



**HAL**  
open science

# Existence of financial equilibrium with differential information: the no-arbitrage characterization

Lionel de Boisdeffre

► **To cite this version:**

Lionel de Boisdeffre. Existence of financial equilibrium with differential information: the no-arbitrage characterization. 2016. halshs-01310860

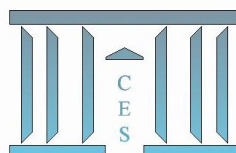
**HAL Id: halshs-01310860**

**<https://shs.hal.science/halshs-01310860>**

Submitted on 3 May 2016

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



**Existence of financial equilibrium with differential  
information: The no-arbitrage characterization**

Lionel De BOISDEFFRE

2016.25



EXISTENCE OF FINANCIAL EQUILIBRIUM WITH DIFFERENTIAL INFORMATION: THE NO-ARBITRAGE CHARACTERIZATION
--

*Lionel de Boisdeffre*,<sup>1</sup>

(March 2016)

***Abstract***

*In the classical approach to asymmetric information, agents are all endowed with a price model a la Radner (Econometrica 47: 655-678, 1979). That is, they are assumed to know exactly how equilibrium prices are determined and would only infer information from markets with reference to that price model. Radner (1979) showed that, under asymmetric information, equilibrium only existed generically in this setting. We now drop that so-called "rational expectation" assumption and study the existence of financial equilibrium when assets are numeraire. We show the existence of equilibrium is, then, characterized by the no-arbitrage condition on financial markets, as in De Boisdeffre (Econ Theory 31: 255-269, 2007), where assets are nominal. This result extends Geanakoplos-Polemarchakis' (Essays in Honor of K.J. Arrow, Starr & Starrett ed., Cambridge UP Vol. 3, 65-96, 1986) to the case of asymmetric information. Contrasting with Radner's, it shows that symmetric and asymmetric information economies can be treated as two applications of a same model, where they share similar properties.*

*Key words:* general equilibrium, asymmetric information, arbitrage, existence.

*JEL Classification:* D52.

---

<sup>1</sup> CATT-University of Pau, 1 Av. du Doyen Poplawski, 64000 Pau, France.  
CES-University of Paris 1, 106-112 Bd. de l'Hôpital, 75013 Paris, France.  
Email: lionel.de.boisdeffre@wanadoo.fr

# 1 Introduction

When agents are asymmetrically informed, they may infer information from observing prices or trade volumes on markets. A traditional approach to that problem is given by Radner's (1979) rational expectation assumption, along which "*agents have a 'model' or 'expectations' of how equilibrium prices are determined*". Agents may then infer private information of other agents from comparing actual prices and price expectations with their theoretical values at a price revealing equilibrium. As is well known since Radner (1979), this demanding assumption is only consistent with the generic existence of equilibrium under asymmetric information.

In the existing literature, the full existence property of equilibrium may only hold on financial markets where agents are symmetrically informed, and in the two cases of nominal assets, as demonstrated by Cass (1984), and numeraire assets, along Geanakoplos and Polemarchakis (1986). When assets are real, equilibrium may fail to exist, as shown by Hart (1975), even with symmetric information.

Assuming that agents have no price model a la Radner, Cornet and De Boisdeffre (2002) introduce basic tools, concepts and properties for an arbitrage theory, embedding jointly the symmetric and asymmetric information settings, into the same model. De Boisdeffre (2007) proves that a financial equilibrium with nominal assets exists, in this model, not only generically - as with rational expectations - but under the same no-arbitrage condition, whether agents had symmetric or asymmetric information. This no-arbitrage condition characterizes the existence of equilibrium, as already known in the symmetric information case, since Cass (1984).

Cornet and De Boisdeffre (2009) show that agents endowed with no price model can still infer enough information, from observing trade, to preclude arbitrage from

purely financial markets. Whence inferred, this arbitrage-free information structure can no longer be refined. It is always consistent with the existence of equilibrium, along De Boisdeffre (2007), but equilibrium prices convey no additional information. We now show the no-arbitrage condition may also characterize the existence of financial equilibrium on numeraire asset markets. This result, which departs from the Radner classical outcome, extends the standard Geanakoplos-Polemarchakis (1986) theorem of symmetric information to the asymmetric case.

With asymmetric information, nothing guarantees a positive lower bound to the value of the numeraire across states at clearing market prices. Yet, this condition is required to bound attainable transfers. A specific paper was, hence, needed to show how attainable transfers could be bounded with numeraire assets. Indeed, De Boisdeffre (2007) only uses the no-arbitrage condition to bound such transfers, which is no longer possible with numeraire assets. To solve the problem, we now introduce a specific price set and compactness argument (see Claim 1, below), which applies to standard preferences, under symmetric information (Remark 3), and to ordered separable preferences, under asymmetric information. The proof is detailed in the latter setting, which is the more general and the paper's purpose. Searching a counterexample to existence, or new fixed-point arguments, goes beyond.

Formally, the model we present is a two-period pure exchange economy, where agents, possibly asymmetrically informed, face uncertainty, at the first period, on which state of nature will randomly prevail tomorrow, out of a finite state space. Agents may exchange consumption goods on spot markets, and securities on financial markets, which pay off in numeraire, i.e., in a given commodity (bundle).

The paper is organized as follows: Section 2 presents the model and its existence Theorem and Section 3 proves the Theorem.

## 2 The basic model

We consider a pure-exchange financial economy with two periods,  $t \in \{0, 1\}$ , finitely many agents,  $i \in I := \{1, \dots, m\}$ , commodities,  $l \in \{1, \dots, L\}$ , states of nature,  $s \in S$ , and assets,  $j \in \{1, \dots, J\}$ , which pay off in numeraire, that is, in a fixed commodity bundle, henceforth denoted by  $e \in \mathbb{R}_+^L$ . Agents face uncertainty at the first period ( $t = 0$ ) about which state,  $s \in S$ , will prevail at the second period ( $t = 1$ ). We let  $s = 0$  be the unique state at  $t = 0$  and  $\Sigma' := \{0\} \cup \Sigma$ , for any subset  $\Sigma \subset S$ .

At  $t = 0$ , each agent,  $i \in I$ , receives a private signal, or information set  $S_i \subset S$ , which correctly informs her that an arbitrary state,  $s \in S_i$ , will prevail at  $t = 1$ . The set collection,  $(S_i)$ , represents agents' information structure. We assume costlessly that  $\cup_{i=1}^m S_i = S$ , and denote by  $\underline{S} := \cap_{i=1}^m S_i$  agents' pooled information set.

We shall present, successively, the notations that will be used throughout, the asset market, the commodity market and the concept of equilibrium.

### 2.1 The notations

Throughout, we denote by  $\mathbb{R}_+$  the set of non-negative real numbers, and  $\mathbb{R}_{++}$  the subset of strictly positive numbers. We denote, for every  $S \times J$ -real-matrix,  $V$ , and vector pair  $(p, q) \in (\mathbb{R}^L)^{S'} \times \mathbb{R}^J$ , for every subset  $\Sigma \subset S'$ , every state,  $s \in \Sigma$ , and all pairs  $(l, j) \in \{1, \dots, L\} \times \{1, \dots, J\}$ ,  $(x, x') \in (\mathbb{R}^L)^{S'} \times \mathbb{R}^{S'}$ ,  $(y, y') \in \mathbb{R}^\Sigma \times \mathbb{R}^\Sigma$  and  $(z, z') \in (\mathbb{R}^L)^\Sigma \times (\mathbb{R}^L)^\Sigma$ :

- 1)  $y_s, z_s$ , respectively, the scalar and vector, indexed by  $s \in \Sigma$ , of  $y$  and  $z$ ;
- 2)  $z_s^l \in \mathbb{R}$  the  $l^{th}$  component of  $z_s \in \mathbb{R}^L$ ;
- 3)  $z \square z'$  the vector  $(z_s \cdot z'_s) \in \mathbb{R}^\Sigma$  and  $y \square z$  the vector  $(y_s z_s) \in (\mathbb{R}^L)^\Sigma$ ;
- 4)  $x[\Sigma]$  and  $x'[\Sigma]$ , respectively, the truncations of  $x$  on  $(\mathbb{R}^L)^\Sigma$  and of  $x'$  on  $\mathbb{R}^\Sigma$ ;
- 5)  $V_s$  the row vector indexed by  $s \in \Sigma$ , of  $V$ , and  $v_s^j$  its  $j^{th}$  component;

- 6)  $V[\Sigma]$  (whenever  $0 \notin \Sigma$ ) the  $\Sigma \times J$ -matrix defined by  $V[\Sigma]_s := V_s$  for each  $s \in \Sigma$ ;
- 7)  $W[\Sigma, q]$  (whenever  $0 \notin \Sigma$ ) the  $\Sigma' \times J$ -matrix defined by  $W[\Sigma, q]_0 := -q$  and  $W[\Sigma, q]_s := V_s$ , for every  $s \in \Sigma$ . We also let  $W(q) := W[S, q]$ ;
- 8)  $V[\Sigma, p]$  (whenever  $0 \notin \Sigma$ ) the  $\Sigma \times J$ -matrix defined by  $V[\Sigma, p]_s := (p_s \cdot e)V_s$ , for each  $s \in \Sigma$ , recalling that  $e \in \mathbb{R}^L$  is the numeraire. We also let  $V(p) := V[S, p]$ ;
- 9)  $W[\Sigma, p, q]$  (whenever  $0 \notin \Sigma$ ) the  $\Sigma' \times J$ -matrix defined by  $W[\Sigma, p, q]_0 := -q$  and  $W[\Sigma, p, q]_s := V(p)_s$ , for every  $s \in \Sigma$ . We also let  $W(p, q) := W[S, p, q]$ .

## 2.2 The financial market

The financial market permits limited transfers across periods and states, via  $J$  numeraire assets  $j \in \{1, \dots, J\}$ , whose contingent payoffs, in each state  $s \in S$ , are denoted by  $v_s^j e$ , where  $e \in \mathbb{R}_+^L \setminus \{0\}$  is the numeraire, or commodity bundle ( $\|e\| = 1$  for simplicity), and  $v_s^j$  is a state-dependent quantity. These quantities, defined for each  $(s, j) \in S \times \{1, \dots, J\}$ , yield a  $S \times J$ -matrix  $V := (v_s^j)$ , which is of full column-rank (i.e.,  $J = \text{rank}V$ ) and known by all agents. Thus, for all price,  $p \in (\mathbb{R}^L)^{S'}$ , the quantities  $p_s \cdot e v_s^j$ , for each  $(s, j) \in S \times \{1, \dots, J\}$ , define a  $S \times J$  price-dependent payoff matrix  $V(p) := (p_s \cdot e v_s^j)$ , in units of account, which is of full column-rank, whenever  $p_s \cdot e > 0$  for each  $s \in S$ . Given the asset price,  $q \in \mathbb{R}^J$ , a portfolio is a vector,  $z \in \mathbb{R}^J$ , which costs  $q \cdot z$  units of account at  $t = 0$ , and delivers a flow,  $Vz$ , of contingent payoffs in numeraire, across states, at  $t = 1$ . We now define the no-arbitrage condition.

**Definition 1** *A price,  $q \in \mathbb{R}^J$ , is said to be a common no-arbitrage price of the structure  $(S_i)$ , or  $(S_i)$  to be  $q$ -arbitrage-free, if the following Condition holds:*

- (a)  $\nexists (i, z, p) \in I \times \mathbb{R}^J \times (\mathbb{R}^L)^S : W[S_i, p, q]z > 0$  and  $p_s \cdot e > 0, \forall s \in S$ .

*The structure  $(S_i)$  is said to be arbitrage-free if the set of common no-arbitrage prices of  $(S_i)$ , denoted by  $Q_c[(S_i)]$ , is non-empty.*

*Remark 1* It follows from Cornet-De Boisdeffre (2009), Theorem 3, that agents endowed with no price model can always infer an arbitrage-free refinement of their information structure, from observing trade opportunities on financial markets. For this reason, we henceforth assume, at no cost, that  $(S_i)$  is arbitrage-free.

### 2.3 The commodity market

Commodities may be traded on spot markets, or consumed, at both dates. The generic agent,  $i \in I$ , has  $(\mathbb{R}_+^L)^{S'_i}$  for consumption set, an endowment,  $e_i \in (\mathbb{R}_+^L)^{S'_i}$ , and preferences represented by a utility function,  $u_i : (\mathbb{R}_+^L)^{S'_i} \rightarrow \mathbb{R}$ . To set a positive lower bound to the value of the numeraire in any state in Section 3, we assume there exist indexes,  $v_i^s : \mathbb{R}_+^2 \rightarrow \mathbb{R}$  (for  $s \in S_i$ ), such that  $u_i(x) := \sum_{s \in S_i} v_i^s(x_0, x_s)$ , for all  $x \in (\mathbb{R}_+^L)^{S'_i}$ .

Given prices,  $(p, q) \in (\mathbb{R}^L)^{S'} \times \mathbb{R}^J$ , all agents,  $i \in I$ , have for budget sets and set of attainable strategies, respectively:

$$B_i(S_i, p, q) := \{ (x, z) \in (\mathbb{R}_+^L)^{S'_i} \times \mathbb{R}^J : p[S'_i] \square (x - e_i) \leq W(S_i, p, q)z \};$$

$$\mathcal{A} := \{(x_i) \in \times_{i=1}^m (\mathbb{R}_+^L)^{S'_i} : \sum_{i=1}^m (x_i - e_i)[\underline{S}'] = 0\}.$$

### 2.4 Equilibrium

The economy described above for a given payoff matrix,  $V$ , and structure,  $(S_i)$ , of information signals,  $S_i \subset S$ , which each agent,  $i \in I$ , receives (or infers) privately, is denoted by  $\mathcal{E} = ((u_i, e_i, S_i)_{i \in I}, V)$ . Its equilibrium concept is as follows:

**Definition 2** *A price and strategy collection,  $((p^*, q^*), [(x_i^*, z_i^*)]) \in (\mathbb{R}^L)^{S'} \times \mathbb{R}^J \times (\times_{i=1}^m B_i(S_i, p^*, q^*))$ , is an equilibrium of the economy  $\mathcal{E} = ((u_i, e_i, S_i)_{i \in I}, V)$  if the following Conditions hold:*

- (a)  $\forall i \in I, x_i^* \in \arg \max u_i(x)$  for  $(x, z) \in B_i(S_i, p^*, q^*)$ ;
- (b)  $\sum_{i=1}^m (x_i^* - e_i)[\underline{S}'] = 0$ ;
- (c)  $\sum_{i=1}^m z_i^* = 0$ .



*Remark 2* We have assumed, to simplify, that agents had the same anticipations of all spot prices. Indeed, all the model's definitions and results hold if agents have idiosyncratic price anticipations in the unrealizable states (i.e.,  $s \in S \setminus \underline{\mathbf{S}}$ ).

The above economy is called standard if it meets the following Assumptions:

**Assumption A1** (*non satiation in the numeraire in any state*):

$$\forall (i, x, s) \in I \times (\mathbb{R}_+^L)^{S'_i} \times S'_i, u_i(x + e_s) > u_i(x), \text{ where } e_s \in (\mathbb{R}_+^L)^{S'_i}, e_s[\{s\}] = e, e_s[S'_i \setminus \{s\}] = 0;$$

**Assumption A2** (*strong survival*):  $\forall i \in I, e_i \in (\mathbb{R}_{++}^L)^{S'_i}$ ;

**Assumption A3** (*continuity*):

$$\forall i \in I, \forall x \in (\mathbb{R}_+^L)^{S'_i}, \forall \varepsilon > 0, \exists \eta > 0 : [y \in (\mathbb{R}_+^L)^{S'_i} \text{ and } \|y - x\| < \eta] \implies |u_i(y) - u_i(x)| < \varepsilon;$$

**Assumption A4** (*strict quasi-concavity*):

$$\forall i \in I, \forall (x, y, \lambda) \in (\mathbb{R}_+^L)^{S'_i} \times (\mathbb{R}_+^L)^{S'_i} \times ]0, 1[, u_i(x + \lambda(y - x)) > \min(u_i(x), u_i(y)).$$

From Claim 1 in De Boisdeffre (2007), the existence of the above equilibrium requires  $(S_i)$  to be arbitrage-free (as assumed). That condition also insures existence:

**Theorem 1** *A standard economy,  $\mathcal{E} = ((u_i, e_i, S_i)_{i \in I}, V)$ , admits an equilibrium.*

### 3 The existence proof

The proof results from three Claims, which refer to the sets defined hereafter.

$$Z_i^o := \{z \in \mathbb{R}^J : V_s \cdot z = 0, \forall s \in S_i\} \text{ and } Z_i^{o\perp}, \text{ its orthogonal, for each } i \in I;$$

$$Z^o := \sum_{i=1}^m Z_i^o \text{ and } Z^{o\perp} = \cap_{i \in I} Z_i^{o\perp}, \text{ its orthogonal.}$$

We set as given  $\varepsilon \in ]0, \frac{1}{L}[$  and let:

$$\Delta := \{p \in (\mathbb{R}^L)^{S'} : \|p_s\| \leq 1, \forall s \in S', p_{\bar{s}}^L \geq \varepsilon, \forall (l, \bar{s}) \in \{1, \dots, L\} \times S \setminus \underline{\mathbf{S}}\};$$

$$Q := \{q \in Z^{o\perp} : \|q\| \leq 1\} \text{ and } \Pi := \Delta \times Q$$

$\Delta_\delta := \{p \in \Delta : p_s \cdot e \geq \delta, \forall s \in \underline{\mathbf{S}}\}$  and  $\Pi_\delta := \Delta_\delta \times Q$ , for every  $\delta \in ]0, \varepsilon[$ ;

$P := \{p \in \Delta : p_s \in P_s, \forall s \in \underline{\mathbf{S}}\}$ , where, for each  $s \in \underline{\mathbf{S}}$ ,

$$P_s := \{p_s \in \mathbb{R}^L, \|p_s\| = 1 : (\exists i \in I, \exists (x_i) \in \mathcal{A}, \text{ s.t. } \left[ \begin{array}{l} (y \in (\mathbb{R}_+^L)^{S'_i}, (y - x_i)[S'_i \setminus \{s\}] = 0 \text{ and } u_i(y) > u_i(x_i)) \\ \implies (p_s \cdot y_s \geq p_s \cdot x_{is} \geq p_s \cdot e_{is}) \end{array} \right] \}.$$

Finally, denoting by  $\mathbf{1}$  the vector of  $\mathbb{R}^{S'}$  whose components are all equal to one, we consider, for each  $i \in I$  and every  $(p, q) \in \Pi$ , the following strategy sets:

$$\bar{B}_i(p, q) := \{ (x, z) \in (\mathbb{R}_+^L)^{S'_i} \times Z_i^{\circ\perp} : p[S'_i] \square (x - e_i) \leq W(S_i, p, q)z + \mathbf{1}[S'_i] \};$$

$$\bar{\mathcal{A}}(p, q) := \{ [(x_i, z_i)] \in \times_{i=1}^m \bar{B}_i(p, q) : (x_i) \in \mathcal{A}, \sum_{i=1}^m z_i \in Z^{\circ} \}.$$

**Claim 1** *For the above sets, the following Assertions hold:*

(i)  $P$  is a compact set;

(ii)  $\exists \underline{\delta} \in ]0, \varepsilon[ : P \subset \Delta_{\underline{\delta}}$ ;

(iii)  $\exists \underline{r} > 0 : [(p, q) \in \Pi_{\underline{\delta}} \text{ and } [(x_i, z_i)] \in \bar{\mathcal{A}}(p, q)] \implies [\sum_{i=1}^m (\|x_i\| + \|z_i\|) < \underline{r}]$ ;

(iv)  $\exists \underline{r} > \underline{r} : [(p, q) \in \Pi, [(x_i, z_i)] \in \bar{\mathcal{A}}(p, q) \text{ and } \|z_i\| \leq \underline{r}, \forall i \in I] \implies [\sum_{i=1}^m (\|x_i\| + \|z_i\|) < \underline{r}]$ .

**Proof of Claim 1-**(i) Let  $s \in \underline{\mathbf{S}}$  and a converging sequence  $\{p^k\}_{k \in \mathbb{N}} := \{(p_s^k)_{s \in S'}\}_{k \in \mathbb{N}}$  of elements of  $P$  be given. Since  $\Delta$  is closed,  $p = \lim_{k \rightarrow \infty} p^k$  belongs to  $\Delta$  and satisfies  $p_s := \lim_{k \rightarrow \infty} p_s^k$ . We may assume there exist  $i \in I$  and a sequence,  $\{x^k\}_{k \in \mathbb{N}} := \{(x_i^k)\}_{k \in \mathbb{N}}$ , of elements of  $\mathcal{A}$ , converging to some  $x := (x_i)$  in the closure of  $\mathcal{A}$  in  $\times_{i=1}^m (\mathbb{R}_+ \cup \{+\infty\})^{L S'_i}$ , such that, for each  $k \in \mathbb{N}$ ,  $(p_s^k, i, x^k)$  satisfies the conditions of the definition of  $P_s$ . We let the reader check, as standard, from market clearance conditions, that the sequence,  $\{x^k[\underline{\mathbf{S}}']\}_{k \in \mathbb{N}} := \{(x_i^k[\underline{\mathbf{S}}'])\}_{k \in \mathbb{N}}$ , is bounded, hence,  $x[\underline{\mathbf{S}}'] := (x_i[\underline{\mathbf{S}}'])$  is finite.

For every  $k \in \mathbb{N}$ , let  $\tilde{x}^k := (\tilde{x}_i^k) \in \mathcal{A}$  be defined (for each  $i \in I$ ) by  $\tilde{x}_i^k[\{0, s\}] := x_i[\{0, s\}]$  and  $\tilde{x}_i^k[S_i \setminus \{s\}] := x_i^k[S_i \setminus \{s\}]$ . Then, the relations  $p_s^k \cdot (x_i^k - e_i)_s \geq 0$ , for every  $k \in \mathbb{N}$ , yield, in the limit,  $p_s \cdot (\tilde{x}_i^k - e_i)_s := p_s \cdot (x_i - e_i)_s \geq 0$ . We now show there exists  $k \in \mathbb{N}$ , such

that  $(p_s, i, \tilde{x}^k)$  satisfies the conditions of the definition of  $P_s$  (hence,  $p_s := \lim p_s^k \in P_s$ ). By contraposition, assume the contrary, i.e., for each  $k \in \mathbb{N}$ , there exists  $y^k \in (\mathbb{R}_+^L)^{S'_i}$ , such that  $y^k[S'_i \setminus \{s\}] = \tilde{x}_i^k[S'_i \setminus \{s\}]$ ,  $u_i(y^k) > u_i(\tilde{x}_i^k)$  (that is,  $v_i^s(x_{i0}, y_s^k) > v_i^s(x_{i0}, x_{is})$ ) and  $p_s \cdot (y^k - x_i)_s < 0$ . Then, given  $k \in \mathbb{N}$ , there exists (from Assumption  $A\mathcal{B}$  and separability)  $K \geq k$ , such that, for every  $k' \geq K$ ,  $u_i(y^k) > u_i(x_i^{k'})$ . The latter relations imply, by construction of each  $x^{k'}$  (for  $k' \geq K$ ),  $p_s^{k'} \cdot (y^k - x_i^{k'})_s \geq 0$ , hence, in the limit ( $k' \rightarrow \infty$ ),  $p_s \cdot (y^k - x_i)_s \geq 0$ , contradicting the inequality,  $p_s \cdot (y^k - x_i)_s < 0$ , assumed above. This contradiction proves that  $p_s \in P_s$ , hence,  $P_s$  and  $P$  are closed, therefore, compact.  $\square$

**Proof of Claim 1-(ii)** Let  $s \in \underline{\mathbf{S}}$  and  $p \in P$  be given. We prove, first, that  $p_s \cdot e > 0$ . Indeed, let  $(p_s, i, x) \in P_s \times I \times \mathcal{A}$  meet the conditions of the definition of  $P_s$ . From Assumption  $A\mathcal{B}$ , there exists  $a_i \in (\mathbb{R}_+^L)^{S'_i}$  such that,  $a_i[S'_i \setminus \{s\}] := 0$ , and  $p_s \cdot a_{is} < p_s \cdot e_{is} \leq p_s \cdot x_{is}$ . Then, for every  $n > 1$ , we let  $x_i^n := (\frac{1}{n}a_i + (1 - \frac{1}{n})x_i) \in (\mathbb{R}_+^L)^{S'_i}$ , which satisfies  $p_s \cdot (x_i^n - x_i)_s < 0$  by construction. Referring to Assumptions  $A1$ - $A\mathcal{B}$  and their notations, there exists  $n > 1$ , such that  $y := (x_i^n + (1 - \frac{1}{n})e_s)$  satisfies  $u(y) > u(x_i)$ , which implies,  $p_s \cdot x_{is} \leq p_s \cdot y_s = p_s \cdot (x_i^n + (1 - \frac{1}{n})e) < p_s \cdot x_{is} + (1 - \frac{1}{n})p_s \cdot e$ . Hence,  $p_s \cdot e > 0$ , that is,  $p_s \cdot e > 0$ , for every pair  $(p, s) \in P \times \underline{\mathbf{S}}$ . The mapping  $\varphi : P \times \underline{\mathbf{S}} \rightarrow \mathbb{R}_{++}$ , defined by  $\varphi(p, s) := p_s \cdot e$  is continuous and attains its minimum for some element  $(\underline{p}, \underline{s})$  on the compact set  $P \times \underline{\mathbf{S}}$ . Then,  $\delta = \varphi(\underline{p}, \underline{s})$  satisfies  $P \subset \Delta_\delta$ , which proves Claim 1-(ii).  $\square$

**Proof of Claim 1-(iii)-(iv)** The proofs of Assertions (iii) and (iv) are similar to those of Lemma 1, p. 266, in De Boisdeffre (2007), to which we refer the reader.  $\square$

We let  $r > 0$  and  $\underline{r} > 0$  be given, along Claim 1, and, for every  $(i, (p, q)) \in I \times \Pi$ :

$$X_i := \{x \in (\mathbb{R}_+^L)^{S'_i} : \|x\| \leq r\} \text{ and } Z_i := \{z \in Z_i^{o\perp} : \|z\| \leq \underline{r}\};$$

$$B_i(p, q) := B_i(S_i, p, q) \cap (X_i \times Z_i);$$

$$\mathcal{A}(p, q) := \{[(x_i, z_i)] \in \times_{i=1}^m B_i(p, q) : (x_i) \in \mathcal{A}, (\sum_{i=1}^m z_i) \in Z^o\}.$$

**Claim 2** *The economy,  $\mathcal{E} = ((u_i, e_i, S_i)_{i \in I}, V)$ , admits prices,  $(p^*, q^*) \in \Pi$ , such that  $\|p_s^*\| = 1$ , for each  $s \in \underline{\mathbf{S}}$ , and strategies,  $[(x_i^*, z_i)] \in \mathcal{A}(p^*, q^*)$ , such that, for each  $i \in I$ ,  $x_i^*$  is optimal in  $B_i(p^*, q^*)$ , that is,  $x_i^* \in \arg \max u_i(x)$  for  $(x, z) \in B_i(p^*, q^*)$ .*

**Proof** The proof is given in De Boisseffre (2007), under Claims 3 to 13, pp. 262-266. Indeed, for every pair  $(i, (p, q)) \in I \times \Pi$ , all latter Claims' proofs apply to the above sets  $X_i, Z_i, B_i(p, q), \mathcal{A}(p, q)$ , and subsequent definitions used in De Boisseffre (2007). Whenever information is asymmetric, admissible prices may also be restricted to any subset of  $\Pi$ , setting fixed the value,  $p[S \setminus \underline{\mathbf{S}}] \in (\mathbb{R}_{++}^L)^{S \setminus \underline{\mathbf{S}}}$ , of commodity prices.  $\square$

We, henceforth, set as given  $(p^*, q^*) \in \Pi$  and  $[(x_i^*, z_i)] \in \mathcal{A}(p^*, q^*)$ , along Claim 2, and let  $(z'_i) \in \times_{i=1}^m Z_i^o$  be such that  $\sum_{i=1}^m z_i = \sum_{i=1}^m z'_i$ . Then, for each  $i \in I$ , we let  $z_i^* = z_i - z'_i$  satisfy, by construction,  $W(S'_i, p^*, q^*)z_i^* = W(S'_i, p^*, q^*)z_i$  and  $\sum_{i=1}^m z_i^* = 0$ .

**Claim 3**  *$p^* \in \Delta_{\underline{\delta}}$ , hence,  $p_s^* \cdot e \geq \underline{\delta}$ ,  $\forall s \in S$ , and  $((p^*, q^*), [(x_i^*, z_i^*)])$ , is an equilibrium.*

**Proof** We show, first, that  $p^* \in P$ , i.e.,  $p_s^* \in P_s$ , for every  $s \in \underline{\mathbf{S}}$ . Indeed, from Claim 2,  $p^* \in \Delta^* := \{p \in \Delta : \|p^*[s]\| = 1, \forall s \in \underline{\mathbf{S}}\}$  and  $(x_i^*) \in \mathcal{A}$ . Let  $s \in \underline{\mathbf{S}}$  be given. From Claim 2, Assumption A1 and above, the relations  $V(p^*)_s \cdot z_i^* = V(p^*)_s \cdot z_i = p_s^* \cdot (x_i^* - e_i)_s$  hold, for each  $i \in I$ , with  $\sum_{i=1}^m z_i^* = 0$ . Thus, there exists  $i \in I$ , such that  $V(p^*)_s \cdot z_i^* \geq 0$ , and we let the reader check from Claim 2 and above that the triple  $(p_s^*, (x_i^*), i)$  meets the conditions of the definition of  $P_s$ . Thus,  $p^* \in P$  and, from Claims 1 & 2,  $p^* \in \Delta_{\underline{\delta}}$ , hence,  $p_s^* \cdot e \geq \underline{\delta}$  for each  $s \in S$ , and  $\sum_{i=1}^m (\|x_i^*\| + \|z_i\|) < \underline{r} < r$ .

From Claim 2 and above, the collection  $((p^*, q^*), [(x_i^*, z_i^*)])$  belongs to  $\Pi \times (\times_{i=1}^m B_i(S_i, p^*, q^*))$  and meets Conditions (b)-(c) of Definition 2 of equilibrium. Assume, by contraposition, that it does not meet Condition (a). Then, there exist  $i \in I$  and  $(x_i, \tilde{z}_i) \in B_i(S_i, p^*, q^*)$ , such that  $u_i(x_i) > u_i(x_i^*)$  and  $\tilde{z}_i \in Z_i^{0\perp}$ . From Assumptions A3-A4, the

above relations,  $\|x_i^*\| + \|z_i\| < \underline{r} < r$ , and the convexity of  $B_i(p^*, q^*)$ , we can take  $(x_i, \tilde{z}_i)$  close enough to  $(x_i^*, z_i)$ , so that  $(x_i, \tilde{z}_i) \in B_i(p^*, q^*)$ , which contradicts Claim 2.  $\square$

*Remark 3* The Assumption of separable utilities was necessary in Claim 1. This is because, with asymmetric information, the no-arbitrage condition suffices to bound attainable consumptions, transfers and strategies with nominal, but not with numeraire assets, due to a possible fall of the numeraire value in realizable states at clearing market prices. With symmetric information, the reader will readily check from the proofs above that separability is not required (since attainable allocations are bounded) and that all the paper's Claims hold if we replace, for each  $i \in I$ , the mapping,  $u_i$ , by a preference correspondence,  $P_i$ , on  $(\mathbb{R}_+^L)^{S'_i}$ , which is open, convex-valued and such that  $(\lambda y + (1 - \lambda)x) \in P_i(x)$ , whenever  $(\lambda, x, y) \in ]0, 1] \times (\mathbb{R}_+^L)^{S'_i} \times P_i(x)$ .

## References

- [1] Cass, D. Competitive equilibrium with incomplete financial markets. University of Pennsylvania. CARESS Working Paper No. 84-09 (1984).
- [2] Cornet, B., De Boisdeffre, L. Arbitrage and price revelation with asymmetric information and incomplete markets. J.Math Econ 38, 393-410 (2002).
- [3] Cornet, B., De Boisdeffre, L. Elimination of arbitrage states in asymmetric information models. Econ Theory 38, 287-293 (2009).
- [4] De Boisdeffre, L.: No-arbitrage equilibria with differential information: an existence proof. Econ Theory 31, 255-269 (2007).
- [5] Geanakoplos, J., Polemarchakis, H. Existence, regularity, and constrained suboptimality of competitive allocations when the asset market is incomplete. W. Haller et alii eds., Essays in honor of K. Arrow, Vol. 3, 65-95, Cambridge U.P. (1986).
- [6] Radner, R. Rational expectations equilibrium: Generic existence and the information revealed by prices. Econometrica 47, 655-678 (1979).