim_{t \rightarrow \infty} -q(t) \lambda_1(t) K(t) = 0$, where $K(t) = K(0) e^{\int_0^t \frac{q(s)-1}{h} ds}$ and $\lambda_1(t) = \lambda_1(0) e^{-\hat{r}t}$. In fact the stable adjustment path is $q(t) = q_2 + (q(0) - q_2) e^{\mu t}$, where $\mu < 0$ is the stable eigenvalue. Hence $\int_0^t \frac{q(s)-1}{h} ds = \frac{q_2-1}{h} t + \frac{q(0)-q_2}{\mu} (e^{\mu t} - 1)$. By substituting this expression into the transversality condition it is possible to check that it is violated. In the case of the unstable root q_1 , there is no transitional dynamics and the transversality condition is satisfied for $q = q_1$. Hence q is always at its steady state value q_1 . This implies that capital is always on its balanced growth path, with a growth rate given by

$$\Lambda = \frac{q_1 - 1}{h} \quad (2.6)$$

By substituting (2.4), (2.5) and (2.6) into the economy's current account, we can solve for the stock of debt:

$$B(t) = -\frac{C(0)}{\rho} e^{(\hat{r}-\rho)t} - \theta K(0) e^{\frac{q_1-1}{h}t} + \left(B(0) + \frac{C(0)}{\rho} + \theta K(0)\right) e^{\hat{r}t}$$

where $\theta \equiv \frac{\left[\frac{q_1-1}{h} + A^{\frac{1}{1-\alpha}} \alpha^{\frac{\alpha}{1-\alpha}} + \frac{(q_1-1)^2}{2h} - A^{\frac{1}{1-\alpha}} \alpha^{\frac{\alpha}{1-\alpha}}\right]}{(\hat{r} - \frac{q_1-1}{h})}$. In order to ensure intertemporal solvency, the transversality condition $\lim_{t \rightarrow \infty} \lambda_1(t) B(t) = \lim_{t \rightarrow \infty} \lambda_1(0) e^{-\hat{r}t} B(t) = 0$ must be satisfied and this hold if and only if

$$C(0) = -\rho(B(0) + \theta K(0))$$

⁸Turnovsky (1996) assumes that the country may accumulate foreign bonds. This difference does not affect the properties of the model.

⁹ λ_1 and λ_2 are the two co-state variables of the optimization problem attached to debt (negative) and capital (positive), respectively.

If $C(0) > 0$, it must be $B(0) + \theta K(0) < 0$. Given the definition of θ and given that the transversality condition implies $\hat{r} > \frac{q_1-1}{h}$, a necessary condition is $\frac{q_1-1}{h} + A^{\frac{1}{1-\alpha}} \alpha^{\frac{1}{1-\alpha}} + \frac{(q_1-1)^2}{2h} - A^{\frac{1}{1-\alpha}} \alpha^{\frac{\alpha}{1-\alpha}} < 0$.

The equilibrium stock of debt is hence given by

$$B(t) = (B(0) + \theta K(0)) e^{(\hat{r}-\rho)t} - \theta K(0) e^{\frac{q_1-1}{h}t} \quad (2.7)$$

Turnovsky's model has some important implications. First, the small open economy assumption implies that the growth rate of domestic consumption can be different from the growth rate of domestic output (capital), which is in sharp contrast with the closed economy version of the model, where all the variables grow at the same rate. Second, the growth rate of the stock of debt varies over time and asymptotically converges to $[\frac{q_1-1}{h}, \hat{r} - \rho]$. In particular if $\frac{q_1-1}{h} < \hat{r} - \rho$, $\lim_{t \rightarrow +\infty} B(t) = -\infty$. On the contrary, if $\frac{q_1-1}{h} > \hat{r} - \rho$, $\lim_{t \rightarrow +\infty} B(t) = +\infty$ and the country would accumulate a forever increasing debt. Finally, the fact that the economy can borrow from the rest of the world does not influence the part of domestic output invested in public expenditures, which is equal to the income share of public goods, α , as in Barro's (1990) model.

2.2 The model with GDP-based collaterals

We assume now that the country's debt capacity is bounded by GDP. The external debt constraint is:

$$0 \leq B \leq \phi Y, \phi \in [0, 1] \quad (2.8)$$

The case $\phi = 0$ corresponds to autarky,¹⁰ while $\phi > 0$ describes international financial integration. Since financial markets are imperfect and the highest obtainable collaterals is the entire GDP, it must be $\phi \leq 1$ ¹¹ in order to take into account the presence of enforcement costs.¹² Nevertheless, this restriction rules out debt-GDP ratios greater than 1. It is possible to show that the analysis remains valid for a wide range of parameters.¹³

The social planner must choose the level of consumption, public expenditures, investment,

¹⁰In this case the constraint (2.8) is binding.

¹¹See Cohen and Sacs (1986) and Boucekine and Pintus (2012). Djankov et al. (2008) estimates that the values of ϕ range from 35% in middle-income countries to 85% in OECD countries.

¹²Imposing a flow variable as a constraint on a stock variable is reasonable in an endogenous growth model because GDP tends to infinity in the long run.

¹³In particular, ϕ must be such that $\phi < 1/(\hat{r} - \rho)$.

external debt and capital which solve the problem

$$\begin{aligned}
& \max_{\{C,G,I\}} \quad U(C) = \int_0^{+\infty} \ln C e^{-\rho t} dt \\
& \text{s. t.} \quad \dot{K} = I \\
& \quad \quad \dot{B} = C + I + G + \frac{h}{2} \frac{I^2}{K} + \hat{r}B - Y \\
& \quad \quad B \leq \phi Y, B \geq 0, \forall t \\
& \quad \quad K(0), B(0) \text{ given}
\end{aligned} \tag{2.9}$$

where Y is given by (2.1). The collateral constraint, $0 \leq B \leq \phi Y$, defines three economic regimes. First, the stock of debt B can be equal to zero (autarky case) or positive. In the latter case the constraint (2.8) can be non binding or binding. The identification of the optimal autarky cases is quite straightforward. Hence we will focus on the two other regimes.

In order to endogenously find the date at which the constraint becomes binding, we solve the model by using the two-stage optimal control technique. Optimal switching time problems that can be found in the literature typically involve technological or ecological states of the world that are exogenously given. For example Tomiyama (1985) and Boucekkine et al. (2004) index the successive regimes by a finite number of discrete parameter values: a newly available technological regime may exhibit a higher productivity parameter. A more recent branch of the literature applies the optimal switching problem to the analysis of irreversibilities (see Boucekkine et al. 2013a, Boucekkine et al. 2013b, Boucekkine et al. 2013c, Prieur et al. 2011): for example when the state variable (the stock of pollution) exceeds a certain threshold value, the natural decay rate goes down permanently. One can reformulate this as an optimal timing problem because the date at which the stock variable reaches the threshold becomes a control variables.

In this paper we tackle a different type of optimal regime switching problems. The economy starts from a situation in which the stock of debt is such that the collateral constraint is not binding and it may become binding at a certain date (this is proved in Proposition 6) which has to be optimally determined. This modifies one of the state equations. For this reason the problem can be set as an optimal switching time problem.¹⁴ The two possible states of the economy can be summarized as follows

$$\left\{ \begin{array}{ll} \dot{B} = C + I + G + \frac{h}{2} \frac{I^2}{K} + \hat{r}B - AG^\alpha K^{1-\alpha} & \text{if } B < \phi Y \quad t \in [0, T] \\ \dot{B} = C + I + \left(\frac{1}{\phi}\right)^{\frac{1}{\alpha}} A^{-\frac{1}{\alpha}} B^{\frac{1}{\alpha}} K^{\frac{\alpha-1}{\alpha}} + \hat{r}B + \frac{h}{2} \frac{I^2}{K} - \frac{1}{\phi}B & \text{if } B = \phi Y \quad t \in (T, +\infty) \end{array} \right. \tag{2.10}$$

Given (2.10), a natural approach is to decompose the problem into two sub-problems, solve them and finally identify the optimal timing T . In our case since

$$V(K(0), B(0)) = \int_0^T \ln C e^{-\rho t} dt + \int_T^{+\infty} \ln C e^{-\rho t} dt$$

¹⁴Of course the problem can be solved by using the traditional method of Lagrange, but this does not allow to determine the endogenous switching time in our case.

one can view the problem sequentially starting from the second regime, for $t \in (T, +\infty)$ and then solving for the first regime. In the binding regime the problem becomes:

$$\begin{aligned} \max_{\{C, I\}} \quad & V_2 = \int_T^{+\infty} \ln C e^{-\rho t} dt \\ \text{s. t.} \quad & \dot{K} = I \\ & \dot{B} = C + I + \left(\frac{1}{\phi}\right)^{\frac{1}{\alpha}} A^{-\frac{1}{\alpha}} B^{\frac{1}{\alpha}} K^{\frac{\alpha-1}{\alpha}} + \hat{r}B + \frac{h}{2} \frac{I^2}{K} - \frac{1}{\phi} B \end{aligned} \quad (2.11)$$

for a given $T, K(T), B(T)$

The corresponding hamiltonian is

$$H^{(2)} = \ln C e^{-\rho t} + \lambda_1^{(2)} \left[C + I + \left(\frac{1}{\phi}\right)^{\frac{1}{\alpha}} A^{-\frac{1}{\alpha}} B^{\frac{1}{\alpha}} K^{\frac{\alpha-1}{\alpha}} + \hat{r}B + \frac{h}{2} \frac{I^2}{K} - \frac{1}{\phi} B \right] + \lambda_2^{(2)} I \quad (2.12)$$

where the upper-script “⁽²⁾” means that the constraint binds. $\lambda_1^{(2)} < 0$ is the co-state variable associated to the stock of debt and $\lambda_2^{(2)} > 0$ is the co-state variable associated to capital when the economy is in the second regime.

Then we solve

$$\begin{aligned} \max_{\{C, G, I\}} \quad & V_1 = \int_0^T \ln C e^{-\rho t} dt + V_2^*(T, B(T), K(T)) \\ \text{s. t.} \quad & \dot{K} = I \\ & \dot{B} = C + I + G + \frac{h}{2} \frac{I^2}{K} + \hat{r}B - AG^\alpha K^{1-\alpha} \\ & K(0), B(0) \text{ given, } T \text{ free} \end{aligned} \quad (2.13)$$

and the corresponding hamiltonian is

$$H^{(1)} = \ln C e^{-\rho t} + \lambda_1^{(1)} \left[C + I + G + \frac{h}{2} \frac{I^2}{K} + \hat{r}B - AG^\alpha K^{1-\alpha} \right] + \lambda_2^{(1)} I \quad (2.14)$$

where the upper-script “⁽¹⁾” means that the regime is non binding.

This recursive scheme works exactly as a dynamic programming problem: the Bellman principle applies to the two sub-intervals implied by the timing problem.

We will discuss the Pontryagin optimality condition in the subsequent sections.¹⁵ Here we shall focus on the optimality conditions with respect to the timing variable T . They are summarized by the following theorem:

Theorem 1. *Let $0 < T < +\infty$ the optimal timing. Then*

$$\lambda_1^{(1)}(T) = \lambda_1^{(2)}(T) \quad (2.15)$$

$$\lambda_2^{(1)}(T) = \lambda_2^{(2)}(T) \quad (2.16)$$

$$H^{(1)}(T) = H^{(2)}(T) \quad (2.17)$$

¹⁵The problem is concave hence each the two sub-intervals problems is well behaved.

A detailed proof is contained in Appendix A. Some comments are in order here. First, the above conditions can be read as continuity or matching conditions at the junction point: condition (2.17) imposes the continuity of the hamiltonian at the optimal junction time and conditions (2.15) and (2.16) impose the continuity of the co-state variables. Here both the co-state variables are continuous because the state variables do not jump at the switching time: the state variables do not have to satisfy any pure state constraint and both can be freely chosen. Second, one can interpret the matching conditions (2.15)-(2.17) as first order optimal timing conditions with respect to the switching date T . In fact condition (2.17) can be rewritten as $H^{(1)}(T) - H^{(2)}(T) = 0$. The left hand side can be interpreted as the marginal gain from extending the first regime at the expense of the regime associated to the next interval. The right hand side represents the marginal cost. Since there is no direct switching cost, this cost is zero. Hence this condition equalizes the marginal benefit to the marginal cost of delaying the switching time.

2.3 Preliminary analysis of economic regimes

In this section we will solve the model by starting with the Pontryagin problem of the binding regime and then by proceeding backward to the non binding case.

2.3.1 The binding regime

For a given T , $B(T)$ and $K(T)$, the Pontryagin necessary conditions of problem (2.11) are

$$\frac{1}{C}e^{-\rho t} + \lambda_1 = 0 \quad (2.18)$$

$$\lambda_1 \left[1 + h \frac{I}{K} \right] + \lambda_2 = 0 \quad (2.19)$$

$$\lambda_1 \left[\frac{1}{\alpha} \left(\frac{1}{\phi} \right)^{\frac{1}{\alpha}} A^{-\frac{1}{\alpha}} \left(\frac{B}{K} \right)^{\frac{1-\alpha}{\alpha}} + \hat{r} - \frac{1}{\phi} \right] = -\dot{\lambda}_1 \quad (2.20)$$

$$-\lambda_1 \left[\frac{h}{2} \frac{I^2}{K^2} + \frac{1-\alpha}{\alpha} \left(\frac{1}{\phi} \right)^{\frac{1}{\alpha}} A^{-\frac{1}{\alpha}} \left(\frac{B}{K} \right)^{\frac{1}{\alpha}} \right] = -\dot{\lambda}_2 \quad (2.21)$$

$$\lim_{t \rightarrow \infty} \lambda_1 B = 0 \quad \lim_{t \rightarrow \infty} \lambda_2 K = 0$$

By using the shadow value of debt λ_1 as the numeraire and defining $q = -\frac{\lambda_2}{\lambda_1}$, we find the growth rate of capital $\frac{I}{K} = \frac{q-1}{h}$ from equation (2.19) and by combining it with (2.20) and (2.21) we obtain the arbitrage equation

$$\frac{\dot{q}}{q} = \frac{1}{\alpha} \left(\frac{1}{\phi} \right)^{\frac{1}{\alpha}} A^{-\frac{1}{\alpha}} \left(\frac{B}{K} \right)^{\frac{1-\alpha}{\alpha}} + \hat{r} - \frac{1}{\phi} - \frac{1}{q} \frac{1-\alpha}{\alpha} \left(\frac{1}{\phi} \right)^{\frac{1}{\alpha}} A^{-\frac{1}{\alpha}} \left(\frac{B}{K} \right)^{\frac{1}{\alpha}} - \frac{(q-1)^2}{2hq}$$

We find it convenient to rewrite the arbitrage equation as a function of $\left(\frac{C}{Y}\right)$ in order to compare the results with Barro (1990) and Turnovsky's (1996) models. From the constraint

we know that the ratio $\left(\frac{B}{K}\right)$ is linked to the ratio $\left(\frac{G}{K}\right)$ in the following way: $\frac{B}{K} = \phi A \left(\frac{G}{K}\right)^\alpha$. Moreover, by using the production function we know that: $\frac{G}{K} = A^{\frac{1}{1-\alpha}} \left(\frac{G}{Y}\right)^{\frac{1}{1-\alpha}}$. Hence the arbitrage equation becomes

$$\frac{\dot{q}}{q} = \frac{1}{\alpha} \left(\frac{1}{\phi}\right) \left(\frac{G}{Y}\right) + \hat{r} - \frac{1}{\phi} - \frac{1}{q} \frac{1-\alpha}{\alpha} A^{\frac{1}{1-\alpha}} \left(\frac{G}{Y}\right)^{\frac{1}{1-\alpha}} - \frac{(q-1)^2}{2hq} \quad (2.22)$$

Log-differentiating equation (2.18) and using equation (2.20) we derive the growth rate of consumption as a function of $\left(\frac{G}{Y}\right)$:

$$\frac{\dot{C}}{C} = \hat{r} - \frac{1}{\phi} + \frac{1}{\alpha} \frac{1}{\phi} \left(\frac{G}{Y}\right) - \rho \quad (2.23)$$

We find it convenient to rewrite the dynamical system made up of the aggregate resource constraint, (2.22), and of (2.23) by defining $b \equiv \frac{B}{K}$ and $c \equiv \frac{C}{K}$, so that

$$\begin{cases} \frac{\dot{q}}{q} = \frac{1}{\alpha} \left(\frac{1}{\phi}\right) \left(\frac{G}{Y}\right) + \hat{r} - \frac{1}{\phi} - \frac{1}{q} \frac{1-\alpha}{\alpha} A^{\frac{1}{1-\alpha}} \left(\frac{G}{Y}\right)^{\frac{1}{1-\alpha}} - \frac{(q-1)^2}{2hq} \\ \frac{\dot{c}}{c} = \hat{r} - \frac{1}{\phi} + \frac{1}{\alpha} \frac{1}{\phi} \left(\frac{G}{Y}\right) - \rho - \frac{q-1}{h} \\ \frac{\dot{b}}{b} = \frac{1}{b} \left[c + \frac{q-1}{h} + A^{\frac{1}{1-\alpha}} \left(\frac{G}{Y}\right)^{\frac{1}{1-\alpha}} + \frac{(q-1)^2}{2h} + \hat{r}b - A^{\frac{1}{1-\alpha}} \left(\frac{G}{Y}\right)^{\frac{\alpha}{1-\alpha}} \right] - \frac{q-1}{h} \end{cases} \quad (2.24)$$

2.3.2 The non binding regime

For a given T , $B(T)$ and $K(T)$, the first order necessary conditions of problem (2.13) are¹⁶

$$\frac{1}{C} e^{-\rho t} + \lambda_1 = 0 \quad (2.25)$$

$$\lambda_1 \left(1 + h \frac{I}{K}\right) + \lambda_2 = 0 \quad (2.26)$$

$$\frac{G}{K} = (A\alpha)^{\frac{1}{1-\alpha}} \iff \frac{G}{Y} = \alpha \quad (2.27)$$

$$\lambda_1 \hat{r} = -\dot{\lambda}_1 \quad (2.28)$$

$$-\lambda_1 \left[\frac{h}{2} \frac{I^2}{K^2} + A(1-\alpha) \left(\frac{G}{K}\right)^\alpha \right] = -\dot{\lambda}_2 \quad (2.29)$$

where $q = -\frac{\lambda_2}{\lambda_1}$ and $\frac{I}{K} = \frac{q-1}{h}$. The public expenditures-to-GDP ratio is equal to α .

By using (2.29), (2.28) and the definition of q we get the arbitrage equation

$$\frac{\dot{q}}{q} = \hat{r} - \frac{(q-1)^2}{2hq} - (1-\alpha) A^{\frac{1}{1-\alpha}} \alpha^{\frac{\alpha}{1-\alpha}} \frac{1}{q} \quad (2.30)$$

Combining equation (2.25) with equation (2.28) we obtain the growth rate of consumption

$$\frac{\dot{C}(t)}{C(t)} = \hat{r} - \rho \quad (2.31)$$

¹⁶This regime is the same as Turnovsky's (1996) model, with exception of the transversality conditions: in our case the boundary conditions are given by the continuity conditions of Theorem 1.

Note that, in order to have a positive consumption growth rate, it must be $\hat{r} > \rho$. By defining $c = \frac{C}{K}$, $b = \frac{B}{K}$ we can write the reduced system as

$$\begin{cases} \dot{q} = \hat{r} - \frac{(q-1)^2}{2hq} - (1-\alpha) A^{\frac{1}{1-\alpha}} \alpha^{\frac{\alpha}{1-\alpha}} \frac{1}{q} \\ \dot{c} = (\hat{r} - \rho) - \frac{q-1}{h} \\ \dot{b} = \frac{1}{b} \left[c + \frac{q-1}{h} + A^{\frac{1}{1-\alpha}} \alpha^{\frac{1}{1-\alpha}} + \frac{(q-1)^2}{2h} + \hat{r}b - A^{\frac{1}{1-\alpha}} \alpha^{\frac{\alpha}{1-\alpha}} \right] - \frac{q-1}{h} \end{cases} \quad (2.32)$$

3 Balanced growth path analysis

In this section we study the balanced growth path of the economy (BGP hereafter) in the case where the economy ends up in the binding regime. The condition for optimal switching will be given in Section 4. Along the BGP, all the endogenous variables grow at the same, positive and constant growth rate, given by equation (2.23), and the endogenous variables ($c, G/Y, q$) of system (2.24) grow at a zero rate. This result can be verified by considering that when the collateral constraint binds, Y and B grow at the same rate. From the production function it can be seen that capital grows at the same rate of output. Finally, the terms of equation (2.10) are all proportional to K or B ,¹⁷ so that also C grows at the same rate.

By setting $\dot{c} = 0$ in (2.24), we obtain $q = h \left[\hat{r} - \frac{1}{\phi} + \frac{1}{\alpha} \frac{1}{\phi} \left(\frac{G}{Y} \right) - \rho \right] + 1$. By plugging it into the $\dot{q} = 0$ equation and rearranging terms we end up with the equation determining the equilibrium (G/Y) ratio

$$\begin{aligned} -\frac{1-\alpha}{\alpha} A^{\frac{1}{1-\alpha}} \phi \left(\frac{G}{Y} \right)^{\frac{1}{1-\alpha}} + \frac{h}{2} \left(\frac{1}{\phi} \right) \frac{1}{\alpha^2} \left(\frac{G}{Y} \right)^2 + [h(\hat{r}\phi - 1) + \phi] \frac{1}{\phi} \frac{1}{\alpha} \left(\frac{G}{Y} \right) \\ + \frac{1}{\phi} \left[\frac{h}{2} (\hat{r}\phi - 1)^2 + \phi(\hat{r}\phi - 1) - \frac{h}{2} (\rho\phi)^2 \right] = 0 \end{aligned} \quad (3.1)$$

Note that by setting $\phi = 0$ we obtain Barro (1990) and Turnovsky's (1996) optimal public expenditures-to-GDP ratio: $(G/Y) = \alpha$. In the following proposition we give the conditions for the existence of a solution for the empirically interesting case $\alpha \leq \frac{1}{2}$.¹⁸

Proposition 2. *The dynamical system (2.24) admits one steady state.*

The proof is given in Appendix B, where we also show that it satisfies the transversality condition.

In order to obtain analytical solutions for the ratio (G/Y) and to study analytically the local stability of the steady state, we focus on the $\alpha = \frac{1}{2}$ case.¹⁹ This case is perfectly equivalent to the case $\alpha < 1/2$, hence the conditions for the existence of a BGP do not change.

¹⁷Equation (2.10) can be rewritten as $\frac{\dot{B}}{B} B = C + \frac{\dot{K}}{K} K + \left(\frac{1}{\phi} \right)^{\frac{1}{\alpha}} A^{-\frac{1}{\alpha}} \left(\frac{B}{K} \right)^{\frac{1}{\alpha}} K + \hat{r}B + \frac{h}{2} \frac{I^2}{K^2} K - \frac{1}{\phi} B$

¹⁸This range of values is consistent with the empirical estimations existing in the literature: Erberts's (1986) estimation for the income share of public expenditures is around 0.43, Aschauer (1989) finds that it is 0.39, Ai and Cassou (1995) find estimations which range between 0.15 and 0.26, finally Chatterjee et al. (2003) assume a value of 0.2.

¹⁹We analyze this case separately since the closed form solutions of the steady state variables are necessary to compute the Jacobian matrix for the stability analysis. Otherwise it is not possible to determine the sign of the trace and the determinant.

Equation (3.1) becomes

$$\begin{aligned} & \left(\frac{2h}{\phi} - A^2\phi\right) \left(\frac{G}{Y}\right)^2 + 2[h(\hat{r}\phi - 1) + \phi] \frac{1}{\phi} \left(\frac{G}{Y}\right) \\ & + \frac{1}{\phi} \left[\frac{h}{2}(\hat{r}\phi - 1)^2 + \phi(\hat{r}\phi - 1) - \frac{h}{2}(\rho\phi)^2\right] = 0 \end{aligned} \quad (3.2)$$

The unique admissible solution is

$$\left(\frac{G}{Y}\right)^* = \frac{-[h(\hat{r}\phi - 1) + \phi] \frac{1}{\phi} + \sqrt{\frac{1}{2}(2 + 2h^2\rho^2 + A^2(h - 2(1 + h\hat{r})\phi + (r(2 + h\hat{r}) - h\rho^2)\phi^2))}}{\left(\frac{2h}{\phi} - A^2\phi\right)} \quad (3.3)$$

The next step is to study the stability of the BGP. System (2.24) allows transitional dynamics. This is due to the presence of investment adjustment costs: the economy does not reach immediately its steady state because the adjustment costs do not allow capital to immediately jump to its BGP. Proposition 3 describes the local stability properties of $\left(\frac{G}{Y}\right)^*$:

Proposition 3. *If $A^2 > -\frac{2}{h-2(1+\hat{r}h)\phi+(\hat{r}(2+h\hat{r})-h\rho^2)\phi^2}$, the balanced growth path solution is saddle-point stable.*

The proof is given in Appendix C.

Numerical analysis shows that the BGP is stable under the $\alpha \leq 1/2$ case, for standard calibrations of the parameters.

To conclude this section we will compare the optimal (G/Y) of our model with that of Barro (1990) and Turnovsky's (1996; 1997) model.

Proposition 4. *The optimal public expenditures-to-GDP ratio of the model with GDP-based collaterals is greater than Barro (1990) and Turnovsky's (1996) solution.*

In order to prove this result, consider the first order condition with respect to public goods of the maximization problem (2.9), solved by attaching a Lagrange multiplier $\eta \geq 0$ to the inequality constraint (2.8).²⁰

$$\frac{\partial Y}{\partial G} - \frac{\eta}{\lambda_1} \frac{\partial Y}{\partial G} = 1 \quad (3.4)$$

where $\eta = \frac{1}{\phi} \left[1 - \frac{1}{\alpha} \left(\frac{G}{Y}\right)\right] \lambda_1$. Being $\eta > 0$ when the constraint binds, and $\lambda_1 < 0$, equation (3.4) implies that in the binding regime (G/Y) must be greater than α , i. e. greater than Barro (1990) and Turnovsky's (1996; 1997) value. Equation (3.4) states that the social planner will choose the optimal (G/Y) by equating the marginal benefit of increasing G to the marginal cost, which, by the economy's resource constraint, is equal to one. In the literature's models the corresponding expression is $\frac{\partial Y}{\partial G} = 1$. Hence we can conclude that in the model with a GDP-based collateral constraint the overall marginal benefit is equal to the marginal benefit of increasing G on production augmented by the term $-\frac{\eta}{\lambda_1} \frac{\partial Y}{\partial G}$, which represents the marginal benefit of relaxing the collateral constraint, since a greater public expenditures increases the country's borrowing capacity. This is the reason why the public expenditures-GDP ratio is greater than that of the literature's models.

²⁰This allows to carry out an economic interpretation.

3.1 Comparative statics

The aim of this section is to run a comparative statics analysis. The analytical results are included in the following proposition:

Proposition 5. *The optimal public expenditures-to-GDP ratio is*
(i) *decreasing in the world interest rate \hat{r} ;*
(ii) *increasing in the technology parameter A .*
(iii) *The long run growth rate is increasing in the TFP parameter.*

Proof. The proof is contained in Appendix D. □

An increase in the international interest rate makes borrowing more costly and given that public expenditures are financed by debt, the optimal (G/Y) ratio decreases. Results (ii) and (iii) stem from the double role played by the TFP in our model. On the one hand, A rises the economy's productivity and this boosts growth. On the other hand, an increase in A , by making the economy richer, has the effect of increasing the collaterals, allowing the economy to borrow more from the rest of the world. The possibility to increase the financing capacity enhances growth.

Unfortunately we cannot obtain analytical results for the main comparative statics, so we run numerical simulations. We fix $\alpha \leq 1/2$,²¹ $h = 15$ (Barro and Sala-i-Martin 2004, Chatterjee et al. 2003), $\hat{r} = 0.035$ and $\rho = 0.03$ (these values allow to satisfy the conditions needed to move from the non binding to the binding regime that will be introduced later in Section 4). We set $A = 1$ in order to obtain a value for the capital-output ratio consistent with the real one, which is around 2.²² We hence obtain the following results:

(iv) *The optimal (G/Y) and the economy's growth rate are increasing in ϕ .* A higher ϕ implies higher growth because a greater financial openness relaxes the borrowing constraint (growth enhancing effect of foreign borrowing).

(v) *The growth rate is decreasing in the world interest rate \hat{r} .* Since (G/Y) is decreasing in \hat{r} , the same happens for the long run growth rate.

(vi) *The optimal (G/Y) and the economy's growth rate are decreasing functions of the installation cost parameter h .* Higher capital adjustment costs reduce the (G/Y) ratio and the growth rate because more resources are spent in installation.

The numerical analysis also shows that the growth rate of GDP is smaller than that of Turnovsky (1996). Recall that in his model the country can borrow as much as it wants because there is no collateral constraint and the growth rate of GDP coincides with the growth rate of capital, which can be different from the growth rate of consumption. This latter is set by the difference between the exogenous international interest rate and the discount rate, and it is independent from any technological parameter. The growth rate in Turnovsky turns out to be greater than that of our model due to the absence of growth-limiting borrowing constraints.

²¹We run the simulations for $\alpha = 0.2$ (Chatterjee et al. 2003) and $\alpha = 1/2$.

²²We repeated the exercise for other values of the parameter and the results are robust.

4 Determination of the optimal switching date

4.1 Optimal switching

4.1.1 The convergence to the binding regime

In this section we study the conditions for having the convergence to the binding regime. We suppose that the economy is in the non binding regime and we assume that the system will be unconstrained forever, that is $T = +\infty$. Hence the model turns out to be the same as Turnovsky's (1996). Here we extend his analysis in order to show when the corner solution $T = +\infty$ is possible.

In Section 2.1 we show that the dynamics of $B(t)$, as given by equation (2.7), depends on the difference between $(\hat{r} - \rho)$ and $\frac{q_1 - 1}{h}$, whose sign is determined by q_1 , which is given by (2.5). These expressions represent the growth rates of consumption and capital (output), respectively, in Turnovsky's model. The growth rate of capital (and output) is positive if $q_1 > 1$, which happens when $\hat{r} < (1 - \alpha) A^{\frac{1}{1-\alpha}} \alpha^{\frac{\alpha}{1-\alpha}}$. We shall concentrate on this case.

Proposition 6. *Suppose that $\hat{r} - \rho < \frac{q_1 - 1}{h}$. Then there exists a date $T < +\infty$ at which the economy hits the constraint.*

In the case $\hat{r} - \rho < \frac{q_1 - 1}{h}$,²³ $B(t)$ is an increasing function of t and $\lim_{t \rightarrow +\infty} B(t) = +\infty$. Hence in this case $B(t)$ hits the constraint at a finite time.

Otherwise,

Proposition 7. *When $\hat{r} - \rho \geq \frac{q_1 - 1}{h}$, $\lim_{t \rightarrow +\infty} B(t) = -\infty$: the economy will end up in autarky within a finite time.*

The case $\hat{r} - \rho \geq \frac{q_1 - 1}{h}$ is more complex. When this condition is satisfied, $\lim_{t \rightarrow +\infty} B(t) = -\infty$, but $B(t)$ can be either always decreasing or first increasing and then decreasing, depending on the values of $\frac{B(0)}{K(0)}$. At $t = 0$, $B'(0)$ is equal to:

$$B'(0) = \theta K(0) \left[\hat{r} - \rho - \frac{q_1 - 1}{h} \right] + (\hat{r} - \rho) B(0)$$

which is negative if $\frac{B(0)}{K(0)} < -\theta \left[1 - \frac{(q_1 - 1)/h}{\hat{r} - \rho} \right]$.

The other condition on $\frac{B(0)}{K(0)}$ is given by the collateral constraint: $\frac{B(0)}{K(0)} < \phi A^{\frac{1}{1-\alpha}} \alpha^{\frac{\alpha}{1-\alpha}}$.

If $\phi A^{\frac{1}{1-\alpha}} \alpha^{\frac{\alpha}{1-\alpha}} < -\theta \left[1 - \frac{(q_1 - 1)/h}{\hat{r} - \rho} \right]$, then $B(t)$ is always decreasing and never hits the constraint.

If $\phi A^{\frac{1}{1-\alpha}} \alpha^{\frac{\alpha}{1-\alpha}} \geq -\theta \left[1 - \frac{(q_1 - 1)/h}{\hat{r} - \rho} \right]$, then when $\frac{B(0)}{K(0)} < -\theta \left[1 - \frac{(q_1 - 1)/h}{\hat{r} - \rho} \right]$, $B(t)$ is decreasing and never hits the constraint. When $\frac{B(0)}{K(0)} \geq -\theta \left[1 - \frac{(q_1 - 1)/h}{\hat{r} - \rho} \right]$ and $\frac{B(0)}{K(0)}$ is close to the collateral constraint, $B(t)$ is first increasing then decreasing, hence it may hit the constraint twice.

²³Since $(\hat{r} - \rho)$ is the growth rate of consumption in the non binding regime, we set $\hat{r} > \rho$. It turns out that $\hat{r} - \rho < \frac{q_1 - 1}{h}$ is possible only if $q_1 > 1$.

One might think that when $\hat{r} - \rho \geq \frac{q_1 - 1}{h}$ the binding regime starts at $t = 0$. However we have checked numerically that $\hat{r} - \rho \geq \frac{q_1 - 1}{h}$ is incompatible with the binding regime optimal from $t = 0$. Hence, the results of Proposition 6 translate into some threshold conditions that the parameters must satisfy for the economy to switch to the binding regime and hence to benefit from financial openness. Notably, this happens if the interest rate and the adjustment costs are small and the productivity is high. In the opposite case, an economy starting in the non binding regime will end up in autarky. A detailed numerical discussion is contained in Section 4.3.

4.1.2 The existence of the non binding regime: the role of investment adjustment costs

As previously shown, the assumption of investment adjustment costs in open economy models is necessary to rule out the unrealistic knife-edge condition on the marginal return of domestic capital. But this assumption has another important implications in our two-regimes model. In the absence of capital adjustment costs, if the knife-edge condition is not satisfied, the regime with slack constraint does not exist, because the country would choose to invest or disinvest at an infinite rate if the internal productivity is, respectively, greater or smaller than the interest rate (see Cohen and Sachs 1986). Since we assume that the country is a net debtor, the internal return must be greater than the world interest rate, so that the economy switches to the binding regime since the initial date. Hence the presence of adjustment costs of investment enriches the inter-regime transitional dynamics.

4.2 The inter-regime transitional dynamics

In this section we simulate the two-regimes model in order to find the endogenous switching date. The existence of transitional dynamics in the binding regime does not allow us to find analytical solutions for the date T . Our strategy is to apply the matching conditions of Theorem 1 in order to link two two-points boundary value problems, one for each regime. Final boundary problems are commonly used for the simulation of the dynamics of growth models. We follow the method proposed by Trimborn et al. (2008a, 2008b).

We start by defining the matching conditions in our case. The conditions (2.15) and (2.16) of Theorem 1 require the equality of the co-state variables between the two regimes. Hence at T the following condition must hold:

$$q^{(1)}(T) = q^{(2)}(T) = q(T) \quad (4.1)$$

We replace (2.17), which is difficult to implement when a regime cannot be solved in closed form, with two other conditions, which emerge from our model. Given that at T the state variables can not jump, it must be $b^{(1)}(T) = b^{(2)}(T) = b(T)$. This value can be obtained by the collateral constraint $\frac{B}{K} = b = \phi A \left(\frac{G}{K}\right)^\alpha = \phi A^{\frac{1}{1-\alpha}} \left(\frac{G}{Y}\right)^{\frac{\alpha}{1-\alpha}}$. Given that b must be continuous at T , the same applies for the (G/Y) ratio. In the non binding regime $\left(\frac{G}{Y}\right) = \alpha$, hence when $b^{(1)}(t)$ hits the constraint it must be

$$b^{(1)}(T) = b^{(2)}(T) = b(T) = \phi A^{\frac{1}{1-\alpha}} \alpha^{\frac{\alpha}{1-\alpha}} \quad (4.2)$$

As a consequence the value of $b(T)$ is known.

The last condition stems from equation (2.15), the equality between the shadow values of external debt across regimes, and from the first order conditions (2.18) and (2.25). These together implies that consumption is continuous at T , so that

$$c^{(1)}(T) = c^{(2)}(T) = c(T) \quad (4.3)$$

Equation (4.3) also derives from the continuity of $\left(\frac{G}{Y}\right)$ at T and from the expression of the growth rate of consumption in the binding regime: $\frac{\dot{C}}{C} = \hat{r} - \frac{1}{\phi} \left(1 - \frac{1}{\alpha} \frac{G}{Y}\right) - \rho$.

The adopted algorithm to simulate the two regimes separately is the relaxation algorithm proposed by Trimborn et al. (2008a, 2008b). Our numerical procedure starts backward with the simulation of the binding regimes and then of the non binding one. The goal of this exercise is to find the switching date which satisfies the matching conditions (4.1), (4.2) and (4.3). The method employed to solve the inter-regime transitional dynamics is described in Appendix E.

We assume the same parameters' values of the baseline calibration of Section 4. We set $\alpha = 1/2$ because in this case we know what conditions the parameters need to satisfy for the BGP to be stable (cf. Proposition 3) and $b(0) = 0.0145$.²⁴ This parametrization satisfies the convergence conditions stated in Proposition 6. The figures below (Figure 4.1, Figure 4.2 and Figure 4.3) show the transition of the economy from the non binding to the binding regime. As indicated by the red dotted-line, the switching date is equal to 2.5 periods.

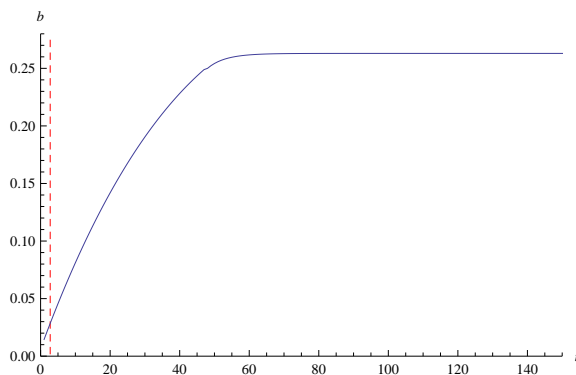


Figure 4.1: The transition inter regime: the debt-to-capital ratio

²⁴The chosen value of initial conditions affects the switching date T : the smaller is $b(0)$, the higher is T .

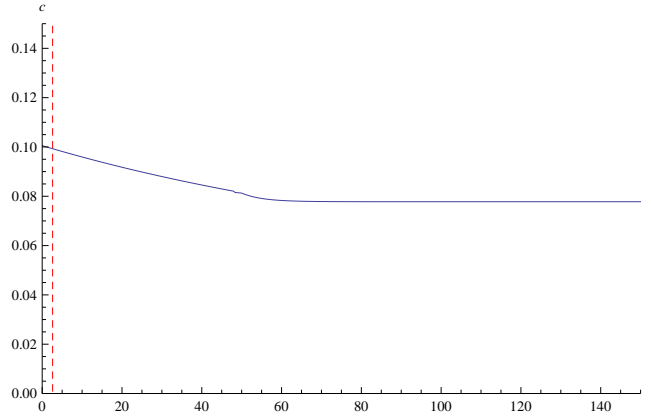


Figure 4.2: The transition inter regime: the consumption-to-capital ratio

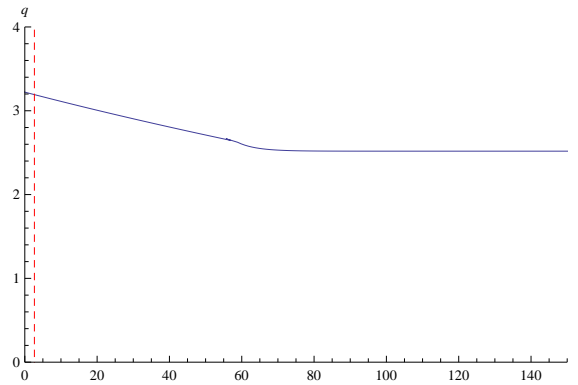


Figure 4.3: The transition inter regime: the Tobin's q

These figures show that the transition occurs in a continuous way: the country starts in the non binding regime and moves to the binding one at $T = 2.5$. Namely, the endogenous variables do not jump: as stated in Section 2.2, the borrowing constraint is not a pure state constraint.

Note also that the non binding regime, described by system (2.32), does not admit a steady state. As shown in Section 2, the growth rates of the endogenous variables in level are different if the constraint does not bind.

During the transition the debt-to-capital ratio follows an increasing path, while the consumption-to-capital ratio and the Tobin's q are decreasing, until the economy attains its BGP. In order to interpret this result, we refer to the economy's current account, as given by equation (2.2). Throughout the transition, the stock of external debt is growing more than the stock of capital. The increasing international capital inflows are spent in physical capital, installation costs, public goods and increasing interest payments. This diverts resources from consumption, reducing the consumption-to-capital ratio. The dynamics

of the Tobin's q is consistent with that of capital. Once the switch to the binding regime occurs, the stock of debt is constrained, forcing the economy to reach an equilibrium where all the variables in level grow at the same rate.

The economy's dynamics is consistent with Turnovsky's (1997) model, where financial market imperfections are introduced by assuming that the cost of borrowing is an increasing function of debt, relative to capital. In particular, a small open economy with unlimited access to international financial markets can sustain differential growth rates of consumption and output along the BGP, in sharp contrast to a closed economy where the economy's resources constraint all the variables to grow at the same rate. For a net debtor country, consumption grows less than capital (output). When a collateral constraint is added, the growth rates differ along the transition. However, once the steady state is reached, all the variables grow at the same rate.

4.3 Comparative statics

In this section we investigate the effects of some relevant parameters on the transition between the non binding and the binding regime.²⁵ The main results are summarized in the findings reported below.

(i) The optimal switching date T is increasing with respect to: financial openness (ϕ), the international interest rate (\hat{r}), the investment adjustment costs (h); it is decreasing with respect to the total factor productivity (A).

An increase of ϕ relaxes the collaterals, hence the initial value of the debt-to-capital ratio, $b(0)$, will be farther away from the constraint and the economy will spend more time in the non binding regime. A rise of the international interest rate and of the adjustment costs makes the transition slower. The first result is linked to the amount of interests to be paid: since debt cost is high, the country won't push the stock of debt until its upper bound immediately. Higher installation costs slow down the accumulation rate of external debt (see Section 4.1.2). Finally, for a greater productivity, the economy moves faster to the binding regime. When A becomes very large, the optimal switching date approaches zero. In our model the parameter A plays a double role. A greater TFP rises the economy's productivity making the country richer. This has in turns the effect of increasing the collaterals, allowing the country to borrow more from the rest of the world. Hence the economy will spend less time in the non binding regime, reaching faster the advantages of financial globalization.

(ii) If the interest rate, the installation costs are too high and/or the total factor productivity is too low, the economy will stay forever in the non binding regime and the debt-to-capital ratio $b(t)$ will tend to zero (autarky).

If the economy has to actually switch to the binding regime, the convergence conditions of Proposition 6 must be satisfied, that is: $\hat{r} - \rho < (q_1 - 1)/h$, where q_1 is given by (2.5). This translates into some threshold conditions on the relevant parameters that the economy has to satisfy in order to hit the constraint and to fully benefit from financial liberalization.

²⁵The assumed parametrization is the baseline calibration of Section 4.2.

The numerical simulations show that when \hat{r} and h are greater than their threshold and A is smaller than its threshold, the BGP value of (G/Y) is smaller than or equal to α , thus implying that the binding regime is never optimal. Remember that the Kuhn-Tucker slackness condition (3.4) shows that the BGP value of (G/Y) must be greater than α in the binding regime (see Proposition 4). The stock of debt will become zero at a certain point in time, and the economy will stay in autarky.

The above results can be linked to the recent empirical literature on financial integration. There is no consensus on the effects of financial globalization on economic growth: the benefits depend on the countries' situation. In particular, Kose et al. (2011) show that there exists a certain threshold of financial and institutional development and economic features that an economy must attain in order to benefit from financial liberalization. Similarly, Aizenman et al. (2011) show that economy's GDP growth is lower in countries with high levels of past debt and weak institutions.

This is consistent with our model's prediction: a country with a low level of productivity, with high adjustment costs of capital or facing a high interest rate will not benefit from financial liberalization, staying in autarky forever. In order to enjoy the advantages of international borrowing, the economy must reach a certain level of financial and institutional development coupled with other favorable economic features. Moreover a high interest rate can be interpreted by the investors as an index of high risk and low institutions development, so that the country will be excluded from the main international capital flows. To conclude, for countries which are not developed enough it would be optimal to do not increase debt until its upper bound and rather stay in autarky.

5 Conclusions

This study analyzes the consequences of external debt collaterals on the optimal growth path of a country. We depart from the existing growth literature with collateral constraints which assumes that the collaterals are capital stock. We extended Barro (1990)-Turnovsky's (1996) small open economy model by assuming that public expenditures can be financed by borrowing on imperfect international financial markets, where external debt is bounded by GDP.

The collateral constraint defines three possible states of the economy: autarky, non binding regime and binding regime, where a balanced growth path exist. We determine the conditions that the economy must satisfy in order to switch from the non-binding to the binding regime, which also define the corner solutions of the problem. First, in the absence of investment adjustment costs, the economy switches to the binding regime at the initial date. Second, if installation costs are positive, for the economy to switch to the binding regime the TFP should be sufficiently high and/or the adjustment costs and the world interest rate should be low enough.

We then compare our results with those of the existing endogenous growth literature. The optimal public expenditure-to-GDP ratio is greater than that of Barro-Turnovsky's model: the planner has incentive to finance public expenditures by debt because this relaxes the collateral constraint.

Acknowledgments

I wish to thank E. Marchetti, C. Garcia-Peñalosa, M. Giannini, F. Mariani, R. Becker, P. Villieu, M. Tidball, T. Trimborn, R. Dos Santos Ferreira, A. Venditti, and the participants in seminars and conferences at Université catholique de Louvain, GREQAM Marseilles, Université d'Orléans. Finally, I am grateful to the anonymous referees for their suggestions, which have significantly improved the paper. The usual disclaimer applies.

Appendix

A Proof of Theorem 1

In this section we prove Theorem 1. We make use of the calculus of variations techniques applied to the sequence of the two control sub-problems shown in the main text.

Second Regime. The problem is (2.11) and the associated hamiltonian is (2.12). We denote by the upper-script “*” the paths which satisfy the first order conditions. The corresponding value function depends on T , $B(T)$ and $K(T)$: $V_2^*(T, B(T), K(T))$. Moreover, since the problem is concave, the following envelope conditions apply

$$\frac{\partial V_2^*}{\partial T} = -H^{(2)*}(T) \quad (\text{A.1})$$

$$\frac{\partial V_2^*}{\partial B(T)} = \lambda_1^{(2)}(T) \quad (\text{A.2})$$

$$\frac{\partial V_2^*}{\partial K(T)} = \lambda_2^{(2)}(T) \quad (\text{A.3})$$

First regime. The corresponding control sub-problem is (2.13) and the corresponding hamiltonian is (2.14).

The value function can be rewritten in terms of the hamiltonian

$$V_1 = \int_0^T \left[H^{(1)} - \lambda_1^{(1)} \dot{B} - \lambda_2^{(1)} \dot{K} \right] dt + V_2^*(T, B(T), K(T))$$

Standard integrations by parts yield

$$\begin{aligned} \int_0^T \lambda_1^{(1)} \dot{B} &= \lambda_1^{(1)}(T) B(T) - \lambda_1^{(1)}(0) B(0) - \int_0^T \dot{\lambda}_1^{(1)} B dt \\ \int_0^T \lambda_2^{(1)} \dot{K} &= \lambda_2^{(1)}(T) K(T) - \lambda_2^{(1)}(0) K(0) - \int_0^T \dot{\lambda}_2^{(1)} K dt \end{aligned}$$

which allows to rewrite the value function as

$$\begin{aligned} V_1 = \int_0^T \left[H^{(1)} + \dot{\lambda}_1^{(1)} B + \dot{\lambda}_2^{(1)} K \right] dt &+ V_2^*(T, B(T), K(T)) - \lambda_1^{(1)}(T) B(T) \\ &+ \lambda_1^{(1)}(0) B(0) - \lambda_2^{(1)}(T) K(T) + \lambda_2^{(1)}(0) K(0) \end{aligned}$$

The first order variation of V_1 with respect to the state and the control variables' paths yields

$$\begin{aligned} \partial V_1 = \int_0^T \left[\frac{\partial H^{(1)}}{\partial B} \partial B + \frac{\partial H^{(1)}}{\partial K} \partial K + \frac{\partial H^{(1)}}{\partial C} \partial C + \frac{\partial H^{(1)}}{\partial G} \partial G + \frac{\partial H^{(1)}}{\partial I} \partial I + \dot{\lambda}_1^{(1)} \partial B + \dot{\lambda}_2^{(1)} \partial K \right] dt \\ + \left(H^{(1)}(T) + \dot{\lambda}_1^{(1)}(T) B(T) + \dot{\lambda}_2^{(1)}(T) K(T) \right) \partial T \\ + \frac{\partial V_2^*(T, B(T), K(T))}{\partial T} \partial T + \frac{\partial V_2^*(T, B(T), K(T))}{\partial B(T)} \partial B(T) + \frac{\partial V_2^*(T, B(T), K(T))}{\partial K(T)} \partial K(T) \\ - \dot{\lambda}_1^{(1)}(T) B(T) \partial T - \lambda_1^{(1)}(T) \partial B(T) - \dot{\lambda}_2^{(1)}(T) K(T) \partial T - \lambda_2^{(1)}(T) \partial K(T) \end{aligned}$$

Reorganizing the terms

$$\begin{aligned} \partial V_1 = & \int_0^T \left[\left(\frac{\partial H^{(1)}}{\partial B} + \dot{\lambda}_1^{(1)} \right) \partial B + \left(\frac{\partial H^{(1)}}{\partial K} + \dot{\lambda}_2^{(1)} \right) \partial K + \frac{\partial H^{(1)}}{\partial C} \partial C + \frac{\partial H^{(1)}}{\partial G} \partial G + \frac{\partial H^{(1)}}{\partial I} \partial I \right] dt \\ & + \left(H^{(1)}(T) + \frac{\partial V_2^*(T, B(T), K(T))}{\partial T} \right) \partial T + \left(\frac{\partial V_2^*(T, B(T), K(T))}{\partial B(T)} - \lambda_1^{(1)}(T) \right) \partial B(T) \\ & + \left(\frac{\partial V_2^*(T, B(T), K(T))}{\partial K(T)} - \lambda_2^{(1)}(T) \right) \partial K(T) \end{aligned}$$

A trajectory is (locally) optimal if any (local) departure from it decreases the value function: $\partial V_1 \leq 0$ for any $\partial B(t)$ and $\partial K(t)$, $t \in (0, T)$, for any $\partial C(t)$ and $\partial G(t)$, $t \in [0, T]$, and for any ∂T , $\partial B(T)$ and $\partial K(T)$. This gives the following necessary conditions for an interior maximizer:

$$\left\{ \begin{array}{l} \frac{\partial H^{(1)}}{\partial C} = 0 \\ \frac{\partial H^{(1)}}{\partial G} = 0 \\ \frac{\partial H^{(1)}}{\partial I} = 0 \\ \frac{\partial H^{(1)}}{\partial B} = -\dot{\lambda}_1^{(1)} \\ \frac{\partial H^{(1)}}{\partial K} = -\dot{\lambda}_2^{(1)} \\ H^{(1)}(T) + \frac{\partial V_2^*(T, B(T), K(T))}{\partial T} = 0 \\ \frac{\partial V_2^*(T, B(T), K(T))}{\partial B(T)} - \lambda_1^{(1)}(T) = 0 \\ \frac{\partial V_2^*(T, B(T), K(T))}{\partial K(T)} - \lambda_2^{(1)}(T) = 0 \end{array} \right.$$

The first five equations are the standard Pontryagin conditions, the last three may be interpreted as the optimality conditions with respect to the switching time T and the free state values $B(T)$ and $K(T)$. Combining these last three equations with equations (A.1), (A.2) and (A.3) obtained in the first sub-problem we get the matching conditions stated in Theorem 1.

B Proof of Proposition 2

Consider equation (3.1) and define

$$\begin{aligned} f\left(\frac{G}{Y}\right) &\equiv \frac{h}{2} \left(\frac{1}{\phi}\right) \frac{1}{\alpha^2} \left(\frac{G}{Y}\right)^2 + [h(\hat{r}\phi - 1) + \phi] \frac{1}{\phi} \frac{1}{\alpha} \left(\frac{G}{Y}\right) + \frac{1}{\phi} \left[\frac{h}{2} (\hat{r}\phi - 1)^2 + \phi(\hat{r}\phi - 1) - \frac{h}{2} (\rho\phi)^2 \right] \text{ and} \\ g\left(\frac{G}{Y}\right) &\equiv \frac{1-\alpha}{\alpha} A^{\frac{1}{1-\alpha}} \phi \left(\frac{G}{Y}\right)^{\frac{1}{1-\alpha}}. \end{aligned}$$

$f\left(\frac{G}{Y}\right)$ is a convex parabola, while $g\left(\frac{G}{Y}\right)$ is a convex increasing function such that $g(0) = 0$ and $\lim_{\frac{G}{Y} \rightarrow +\infty} g\left(\frac{G}{Y}\right) = +\infty$. The assumption $\alpha < \frac{1}{2}$ guarantees that $\lim_{\frac{G}{Y} \rightarrow +\infty} \frac{f\left(\frac{G}{Y}\right)}{g\left(\frac{G}{Y}\right)} = 0$.

We study the intersections of $f\left(\frac{G}{Y}\right)$ with the positive $\left(\frac{G}{Y}\right)$ - *axis* because this helps in determining the number of intersections between $g\left(\frac{G}{Y}\right)$ and $f\left(\frac{G}{Y}\right)$ (the assumption $\alpha < \frac{1}{2}$ limits the number of possible intersections to two). The discriminant of the second degree equation defined by $f\left(\frac{G}{Y}\right) = 0$ is positive, hence it can have either two negative solutions, one positive and one negative solution or two positive solutions:

	$\frac{h}{2} \left(\frac{1}{\phi}\right) \frac{1}{\alpha^2}$	$[h(\hat{r}\phi - 1) + \phi\tau] \frac{1}{\phi} \frac{1}{\alpha}$	$\frac{1}{\phi\tau} \left[\frac{h}{2}(\hat{r}\phi - 1)^2 + \phi(\hat{r}\phi - 1) - \frac{h}{2}(\rho\phi)^2\right]$	$f\left(\frac{G}{Y}\right) = 0$
1	+	+	+	2 negative solutions
2	+	+	-	1 positive and one negative
3	+	-	-	1 positive and one negative
4	+	-	+	2 positive solutions

We define $D \equiv [h(\hat{r}\phi - 1) + \phi] \frac{1}{\phi} \frac{1}{\alpha}$ and $E \equiv \frac{1}{\phi} \left[\frac{h}{2}(\hat{r}\phi - 1)^2 + \phi(\hat{r}\phi - 1) - \frac{h}{2}(\rho\phi)^2\right]$. The following cases are possible:

1. $D \geq 0$ if $h \leq \frac{\phi}{1-\hat{r}\phi}$ and $E > 0$ if $h > \frac{2\phi(1-\hat{r}\phi)}{(\hat{r}^2-\rho^2)(\phi)^2+1-2\hat{r}\phi}$. Since $\frac{\phi}{1-\hat{r}\phi} < \frac{2\phi(1-\hat{r}\phi)}{(\hat{r}^2-\rho^2)(\phi)^2+1-2\hat{r}\phi}$ the two conditions are incompatible.

2. $D \geq 0$ if $h \leq \frac{\phi}{1-\hat{r}\phi}$, $E \leq 0$ if $h \leq \frac{2\phi(1-\hat{r}\phi)}{(\hat{r}^2-\rho^2)(\phi)^2+1-2\hat{r}\phi}$. Hence to have case 2 we must require $h \leq \frac{\phi}{1-\hat{r}\phi}$.

3. $D < 0$ if $h > \frac{\phi}{1-\hat{r}\phi}$, $E \leq 0$ if $h \leq \frac{2\phi(1-\hat{r}\phi)}{(\hat{r}^2-\rho^2)(\phi)^2+1-2\hat{r}\phi}$. Hence $\frac{\phi}{1-\hat{r}\phi} < h \leq \frac{2\phi(1-\hat{r}\phi)}{(\hat{r}^2-\rho^2)(\phi)^2+1-2\hat{r}\phi}$.

4. $D < 0$ if $h > \frac{\phi}{1-\hat{r}\phi}$, $E > 0$ if $h > \frac{2\phi(1-\hat{r}\phi)}{(\hat{r}^2-\rho^2)(\phi)^2+1-2\hat{r}\phi}$. This implies $h > \frac{2\phi(1-\hat{r}\phi)}{(\hat{r}^2-\rho^2)(\phi)^2+1-2\hat{r}\phi}$.

Cases 2 and 3 can be summarized by requiring $h \leq \frac{2\phi(1-\hat{r}\phi)}{(\hat{r}^2-\rho^2)(\phi)^2+1-2\hat{r}\phi}$.

Hence, the dynamical system admits

(a) One solution iff $h \leq \frac{2\phi(1-\hat{r}\phi)}{(\hat{r}^2-\rho^2)(\phi)^2+1-2\hat{r}\phi}$;

(b) Two solutions iff $h > \frac{2\phi(1-\hat{r}\phi)}{(\hat{r}^2-\rho^2)(\phi)^2+1-2\hat{r}\phi}$.

Both the solutions satisfy the transversality condition:

$\lim_{t \rightarrow +\infty} -q^* \lambda_1^{(2)}(t) K(t) = -q^* \lambda_1^{(2)}(T) \exp[-\rho t]$, where q^* is the BGP value of the Tobin's q .

We can prove that case (b) implies just a false multiplicity.

A solution of equation (3.1) is a balanced growth path if the growth rate (2.23) is positive, that is

$$\hat{r}\phi - 1 + \frac{1}{\alpha} \left(\frac{G}{Y}\right) \geq \rho\phi$$

This allows to write the following inequality

$$\begin{aligned} & \frac{h}{2} \left(\frac{1}{\phi}\right) \frac{1}{\alpha^2} \left(\frac{G}{Y}\right)^2 + h(\hat{r}\phi - 1) \frac{1}{\phi} \frac{1}{\alpha} \left(\frac{G}{Y}\right) + \frac{1}{\phi} \left[\frac{h}{2}(\hat{r}\phi - 1)^2 - \frac{h}{2}(\rho\phi)^2\right] + \rho\phi \\ & \leq \frac{h}{2} \left(\frac{1}{\phi}\right) \frac{1}{\alpha^2} \left(\frac{G}{Y}\right)^2 + [h(\hat{r}\phi - 1) + \phi] \frac{1}{\phi} \frac{1}{\alpha} \left(\frac{G}{Y}\right) + \frac{1}{\phi} \left[\frac{h}{2}(\hat{r}\phi - 1)^2 + \phi(\hat{r}\phi - 1) - \frac{h}{2}(\rho\phi)^2\right] \\ & = \frac{1-\alpha}{\alpha} A^{\frac{1}{1-\alpha}} \phi \left(\frac{G}{Y}\right)^{\frac{1}{1-\alpha}} \end{aligned}$$

By defining $l\left(\frac{G}{Y}\right)$ the first function, $g\left(\frac{G}{Y}\right)$ the second one and $f\left(\frac{G}{Y}\right)$ the third one, each solution of equation (3.1) must be such that

$$l\left(\frac{G}{Y}\right) \leq g\left(\frac{G}{Y}\right) = f\left(\frac{G}{Y}\right)$$

in order to be a BGP. In what follows we are going to give a geometrical proof. We can show that

$$-l(0) > g(0), \text{ for } \phi < \frac{1}{\hat{r}-\rho};$$

- $l\left(\frac{G}{Y}\right)$ and $g\left(\frac{G}{Y}\right)$ cross just once, $\left(\frac{G}{Y}\right) \Big|_{l\left(\frac{G}{Y}\right)=g\left(\frac{G}{Y}\right)} = \alpha(\rho\phi - \hat{r}\phi + 1) > 0$, for $\phi < \frac{1}{\hat{r}-\rho}$;
- $\left(\frac{G}{Y}\right) \Big|_{l\left(\frac{G}{Y}\right)=g\left(\frac{G}{Y}\right)} > \left(\frac{G}{Y}\right)_1 \Big|_{g\left(\frac{G}{Y}\right)=0} = \frac{h\alpha - \alpha\phi - h\hat{r}\phi - \sqrt{\alpha^2\phi^2 + h^2\alpha^2\rho^2\phi^2}}{h}$;
- $\left(\frac{G}{Y}\right) \Big|_{l\left(\frac{G}{Y}\right)=g\left(\frac{G}{Y}\right)} > \left(\frac{G}{Y}\right)_2 \Big|_{g\left(\frac{G}{Y}\right)=0} = \frac{h\alpha - \alpha\phi - h\hat{r}\phi + \sqrt{\alpha^2\phi^2 + h^2\alpha^2\rho^2\phi^2}}{h}$.

These results exclude the smallest solution of case 2, as it is shown by the figure below (Figure B .1):

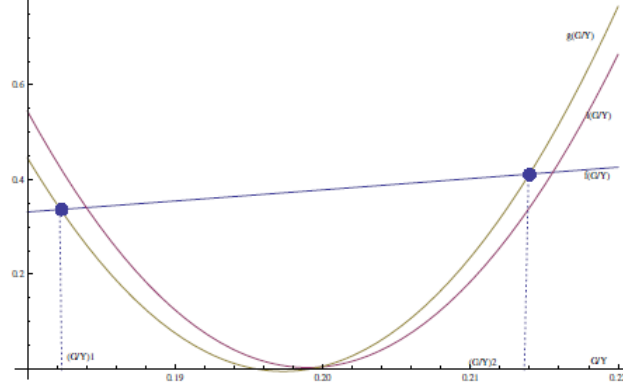


Figure B.1: Existence and Uniqueness of the BGP

In the figure, $\left(\frac{G}{Y}\right)_1$ and $\left(\frac{G}{Y}\right)_2$ are the solutions of equation (3.1). For $\left(\frac{G}{Y}\right)_1$, $l\left(\left(\frac{G}{Y}\right)_1\right) > g\left(\left(\frac{G}{Y}\right)_1\right) = f\left(\left(\frac{G}{Y}\right)_1\right)$, hence the growth rate is negative. $\left(\frac{G}{Y}\right)_1$ is not a BGP. For $\left(\frac{G}{Y}\right)_2$, if, as in the figure, $l\left(\left(\frac{G}{Y}\right)_2\right) < g\left(\left(\frac{G}{Y}\right)_2\right) = f\left(\left(\frac{G}{Y}\right)_2\right)$ it is a BGP ($l\left(\frac{G}{Y}\right)$ should cross $g\left(\frac{G}{Y}\right)$ before it crosses $f\left(\frac{G}{Y}\right)$). It should be remarked that the parameter A , which rules the position of $f\left(\frac{G}{Y}\right)$ on the graphic, should not be too low, otherwise we should reject also the larger solution of case 2) in Proposition 8.²⁶

C Proof of Proposition 3

The dynamical system is

$$\begin{cases} \dot{c} = \left(\hat{r} - \frac{1}{\phi} + 2 \left(\frac{1}{\phi} \right)^2 A^{-2}b - \rho - \frac{q-1}{h} \right) c \\ \dot{q} = \left(\hat{r} - \frac{1}{\phi} \right) q + 2 \left(\frac{1}{\phi} \right)^2 A^{-2}bq - \left(\frac{1}{\phi} \right)^2 A^{-2}b^2 - \frac{(q-1)^2}{2h} \\ \dot{b} = c + \frac{q-1}{h} + \left(\frac{1}{\phi} \right)^2 A^{-2}b^2 + \hat{r}b + \frac{(q-1)^2}{2h} - \frac{1}{\phi}b - \frac{q-1}{h}b \end{cases}$$

and the Jacobian matrix, J_1 , is given by

²⁶Numerical analysis has shown that the result is valid for values of A commonly used in the simulations.

$$\begin{bmatrix} \hat{r} - \frac{1}{\phi} + 2 \left(\frac{1}{\phi}\right)^2 A^{-2} b_1 - \rho - \frac{q_1 - 1}{h} & -\frac{1}{h} c_1 & 2 \left(\frac{1}{\phi}\right)^2 A^{-2} c_1 \\ 0 & 2 \left(\frac{1}{\phi}\right)^2 A^{-2} b_1 + \hat{r} - \frac{1}{\phi} - \frac{q_1 - 1}{h} & 2 \left(\frac{1}{\phi}\right)^2 A^{-2} q_1 - 2 \left(\frac{1}{\phi}\right)^2 A^{-2} b_1 \\ 1 & \frac{1}{h} + \frac{q_1 - 1}{h} - \frac{1}{h} b_1 & 2 \left(\frac{1}{\phi}\right)^2 A^{-2} b_1 + \hat{r} - \frac{1}{\phi} - \frac{q_1 - 1}{h} \end{bmatrix}$$

where

$$b_1 = \phi A^2 \frac{-[h(\hat{r}\phi - 1) + \phi] \frac{1}{\phi} + \sqrt{\frac{1}{2}(2 + 2h^2\rho^2 + A^2(h - 2(1 + h\hat{r})\phi + (r(2 + h\hat{r}) - h\rho^2)\phi^2))}}{\left(\frac{2h}{\phi} - A^2\phi\right)}$$

$$q_1 = h \left[\hat{r} - \frac{1}{\phi} + 2 \left(\frac{1}{\phi}\right)^2 A^{-2} b_1^2 - \rho \right] + 1$$

$$c_1 = \frac{q_1 - 1}{h} b_1 + \frac{1}{\phi} b_1 - \frac{q_1 - 1}{h} - \left(\frac{1}{\phi}\right)^2 A^{-2} b_1^2 - \hat{r} b_1 - \frac{(q_1 - 1)^2}{2h}$$

$$\det(J_1) = \frac{-\frac{1}{\sqrt{2}A^2h(\phi)^2} \left[\rho \sqrt{2 + 2h^2\rho^2 + A^2(h - 2(1 + \hat{r}h)\phi + (\hat{r}(2 + h\hat{r}) - h\rho^2)(\phi)^2)} \right.}{\left. \left(-2h\rho + \sqrt{2} \sqrt{2 + 2h^2\rho^2 + A^2(h - 2(1 + \hat{r}h)\phi + (\hat{r}(2 + h\hat{r}) - h\rho^2)(\phi)^2)} \right) \right]}$$

$$\text{tr}(J_1) = 2\rho$$

The trace is positive.

The determinant is negative if $A^2 > -\frac{2}{h - 2(1 + \hat{r}h)\phi + (\hat{r}(2 + h\hat{r}) - h\rho^2)\phi^2}$.

The negative determinant implies that the product of the three eigenvalues is negative, implying that the number of negative eigenvalues is either one or three. However, the positive trace, being the sum of the three eigenvalues, rules out the latter case, thus implying one negative eigenvalue. Hence, the steady state is saddle-point stable.

D Proof of Proposition 4

In order to prove this proposition note that an increase in \hat{r} moves the function $g\left(\frac{G}{Y}\right)$ of Figure B.1 up in the graph, so that $(G/Y)_2$ decreases while an increase in A moves the $f\left(\frac{G}{Y}\right)$ curve of Figure B.1 up in the graph, so that $(G/Y)_2$ increases. Given that (2.23) is increasing in (G/Y) , the growth rate increases as well.

E The inter-regime transitional dynamics: methodology

In this section we illustrate the method which has been employed to simulate the inter-regime transitional dynamics. As we anticipated in Section 4.2, the aim of our numerical exercise is to link two two-points boundary problems (see Trimborn et al. 2008a, 2008b), in order to find the switching date T .

The simulation proceeds as follows :

Step_1 *Guess a final date T*

Step_2 *Run the binding regime. Set:*

1. The initial conditions. In our model they are given by the value of $b(T)$ given by condition (4.2).
2. The boundary conditions. The transversality conditions are replaced by the steady state values of the variables.
3. Run the simulations and write down the values of $c(T)$ and $q(T)$.²⁷

Step_3 *Run the non binding regime. Set:*

1. The initial conditions. In our case, they are represented by the initial value of the state variable, $b(0)$.
2. The boundary conditions, that is the values of $c(T)$ and $q(T)$ found in Step_2, as well as the value of $b(T)$ given by condition (4.2).
3. Run the simulations. If the variables $c(T)$, $q(T)$ and $b(T)$ of the non binding regime converge to the final conditions, as set in 2), the algorithm stops. Otherwise, it comes back to Step_1.

²⁷Condition (4.2) shows also that the values of the variables at T do not depend on the choice of T . This choice will only affect the duration of the transition to the steady state.

References

- [1] Ai, C., Cassou, S. (1995). A normative analysis of public capital. *Applied Economics*, 27, 1201-09.
- [2] Aizenman, J., Kletzer, K., Pinto, B. (2007). Economic growth with constraints on tax revenues and public debt: implications for fiscal policy and cross-country differences. NBER Working Paper 12750.
- [3] Aizenman, J., Hutchinson, M. and Jinjarak., Y., (2011). What is the risk of European sovereign debt defaults? Fiscal space, CDS spreads and market pricing of risk. NBER Working Paper 17407.
- [4] Aschauer, D. (1989). Is public expenditure productive?. *Journal of Monetary Economics*, 23, 177-200.
- [5] Barro, R. J., (1990). Government spending in a simple model of endogenous growth. *Journal of Political Economy*, 98, S103–S125.
- [6] Barro, R., Sala-i-Martin, X., (2004). *Economic Growth*, New York, McGraw-Hill.
- [7] Boucekkine, R., Saglam, C. & Vallee, T. (2004). Technology Adoption Under Embodiment: A Two-Stage Optimal Control Approach, *Macroeconomic Dynamics*, 8, 250-271.
- [8] Boucekkine, R., Pintus, P. (2012), History’s curse: leapfrogging, growth breaks and growth reversals under international borrowing without commitment. *Journal of Economic Growth*, 17, 27-47.
- [9] Boucekkine, R., Pommeret, A., Prieur, F. (2013a). Technological vs. ecological switch and the environmental Kuznets Curve. *American Journal of Agricultural Economics*, 95, 252-60.
- [10] Boucekkine, R., Pommeret, A., Prieur, F. (2013b). On the timing and optimality of capital controls: public expenditures, debts dynamics and welfare. *International Journal of Economic Theory*, 9, 101-112.
- [11] Boucekkine, R., Pommeret, A., Prieur, F. (2013c). Optimal regime switching and threshold effects. *Journal of Economic Dynamics and Control*, forthcoming.
- [12] Brock, L. P. (2009). Collateral Constraints and Macroeconomic Adjustment in an Open Economy, University of Washington Economics Working Paper UWEC 2009-3.
- [13] Chatterjee, S., Sakoulis, G., Turnovsky, S. (2003). Unilateral capital transfers, public investment, and economic growth. *European Economic Review* 47, 1077-1103.
- [14] Cohen, D., Sachs, J. (1986). Growth and external debt under risk of debt repudiation. *European Economic Review*, 30, 526-60.
- [15] Djankov, S., Hart, O., McLiesh, C., Shleifer, A. (2008). Debt enforcement around the world. *Journal of Political Economy* 116, 1105-1149.

- [16] Eaton, J., Gersovitz, M. (1981). Debt with potential repudiation: theoretical and empirical analysis. *Review of Economic Studies*, 48, 289-309.
- [17] Eberts, R. (1986). Estimating the contribution of urban public infrastructure to regional growth. Working Paper 8610, Federal Reserve Bank of Cleveland.
- [18] Futagami, K., Iwaisako, T. Ohdoi, R. debt poly rules, productive government spending, and multiple growth paths. *Macroeconomic Dynamics*, 12, 445-462.
- [19] Hayashi, F. (1982). Tobin's marginal q, average q: a neoclassical interpretation. *Econometrica*, 50, 213-224.
- [20] Kose, M. A., Prasad, E. S., Taylor, A. S. (2011). Thresholds in the process of financial integration. *Journal of International Money and Finance*, 30, 147-179.
- [21] Prieur, F., Tidball, M. and Withagen, C. (2011). Optimal emission-extraction policy in a world of scarcity and irreversibility, CESifo working paper 3512.
- [22] Tomiyama, K. (1985). Two-stage optimal control problems and optimality conditions. *Journal of Economic Dynamics and Control* 9, 317-337.
- [23] Trimborn, T., Koch, K., Steger, S. (2008a). Multidimensional Transitional Dynamics: A Simple Numerical Procedure. *Macroeconomic Dynamics*, vol. 12(03), 301-19.
- [24] Trimborn, T., Koch, K., Steger, S. (2008b). Multidimensional Transitional Dynamics: A Simple Numerical Procedure (Mathematica). QM&RBC Codes 193, Quantitative Macroeconomics & Real Business Cycles.
- [25] Turnovsky, S. J., (1996). Fiscal Policy, Growth, and Macroeconomic Performance in a Small Open Economy, *Journal of International Economics*, 40, 41-66.
- [26] Turnovsky, S. J., (1997). Equilibrium Growth in a Small Economy Facing an Imperfect World Capital Market, *Review of Development Economics*, 1, 1-22.
- [27] Turnovsky, S. J., (2002). Knife-edge conditions and the macrodynamics of small open economies. *Macroeconomic Dynamics*, 6, 307-335.